TRANSPORT PHENOMENA IN VERY DILUTE GASES

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I. Introduction

In the study of the transport phenomena in gases one usually considers only the two extreme cases; the so-called Clausius gas and the so-called Knudsen gas. The former corresponds to a comparatively high density (but density still low enough that only binary collisions need to be considered), or rather to a Knudsen number $M = d/\lambda$ much larger than unity, where d is the dimension of the container or the body in question and λ is the mean free path. In this case the heat flux is given by

$$\vec{q} \sim -\rho \bar{C} \lambda c_v \operatorname{grad} T$$

where ρ is the density of the gas, $\bar{C}$ the mean molecular velocity and c_v the specific heat at constant volume. Thus the heat flux is proportional to the temperature gradient and independent of the pressure. The heat flow between two parallel plates at different temperatures is proportional to the temperature difference, inversely proportional to the distance, d, between the plates and independent of the pressure. Analogously the pressure tensor is proportional to the velocity gradient and independent of the pressure.

The Knudsen gas is a gas of such low density (or so small value of M) that the collisions between the gas molecules can be neglected completely. In this case there is no temperature or velocity gradient in the proper sense. The heat flux between two parallel plates is proportional to the temperature difference between the plates and the pressure, but it is independent of d. The force on a moving plate has similar behavior, i.e. it is proportional to the velocity of the moving plate, the pressure, but is independent of d.

We are interested mainly in the transition region between the Knudsen gas regime and the Clausius gas regime. Mathematically the starting point is always the Boltzmann equation for the distribution function $f(x,y,z;c_x,c_y,c_z;t)$:

$$\frac{\partial f}{\partial t} + c_\alpha \frac{\partial f}{\partial x_\alpha} + X_\alpha \frac{\partial f}{\partial c_\alpha} = \int d\vec{c}_1 \int d\Omega\, g I(g,\theta)(f'f_1' - ff_1) \tag{1}$$

where the second term on the left is the time rate of change of f by streaming in the coordinate space and the third term is the time rate of change of f by streaming in the velocity space, X_α being the αth component of the acceleration due of the outside forces. The right hand side is the time rate of change of f by collision; $I(g, \theta)\, d\Omega$ is the differential collision cross-section.

For the Clausius regime a systematic successive approximation method is given by the well-known Hilbert Enskog development. One starts as the zeroth approximation by neglecting the streaming terms, in which case one gets the local Maxwell-Boltzmann distribution. The first approximation is obtained by putting $f = f^{(0)} + f^{(1)}$ and the streaming is taken into account in the perturbation of $f^{(1)}$. We have given a summary of this development and some calculations in a previous report[1].

In the present report we shall concern ourselves with the development starting from the Knudsen gas regime. A systematic successive approximation method will be the successive collision method which corresponds mathematically to the Neumann development for the solution of the integral equation (1). We start again with equation (1) and neglect, contrary to the Clausius' case, in the zeroth approximation the collision terms. To make the calculation easier we restrict ourselves to the two simple cases: the heat flux between two parallel plates at different temperatures, and the drag on one of two parallel plates when one of them is moving with a given velocity while the other is kept stationary. To further simplify the calculation we assume in the former case that the ratio of the temperature difference, ΔT, to the mean temperature, T, is much smaller than unity so that one can make a series development in the parameter $\Delta T/T$ and drop all terms of order higher than the first. In the latter case the assumption is that the velocity of the plate is much smaller than the mean molecular velocity and a similar development is made.

In the next section we shall derive the zeroth and the first order distribution function for a gas between two parallel plates at different temperatures. We shall then derive the expression for the heat flux up to the first approximation. The integrals will be evaluated as far as possible for a general r^{-s} force law between the molecules and

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(1) C. S. Wang Chang and G. E. Uhlenbeck, On the Transport Phenomena in Rarified Gases, APL/JHU CM-443 UMH-3-F, Feb. 20, 1948.

specialization to the Maxwell and the elastic sphere models shall be made at the end of the calculation. To shorten the writing, we shall give the calculation for the case of perfect accomodation, i.e. molecules which hit the solid are reemitted by the solid with no memory of their impinging speed and thus with a distribution corresponding to the temperature and velocity of the solid.

The third section will deal with analogous developments for the force on one of the two plates when it is moving with a given velocity. The results obtained in both sections II and III will be discussed in the last section. The effect of imperfect accomodation will also be mentioned there.

II HEAT FLUX BETWEEN PLATES AT DIFFERENT TEMPERATURES

We consider two plates parallel to each other and to the y-z plane at a distance d apart. The upper plate is kept at a temperature T_2 and the lower one at a temperature $T_1 > T_2$. Assuming perfect accomodation each plate will emit molecules with a velocity distribution corresponding to the temperature of the plate.

A. Zeroth approximation.

In this approximation, we neglect the collisions in between the plates. The velocity distribution function between the plates will then be given by:

$$f^{(0)} = A_1^{(0)} e^{-\frac{mc^2}{2kT_1}} \frac{1+\operatorname{sign} c_x}{2} + A_2^{(0)} e^{-\frac{mc^2}{2kT_2}} \frac{1-\operatorname{sign} c_x}{2} \qquad (2)$$

where

$$\operatorname{sign} c_x = \frac{c_x}{|c_x|} = \begin{cases} +1 & \text{when } c_x \text{ is positive} \\ -1 & \text{when } c_x \text{ is negative} \end{cases}$$

$A_1^{(0)}$ and $A_2^{(0)}$ are constants to be determined by the conditions that:

1) The total number of particles (nd) is given.

2) In the steady state, the total number passing through (3) unit area per second upwards and downwards are equal.

These conditions lead to the following expressions for $A_1^{(0)}$ and $A_2^{(0)}$;

$$A_1^{(0)} = 2n\left(\frac{m}{2\pi k T_1}\right)^{3/2} \frac{\sqrt{T_2}}{\sqrt{T_1}+\sqrt{T_2}}$$

$$A_2^{(0)} = 2n\left(\frac{m}{2\pi k T_2}\right)^{3/2} \frac{\sqrt{T_1}}{\sqrt{T_1}+\sqrt{T_2}}$$

With the distribution function (2), one finds easily the Knudsen expression for the heat flux:

$$q^{(0)} = \frac{m}{2}\int d\vec{c}\; c^2 c_x f^{(0)}$$

$$= \frac{m}{2}\Big[A_1^{(0)} \int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dc_x\, dc_y\, dc_z\; c_x c^2 e^{-\frac{mc^2}{2kT_1}}$$

$$+ A_2^{(0)} \int_{-\infty}^{0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dc_x\, dc_y\, dc_z\; c_x c^2 e^{-\frac{mc^2}{2kT_2}} \Big]$$

$$= 2nk\sqrt{\frac{2k}{\pi m}}\; \frac{\sqrt{T_1 T_2}}{\sqrt{T_1}+\sqrt{T_2}}\,(T_1 - T_2)$$

It is also simple to calculate the temperature and the pressure between the plates:

$$T = \sqrt{T_1 T_2}$$

and

$$p = nk\sqrt{T_1 T_2}$$

Thus there is a temperature jump at both the upper and the lower plate. All the above results are in agreement with those of Knudsen[2].

B. First approximation.

In the first approximation we take into account the collision terms due to the zeroth order distribution. The Boltzmann equation for the steady state with no outside forces and with temperature variation along the x-axis only is given by:

$$c_{x+}\frac{\partial f_+^{(1)}}{\partial x} = \int\cdots\int d\vec{c}_1\, d\Omega\, g I(g,\theta)\,(f'^{(0)}f_1'^{(0)} - f^{(0)}f_1^{(0)})$$
$$\equiv c_{x+}F_+(c_x, c_y, c_z) \qquad (4a)$$

where the integration is limited to all direct collisions where c_x is positive and all restituting collisions leading to a positive velocity component c_x. The integral in Equation (4a) we define as $c_{x+}F_+\ (c_x, c_y, c_z)$. There is an analogous expression for $f_-^{(1)}$. The F's are complicated functions of the velocities but because of the conservation theorems, they satisfy certain integral relationships:

a) Conservation of total number

$$\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty dc_x dc_y dc_z\, c_x\left[F_+(c_x\, c_y\, c_z) - F_-(-c_x\, c_y\, c_z)\right] \qquad (5a)$$

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(2) See, for instance, Lorentz: Lectures in Theoretical Physics, Vol. I. Kinetic Problems, Ch. III.

b) Conservation of momentum

$$\int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dc_x\,dc_y\,dc_z\,c_x^2\,[F_+ + F_-(-)] = 0 \qquad (5b_1)$$

$$\int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dc_x\,dc_y\,dc_z\,c_x\,c_j\,[F_+ - F_-(-)] = 0 \qquad j = 2,3 \quad (5b_2)$$

c) Conservation of energy

$$\int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dc_x\,dc_y\,dc_z\,c_x\,c^2\,[F_+ - F_-(-)] = 0 \quad . \qquad (5c)$$

To find the first order distribution function, one puts in the right hand side of Equation (4) the zeroth order distribution functions. Choosing the origin at the middle point between the plates, one may write:

$$f_+^{(1)} = A_1^{(1)} e^{-\frac{mc_+^2}{2kT_1}} + F_+(c_x\,c_y\,c_z)\,(x+\tfrac{d}{2}) \; .$$

Analogously

$$f_-^{(1)} = A_2^{(0)} e^{-\frac{mc_-^2}{2kT_2}} + F_-(c_x\,c_y\,c_z)\,(x-\tfrac{d}{2}) \; .$$

The A's are again determined by the two conditions (3). Using the conservation theorem (5a) one obtains:

$$A_1^{(1)} = A_1^{(0)} - d\left(\frac{m}{2\pi kT_1}\right)^{3/2} \frac{\sqrt{T_2}}{\sqrt{T_1}+\sqrt{T_2}}\left[I_0 + \sqrt{\frac{\pi m}{2kT_2}}\; I_1 \right]$$

$$A_2^{(1)} = A_2^{(0)} + d\left(\frac{m}{2\pi kT_2}\right)^{3/2} \frac{\sqrt{T_1}}{\sqrt{T_1}+\sqrt{T_2}}\left[-I_0 + \sqrt{\frac{\pi m}{2kT_1}}\; I_1 \right]$$

where

$$I_0 = \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty dc_x dc_y dc_z \left[F_+(c_x c_y c_z) - F_-(-c_x c_y c_z) \right]$$

$$I_1 = \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty dc_x dc_y dc_z \left[F_+(c_x c_y c_z) + F_-(-c_x c_y c_z) \right]$$

All macroscopic quantities, especially the density n and the temperature T become now linear functions of x.

One can calculate the heat flux to the first approximation:

$$q = \frac{m}{2} \int d\vec{c}\, c^2 c_x f^{(1)}$$

$$= 2nk \sqrt{\frac{2k}{\pi m}} \frac{\sqrt{T_1 T_2}}{\sqrt{T_1} + \sqrt{T_2}} (T_1 - T_2) \left\{ 1 + \frac{d}{2n} I_0 \right.$$

$$\left. - \frac{d}{2n} \sqrt{\frac{\pi m}{2k}} \frac{T_1^{3/2} + T_2^{3/2}}{\sqrt{T_1 T_2}(T_1 - T_2)} I_1 + \frac{d}{4k} \frac{m}{2n} \sqrt{\frac{\pi m}{2k}} \frac{\sqrt{T_1} + \sqrt{T_2}}{\sqrt{T_1 T_2}(T_1 - T_2)} I_2 \right\}$$

where

$$I_2 = \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty dc_x dc_y dc_z\, c^2 c_x \left[F_+(c_x c_y c_z) + F_-(-c_x c_y c_z) \right]$$

and the equation for the energy conservation (5c) is used. The heat flux, q, comes out independent of x as it should be. Each of the integrals I are eight fold integrals. In addition it is convenient to use the well-known integral representations for the sign functions, thus the multiplicity of the integral is increased to 10. To make the calculation easier we assume that the temperature difference $T_1 - T_2$ is small compared with the average temperature $T = (T_1 + T_2)/2$. Or calling $T_1 = T + \Delta T$, $T_2 = T - \Delta T$, we assume that $2\Delta T/T$ is much smaller than unity. Further we assume the molecules interact with each other with the r^{-s} force law which has the Maxwell molecules ($s = 5$) and the elastic spheres ($s = \infty$) as special cases. With these approximations and keeping terms of the first order in $\Delta T/T$, one has

$$f^{(0)} = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mc^2}{2kT}}\left[1+\left(\frac{mc^2}{2kT}-2\right)\frac{\Delta T}{T}\,\mathrm{sign}\,c_x\right] \quad (6)$$

$$q = 2nkT\sqrt{\frac{2kT}{\pi m}}\frac{\Delta T}{T}\left\{1+\frac{d}{2n}I_0-\frac{d}{8n}\sqrt{\frac{\pi m}{2kT}}\frac{T}{\Delta T}\left(4I_1-\frac{m}{kT}I_2\right)\right\}$$

$$\equiv q^{(0)}+q^{(1)} \quad (7)$$

where $q^{(0)}$ is the Knudsen expression and

$$q^{(1)} = q^{(0)}\left\{\frac{d}{2n}I_0-\frac{d}{8n}\sqrt{\frac{\pi m}{2kT}}\frac{T}{\Delta T}\left(4I_1-\frac{m}{kT}I_2\right)\right\}$$

It will be seen later that all the I's are proportional to $\Delta T/T$, thus we can drop the term containing I_o in this approximation. $q^{(1)}$ actually is also proportional to $\Delta T/T$.

For a force law $F = \chi r^{-s}$, the collision cross section can depend only on the force constant κ, the relative velocity g, the mass m, and the value of s. Out of these four quantities, the dimension of an area can be formed only in one way, namely, $(mg^2/\chi)^{-2/(s-1)}$. We shall write

$$I(g,\theta)\,d\Omega = g^{-\frac{4}{s-1}}\,F(\theta,\kappa,s)\sin\theta\,d\theta\,d\varepsilon$$

where θ is the angle through which the relative velocity has turned, and ε is the azimuthal angle the plane containing the relative velocities before and after collision makes with a fixed reference plane. F is now independent of the relative velocity. For elastic spheres $F = \sigma^2/4$, where σ is the molecular diameter. For Maxwell molecules $gI(g, \theta) = F(\theta, \varkappa, s)$ is independent of the relative velocity g.

As stated before, the contribution due to the term containing I_o is of higher order in $\Delta T/T$. The calculation of the quantity $q^{(1)}/q^{(0)}$ thus amounts to the evaluation of the integral $4I_1 - \frac{m}{kT} I_2$.

$$\frac{q^{(1)}}{q^{(0)}} = -\frac{d}{8n}\sqrt{\frac{\pi m}{2kT}}\frac{T}{\Delta T}\left\{4\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty dc_x dc_y dc_z\, c_x\left[F_+ + F_-(-)\right]\right.$$

$$\left. - \frac{m}{kT}\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty dc_x dc_y dc_z\, c_x c^2\left[F_+ + F_-(-)\right]\right\}$$

where F_+ is defined by (4a), and F_- by a similar expression.

$$\frac{q^{(1)}}{q^{(0)}} = -\frac{d}{8n}\sqrt{\frac{\pi m}{2kT}}\frac{T}{\Delta T}\left\{4\int_{-\infty}^{\infty}\cdots\int d\vec{c}\, d\vec{c}_1 \operatorname{sign} c_x \iint d\Omega\, g I(g,\theta)\left(f'^{(0)} f_1'^{(0)} - f^{(0)} f_1^{(0)}\right)\right.$$

$$\left. - \frac{m}{kT}\int_{-\infty}^{\infty}\cdots\int d\vec{c}\, d\vec{c}_1 \operatorname{sign} c_x\, c^2 \iint d\Omega\, g I(g,\theta)\left(f'^{(0)} f_1'^{(0)} - f^{(0)} f_1^{(0)}\right)\right\}.$$

Using (6), one has up to the order $\Delta T/T$

$$f'^{(0)} f_1'^{(0)} - f^{(0)} f_1^{(c)} = n^2\left(\frac{m}{2\pi kT}\right)^3 e^{-\frac{mc^2}{2kT} - \frac{mc_1^2}{2kT}}\,\Delta\left[\left(\frac{mc^2}{2kT} - 2\right)\frac{\Delta T}{T} \operatorname{sign} c_x\right]$$

where $\Delta[\phi] = \phi' + \phi_1' - \phi - \phi_1$. Since $f'^{(0)} f_1'^{(0)} - f^{(0)} f_1^{(0)}$ enters in I_0 as well as I_1 and I_2, we see that all the I's are proportional to $\Delta T/T$.

$$\frac{q^{(1)}}{q^{(0)}} = -\frac{\pi n d}{8}\left(\frac{m}{2\pi kT}\right)^{7/2}\int_{-\infty}^{\infty}\cdots\int d\vec{c}\, d\vec{c}_1 \iint d\Omega\, g\, I(g,\theta)\, \mathrm{sign}\, c_x \cdot$$

$$\cdot\; e^{-\frac{mc^2}{2kT}-\frac{mc_1^2}{2kT}}\left(4-\frac{mc^2}{kT}\right)\Delta\left[\left(\frac{mc^2}{2kT}-2\right)\mathrm{sign}\, c_x\right]$$

$$= -\frac{\pi n d}{8}\left(\frac{m}{2\pi kT}\right)^{7/2}\iiint_{-\infty}^{\infty} d\vec{c}\, \mathrm{sign}\, c_x\, e^{-\frac{mc^2}{2kT}}\left(4-\frac{mc^2}{kT}\right)\cdot$$

$$\cdot\iiint_{-\infty}^{\infty} d\vec{c}_1\, e^{-\frac{mc_1^2}{2kT}}\int_0^{\pi} d\theta \sin\theta\, g^{\frac{s-5}{s-1}} F(\theta,x,s)\int_0^{2\pi} d\varepsilon\, \Delta\left[\left(\frac{mc^2}{2kT}-2\right)\mathrm{sign}\, c_x\right].$$

We introduce now the center of gravity and the relative velocities ($\vec{G}$ and $\vec{g}$)

$$\vec{c} = \vec{G} - \frac{\vec{g}}{2} \qquad\qquad \vec{c}\,' = \vec{G} - \frac{\vec{g}\,'}{2}$$

$$\vec{c}_1 = \vec{G} + \frac{\vec{g}}{2} \qquad\qquad \vec{c}_1' = \vec{G} + \frac{\vec{g}\,'}{2}$$

and we specify the orientation by choosing the plane containing g and the x axis as the reference plane from which ε is measured, then[3]:

$$g'_x = g_x \cos\theta - \sqrt{g^2 - g_x^2}\,\sin\theta\cos\varepsilon$$

$$g'_y = g_y \cos\theta + \frac{g_x g_y}{\sqrt{g^2-g_x^2}}\sin\theta\cos\varepsilon - \frac{g g_z}{\sqrt{g^2-g_x^2}}\sin\theta\sin\varepsilon \qquad (9)$$

$$g'_z = g_z \cos\theta + \frac{g_x g_z}{\sqrt{g^2-g_x^2}}\sin\theta\cos\varepsilon + \frac{g g_y}{\sqrt{g^2-g_x^2}}\sin\theta\sin\varepsilon$$

$$c'^2 = G^2 - G\cdot g' + \frac{g^2}{4}$$

$\Delta\left[\left(\frac{mc^2}{2kT} - 2\right)\operatorname{sign} c_x\right]$ consists of the sum of four terms. It is clear that one needs to calculate one term only, say $\left(\frac{mc'^2}{2kT} - 2\right)\operatorname{sign} c'_x$ the other terms are obtainable by replacing θ by $\pi+\theta$, 0, and π respectively together with proper signs in front. We shall call these terms A, B, C, and D respectively, then:

$$\left(\frac{q^{(1)}}{q^{(0)}}\right)_A = -\frac{\pi n d}{8}\left(\frac{m}{2\pi kT}\right)^{7/2}\int\cdots\int d\vec{c}\,d\vec{c}_1\,\sin\theta\,d\theta\,d\varepsilon\,\operatorname{sign} c_x\, g^{\frac{s-5}{s-1}}\,F(\theta,\kappa,s)$$
$$e^{-\frac{mc^2}{2kT}-\frac{mc_1^2}{2kT}}\left(4-\frac{mc^2}{kT}\right)\left(\frac{mc'^2}{2kT}-2\right)\operatorname{sign} c'_x$$

Introducing the integral representation for the sign function:

$$\operatorname{sign} x = -\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{e^{itx}}{t}\,dt$$

making use of Equations (9), letting the polar angles of g be α, β, and

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(3) Jeans: The Dynamical Theory of Gases 4th Ed. p. 217.

using the cylindrical coordinates for G: G_x, G_r, and ϕ ; one finds:

$$\left(\frac{q^{(1)}}{q^{(0)}}\right)_A = \frac{nd}{8\pi}\left(\frac{m}{2\pi kT}\right)^{7/2}\int_0^{\pi} F(\theta,\kappa,s)\sin\theta\, d\theta \int_{-\infty}^{\infty}\frac{dv}{v}\int_{-\infty}^{\infty}\frac{dt}{t}\int_0^{\infty} dg\, g^{\frac{3s-7}{s-1}}\, e^{-\frac{mg^2}{4kT}}$$

$$\int_0^{\pi}\sin\alpha\, d\alpha \int_0^{2\pi} d\varepsilon \int_{-\infty}^{\infty} dG_x\, e^{-\frac{mG_x^2}{kT}} \int_0^{\infty} dG_r\, G_r\, e^{-\frac{mG_r^2}{kT}} \int_0^{2\pi} d\varphi \int_0^{2\pi} d\beta \cdot$$

$$e^{iv(G_x-\frac{g}{2})\cos\alpha + it(G_x - \frac{g}{2}\cos\alpha\cos\theta + \frac{g}{2}\sin\alpha\sin\theta\cos\varepsilon)} \cdot$$

$$\left[4-\frac{m}{kT}\left(G^2 - G\cdot g + \frac{g^2}{4}\right)\right]\left[\frac{m}{2kT}\left(G_x^2 + G_r^2 - G_x g\cos\theta\cos\alpha\right.\right.$$

$$\left.\left. + G_x g\sin\alpha\sin\theta\cos\varepsilon - G_y g_y' - G_z g_z' + \frac{g^2}{4}\right) - 2\right] .$$

Next we introduce dimensionless variables

$$G'' = \sqrt{\frac{m}{kT}}\; G \qquad\qquad g'' = \sqrt{\frac{m}{4kT}}\; g$$

$$v'' = \sqrt{\frac{kT}{4m}}\; v \qquad\qquad t'' = \sqrt{\frac{kT}{4m}}\; t$$

and then drop the primes. Integrations over β, φ, G_r, and G_x lead to

$$\frac{q^{(1)}}{q^{(0)}} = \frac{nd}{\pi^2}\, 2^{-\frac{3s+5}{2(s-1)}} \left(\frac{m}{kT}\right)^{-\frac{2}{s-1}} \int_0^{\pi} d\theta \sin\theta\, F(\theta,\pi,s) \int_{-\infty}^{\infty} \frac{dv}{v} \int_{-\infty}^{\infty} \frac{dt}{t}\, e^{-(t+v)^2} .$$

$$\cdot \int dg\, g^{\frac{3s-7}{s-1}} e^{-g^2} \int_0^{\pi} d\alpha \sin\alpha\, e^{-2ivg\cos\alpha - 2itg\cos\alpha\cos\theta}$$

$$\cdot \int_0^{\pi} d\varepsilon\, e^{2itg\sin\alpha\sin\theta\cos\varepsilon} \Big\{ \frac{31}{8} + \frac{3}{2}(t+v)^2 + \frac{1}{2}(t+v)^4 - \frac{5g^2}{2} + g^2\cos\theta \tag{10}$$

$$- g^2(t+v)^2 + \frac{g^4}{2} + \frac{3}{2} ig(t+v)\cos\alpha(1+\cos\theta) + ig(t+v)^3\cos\alpha(1+\cos\theta)$$

$$- 2g^2(t+v)^2\cos^2\alpha\cos\theta - ig^3(t+v)\cos\alpha(1+\cos\theta)$$

$$+ \sin\alpha\sin\theta\cos\varepsilon\Big[-\frac{3ig}{2}(t+v) - ig(t+v)^3 + 2g^2(t+v)^2\cos\alpha + ig^3(t+v)^3\Big\} .$$

For the integration over ε we have:

$$\int_0^{2\pi} d\varepsilon\, e^{iz\cos\varepsilon} = 2\pi J_0(z)$$

$$\int_0^{2\pi} d\varepsilon \cos\varepsilon\, e^{iz\cos\varepsilon} = 2\pi i J_1(z) \tag{11}$$

where J(z) is the Bessel Function of argument z.

The α integrations involved are of the forms*:

1) $$\int_0^{\pi} d\alpha \sin\alpha \,(A + B\cos\alpha)\, e^{-ia\cos\alpha}\, J_0(b\sin\alpha)$$

$$= \sqrt{2\pi}\left\{ \frac{A J_{\frac{1}{2}}(\sqrt{a^2+b^2})}{(a^2+b^2)^{1/4}} - \frac{iaB J_{3/2}(\sqrt{a^2+b^2})}{(a^2+b^2)^{3/4}} \right\}$$

2) $$\int_0^{\pi} d\alpha \sin\alpha \cos^2\alpha \, e^{-ia\cos\alpha}\, J_0(b\sin\alpha)$$

$$= \sqrt{2\pi}\left\{ \frac{J_{3/2}(\sqrt{a^2+b^2})}{(a^2+b^2)^{3/4}} - \frac{a^2 J_{5/2}(\sqrt{a^2+b^2})}{(a^2+b^2)^{5/4}} \right\} \qquad (12)$$

3) $$\int_0^{\frac{\pi}{2}} d\alpha \sin^2\alpha \, e^{-ia\cos\alpha}\, J_1(b\sin\alpha)$$

$$= \sqrt{2\pi}\, \frac{b J_{3/2}(\sqrt{a^2+b^2})}{(a^2+b^2)^{3/4}}$$

4) $$\int_0^{\frac{\pi}{2}} d\alpha \sin^2\alpha \cos\alpha \, e^{-ia\cos\alpha}\, J_1(b\sin\alpha)$$

$$= -i\sqrt{2\pi}\, \frac{ab J_{5/2}(\sqrt{a^2+b^2})}{(a^2+b^2)^{5/4}}$$

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* For proof, see Appendix I.

Putting all these in (10), one obtains

$$\left(\frac{q^{(1)}}{q^{(0)}}\right)_A = -\frac{nd}{\sqrt{\pi}}\left(\frac{4kT}{m}\right)^{-\frac{2}{s-1}}\int_0^{\pi} F(\theta,x,s)\sin\theta\, d\theta \int_{-\infty}^{\infty}\frac{dv}{v}\int_{-\infty}^{\infty}\frac{dt}{t}\, e^{-(t+v)^2}\cdot$$

$$\cdot\int dg\, g^{\frac{3s-7}{s-1}} e^{-g^2}\Big\{\Big[\tfrac{31}{8}+\tfrac{3}{2}(t+v)^2+\tfrac{1}{2}(t+v)^4-\tfrac{5}{2}g^2-g^2(t+v)^2+\cancel{g^4}+g^2\cos\theta\Big]\frac{J_{\frac{1}{2}}(2gz)}{\sqrt{2gz}}$$

$$+\Big[-2g^2(t+v)^2\cos\theta+3g^2(t+v)^2(1+\cos\theta)+2g^2(t+v)^4(1+\cos\theta)-2g^4(t+v)^2(1+\cos\theta)\Big]\frac{J_{3/2}(2gz)}{(2gz)^{3/2}}$$

$$+\,8g^4(t+v)^2\Big[(v+t)^2\cos\theta+vt(1-\cos\theta)^2\Big]\frac{J_{5/2}(2gz)}{(2gz)^{5/2}}\Big\}$$

where $z^2 = v^2+2vt\cos\theta+t^2$.

The g integrals are special cases of the Hankel's exponential integrals(4):

$$\int_0^{\infty} J_\nu(at)\, e^{-p^2t^2}\, t^{\mu-1}\, dt$$

$$= \frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}\mu)}{2p^{\mu}\Gamma(\nu+1)}\, e^{-a^2/4p^2}\left(\frac{a}{2p}\right)^{\nu} F\left(\frac{\nu}{2}-\frac{\mu}{2}+1;\ \nu+1;\ \frac{a^2}{4p^2}\right)$$

where F(a, b, x) is the confluent hypergeometric function defined by:

$$F(a,b,x) = 1+\frac{a}{b}x+\frac{a(a+1)}{b(b+1)}\frac{x^2}{2!}+\cdots$$

In our case t = g, p = 1, and $a = 2z = 2(v^2+2vt\cos\theta+t^2)^{1/2}$. There are six different sets of values for μ and ν. They are tabulated

(4) Watson, Theory of Bessel Functions, p. 393.

as follows:

ν	μ	$\frac{\nu+\mu}{2}$	$\frac{\nu-\mu}{2}+1$			$\nu+1$
			general in s	s=5	s=∞	
$\frac{1}{2}$	$\frac{3s-7}{s-1}+1-\frac{1}{2}$	$\frac{2s-4}{s-1}$	$-\frac{s-5}{2(s-1)}$	0	$-\frac{1}{2}$	$\frac{3}{2}$
$\frac{1}{2}$	$\frac{3s-7}{s-1}+3-\frac{1}{2}$	$\frac{3s-5}{s-1}$	$-\frac{3s-7}{2(s-1)}$	-1	$-\frac{3}{2}$	$\frac{3}{2}$
$\frac{1}{2}$	$\frac{3s-7}{s-1}+5-\frac{1}{2}$	$\frac{4s-6}{s-1}$	$-\frac{5s-9}{2(s-1)}$	-2	$-\frac{5}{2}$	$\frac{3}{2}$
$\frac{3}{2}$	$\frac{3s-7}{s-1}+3-\frac{3}{2}$	$\frac{3s-5}{s-1}$	$-\frac{s-5}{2(s-1)}$	0	$-\frac{1}{2}$	$\frac{5}{2}$
$\frac{3}{2}$	$\frac{3s-7}{s-1}+5-\frac{3}{2}$	$\frac{4s-6}{s-1}$	$-\frac{3s-7}{2(s-1)}$	-1	$-\frac{3}{2}$	$\frac{5}{2}$
$\frac{5}{2}$	$\frac{3s-7}{s-1}+5-\frac{5}{2}$	$\frac{4s-6}{s-1}$	$-\frac{s-5}{2(s-1)}$	0	$-\frac{1}{2}$	$\frac{7}{2}$

Since $F(a, b, x)$ breaks off if a is zero or a negative integer, it is seen that for Maxwell molecules where a has only zero or negative integral values, the expression for $q^{(1)}/q^{(0)}$ takes a particularly simple form. We shall see later that for elastic spheres where a has negative half integral and b has positive half integral values $q^{(1)}/q^{(0)}$ is also very simple.

After the g integration and writing $\alpha = -\frac{s-5}{2(s-1)}$, $\beta = \frac{3}{2}$

$$\left(\frac{q^{(1)}}{q^{(0)}}\right)_A = -\frac{nd}{8\sqrt{2\pi}}\frac{\Gamma(\beta-\alpha)}{\Gamma(\beta+2)}\left(\frac{4kT}{m}\right)^{-\frac{2}{s-1}}\int_0^{\pi} d\theta \sin\theta\, F(\theta, \kappa, s)\int_{-\infty}^{\infty}\frac{dv}{v}\int_{-\infty}^{\infty}\frac{dt}{t}\,\cdot$$

$$\cdot\, e^{-(t+v)^2-(v^2+2vt\cos\theta+t^2)}\Big\{4\beta(\beta+1)\left[\tfrac{31}{8}+\tfrac{3}{2}(t+v)^2+\tfrac{1}{2}(t+v)^4\right]F(\alpha,\beta,\zeta^2)$$

$$+4\beta(\beta+1)(\beta-\alpha)\left[-\tfrac{5}{2}+\cos\theta-(t+v)^2\right]F(\alpha-1,\beta,\zeta^2)+2\beta(\beta+1)(\beta-\alpha)(\beta+1-\alpha)F(\alpha-2,\beta,\zeta^2)$$

$$+2(\beta+1)(\beta-\alpha)\left[(t+v)^2(3+\cos\theta)+2(t+v)^4(1+\cos\theta)\right]F(\alpha,\beta+1,\zeta^2) \tag{13}$$

$$+2(\beta+1)(\beta-\alpha)(\beta+1-\alpha)\left[-2(t+v)^2(1+\cos\theta)\right]F(\alpha-1,\beta+1,\zeta^2)$$

$$+8(\beta-\alpha)(\beta+1-\alpha)(t+v)^2\left[(t+v)^2\cos\theta+tv(1+\cos\theta)^2\right]F(\alpha,\beta+2,\zeta^2)\Big\}$$

Since F is in general an infinite series in x, the integrations over v and t involve integrals of the form*:

$$\int_{-\infty}^{\infty}\frac{dv}{v}\int_{-\infty}^{\infty}\frac{dt}{t}\, e^{-(t+v)^2-(t^2+v^2+2vt\cos\theta)}(t+v)^{2m}(t^2+v^2+2vt\cos\theta)^n$$

$$= (-1)^{n+m+1}\cdot 2\pi\,\frac{\partial^n}{\partial q^n}\frac{\partial^m}{\partial p^m}\sin^{-1}\frac{p+q\cos\theta}{p+q}\Bigg|_{p=q=1} \tag{14}$$

$$\int_{-\infty}^{\infty}dv\int_{-\infty}^{\infty}dt\; e^{-(t+v)^2-(t^2+v^2+2vt\cos\theta)}(t+v)^2(t^2+v^2+2vt\cos\theta)^n$$

$$= (-1)^{n+1}\cdot\pi\,\frac{\partial^n}{\partial q^n}\frac{\partial}{\partial p}\frac{1}{\sqrt{(p+q)^2-(p+q\cos\theta)^2}}\Bigg|_{p=q=1}$$

*Derivation given in Appendix II.

Using these formulas, we find:

$$\left(\frac{q^{(1)}}{q^{(0)}}\right)_A = \frac{nd}{8}\sqrt{2\pi}\,\frac{\Gamma(\beta-\alpha)}{\Gamma(\beta+2)}\left(\frac{4kT}{m}\right)^{-\frac{2}{s-1}}\int_0^{\pi} d\theta \sin\theta\, F(\theta,\kappa,s)$$

$$\Big\{\Big[4\beta(\beta+1)\left(\frac{31}{8}-\frac{3}{2}\frac{\partial}{\partial p}+\frac{1}{2}\frac{\partial^2}{\partial p^2}\right)F\left(\alpha,\beta,-\frac{\partial}{\partial q}\right)$$

$$+4\beta(\beta+1)(\beta-\alpha)\left(-\frac{5}{2}+\cos\theta+\frac{\partial}{\partial p}\right)F\left(\alpha-1,\beta,-\frac{\partial}{\partial q}\right)+2\beta(\beta+1)(\beta-\alpha)(\beta+1-\alpha)F\left(\alpha-2,\beta,-\frac{\partial}{\partial q}\right)$$

$$+2(\beta+1)(\beta-\alpha)\left[-(3+\cos\theta)\frac{\partial}{\partial p}+2(1+\cos\theta)\frac{\partial^2}{\partial p^2}\right]F\left(\alpha,\beta+1,-\frac{\partial}{\partial q}\right) \qquad (15)$$

$$+4(\beta+1)(\beta-\alpha)(\beta+1-\alpha)(1+\cos\theta)\frac{\partial}{\partial p}F\left(\alpha-1,\beta+1,-\frac{\partial}{\partial q}\right)$$

$$+8(\beta-\alpha)(\beta+1-\alpha)\cos\theta\,\frac{\partial^2}{\partial p^2}F\left(\alpha,\beta+2,-\frac{\partial}{\partial q}\right)\sin^{-1}\frac{p+q\cos\theta}{p+q}$$

$$+4(\beta-\alpha)(\beta+1-\alpha)(1-\cos\theta)^2\frac{\partial}{\partial p}F\left(\alpha,\beta+2,-\frac{\partial}{\partial q}\right)\frac{1}{\sqrt{(p+q)^2-(p+q\cos\theta)^2}}\Big\}_{p=q=1}$$

This is as far as one can go for a general r^{-s} force law.

<u>Specialization to Maxwell Molecules</u>:

$s = 5, \qquad \alpha = 0, \qquad \beta = 3/2.$

All the F's in Equation (16) become polynomials of at most three terms. Collecting terms, one finds.

$$\left(\frac{q^{(1)}}{q^{(0)}}\right)_A = \frac{nd}{2}\sqrt{\frac{\pi m}{2kT}}\int_0^{\pi} d\theta\sin\theta\, F(\theta,\kappa,s)\Big\{\Big[\left(2+\frac{3}{2}\cos\theta\right)+(1+2\cos\theta)\frac{\partial}{\partial p}+\left(\frac{3}{2}+3\cos\theta\right)\frac{\partial^2}{\partial p^2}$$

$$+\cos\theta\frac{\partial}{\partial q}+(2+\cos\theta)\frac{\partial^2}{\partial p\,\partial q}+\frac{1}{2}\frac{\partial^2}{\partial q^2}\Big]\sin^{-1}\frac{p+q\cos\theta}{p+q}+(1-\cos\theta)^2\frac{\partial}{\partial p}\frac{1}{\sqrt{(p+q)^2-(p+q\cos\theta)^2}}\Big\}_{p=q=1}$$

$$= +\frac{nd}{2}\sqrt{\frac{\pi m}{2kT}}\int_0^{\pi} d\theta\sin\theta\, F(\theta,\kappa,s)\left\{\left(2+\frac{3}{2}\cos\theta\right)\sin^{-1}\frac{1+\cos\theta}{2}-\frac{\sqrt{1-\cos\theta}\,(2+\cos\theta)(1+\cos\theta)}{2(3+\cos\theta)^{3/2}}\right\}$$

and

$$\frac{q^{(1)}}{q^{(0)}} = -\frac{nd}{8}\sqrt{\frac{\pi m}{2kT}}\int_0^{\pi} d\theta \sin\theta \, F(\theta, \kappa, s)$$

$$\left\{ 7\pi - 2(4-3\cos\theta)\sin\frac{1-\cos\theta}{2} - 2(4+3\cos\theta)\sin^{-1}\frac{1+\cos\theta}{2} \right. \tag{16}$$

$$\left. + \frac{2(2-\cos\theta)(1-\cos\theta)\sqrt{1+\cos\theta}}{(3-\cos\theta)^{3/2}} + \frac{2(2+\cos\theta)(1+\cos\theta)\sqrt{1-\cos\theta}}{(3+\cos\theta)^{3/2}} \right\}$$

<u>**Specialization to Elastic Sphere Model:**</u>

$s = \infty$, $\alpha = -1/2$, $\beta = 3/2$.

$F(\theta, \kappa, s) = \sigma^2/4$, independent of θ.

$$\left(\frac{q^{(1)}}{q^{(0)}}\right)_A = \frac{\sqrt{2}\, n d \sigma^2}{4}\int_0^{\pi} d\theta \sin\theta \left\{ \left[\left(\frac{31}{8} - \frac{3}{2}\frac{\partial}{\partial p} + \frac{1}{2}\frac{\partial^2}{\partial p^2}\right) F\left(-\frac{1}{2}, \frac{3}{2}, -\frac{\partial}{\partial q}\right) \right.\right.$$

$$+ 2\left(-\frac{5}{2} + \cos\theta + \frac{\partial}{\partial p}\right) F\left(-\frac{3}{2}, \frac{3}{2}, -\frac{\partial}{\partial q}\right) + 3F\left(-\frac{5}{2}, \frac{3}{2}, -\frac{\partial}{\partial q}\right) +$$

$$+ \frac{2}{3}\left\{-(3+\cos\theta)\frac{\partial}{\partial p} + 2(1+\cos\theta)\frac{\partial^2}{\partial p^2}\right\} F\left(-\frac{1}{2}, \frac{5}{2}, -\frac{\partial}{\partial q}\right) + \tag{17}$$

$$\left. + 4(1+\cos\theta)\frac{\partial}{\partial p} F\left(-\frac{3}{2}, \frac{5}{2}, -\frac{\partial}{\partial q}\right) + \frac{16}{5}\cos\theta \frac{\partial^2}{\partial p^2} F\left(-\frac{1}{2}, \frac{7}{2}, -\frac{\partial}{\partial q}\right) \right] \sin^{-1}\frac{p+q\cos\theta}{p+q}$$

$$\left. + \frac{8}{5}(1-\cos\theta)^2 \frac{\partial}{\partial p} F\left(-\frac{1}{2}, \frac{7}{2}, -\frac{\partial}{\partial q}\right) \frac{1}{\sqrt{(p+q)^2 - (p+q\cos\theta)^2}} \right\}_{p=q=1}$$

The θ integral can be done. But for doing this we will have to consider the terms A, B, C, and D separately. Term C is especially simple. For $\theta = 0$, $\cos\theta = 1$, and $\sin^{-1}\frac{p+q\cos\theta}{p+q} = \frac{\pi}{2}$,

$$\left(\frac{q^{(1)}}{q^{(0)}}\right)_C = -\frac{nd\sigma^2}{\sqrt{2}}\left[\frac{31}{8}\,\frac{\pi}{2} + 2\left(-\frac{3}{2}\right)\frac{\pi}{2} + 3\cdot\frac{\pi}{2}\right] = -\frac{31\,nd\,\pi\sigma^2}{16\sqrt{2}}$$

Term D is the next simple one. For this we put $\theta = \pi$. Instead of using Equation (17) it is simpler to write the last two terms in a slightly different form. The last term of equation (14) can in this case be written as one term instead of being separated into two:

$$8(\beta-\alpha)(\beta+1-\alpha)(t+v)^2\left[(t+v)^2\cos\theta + vt(1+\cos\theta)^2\right]F(\alpha,\beta+2,\zeta^2)$$

$$= -8(\beta-\alpha)(\beta+1-\alpha)(t+v)^2(t-v)^2\,F(\alpha,\beta+2,(t-v)^2)\ .$$

Thus

$$\left(\frac{q^{(1)}}{q^{(0)}}\right)_D = -\frac{nd\sigma^2}{\sqrt{2}}\left\{\left(\frac{31}{8} - \frac{3}{2}\frac{\partial}{\partial p} + \frac{1}{2}\frac{\partial^2}{\partial p^2}\right)F\left(-\frac{1}{2},\frac{3}{2},-\frac{\partial}{\partial q}\right) + \right.$$

$$+2\left(-\frac{1}{2}+\frac{\partial}{\partial p}\right)F\left(-\frac{3}{2},\frac{3}{2},-\frac{\partial}{\partial p}\right) + 3F\left(-\frac{5}{2},\frac{3}{2},-\frac{\partial}{\partial q}\right)$$

$$\left.-\frac{4}{3}\frac{\partial}{\partial p}F\left(-\frac{1}{2},\frac{5}{2},-\frac{\partial}{\partial q}\right) - \frac{16}{5}\frac{\partial^2}{\partial p\,\partial q}F\left(-\frac{1}{2},\frac{7}{2},-\frac{\partial}{\partial q}\right)\right\}\sin^{-1}\frac{p-q}{p+q}\,\Bigg|_{p=q=1}$$

By the use of the recurrence formulas of the confluent hypergeometric function one can express all the F's we have in terms of $F(3/2, 3/2, -\frac{\partial}{\partial q})$ and $F(1/2, 3/2, -\frac{\partial}{\partial q})$. The values of these functions when operating on $\sin^{-1}\frac{p+q\cos\theta}{p+q}$ are given as follows:*

*For derivation see Appendix III.

$$F\left(\tfrac{3}{2},\tfrac{3}{2},-\tfrac{\partial}{\partial q}\right)\sin^{-1}\frac{p-q}{p+q} = \sin^{-1}\frac{p-(q-1)}{p+(q-1)}$$
$$F\left(\tfrac{1}{2},\tfrac{3}{2},-\tfrac{\partial}{\partial q}\right)\sin^{-1}\frac{p-q}{p+q} = \frac{1}{2}\int_0^1 \frac{d\rho}{\sqrt{\rho}}\sin^{-1}\frac{p-(q-\rho)}{p+(q-\rho)} \tag{18}$$

Making use of these equations, one arrives at the final result:

$$\left(\frac{q^{(1)}}{q^{(0)}}\right)_D = \frac{\sqrt{2}\,nd\sigma^2\pi}{16}(3-2\sqrt{2})$$

The calculations for $(q^{(1)}/q^{(0)})_A$ and $(q^{(1)}/q^{(0)})_B$ follow the same line as that for $(q^{(1)}/q^{(0)})_D$ except that they are a little more complicated on account of the θ integral. We will omit the detail calculation but remark only that it is not difficult to see that $(q^{(1)}/q^{(0)})_A$ and $(q^{(1)}/q^{(0)})_B$ are equal. They are

$$\left(\frac{q^{(1)}}{q^{(0)}}\right)_A = \left(\frac{q^{(1)}}{q^{(0)}}\right)_B = \frac{\sqrt{2}\,nd\sigma^2\pi}{32}(15-6\sqrt{2})$$

giving:

$$\frac{q^{(1)}}{q^{(0)}} = \frac{\sqrt{2}\,nd\pi\sigma^2}{32}(5-16\sqrt{2}) \tag{19}$$

<u>III FORCE ON A MOVING PLATE</u>

In this case we consider two plates placed parallel to each other and to the y-z plane at a distance d apart. They are kept at the same temperature but the lower plate is moving with a velocity w along the z-direction. The zeroth order distribution is now:

$$f^{(0)} = n\left(\frac{m}{2\pi kT}\right)^{3/2}\left[e^{-\frac{mc^2}{2kT}}\,\frac{1+\operatorname{sign} c_x}{2} + e^{-\frac{m}{2kT}(c-\vec{k}w)^2}\,\frac{1-\operatorname{sign} c_x}{2}\right] \tag{20}$$

where $\vec{k}$ is a unit vector in the z-direction. The average x and y velocities are zero. The average z velocity is w/2. There is a jump of the velocity at both the upper and the lower plate. The force on the lower plate per unit area is

$$\begin{aligned} P_{xz}^{(0)} &= mA_1^{(0)}\int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dc_x\,dc_y\,dc_z\; c_x(c_z-w)\,e^{-\frac{mc^2}{2kT}} \\ &\quad + mA_2^{(0)}\int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dc_x\,dc_y\,dc_z\; c_x(c_z-w)\,e^{-\frac{m}{2kT}(c-\vec{k}w)} \\ &= -pw\sqrt{\frac{m}{2\pi kT}} \end{aligned} \tag{21}$$

which is the Knudsen expression.

The calculations for the first order distribution function and the force on the plate go along similar lines as in the previous section. We found:

$$f_+^{(1)} = A_1^{(1)} e^{-\frac{mc^2}{2kT}} + \left(x+\frac{d}{2}\right)F_+(c_x c_y c_z)$$

$$f_-^{(1)} = A_2^{(1)} e^{-\frac{m}{2kT}(c-\vec{k}w)^2} + \left(x-\frac{d}{2}\right)F_-(c_x c_y c_z)$$

where now

$$A_1^{(1)} = A_1^{(0)} - \frac{d}{2}\left(\frac{m}{2\pi kT}\right)^{3/2} I_0 - \frac{d}{2}\left(\frac{m}{2\pi kT}\right)^{3/2}\sqrt{\frac{\pi m}{2kT}}\, I_1$$

$$A_2^{(1)} = A_2^{(0)} - \frac{d}{2}\left(\frac{m}{2\pi kT}\right)^{3/2} I_0 + \frac{d}{2}\left(\frac{m}{2\pi kT}\right)^{3/2}\sqrt{\frac{\pi m}{2kT}}\, I_1 \quad .$$

The average streaming velocity in the z-direction is a linear function of x. There is still a jump of the velocity at both of the plates. Making use of the momentum conservation ($5b_2$), the force on the lower plate has the following expression:

$$p_{xz} = -pw\sqrt{\frac{m}{2\pi kT}}\left\{1 - \frac{d}{2n} I_0 + \frac{d}{2n}\sqrt{\frac{\pi m}{2kT}}\, I_1 - \frac{d}{nw}\sqrt{\frac{\pi m}{2kT}}\, I_3\right\}$$

where

$$I_3 = \int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[F_+(c_x c_y c_z) + F_-(-c_x c_y c_z)\right] c_x c_z \, dc_x \, dc_y \, dc_z$$

Thus

$$\frac{p_{xz}^{(1)}}{p_{xz}^{(0)}} = -\frac{d}{2n} I_0 + \frac{d}{2n}\sqrt{\frac{\pi m}{2kT}}\, I_1 - \frac{d}{nw}\sqrt{\frac{\pi m}{2kT}}\, I_3$$

As in the previous case, we now assume that w is small compared to the average molecular velocity, then making the development and keeping terms linear in w only,

$$f^{(0)} = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mc^2}{2kT}}\left[1 + \frac{m}{kT} w c_z \frac{1 - \operatorname{sign} c_x}{2}\right]$$

$$f'^{(0)} f_1'^{(0)} - f^{(0)} f_1^{(0)} = -n^2 w \frac{m}{2kT}\left(\frac{m}{2\pi kT}\right)^3 e^{-\frac{mc^2}{2kT} - \frac{mc_1^2}{2kT}} \Delta(c_z \operatorname{sign} c_x)$$

the term Δc_z is zero on account of the conservation of momentum. In this case the contribution to $p_{xz}^{(1)}/p_{xz}^{(0)}$ due to the first two terms, i.e. the terms with I_0 and I_1 are of the order $\sqrt{\frac{m}{kT}}\,W$ or higher while the last term givec a contribution independent of this parameter. Using the same notations A, B, C, and D, and to this order of approximation, one has:

$$\left(\frac{p_{xz}^{(1)}}{p_{xz}^{(0)}}\right)_A = -\frac{d}{nW}\sqrt{\frac{\pi m}{2kT}}\,(I_3)_A$$

$$= +nd\sqrt{\frac{\pi m}{2kT}}\left(\frac{m}{2kT}\right)\left(\frac{m}{2\pi kT}\right)^3 \iiint_{-\infty}^{\infty} d\vec{c}\,\mathrm{sign}\,c_x\, e^{-\frac{mc^2}{2kT}} \iiint_{-\infty}^{\infty} d\vec{c}_1\, e^{-\frac{mc_1^2}{2kT}} \cdot$$

$$\cdot \iint g^{\frac{s-5}{s-1}} F(\theta,\kappa,s)\sin\theta\, d\theta\, d\varepsilon\; c_z c_z'\,\mathrm{sign}\,c_x' \;.$$

Making the proper change of variables and introducing the integral representations of the sign function:

$$\left(\frac{p_{xz}^{(1)}}{p_{xz}^{(0)}}\right)_A = 2^{-\frac{s+7}{2(s+1)}}\,\frac{nd}{\pi^{9/2}}\left(\frac{m}{kT}\right)^{\frac{2}{s-1}} \int_0^{\pi} d\theta\sin\theta\, F(\theta,\kappa,s) \int_{-\infty}^{\infty}\frac{dv}{v}\int_{-\infty}^{\infty}\frac{dt}{t}\int_{-\infty}^{\infty} dg\, g^{\frac{3s-7}{s-1}} e^{-g^2} \cdot$$

$$\int_0^{\pi} d\alpha\sin\alpha \int_0^{2\pi} d\varepsilon \int_{-\infty}^{\infty} dG_x\, e^{-G_x^2}\int_0^{\infty} dG_r\, G_r\, e^{-G_r^2}\int_0^{2\pi} d\beta\int_0^{2\pi} d\varphi \;\cdot$$

$$\cdot\, e^{2it(G_x - g\cos\alpha) + 2iv(G_x - g\cos\alpha\cos\theta + g\sin\alpha\sin\theta\cos\varepsilon)}$$

$$\cdot\,(G_r\cos\varphi - g\sin\alpha\cos\beta)(G_r\cos\varphi - g\sin\alpha\cos\beta\cos\theta$$

$$- g\cos\alpha\cos\beta\sin\theta\cos\varepsilon - g\sin\beta\sin\theta\sin\varepsilon)\,.$$

When all the integrations except the last one are carried out, we find:

$$\left(\frac{p_{xz}^{(1)}}{p_{xz}^{(0)}}\right)_A = +2^{-\frac{s+7}{2(s-1)}} nd\sqrt{\pi}\left(\frac{kT}{m}\right)^{-\frac{2}{s-1}} \frac{\Gamma(\beta-\alpha)}{\Gamma(\beta+2)}\int_0^\pi d\theta \sin\theta\, F(\theta,\kappa,s)\cdot$$

$$\Big\{\Big[2\beta(\beta+1)F(\alpha,\beta,-\tfrac{\partial}{\partial q}) + 2\beta(\beta+1)(\beta-\alpha)\cos\theta\, F(\alpha-1,\beta,-\tfrac{\partial}{\partial q})$$

$$-(\beta+1)(\beta-\alpha)\cos\theta\, F(\alpha,\beta+1,-\tfrac{\partial}{\partial q}) \tag{22}$$

$$-2(\beta-\alpha)(\beta+1-\alpha)\cos\theta\frac{\partial}{\partial p}F(\alpha,\beta+2,-\tfrac{\partial}{\partial q})\Big]\sin^{-1}\frac{p+q\cos\theta}{p+q}$$

$$-(\beta-\alpha)(\beta+1-\alpha)(1-\cos\theta)^2 F(\alpha,\beta+2,-\tfrac{\partial}{\partial q})\frac{1}{\sqrt{(p+q)^2-(p+q\cos\theta)^2}}\Big\}$$

Equation (22) gives for Maxwell molecules,

$$\left(\frac{p_{xz}^{(1)}}{p_{xz}^{(0)}}\right) = -nd\sqrt{\frac{\pi m}{2kT}}\int_0^\pi d\theta\sin\theta\, F(\theta,\kappa,s)\Big\{\pi-(1-\cos\theta)\sin^{-1}\frac{1-\cos\theta}{2}$$

$$-(1+\cos\theta)\sin^{-1}\frac{1+\cos\theta}{2}+\frac{1}{2}\frac{\sin^2\theta}{\sqrt{4-(1-\cos\theta)^2}}+\frac{1}{2}\frac{\sin^2\theta}{\sqrt{4-(1+\cos\theta)^2}}\Big\} \tag{23}$$

For elastic sphere, we get again a number:

$$\frac{p_{xz}^{(1)}}{p_{xz}^{(0)}} = -\frac{\sqrt{2}\,nd\pi\sigma^2}{48}\,(5+26\sqrt{2}) \tag{24}$$

IV DISCUSSION

In sections II and III we have given the calculation and the results of the heat flux and the drag on a moving plate for both the Maxwell and the elastic sphere molecules. For elastic spheres both $q^{(1)}/q^{(0)}$ and $p^{(1)}/p^{(0)}$ are negative. This is to be expected. Let us consider the heat flux. When one takes some collisions between the plates into account; it means that some of the molecules that hit the upper plate will not be directly from the lower plate, but only from either of the plates after having suffered a collision with other gas molecules. They will have a velocity distribution corresponding to a lower temperature than that of the lower plate. Hence the heat carried by these molecules will be smaller. This is of course to be expected no matter what the molecular model is. It will probably be difficult to prove it in general. But for the Maxwell molecules, this can be shown even without the evaluation of the last integral.

We have anticipated at the beginning that our development parameter is d/λ . This is clear from physical grounds, but we have so far not put it in evidence in our expressions. For this purpose, we will have first to define a proper mean free path. In the classical theory, the total collision cross-section is infinite for all the r^{-s} models except the elastic sphere. Hence the ordinary definition of the mean free path has to be modified. In transport problem, it seems most natural to use the transport cross section:

$$Q_{Tr} = 2\pi \int_0^{\pi} (1-\cos^2\theta)\, I(g,\theta) \sin\theta \, d\theta \tag{25}$$

$$= 2\pi g^{-\frac{4}{s-1}} \int_0^{\pi} d\theta \sin^3\theta \, F(\theta, \kappa, s)$$

where the last integral has in general to be evaluated by numerical integration. We can then define a corresponding generalized Maxwell mean free path:

$$\lambda_{Tr} = \frac{C}{n\bar{Q}_{Tr}}$$

where

$$\bar{Q}_{Tr} = \frac{\int_0^\infty dg\, g^2 e^{-\frac{mg^2}{4kT}} Q_{Tr}}{\int_0^\infty dg\, g^2 e^{-\frac{mg^2}{4kT}}} \tag{26}$$

is an average transport cross section. The constant C we shall determine by requiring λ_{Tr} to reduce to the Maxwell mean free path for elastic spheres.

For elastic sphere

$$F(\theta,\kappa,s) = \frac{\sigma^2}{4}$$

$$Q_{Tr} = \frac{2\pi\sigma^2}{3} = \bar{Q}_{Tr}$$

$$\lambda_{Tr} = \frac{C}{n\bar{Q}_{Tr}} = \lambda = \frac{1}{\sqrt{2}\,\pi n\sigma^2} \tag{27}$$

$$C = \frac{\sqrt{2}}{3}$$

For a general force law, therefore:

$$\lambda_{Tr} = \frac{\sqrt{2}}{3n\bar{Q}_{Tr}}$$

$$= \frac{\Gamma(\frac{3}{2})}{3\sqrt{2}\,\pi\,\Gamma(\frac{3}{2}-\frac{2}{s-1})}\left(\frac{4kT}{m}\right)^{\frac{2}{s-1}} \frac{1}{n\int_0^\pi d\theta \sin^3\theta\, F(\theta,\kappa,s)} \tag{28}$$

One can in general write

$$q = q^{(0)} F_1\left(\frac{d}{\lambda_{Tr}}\right)$$

and

$$p = p^{(0)} F_2\left(\frac{d}{\lambda_{Tr}}\right) ,$$

For very dilute gases,

$$F_1 = 1 + \alpha_1\left(\frac{d}{\lambda_{Tr}}\right) + \alpha_2\left(\frac{d}{\lambda_{Tr}}\right)^2 + \cdots$$

$$F_2 = 1 + \beta_1\left(\frac{d}{\lambda_{Tr}}\right) + \beta_2\left(\frac{d}{\lambda_{Tr}}\right)^2 + \cdots$$

To our present order of approximation, i.e. $\Delta T/T$, $mw^2/2kT \ll 1$, it can be seen fairly easily that the α's and β's are independent of the temperature and the force constant but are dependent on the power law s. For elastic spheres, Equations (19) and (24) yield

$$\alpha_1(s=\infty) = \frac{5-16\sqrt{2}}{32} = -0.551$$

$$\beta_1(s=\infty) = -\frac{5+26\sqrt{2}}{48} = -0.870$$

For Maxwell molecules one has to do the θ integration numerically. We can write Equation (16) as

$$\frac{q^{(1)}}{q^{(0)}} = -\frac{nd}{8}\sqrt{\frac{\pi m}{2kT}} \int_0^{\pi} d\theta \sin\theta \, F(\theta, x, s)\, G_1(\theta)$$

where $G_1(\theta)$ stands for the long expression in the braces of (16). Since for Maxwell molecules,

$$\lambda_{Tr} = \frac{1}{6}\sqrt{\frac{2kT}{\pi m}} \frac{1}{n\int_0^\pi d\theta \sin^3\theta \, F(\theta,\kappa,s)}$$

one has,

$$\alpha_1 \frac{d}{\lambda_{Tr}} = \frac{q^{(1)}}{q^{(0)}} = -\frac{nd}{8}\sqrt{\frac{\pi m}{2kT}}\int_0^\pi d\theta \sin\theta \, F(\theta,\kappa,s)\, G_1(\theta)$$

giving

$$\alpha_1 = -\frac{1}{48}\frac{\int_0^\pi d\theta \sin\theta \, F(\theta,\kappa,s)\, G_1(\theta)}{\int_0^\pi d\theta \sin^3\theta \, F(\theta,\kappa,s)}$$

$$= -\frac{1}{48}\frac{\int_0^\pi d\theta \sin\theta \, F'(\theta,s)\, G_1(\theta)}{\int_0^\pi d\theta \sin^3\theta \, F'(\theta,s)}$$

where $F'(\theta,s) = \sqrt{\frac{m}{2\kappa}}\, F(\theta,\kappa,s)$. Analogously one finds from the expression for $p_{xz}^{(1)}\ p_{xz}^{(0)}$

$$\beta_1 = -\frac{1}{6}\frac{\int_0^\pi d\theta \sin\theta \, F'(\theta,s)\, G_2(\theta)}{\int_0^\pi d\theta \sin^3\theta \, F'(\theta,s)}$$

The integral $\int_0^\pi d\theta \sin^3\theta \, F'(\theta,s)$ is usually called $A_2(5)$. It has been evaluated by Maxwell[(5)] and others [(6)] and has the value 0.436. We have computed the integrals in the numerators. They give for α_1 and β_1 the values

$$\alpha_1\ (s = 5) = -0.536$$

$$\beta_1\ (s = 5) = -0.734$$

(5) Maxwell Collected Papers II, p. 26.

(6) Chapman and Cowling: The Mathematical Theory of Non-uniform Gases, p. 172.

For a Clausius gas the functions F_1 and F_2 must have the forms:

$$F_1 = \gamma(s,T)\,\frac{\lambda_{Tr}}{d}$$

$$F_2 = \delta(s,T)\,\frac{\lambda_{Tr}}{d}$$

It may be expected that γ and δ will also be independent of T. This is indeed so; from the Hilbert-Enskog-Chapman theory one finds [7]

$$q = -\frac{\frac{75}{4}\frac{k}{m}\sqrt{\frac{mkT}{\pi}}\left(\frac{2kT}{\kappa}\right)^{\frac{2}{s-1}}}{8\Gamma\left(4-\frac{2}{s-1}\right)A_2(s)}\;\frac{2\Delta T}{d} = q^{(0)}\gamma(s,T)\,\frac{\lambda_{Tr}}{d}$$

$$p = -\frac{5}{8}\,\frac{\sqrt{\frac{mkT}{\pi}}\left(\frac{2kT}{\kappa}\right)^{\frac{2}{s-1}}}{\Gamma\left(4-\frac{2}{s-1}\right)A_2(s)}\;\frac{W}{d} = p^{(0)}\delta(s,T)\,\frac{\lambda_{Tr}}{d}$$

giving for and :

$$\gamma = \frac{225\pi}{32}\,\frac{\Gamma\left(\frac{3}{2}-\frac{2}{s-1}\right)}{\Gamma\left(4-\frac{2}{s-1}\right)\Gamma\left(\frac{3}{2}\right)} = \begin{cases} \frac{75\pi}{64} = 3.68^{*} & \text{for elastic spheres} \\ \frac{15}{2} = 7.50 & \text{for Maxwell molecules} \end{cases}$$

$$\delta = \frac{15\pi}{4}\,\frac{\Gamma\left(\frac{3}{2}-\frac{2}{s-1}\right)}{\Gamma\left(4-\frac{2}{s-1}\right)\Gamma\left(\frac{3}{2}\right)} = \begin{cases} \frac{5\pi}{8} = 1.96^{*} & \text{for elastic spheres} \\ 4 & \text{for Maxwell molecules} \end{cases}$$

Thus we know the values of F_1, F_2, dF_1/dM, and dF_2/dM for $M = 0$, and

(7) Chapman and Cowling; the Mathematical Theory of Non-uniform Gases, p. 172.

* For elastic spheres, the exact value of γ is the value given above multiplied by 1.025, the value of δ is to be multiplied by 1.016, See Chapman and Cowling, p. 169.

the behavior of F_1 and F_2 for very large M. In figures 1 and 2 we have drawn the initial values and the initial slopes for F_1 and F_2 for both the Maxwell and the elastic sphere model (A-the straight lines). The curves B are the Clausius' expressions if considered as exact for all values of M. Assuming the functions F_1 and F_2 are monotonic the heavy dotted lines are drawn as plausible interpolations for F's as guided by the initial values and the asympototic behavior.

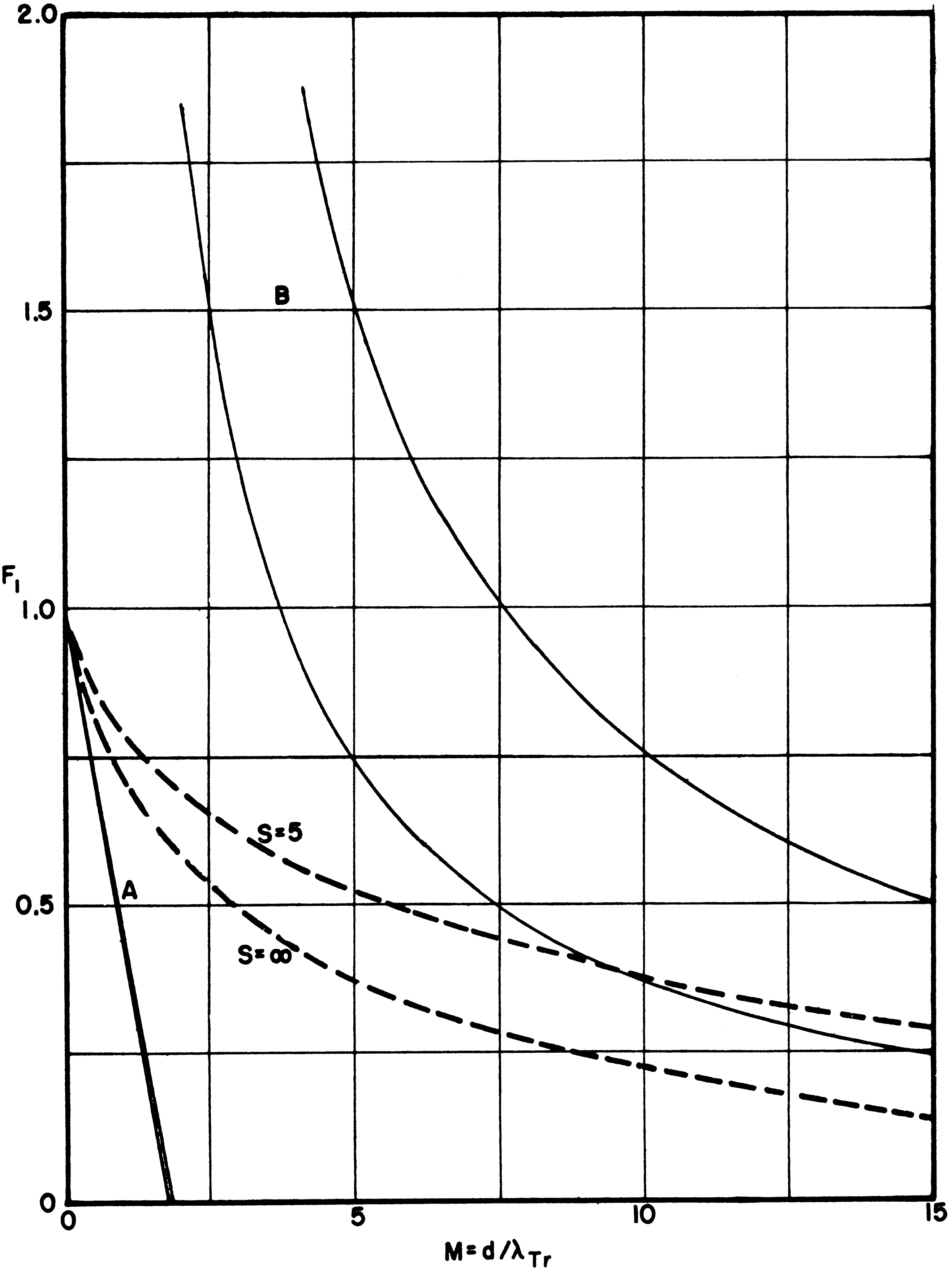

FIG. 1

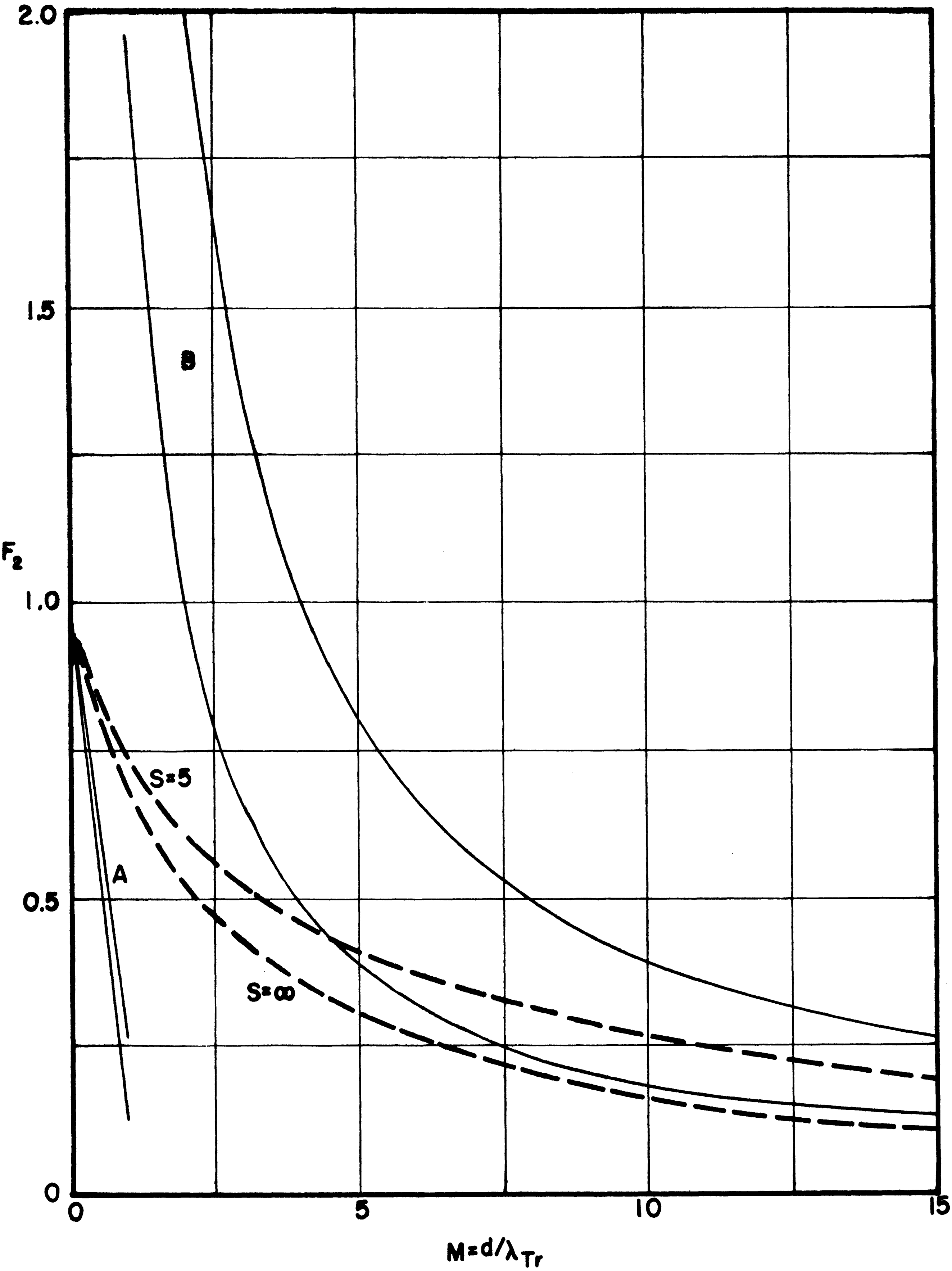

FIG. 2

All the above results are for the case of perfect accomodation. We have also made the calculation for a general value of a, where a is the fraction that is reemitted with the temperature of the solid; (1 - a) is then the fraction that retains its original distribution. For the heat flux problem, the zeroth order distribution function is then

$$f^{(0)} = \left[aA_{1a}^{(0)} e^{-\frac{mc^2}{2kT_1}} + (1-a) A_{2a}^{(0)} e^{-\frac{mc^2}{2kT_2}} \right] \frac{1+\operatorname{sign} c_x}{2}$$

$$+ \left[(1-a) A_{1a}^{(0)} e^{-\frac{mc^2}{2kT_1}} + aA_{2a}^{(0)} e^{-\frac{mc^2}{2kT_2}} \right] \frac{1-\operatorname{sign} c_x}{2}$$

The conditions (3) lead to $A_{1a}^{(0)}$ and $A_{2a}^{(0)}$ same as before, i.e. same as when a = 1. The zeroth order heat flux is however changed

$$q_a^{(0)} = (2a - 1)\, q^{(0)}$$

For perfect accomodation a = 1, $q_1^{(0)} = q^{(0)}$. When a = 1/2, it means that half of the molecules will assume the temperature of the wall they hit while the rest have their original temperature. The molecules going up and down will then have the same velocity distribution. Hence there will be no heat flux.

The first order heat flux is found to be:

$$q_a = q_a^{(0)} \left\{ 1 - \frac{d}{2\eta} \frac{1}{(2a-1)} \sqrt{\frac{\pi m}{2k}} \frac{T_1^{3/2} + T_2^{3/2}}{\sqrt{T_1 T_2}\,(T_1 - T_2)} I_1 + \frac{d}{2\eta} I_0 \right.$$

$$\left. + \frac{md}{8\eta k(2a-1)} \sqrt{\frac{\pi m}{2k}} \frac{\sqrt{T_1} + \sqrt{T_2}}{\sqrt{T_1 T_2}\,(T_1 - T_2)} I_2 \right\} .$$

The I's are to be calculated from $f^{(0)}$ which contains also a. To the first order in $\Delta T/T$:

$$f^{(0)} = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mc^2}{2kT}} \left\{ 1 + (2a-1)\left(\frac{mc^2}{2kT} - 2\right) \frac{\Delta T}{T} \operatorname{sign} c_x \right\}$$

and

$$f'^{(0)} f_1'^{(0)} - f^{(0)} f_1^{(0)} =$$

$$= n^2 \left(\frac{m}{2\pi kT}\right)^3 e^{-\frac{mc^2}{2kT} - \frac{mc_1^2}{2kT}} \Delta\left[\left(\frac{mc^2}{2kT} - 2\right) \operatorname{sign} c_x\right] \frac{\Delta T}{T} (2a-1) .$$

Hence all the I's are smaller than the corresponding values for perfect accomodation by a factor 2a - 1. I_0 remains zero. As a result α_1 is not changed, and neither is β_1.

As a final remark we would like to mention that if one is not interested in the general r^{-s} model but wants only the results for the Maxwell molecules or for the elastic spheres one can perform the ten integrations in slightly different and more natural orders. The manipulation is then considerably simplified and there is no need for so many elaborate integral formulas involving Bessel functions. In the case of the Maxwell molecules, since $gI(g,\theta)$ is independent of g and since there is cylindrical symmetry about the x-axis, instead of the spherical polar coordinates g, α, β, the cylindrical coordinates g_r, g_x, and β are more convenient. For elastic spheres, $I(g,\theta)$ is independent of θ. Instead of integrating according to the order indicated in the formula on page 13 it is more advantageous to integrate over θ right after the ε integration and to do the g integral last. In so doing the work is considerably less. On the other hand, since θ is integrated out earlier, one has to calculate the four terms A, B, C, and D (see p. 12) separately. The additional work, however does not amount to too much, since one can see easily that A = B and that the term C is extremely simple. We carried out the program mentioned in this paragraph first; later to have a check of the results obtained and to get expressions for more general force laws we did the calculations presented in sections II and III.

Appendix I Proof of formulas (12)

To prove Equation (12), one first separates the integral I

$$I = \int_0^{\pi} d\alpha \sin\alpha\, (A + B\cos\alpha)\, e^{-ia\cos\alpha}\, J_0(b\sin\alpha)$$

into the sum of two integrals, the integral from 0 to $\pi/2$ plus the integral from $\pi/2$ to π. In the second integral, writing $\alpha = \pi - \beta$; remembering that the cosine is even while the sine is odd, and J_0 is even in its argument, and making use of

$$\cos(a\cos\alpha) = \sqrt{\frac{\pi a\cos\alpha}{2}}\, J_{-\frac{1}{2}}(a\cos\alpha)$$

$$\sin(a\cos\alpha) = \sqrt{\frac{\pi a\cos\alpha}{2}}\, J_{\frac{1}{2}}(a\cos\alpha)$$

one obtains

$$I = \sqrt{2\pi a} \int_0^{\frac{\pi}{2}} d\alpha \sin\alpha\, J_0(b\sin\alpha) \left[A\sqrt{\cos\alpha}\, J_{-\frac{1}{2}}(a\cos\alpha) - iB\cos^{3/2}\alpha\, J_{\frac{1}{2}}(a\cos\alpha) \right]$$

Using the second Sonine finite integral (Watson, Bessel Functions, p. 376):

$$\int_0^{\frac{\pi}{2}} J_\mu(z\sin\theta) J_\nu(Z\cos\theta) \sin^{\mu+1}\theta \cos^{\nu+1}\theta\, d\theta = \frac{z^\mu Z^\nu J_{\mu+\nu+1}\left(\sqrt{Z^2+z^2}\right)}{(Z^2+z^2)^{\frac{1}{2}(\mu+\nu+1)}}$$

we have the first equation of (12). The proof for the other equations are similar.

Appendix II Proof of Equations (14)

$$I = \int_{-\infty}^{\infty} \frac{dv}{v} \int_{-\infty}^{\infty} \frac{dt}{t}\, e^{-(t+v)^2-(t^2+v^2+2vt\cos\theta)}\,(t+v)^{2m}\,(v^2+t^2+2vt\cos\theta)^n$$

$$= \int_{-\infty}^{\infty} \frac{dv}{v} \int_{-\infty}^{\infty} \frac{dt}{t}\, e^{-p(t+v)^2-q(t^2+v^2+2vt\cos\theta)}\,(t+v)^{2m}\,(v^2+t^2+2vt\cos\theta)^n \qquad p=q=1$$

$$= (-1)^{n+m} \frac{\partial^n}{\partial q^n}\frac{\partial^m}{\partial p^m} \int_{-\infty}^{\infty} \frac{dv}{v} \int_{-\infty}^{\infty} \frac{dt}{t}\, e^{-p(t+v)^2-q(v^2+t^2+2vt\cos\theta)} \qquad p=q=1$$

It is always to be remembered that in the integral representation of the sign function the principal value of the integral is meant.

$$I' \equiv P\int_{-\infty}^{\infty} \frac{dv}{v} \int_{-\infty}^{\infty} \frac{dt}{t}\, e^{-p(t+v)^2-q(v^2+t^2+2vt\cos\theta)}$$

$$= P\int_{-\infty}^{\infty} \frac{dv}{v} \int_{-\infty}^{\infty} \frac{dt}{t}\, e^{-t^2(p+q)-v^2(p+q)-2vt\underbrace{(p+q\cos\theta)}_{\gamma}}$$

$$\frac{dI'}{d\gamma} = -2P\int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} dt\, e^{-t^2(p+q)-v^2(p+q)-2\gamma vt}$$

$$= -\frac{2\pi}{\sqrt{(p+q)^2-\gamma^2}}$$

$$P\int_{-\infty}^{\infty} \frac{dv}{v} \int_{-\infty}^{\infty} \frac{dt}{t}\, e^{-(t^2+v^2)(p+q)-2\gamma vt} = -2\pi \sin^{-1}\frac{\gamma}{p+q} + C$$

when $\gamma = 0$, I' is zero; hence C = 0.

$$I = (-1)^{n+m+1}\, 2\pi \frac{\partial^n}{\partial q^n}\frac{\partial^m}{\partial p^m} \sin^{-1}\frac{p+q\cos\theta}{p+q}$$

Appendix III Derivation of Equations (18)

A.

$$F\left(\tfrac{3}{2}, \tfrac{3}{2}, -\tfrac{\partial}{\partial q}\right) \sin^{-1} \frac{p-q}{p+q} = \sin^{-1} \frac{p-(q-1)}{p+(q-1)}$$

$$F(a, b, x) = e^{x} F(a-b, b, -x)$$

$$F\left(\tfrac{3}{2}, \tfrac{3}{2}, -\tfrac{\partial}{\partial q}\right) \sin \frac{p-q}{p+q} = e^{-\frac{\partial}{\partial q}} F\left(0, \tfrac{3}{2}, \tfrac{\partial}{\partial q}\right) \sin^{-1} \frac{p-q}{p+q}$$

$$= e^{-\frac{\partial}{\partial q}} \sin^{-1} \frac{p-q}{p+q} .$$

But

$$e^{\alpha \frac{d}{dx}} f(x) = f(\alpha + x) .$$

Hence

B.

$$F\left(\tfrac{3}{2}, \tfrac{3}{2}, -\tfrac{\partial}{\partial q}\right) \sin^{-1} \frac{p-q}{p+q} = \sin^{-1} \frac{p-(q-1)}{p+(q-1)}$$

$$F\left(\tfrac{1}{2}, \tfrac{3}{2}, -\tfrac{\partial}{\partial q}\right) \sin^{-1} \frac{p-q}{p+q} = \frac{1}{2} \int_0^1 \frac{d\rho}{\sqrt{\rho}} \sin^{-1} \frac{p-(q-\rho)}{p+(q-\rho)}$$

$$F\left(\tfrac{1}{2}, \tfrac{3}{2}, -x\right) = 1 - \frac{x}{3} + \frac{x^2}{2!5} - \frac{x^3}{3!7} + \cdots$$

Comparing with the series

$$\int_0^x e^{-t^2} dt = x - \frac{x^3}{3} + \frac{x^5}{2!5} \cdots = x\left[1 - \frac{x^2}{3} + \frac{x^4}{2!5} \cdots\right]$$

one finds

$$F\left(\tfrac{1}{2}, \tfrac{3}{2}, -x\right) = \frac{1}{\sqrt{x}} \int_0^{\sqrt{x}} e^{-t^2} dt$$

Putting $t = \sqrt{x\varrho}$, $dt = \frac{1}{2}\sqrt{\frac{x}{\varrho}}\, d\varrho$

$t = 0$ $\varrho = 0$

$t = \sqrt{x}$ $\varrho = 1$

$$F\left(\tfrac{1}{2}, \tfrac{3}{2}, -\tfrac{\partial}{\partial q}\right) = \frac{1}{2}\int_0^1 e^{-\varrho\frac{\partial}{\partial q}} \frac{d\varrho}{\sqrt{\varrho}}$$

Thus

$$F\left(\tfrac{1}{2}, \tfrac{3}{2}, -\tfrac{\partial}{\partial q}\right) f(q) = \frac{1}{2}\int_0^1 \frac{d\varrho}{\sqrt{\varrho}}\, f(q-\varrho) .$$

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