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ON THE THEORY OF THE THICKNESS OF WEAK SHOCK WAVES

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I. Introduction

The two problems often suggested in connection with the Burnett equations are: 1) the velocity of propagation of high frequency sound waves, and 2) the shock wave problem. In the Burnett equations higher order differential quotients appear, thus the application of Burnett equation to most problems calls for a knowledge of additional boundary conditions. The above mentioned two problems are the simplest because they are the two problems that do not involve boundary conditions. The first problem has been treated by Primakoff and Tsien and Schamberg, and recently we have considered especially the calculation of the velocity of sound in helium gas [1]. The problem of the thickness of one dimensional shock waves has been studied by both Becker[2] and Thomas [3] using Stokes-Navier equations as exact, though Thomas included in his paper a discussion as to the changes to be expected when the Burnett terms are taken into account. Becker started with the Stokes-Navier equations and treated the viscosity coefficient, μ, and the heat conduction coefficient, ν, as constants independent of the temperature. For a particular value of the ratio $f=\mu/c_v\nu$, namely $f=4\gamma/3$ where γ is the ratio of the specific heats c_p and c_v of the gas, he was able to solve the hydrodynamical equations exactly and obtained an expression for the velocity distribution. From this velocity distribution the shock thickness was calculated. His results show that for the Mach number, M, slightly larger than unity the shock wave thickness, t, is about a few mean free paths, while for large Mach numbers, say $M\sim35$, t is of the order of 10^{-7} cm, or of the order of the mean distance between the molecules.

(1) C. S. Wang Chang On the Dispersion of Sound in Helium. APL/JHU CM-467
(2) R. Becker, Zeits. f. Phys. 8, 321-362, 1922.
(3) L. H. Thomas, J. Chem. Phys. 12. 449-453, 1944.

There are three objections to this theory:

1) The Stokes-Navier equations were treated as exact, which is against the spirit of the development theory of Enskog.

2) The viscosity and the heat conduction coefficients are certainly not constant especially for very strong shocks where the temperature change is tremendous.

3) His solution is for the one value of $f = 4\gamma/3$. For diatomic gases with $\gamma = 7/5$, this value of f (1.87) is near enough to both the theoretical and the experimental value, so that the calculation seems to be applicable for air. But the Stokes-Navier equations are strictly true only for monoatomic gases where there are no internal degrees of freedom. For polyatomic gases the relaxation time is usually so large that its effect masks the effect of the other two gas coefficients. Thus we do not expect results derived from the Stokes-Navier equations to hold good for air, or any polyatomic gases unless we are sure the relaxation time is very short.

Thomas' calculation removes the second objection. He included in his calculation the temperature dependence of μ and ν . An elastic sphere model was taken for the molecules, so that both μ and ν were taken to be proportional to the square root of the absolute temperature. The equations are still soluble for the same particular value of f. It was found that for $M = 2$ the thickness of the shock wave is about four times the mean free path, λ_m, (the mean free path at the place where the velocity gradient is a maximum). For the infinitely strong shock, t has the value of $1.74\,\lambda_m$. We will discuss his results and calculation further in II.

Recently Mott-Smith* made an essentially different approach to calculate the thickness of shock waves. Using neither the Stokes-Navier

*We are indebted to Commander H. M. Mott-Smith for showing us his manuscript on "Kinetic Theory Treatment of a Shock Wave."

equations nor the idea of Enskog's development, he started directly from the Boltzmann equation and the Boltzmann's equation of transport. For both monoatomic and diatomic gases, he found that the thickness of a shock wave is inversely proportional to (M - 1) for M - 1 small, and that for infinitely strong shock waves, t is of the order of a few mean free paths.

As a summary of the existing results on the calculation of the shock thickness, we have plotted in Figure 1, the values of λ/t against the Mach number M, where λ is the mean free path in the medium before the shock. Curve I is computed from Becker's theory, Curve II follows from the calculation of Thomas except that t is now measured in terms of λ instead of λ_m. Curve III is calculated by using the formulas of Mott-Smith*. Curves I and II have the same initial slope. For high M according to Becker λ/t increases almost linearly with M, whereas according to Thomas λ/t increases much slower with M and approaches a value of $\lambda/t = 1.02$ for $M = \infty$. Curve III has a smaller initial slope and the asymptotic value of λ/t is 0.55. One sees that there is quite a difference between the theories especially for strong shock waves. In our opinion, for strong shock waves the calculation of Mott-Smith is probably the most dependable, but his theory has to be modified since an exact theory should give the initial slope of Becker and Thomas.

We propose to develop a consistent theory for the shock thickness for weak shock waves. We shall follow the idea of the Enskog development and develop the shock thickness in a power series of M - 1. To the first approximation we shall use the Stokes-Navier equations and stop at the first power of M - 1. We shall see that we get the same results as Thomas if we develop Thomas' final

*We differ here from Mott-Smith in one respect. We found that in comformity with Becker's definition of the shock thickness $t = 4\lambda/B$ instead of Mott-Smith's $t = \lambda/B$. For B we have used Mott-Smith's value determined from the consideration of the transport of u^2, where u is the streaming velocity, and $\gamma = 7/5$.

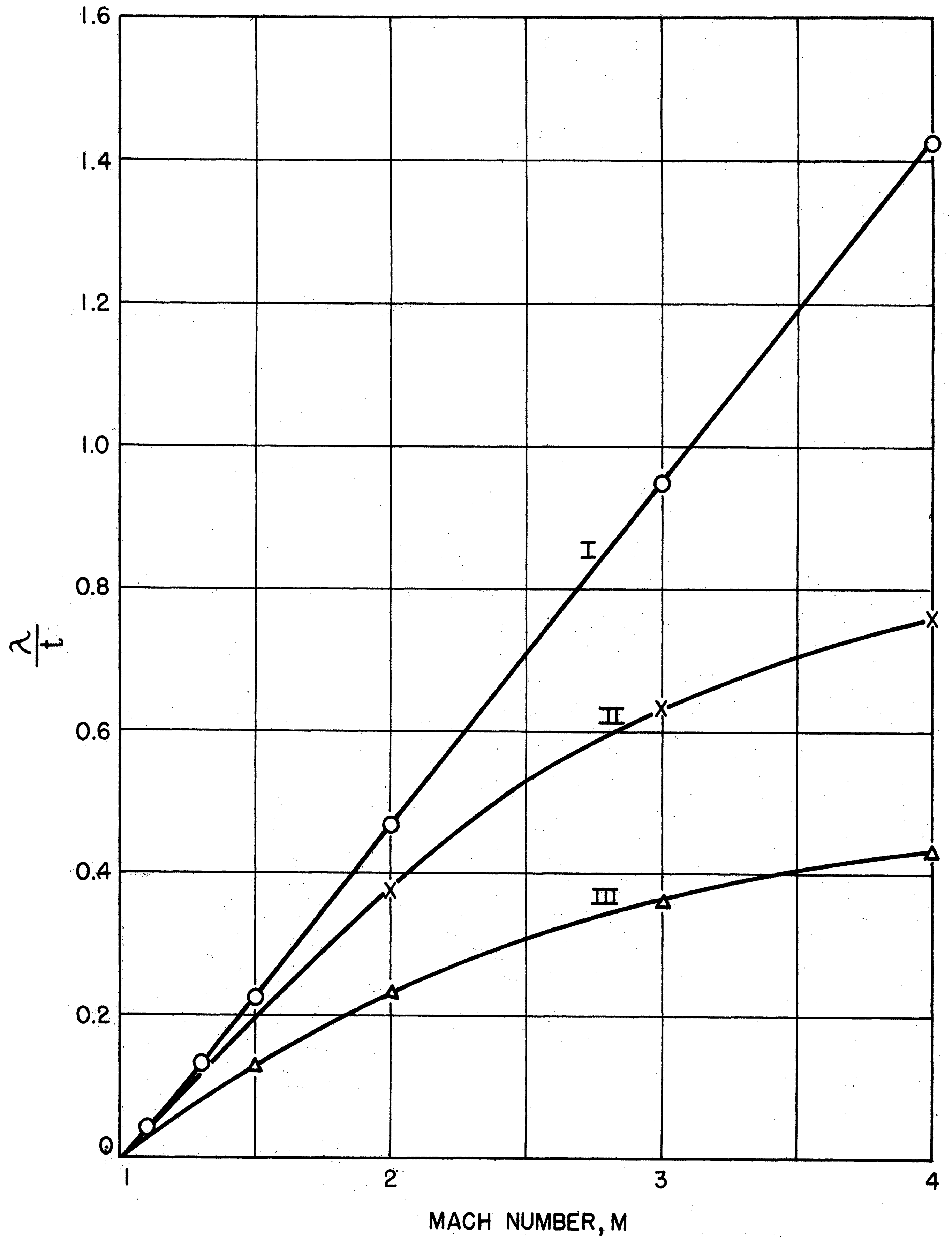

FIG. I
VARIATION OF THE THICKNESS OF SHOCK WAVES
WITH THE MACH NUMBER.

reduce 4 X 3 1/4

result also in power series in $M - 1$. To the second order we include the Burnett terms and carry the calculation to $(M - 1)^2$, etc.

In II we shall restrict ourselves to the first order calculation. We will give Thomas' result and compare it with ours. The calculation will be extended to the second and third order terms in III. The last section will be devoted to the application of the result to a monoatomic gas.

II. The Thickness of Shock Waves from the Stokes-Navier Equations.

We start by writing down the general one dimensional hydrodynamical equations for the steady state,

$$\frac{d}{dx}(\rho u) = 0$$

$$\frac{dp}{dx} + \rho u \frac{du}{dx} = 0$$

$$u\frac{dE}{dx} + E\frac{du}{dx} + \frac{dq}{dx} + p\frac{du}{dx} = 0$$

where ρ is the density, u is the streaming velocity, p is the pressure tensor ($p = p_{xx}$ in the present case) and q is the heat current density. E is the total energy $= nkT/(\gamma - 1)$. These equations can be integrated once immediately yielding.

$$\begin{aligned} \rho u &= \text{const.} = A \\ p + Au &= \text{const.} = B \\ uE + q + Bu - \frac{Au^2}{2} &= \text{const.} = C , \end{aligned} \tag{1}$$

where A, B, and C are constants. Given the velocity and the state of the gas before the shock, equations (1) determine the conditions after the shock uniquely. In the zeroth order (ideal hydrodynamical equations) one must put $p = nkT$, $q = 0$, and equations (1) give then the well known shock conditions. The shock wave appears as a discontinuity in the velocity, the temperature, the density, and the pressure. However, when the first order pressure tensor and heat flux vector are taken into account one finds that the shock waves are really not discontinuities but there are regions in which the velocity, the density, the temperature, and the pressure undergo large changes. In this order of approximation (Stokes-Navier equations)

$$p = nkT - \frac{4}{3}\mu \frac{du}{dx}$$

$$q = -\lambda \frac{dT}{dx}$$

equations (1) become now:

$$\rho u = A \qquad (2a)$$

$$\rho \frac{kT}{m} - \frac{4}{3}\mu \frac{du}{dx} + \rho u^2 = B \qquad (2b)$$

$$\frac{\gamma}{\gamma-1}\rho u \frac{kT}{m} - \frac{4}{3}\mu u \frac{du}{dx} + \frac{\rho u^3}{2} - \lambda \frac{dT}{dx} = C \qquad (2c)$$

The constants A, B, and C can be expressed in terms of the gas quantities before the shock (indicated by the subscript zero):

$$A = \rho_0 u_0$$

$$B = \rho_0 \frac{kT_0}{m} + \rho_0 u_0^2$$

$$C = \frac{\gamma}{\gamma-1}\rho_0 u_0 \frac{kT_0}{m} + \frac{\rho_0 u_0^3}{2}$$

Equations (2) will be expressed in dimensionless forms. For this purpose we introduce the following dimensionless variables,

$$\tau = \frac{T}{T_0}$$

$$v = \frac{u}{u_0}$$

$$b = \frac{x}{t}$$

We will also need the Mach number, M

$$M = \frac{u_0}{a_0}$$

where a_0 is the velocity of sound in the medium before the shock. For the temperature dependence of the viscosity coefficient we will first use the more general expression

$$\frac{\mu}{\mu_0} = \left(\frac{T}{T_0}\right)^{\frac{1}{s}}$$

where s is equal to one or two for the Maxwell molecules (points repelling with a force $\sim 1/r^5$), and the elastic spheres respectively. The specification of the value of s will be made later when we apply our results to definite molecular models. Further we write

$$\mu = \eta \rho_0 a_0 \lambda$$

η is the dimensionless constant $5\sqrt{2\pi}\ /\ (16\sqrt{\gamma})$, and λ is a general Maxwell mean free path in the medium before the shock, defined by

$$\lambda = \frac{\sqrt{2}}{n_0 \int_0^\infty g^7 e^{-g^2} Q_\eta(g)\, dg}$$

n_o being the number density and $Q_\eta(g)$ is the so-called transport cross-section. For elastic spheres

$$q_\eta = \frac{2\pi\sigma^2}{3}$$

where σ is the molecular diameter; λ is simply the ordinary Maxwell mean free path. Dividing the equations (2b) and (2c) by (2a) and expressing the resulting equations in dimensionless variables we obtain:

$$\tau - \frac{4}{3}\eta\gamma M\frac{\lambda}{t}\tau^{\frac{1}{5}} v\frac{dv}{db} + \gamma M^2 v(v-1) - v = 0 \tag{3a}$$

and

$$\begin{aligned} \tau - \frac{4}{3}\eta(\gamma-1)M\frac{\lambda}{t}\tau^{\frac{1}{5}} v\frac{dv}{db} &+ \frac{\eta f}{\gamma}M^{-1}\frac{\lambda}{t}\tau^{\frac{1}{5}}\frac{d\tau}{db} \\ &+ \frac{1}{2}(\gamma-1)M^2v^2 - \frac{1}{2}(\gamma-1)M^2 - 1 = 0 \end{aligned} \tag{3b}$$

For further calculations it is simpler to use equation (3a) and the equation obtained by subtracting (3b) from (3a). These are

$$\begin{aligned} &\tau - \frac{4}{3}\eta\gamma M\frac{\lambda}{t}\tau^{\frac{1}{5}} v\frac{dv}{db} + \gamma M^2 v(v-1) - v = 0 \\ &\frac{4}{3}\eta M\frac{\lambda}{t}\tau^{\frac{1}{5}} v\frac{dv}{db} - \frac{\eta f}{\gamma}M^{-1}\frac{\lambda}{t}\tau^{\frac{1}{5}}\frac{d\tau}{db} + \frac{1}{2}(\gamma-1)M^2v^2 \end{aligned} \tag{4a}$$

$$-\frac{1}{2}(\gamma-1)M^2 - \gamma M^2 v(v-1) + (v-1) = 0 \tag{4b}$$

Making now series expansions of all the quantities in power series in y where y stands for $M - 1$,

$$\begin{aligned} \tau &= 1 + \tau_1 y + \tau_2 y^2 + \cdots \\ v &= 1 + v_1 y + v_2 y^2 + \cdots \\ \frac{\lambda}{t} &= g_1 y + g_2 y^2 + \cdots \end{aligned} \tag{5}$$

The expansions for τ and v start from the constant term 1 while that for λ/t starts with the linear term $g_1 y$ because for y equals to zero there is no shock wave. Substituting (5) into (4a) and equating terms of equal powers of y one finds that the terms independent of y are identically zero while the first order terms yield:

$$\tau_1 = -(\gamma-1)\, v_1$$

From (4b) and the above equation one finds the differential equation for v_1

$$\eta g_1 \left[\frac{4}{3} + f - \frac{f}{\gamma}\right] \frac{dv_1}{db} = 2 v_1 \left(1 + \frac{\gamma+1}{4} v_1\right),$$

the solution of which is:

$$v_1 = \frac{A e^{2\alpha b}}{1 - \frac{\gamma+1}{4} A e^{2\alpha b}} \qquad (6)$$

where

$$\alpha = \frac{1}{\eta g_1 \left(\frac{4}{3} + f - \frac{f}{\gamma}\right)}$$

For $b = -\infty$, $v_1 = 0$, as it should since one is then in the region before the shock. The value of v_1 at $b = +\infty$ is given by the term $\sim y$ of the expansion of the velocity after the shock, v_f, determined by the shock conditions,

$$v_f = \frac{1}{\gamma+1}\left[\gamma - 1 + \frac{2}{M^2}\right] = 1 - \frac{2}{\gamma+1}\left[2y - 3y^2 - \cdots\right] \qquad (7)$$

i.e. $v_1 = -\frac{4}{\gamma+1}$ at $b = +\infty$. Equation (6) satisfies therefore both the boundary conditions. A is the integration constant which remains undetermined because the shock is not localized. If desired one can fix A by requiring that the maximum velocity gradient occurs at $b = 0$. This is, however, arbitrary and

~~and~~ it is not necessary at present.

Following the earlier workers we define the shock wave thickness by

$$t = \frac{u_f - u_o}{\left(\frac{du}{dx}\right)_{max.}} \tag{8}$$

In dimensionless notation this becomes:

$$\left(\frac{dv}{db}\right)_{max} = (v)_f - 1$$

where the right hand side is known and the left hand side involves the parameters g_1, g_2, etc. which measures the thickness. To the present order of approximation we have therefore,

$$\left(\frac{dv_1}{db}\right)_{max.} = -\frac{4}{\gamma+1} .$$

From the velocity distribution (6) it follows that the maximum slope occurs at the value of b satisfying the following equation

$$e^{2\alpha b} = -\frac{4}{(\gamma+1)A} .$$

Using this relation and equations (6) and (8), one can solve for α and g_1;

$$\alpha = 2$$

$$g_1 = \frac{8\sqrt{\gamma}}{5\sqrt{2\pi}} \frac{1}{\frac{4}{3} + f - \frac{f}{\gamma}} \tag{9}$$

We observed that g_1 is independent of s, the temperature dependence of μ and ν. This is not surprising since it is clear that the temperature dependence of μ and ν will contribute only to the second order terms. Up to this order the velocity distribution is symmetric and the velocity at the maximum velocity gradient is $-2y/(\gamma+1)$.

By eliminating x from (2a) and (2c) Becker arrived at a differential equation connecting T and u which can be solved for $f = 4\gamma/3 = 1.87$. This relation of T as a function of u was put back into (2b) and an expression for $\frac{du}{dx}$ was obtained which depends only on the unknown u. Making use of this last expression the calculation of the shock thickness is straight forward. Becker did not take into account of the temperature variation of the viscosity and the heat conduction coefficients, but the essential features of the calculation are not changed by the inclusion of these temperature dependences. Letting μ and γ be proportional to $T^{1/s}$, the generalized Becker's relation between $\frac{du}{dx}$ and u is, in our notation and with λ = the mean free path before the shock:

$$\frac{4}{3}\eta\lambda\frac{dv}{dx} = \frac{1}{\gamma M v}\,\frac{v^2 + v_f - v(1+v_f)}{\left(\frac{1+v_f}{1+\gamma M^2}\right)^{1-\frac{1}{s}}\left\{v_f - \left[\frac{\gamma M^2}{1+\gamma M^2}(1+v_f) - 1\right]v^2\right\}^{\frac{1}{s}}}$$

This reduces to Becker's result if $s = 0$, while for $s = 2$ it is the Thomas' formula. Restricting ourselves to $s = 2$, it can be deduced that $\frac{dv}{dx}$ will be a maximum when v is a solution of the cubic equation:

$$v^3(v_f+1)\left[\frac{\gamma M^2}{1+\gamma M^2}(1+v_f)-1\right] - v^2 v_f\left[2\frac{\gamma M^2}{1+\gamma M^2}(1+v_f)-1\right] + v_f^2 = 0$$

Thomas computed the value of λ/t for several values of the Mach number using (8) as the defining equation for t. His results are exact when the Stokes-Navier equations are considered to be exact. To compare with our results we make again series expansions in powers of y. We find

$$\frac{\lambda}{t} = \frac{6}{5\sqrt{2\pi\gamma}}\,y\left\{1 + \frac{5-3\gamma}{2(1+\gamma)}y + \frac{9\gamma^2-34\gamma+21}{4(\gamma+1)^2}y^2 + \cdots\right\}$$

One sees that, as to be expected, the coefficient of y is the same as g_1 if in equation (9) we put $f = 4\gamma/3$.

III. Higher Approximations.

To extend this calculation so that it is applicable to stronger shock waves, it is necessary to make the development to higher order terms. However, it is not correct merely to solve for v_2, v_3, etc. from equations (4). A theory consistent with Enskog's expansion idea must then include in the pressure tensor and the heat conduction vector the higher order terms resulting from the Enskog development. In this section the calculation will be extended two steps further.

A. The second approximation (the Burnett terms).

It is clear from the method of development that when one goes to the next approximation one needs only to take the linear second order Burnett terms into account. The non-linear terms contribute only to the third or higher order of approximation just as the temperature dependence of μ and λ does not contribute to the first order effect. The linear terms of $p_{xx}^{(2)}$ and $q_x^{(2)}$ are:

$$p_{xx}^{(2)} = \frac{2}{3}\omega_2 \frac{\mu^2}{\rho u}\frac{d^2u}{dx^2} + \frac{2}{3}\frac{\mu^2}{\rho T}(\omega_3 - \omega_2)\frac{d^2T}{dx^2}$$

$$q_x^{(2)} = \frac{2}{3}(\theta_2 + \theta_4)\frac{\mu^2}{\rho}\frac{d^2u}{dx^2}$$

where the ω's and the θ's are slowly varying functions of $T^{(4)}$. They are dimensionless quantities, and for the Maxwellian model they are pure numbers independent of T. The two equations taking the place of (4) are:

$$\tau - \frac{4}{3}\gamma\eta M\frac{\Lambda}{t}v\tau^{\frac{1}{s}}\frac{dv}{db} + \frac{2}{3}\omega_2\gamma\eta^2\frac{\Lambda^2}{t^2}v\tau^{\frac{2}{s}}\frac{d^2v}{db^2}$$
$$+ \frac{2}{3}(\omega_3-\omega_2)\gamma\eta^2\frac{\Lambda^2}{t^2}v^2\tau^{\frac{2-s}{s}}\frac{d^2\tau}{db^2} + \gamma M^2 v(v-1) - v = 0 \qquad (10a)$$

(4) "On the Transport Phenomena in Rarified Gases" by C. S. Wang Chang and G. E. Uhlenbeck. APL/JHU CM-443, Feb. 20, 1948.

and

$$
\begin{aligned}
&\frac{4}{3}\eta\frac{\lambda}{t}v\tau^{\frac{1}{5}}\frac{dv}{db} - \frac{f\eta}{\gamma}\frac{1}{M}\frac{\lambda}{t}\tau^{\frac{1}{5}}\frac{d\tau}{db} - \frac{2}{3}\omega_2\eta^2\frac{\lambda^2}{t^2}v\tau^{\frac{2}{5}}\frac{d^2v}{db^2} \\
&+\frac{2}{3}(\theta_2+\theta_4)\eta^2(\gamma-1)\frac{\lambda^2}{t^2}v\tau^{\frac{2}{5}}\frac{d^2v}{db^2} - \frac{2}{3}(\omega_3-\omega_2)\eta^2\frac{\lambda^2}{t^2}v^2\tau^{\frac{2-5}{5}}\frac{d^2\tau}{db^2} \qquad (10b) \\
&+\frac{v^2}{2}(\gamma-1)M^2 - \frac{1}{2}(\gamma-1)M^2 - \gamma M^2 v(v-1) + v-1 = 0
\end{aligned}
$$

From (10a) it follows that

$$\tau_2 = -(\gamma-1)v_2 - \gamma v_1(v_1+2) + \frac{4}{3}\eta\gamma a_1 v_1'$$

This equation together with (10b) gives the differential equation for v_2 in terms of known functions,

$$
\begin{aligned}
&v_2' - 2\alpha\left[1+\frac{\gamma+1}{2}v_1\right]v_2 = \alpha v_1 + \alpha(\gamma+1)v_1^2 \\
&-\left[\frac{g_2}{g_1} - \frac{\frac{4}{3}+f+\frac{f}{\gamma}}{\frac{4}{3}+f-\frac{f}{\gamma}}\right]v_1' + \left[\frac{\gamma-1}{5} - \frac{2f+\frac{4}{3}}{\frac{4}{3}+f-\frac{f}{\gamma}}\right]v_1 v_1' \qquad (11) \\
&+\frac{\frac{4}{3}f+\varepsilon}{\alpha\left(\frac{4}{3}+f-\frac{f}{\gamma}\right)^2}v_1''
\end{aligned}
$$

where ε is written for

$$\varepsilon = \frac{2}{3}\omega_2 - \frac{2}{3}(\omega_3-\omega_2)(\gamma-1) - \frac{2}{3}(\theta_2+\theta_4)(\gamma-1).$$

If one is only interested in the thickness of the shock waves, it will not be necessary to solve the differential equation (11). Since v_1 is given by (6), by inspection it is seen that the integrating factor of (11) is

$$e^{-2\alpha b}\left(1-\frac{\gamma+1}{4}Ae^{2\alpha b}\right)^2.$$

Thus

$$v_2 = \frac{e^{2\alpha b}}{\left(1-\frac{\gamma+1}{4}Ae^{2\alpha b}\right)^2}\int e^{-2\alpha b}\left(1-\frac{\gamma+1}{4}Ae^{2\alpha b}\right)^2 F\,db$$

with F standing for the right hand members of (11). But

$$\frac{d}{db}\frac{e^{2\alpha b}}{\left(1-\frac{\gamma+1}{4}Ae^{2\alpha b}\right)^2} = 0$$ at the maximum slope,

hence we have

$$(v_2')_{max.} = (F)_{max.},$$

the subscript max. means that all quantities are to be evaluated at b_m where v_2'' is zero. g_2 is then solved by setting

$$(v_2')_{max.} = +\frac{6}{\gamma+1}$$

as follows from equations (7) and (8).

The newly introduced Burnett terms enter in F only as coefficients of v_1'', but $(v_1'')_{max.}$ vanishes; hence the Burnett terms do not contribute to g_2, or the shock wave thickness to this order of approximation. The ratio g_2/g_1 is found to be:

$$\frac{g_2}{g_1} = \frac{3-\gamma}{2(\gamma+1)} - \frac{2(\gamma-1)}{5(\gamma+1)} \qquad (12)$$

For $s=2$, (12) agrees with the coefficient of the y^2 term of the result of Thomas. However, this fact should not be taken too seriously, since it is due to the definition (8) of the shock wave thickness t. If another definition for the thickness was chosen, the Burnett terms would have influenced the coefficient of y^2 since they have an influence on the velocity distribution inside the shock. The velocity distribution is obtained by integrating equation (11) and one finds:

$$
\begin{aligned}
v_2 = \frac{e^{2\alpha b}}{\left(1-\frac{\gamma+1}{4}Ae^{2\alpha b}\right)^2} \Bigg\{ & \alpha A b - \frac{1}{2}\frac{\gamma+1}{4}A^2 e^{2\alpha b} + \frac{A^2}{2}(\gamma+1)e^{2\alpha b} \\
& - \frac{\frac{4}{3}\left(\frac{g_2}{g_1}-1\right) + \frac{8}{3} + \frac{f}{\gamma}(\gamma-1)\left(\frac{g_2}{g_1}-1\right) + 2f}{\frac{4}{3}+f-\frac{f}{\gamma}}\, 2\alpha A b \\
& - \left(\frac{\gamma-1}{5} - \frac{2f+\frac{4}{3}}{\frac{4}{3}+f-\frac{f}{\gamma}}\right)\frac{4A}{\gamma+1}\log\left(1-\frac{\gamma+1}{4}Ae^{2\alpha b}\right) \qquad (13) \\
& + \frac{\frac{4}{3}f+\varepsilon}{\alpha\left(\frac{4}{3}+f-\frac{f}{\gamma}\right)^2}\left[4\alpha^2 A b - 4\alpha A \log\left(1-\frac{\gamma+1}{4}Ae^{2\alpha b}\right)\right] + C \Bigg\}
\end{aligned}
$$

where the integration constant is to be determined from the fact that at b_m v_2'' must vanish. In the last section we will plot the velocity distributions $v_1 y + v_2 y^2$ both from the consistent theory and from the Stokes-Navier equations when treated as exact.

B. The third approximation.

In this approximation not only should all the Burnett terms be taken into account but also the linear terms arising from $f^{(3)}$ are of importance. Dropping all the terms that will not contribute, we write

$$p_{xx}^{(2)} = \frac{2}{3}\omega_2 \frac{\mu^2}{pu}\frac{d^2u}{dx^2} + \frac{2}{3}(\omega_3-\omega_2)\frac{\mu^2}{pT}\frac{d^2T}{dx^2} + \frac{2}{3}(\omega_2-\omega_4)\frac{\mu^2}{puT}\frac{dT}{dx}\frac{du}{dx}$$

$$+ \frac{2}{3}(\omega_4+\omega_5)\frac{\mu^2}{pT^2}\left(\frac{dT}{dx}\right)^2 + \frac{2}{3}\left(\omega_1 - \frac{7}{3}\omega_2 + \frac{4}{9}\omega_6\right)\frac{\mu^2}{p}\left(\frac{du}{dx}\right)^2$$

$$- \frac{2}{3}\omega_2 \frac{\mu^2}{pu^2}\left(\frac{du}{dx}\right)^2$$

$$q_x^{(2)} = \frac{2}{3}(\theta_2+\theta_4)\frac{\mu^2}{p}\frac{d^2u}{dx^2} + \left(\theta_1 + \frac{8}{3}\theta_2 + \frac{2}{3}\theta_3 + \frac{2}{3}\theta_5\right)\frac{\mu^2}{pT}\frac{dT}{dx}\frac{du}{dx}$$

$$- \frac{2}{3}\theta_3 \frac{\mu^2}{pu}\left(\frac{du}{dx}\right)^2$$

$$p_{xx}^{(3)} = \frac{2}{3}(\omega_7+\omega_8)\frac{\mu^3}{p\rho}\frac{d^3u}{dx^3}$$

$$q_x^{(3)} = (\theta_6+\theta_7)\frac{\mu^3}{p^2T}\frac{d^3T}{dx^3} - \theta_6\frac{\mu^3}{p^2u}\frac{d^3u}{dx^3}$$

where ω_7, ω_8, θ_6, and θ_7, like the other ω's and θ's are the new slowly varying temperature dependent dimensionless quantities. For Maxwellian molecules they are simply numbers. The calculation is tedious and lengthy but the method is the same. We shall only present the final result in the most general form;

$$\frac{q_3}{q_1} = \frac{\gamma^2-6\gamma+1}{2(\gamma+1)^2} - \frac{\frac{8}{3}f}{(\gamma+1)B^2} + \frac{(2f+\frac{4}{3})^2}{(\gamma+1)^2B^2} - \frac{(\frac{4}{3}f+\varepsilon)^2}{B^4}$$

$$+ \frac{2(\gamma^2-6\gamma+1)}{5(\gamma+1)^2} + \frac{\frac{8}{3}f(\gamma-1)}{5(\gamma+1)B^2} + \frac{8}{35B} + \frac{3(\gamma-1)^2}{5^2(\gamma+1)^2}$$

$$-\frac{4}{3(\gamma+1)B^2}\Big[\gamma\omega_1-\frac{4}{3}\gamma\omega_2-\frac{(f+\frac{4}{3})\gamma(\gamma+1)}{B}\omega_2+\gamma(\gamma-1)\omega_3+\gamma(\gamma-1)\omega_4$$

$$+(\gamma-1)^2\omega_5+\frac{4}{9}\gamma\omega_6+\frac{3}{2}(\gamma-1)^2\theta_1+4(\gamma-1)^2\theta_2+\gamma(\gamma-1)\theta_3+(\gamma-1)^2\theta_5\Big]$$

$$-\frac{2}{B^3}\Big[\frac{2}{3}\gamma(\omega_7+\omega_8)+\gamma(\gamma-1)\theta_6+(\gamma-1)^2\theta_7\Big]$$

in which we have put B for $\frac{4}{3}+f-\frac{f}{\gamma}$.

IV. Application to Monoatomic Gases.

The calculation will now be applied to monoatomic gases. We shall limit ourselves to the treatment of the Maxwellian model, the molecules interacting with the r^{-5} force law. This choice was made not because this model is any better than other models from the physical point of view but because using this model the task of numerical computation will be much simplified. For Maxwell molecules the constants have the following values:

$f = 5/2$	$\gamma = 5/3$	$B = \frac{4}{3}+f-\frac{f}{\gamma} = 7/3$
$\omega_1 = 10/3$	$\theta_1 = 75/8$	$\omega_7 = -4/3$
$\omega_2 = 2$	$\theta_2 = -45/8$	$\omega_8 = 5/3$
$\omega_3 = 3$	$\theta_3 = -3$	$\theta_6 = -5/8$
$\omega_4 = 0$	$\theta_4 = 3$	$\theta_7 = 21/16$
$\omega_5 = 3$	$\theta_5 = 117/4$	
$\omega_6 = 8$		

The result for the shock thickness is summarized in Table I:

Table I Shock Thickness in Monoatomic Gases (Maxwell molecules)

$$\frac{\lambda}{t} = g_1 y \left(1 + \frac{g_2}{g_1} y + \frac{g_3}{g_1} y^2 + \cdots\right)$$

$$g_1 = \frac{8}{7\sqrt{2\pi\gamma}}$$

λ = the mean free path in the medium before the shock.

$y = M-1$, M = Mach number

	g_2/g_1	g_3/g_1
Stokes-Navier	-1/4	-0.349
Burnett	-1/4	-1.176
Third order equations	-1/4	-1.271

From this table one sees that the development of the thickness of shock waves in powers of (M - 1) converges very slowly and is therefore only applicable for Mach numbers which are only slightly bigger than one*. It should also be remembered that this calculation is valid only for monoatomic gases. The extension to diatomic gases is possible. If the effect of the relaxation time is of the same order of magnitude as the effects of μ and ν, one can

*In fact one sees already from the series expansion of the velocity after the shock, v_f, that one probably would not get fast convergent series for λ/t. v_f is given by

$$v_f = 1 - \frac{2}{\gamma+1}\left(2y - 3y^2 \cdots\right)$$

which is convergent only for $y < 1$ and even then the convergence is very slow. Since the consistent theory is certainly to yield the successive terms in the above formula for v_f in the successive approximations, it is seen that so far as finding the velocity distribution is concerned we can not expect to find better convergence. The shock thickness is derived from the velocity distribution, and hence the slow convergence of our result could have been anticipated.

take this fact into account to the first order of approximation by replacing $\frac{4}{3}\mu$ in the expression for $p_{xx}^{(1)}$ by $\frac{4}{3}\mu + \varkappa$ where $\varkappa$ is the "second viscosity coefficient"#. To higher orders not only the expressions for ω's and θ's will be changed but also new constants will enter. Because of the slow convergence of the development it seems at present hardly worthwhile to go into such calculations.

In III it was remarked that the fact that g_2/g_1 is not changed by taking into account of the Burnett terms is only accidental. There are certainly second order effects due to the Burnett terms, only these effects are not reflected in our calculation of the shock wave thickness because of the particular definition we adopted. To see one of the second order effects of the Burnett terms we have plotted in Fig. II (on following page) the curves with $v_1 y + v_2 y^2$ against b. Curve I is calculated from the Stokes-Navier equations while curve II is obtained from the Burnett equations. The horizontal dotted lines are the asymptotes of the value of $v_f = u_f/u_o$ up to the different orders of approximations as indicated. The plot is made for $y = M - 1 = 0.2$. One sees that the difference of the velocity distributions is appreciable even for such weak shock waves.

The author wishes to express her gratitude to Professor George E. Uhlenbeck for his interest in this work and for his helpful suggestions and discussions.

#See reference (4)

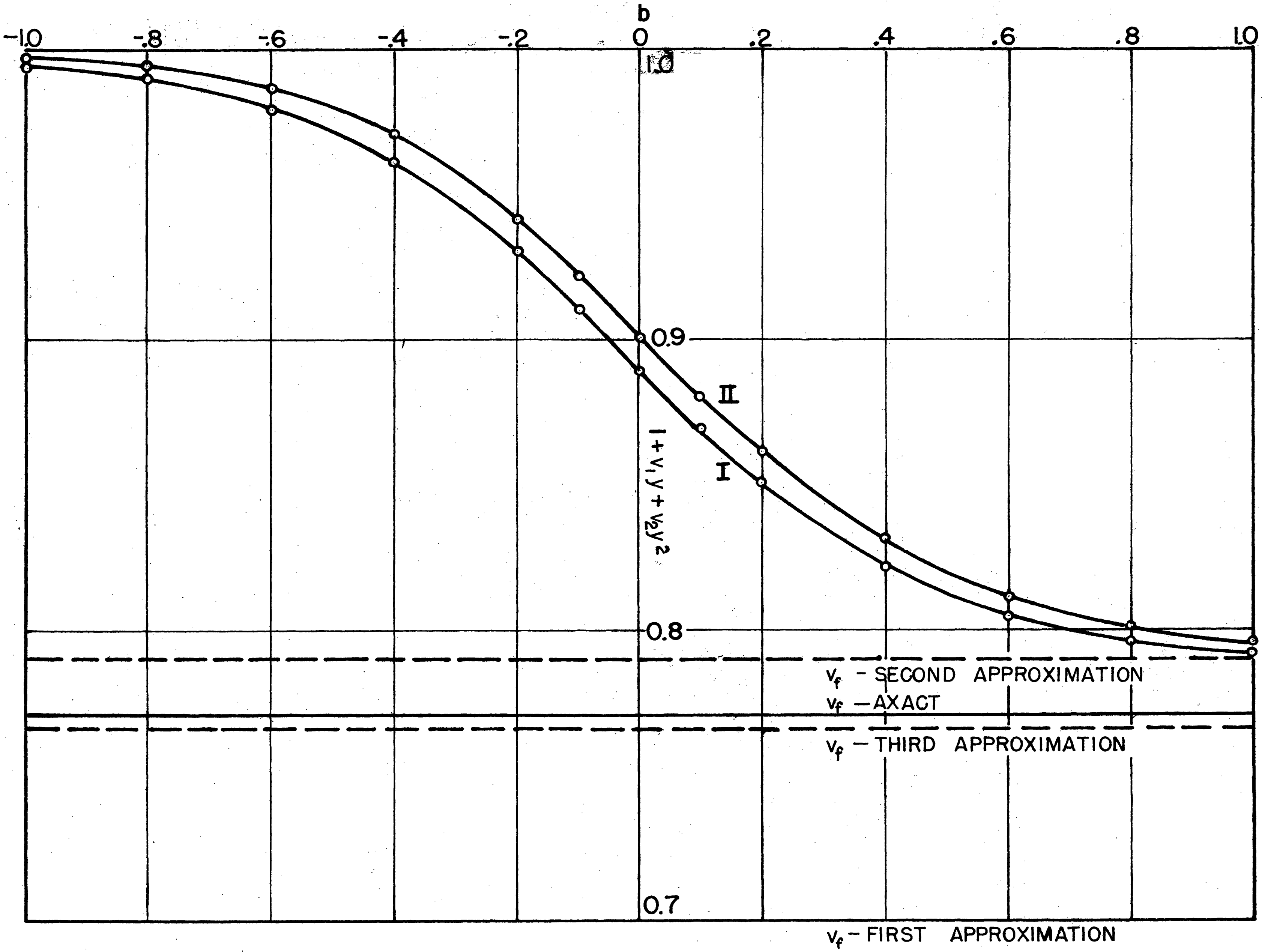

FIG II

SECOND ORDER VELOCITY DISTRIBUTION FOR MACH NUMBERS = 1.2 , y = 0.2

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