# NEW CHARATERIZATIONS OF THE NO-AGING PROPERTY AND THE $l_1$-ISOTROPIC MODEL

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## Abstract

The no-aging property and the $\ell_1$-isotropic model it implies have been introduced to overcome certain shortcomings of the exponential model. However, its definition is abstract and not very useful for practitioners. This paper presents several additional characterizations of the no-aging property. Included are (1) characterizations that appropriately generalize the memoryless property and the constant-failure-rate property of the exponential, (2) behavioral characterizations based on fair bets, and (3) geometric characterizations of the survival and density function and differential-geometric characterizations based on tensor methods.

# 1  Introduction

In this paper, we characterize the no-aging property and the joint probability models it implies. The no-aging property was proposed by Barlow and Mendel in [2]. They consider a finite, exchangeable population of lifetimes and argue that the correct probabilistic model for aging is given by a Schur-concave joint survival function (for the theory of Schur concavity, see e.g. Marshall and Olkin [12]). No-aging is described by a Schur-constant survival function and these functions are also called $\ell_1$-isotropic. Their definition of no-aging is particularly appealing for a Bayesian who wishes to use the unconditional distribution of the lifetimes, rather than a distribution conditional on some parameter.

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The problem with the no-aging property is that its definition in terms of Schur-constancy can be far removed from the type of assessments a practitioner is comfortable in making.

This paper compiles eight characterizations, five of them new, of the no-aging property and the $\ell_1$-isotropic model in an attempt to make these concepts more useful to practitioners. The characterizations are compiled in the theorem in Section 2.

We have opted for characterizations that have a direct behavioral interpretation or an appealing geometric interpretation. The behavioral characterizations are particularly relevant to a Bayesian decision maker. They are stated in terms of fair bets. The geometric characterizations concern symmetries and invariances of the survival function and the density function. We also give several differential-geometric characterizations. The use of differential geometry is not very common in probability (see, however, Santalo [13]). However, these characterizations are very compact and provide perhaps the best pictures.

Two of the characterizations that in the spirit of the classical constant-failure-rate and memoryless properties of the exponential. These two properties have played a central role in characterizing the exponential model; see, e.g., [4-10, 15]. These are useful for seeing how the no-aging property and the $\ell_1$-isotropic model departs from the classical no-aging and the exponential model.

## 2 Characterizations

We state the various characterizations in the form of a theorem. An item-by-item discussion precedes the proof. We first introduce the main notation and we define the empirical failure frequency.

Consider a population

$$\vec{X} = (X_1, \ldots, X_N)$$

of lifetimes, i.e., a sequence of non-negative real-valued random variables. Let $\vec{X} \geq \vec{r}$ mean that $X_i \geq r_i$, for $i = 1, \ldots, N$. Write the joint survival function $F : \mathbb{R}_+^N \to [0, 1]$ as

$$F(\vec{r}) = \text{Prob}(\vec{X} \geq \vec{r}), \tag{1}$$

and the density function $f : \mathbb{R}_+^N \to \mathbb{R}$ as

$$f(\vec{x}) = \frac{(-1)^N \partial^N \bar{F}(\vec{x})}{\partial x_1 \cdots \partial x_N},$$

when it exists. It will be convenient, though, to use the notation $\bar{F}$ and $f$ also for the marginal survival and density functions of any subsequence of $\vec{x}$. This will be indicated by the arguments. Finally, let $\nabla$ denote the gradient operator and call

$$-\nabla \log \bar{F}$$

the hazard gradient (see also Marshall [11])

For the differential-geometric characterizations, let $\partial/\partial x_i$ be the infinitesimal vector in the $i$-th coordinate. The exterior derivative is denoted by $d$, the Lie derivative by $\mathcal{L}()$, the anti-symmetric product for tensors is denoted $\wedge$, and tensor saturation is denoted by $\lrcorner$. (See e.g. Burke [3]).

**Definition:** *For a population* $\vec{X} = (X_1, \ldots, X_N)$ *of lifetimes, the failure frequency is:*

$$\lambda_N(\vec{X}) = \frac{N}{\sum_{i=1}^N X_i}.$$

We use the following definition of the no-aging property originally proposed by Barlow and Mendel [2].

**Definition:** *A population* $\vec{X} = (X_1, \ldots, X_N)$ *has the no-aging property if* $\bar{F}$ *is a function of the failure frequency* $\lambda_X$ *alone.*

The following theorem shows the equivalence of eight characterizations of the no-aging property. The definition appears as Char. 3. The equivalence of Char. 3 and Char. 4 is elementary, and the equivalence of Char. 3 and Char. 5 is due to Barlow and Mendel [2]. The other characterizations are new. Char. 1 can be viewed as a specialization of Spizzichino's characterization of the Schur-concave class [14].

**Theorem:** *The following are equivalent:*

*1. $\bar{F}(x_i + h \mid \vec{X} \geq \vec{x}) = \bar{F}(x_j + h \mid \vec{X} \geq \vec{x})$ for all $\vec{x}$, and all $h > 0$.*

*2. The components of $-\nabla \log \bar{F}(\vec{x})$ are identically-equal functions of $\lambda_N$ alone.*

*3. $\bar{F}$ is a function of $\lambda_N$ alone (definition of the no-aging property).*

*4. $f$ is a function of $\lambda_N$ alone.*

*5. $\bar{F}(\vec{x})$ is an $\ell_1$-isotropic survival function, i.e, for all $n$, $0 < n < N$.*

$$F(x_1, \ldots, x_n) = \int \left[ 1 - \frac{\lambda_N}{N} \sum_{i=1}^{n} x_i \right]^{N-1} P(d\lambda_N).$$

*Let $W_{i,j} = \partial/\partial x_i - \partial/\partial x_j$ and $\mathbf{W}$ the wedge product of any $(N-1)$ linearly independent $W_{i,j}$. Let $\mathbf{p}$ be a probability $N$-form field on $\mathbb{R}_+^N$, and $\mathbf{p}_{|\lambda_N}$ the conditional probability $(N-1)$-form field on the simplex $\{\lambda_N(\vec{X}) = \lambda_N\}$.*

*6. $-d \log \bar{F} = -g(\lambda_N) d\lambda_N$, with $g : \mathbb{R}_+ \to \mathbb{R}_+$.*

*7. $\mathcal{L}_{W_{i,j}}(\mathbf{p}) = 0$ for all $i, j$.*

*8. $\mathbf{p}_{|\lambda_N} \lrcorner \mathbf{W}$ is a function of $\lambda_N$ alone.*

## Remarks:

Char.1 is the appropriate generalization of the memoryless property. From Char. 1 it follows that the lifetimes are exchangeable. However, Char. 1 does *not* imply that the $X_i$'s are independent. If we assume independence *in addition* to Char. 1, we find that:

$$\bar{F}(x_i + h \mid \vec{X} > \vec{x}) = \bar{F}(h), \quad \forall i,$$

which is the memoryless property that characterizes the exponential.

Char. 1 has a behavioral interpretation: 'Given two similar components which have not yet failed, one would bet the same amount for the same return on the event that either component fails during an additional increment of usage, regardless of their ages.'

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Char.2 is the appropriate generalization of the constant-failure-rate property. If
we assume independence in addition to Char. 2, we find that:

$$\frac{d}{dx_i} \log \bar{F}(x_i) = \text{constant}, \quad \forall i,$$

which characterizes i.i.d. exponential lifetimes (see Marshall [11]).

Char. 6 addresses several problems with the the use of the failure gradient
(see remarks below).

Char.3 is the definition of the no-aging property and the $\ell_1$-isotropic model. This
is Schur-constancy of $\bar{F}$: $\bar{F}$ can be moved anywhere along a simplex
$\{\lambda_N(\vec{X}) = \lambda_N\}$ without changing its value. (See further Barlow and
Mendel [2]).

Char. 3 is physically covariant under changes in the scales or units in which
lifetime is measured. These are here the admissible coordinate trans-
formations. Mathematically, a scale change is represented by a smooth
monotone increasing function $Y$ of lifetime:

$$(X_1, \ldots, X_N) \mapsto (Y(X_1), \ldots, Y(X_N)).$$

The new $y$ values can simply be substituted in the expression for $\bar{F}$.

Char.4 states that the density is uniform on simplexes $\{\lambda_N(\vec{X}) = \lambda_N\}$. This
uniformity, however, depends on the scale used for $\vec{X}$. $f$ is *not* physically
covariant, as changes in scale involve a Jacobian factor. See Char. 7 for a
physically covariant version of Char. 4.

Char.5 gives a de Finetti-type representation for the family of $\ell_1$-isotropic survival
functions. For any $N$, this family strictly includes the i.i.d. exponentials,
although the intergrand represents dependent variables. When $N \to \infty$,
it converges to a mixture of i.i.d. exponentials with rate $\lambda_\infty$, the limiting
failure frequency:

$$F(x_1, \ldots x_n) = \int \prod_{i=1}^{n} e^{-\lambda_\infty x_i} P(d\lambda_\infty).$$

(See also Barlow and Mendel [2].)

Char.6 improves upon Char. 2 in two important ways. (see Figure 1.)

First, Char. 6 is physically covariant. There are no explicit references to $\vec{X}$.

Second, the gradient operator is replaced with the exterior-derivative operator $d$, representing the differential or total derivative of $\log \bar{F}$. Unlike the gradient operator, the exterior derivative does not depend on a metric or inner product. This dependence on a metric is a not-so-well-known defect of Marshall's [11] hazard gradient: a gradient give the *direction* of greatest increase, but without a natural metric there is no natural notion of direction. Our definition makes $-d \log \bar{F}$ a 1-form rather than a vector, so a name such as "hazard 1-form" or "hazard differential" would be more appropriate.[1]

Char.7 states that the joint density is uniform on simplexes as in Char. 4, but now using differential $N$-forms, thereby eliminating the references to $\vec{X}$. The $W_{ij}$'s are the vectors that leave $\lambda_N$ invariant. By requiring that the Lie derivative of p vanishes, we obtain the invariant density. (See Figure 2.)

Char.8 states that the conditional density on a simplex is uniform with respect to the natural notion of hyper-area on the simplex. $\mathbf{W}$ gives a unit of hyper-area on the simplex. $p_{|\lambda_N} \lrcorner \mathbf{W}$ is a scalar function that gives the (infinitesimal) amount of probability per unit of hyper-area. If this is constant on the simplex, the conditional density is uniform. This is shown in Figure 3.

This characterization also has an interpretation in terms of fair bets: 'Given the failure frequency, one would bet the same amount for the same return on any outcome for $\vec{X}$.'

## Proofs:

We show how Chars. 1 through 5 are equivalent to the definition in Char. 3. Chars. 6 and 7 are shown to be equivalent to their respective coordinate-based

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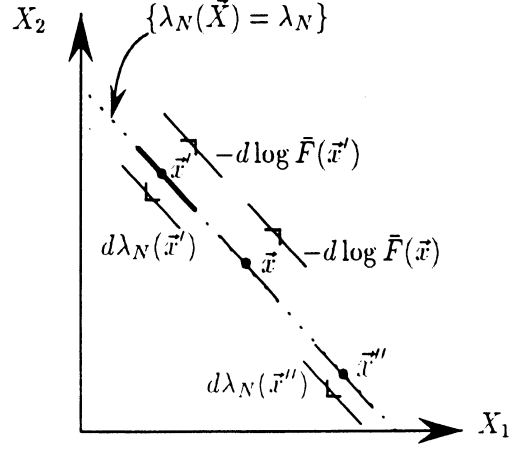[1] Force is a 1-form, so "force of mortality" is appropriate from this perspective.

Figure 1: The hazard differentials at $\vec{x}$ and $\vec{x}'$; they are pictured by two infinitesimally-spaced level lines of the hazard function $-\log \bar{F}$ with the arrowhead flagging the positive direction (see also Burke [3]). Also pictured are the differentials of $\lambda_N$ at $\vec{x}'$ and $\vec{x}''$. $-g$ is the factor with which $d\lambda_N$ is multiplied to obtain the hazard differential and this factor is constant on the simplex.
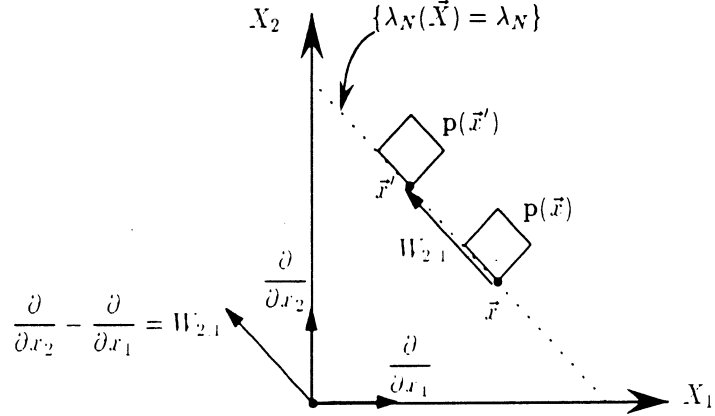


Figure 2: The probability $N$-form ($N = 2$) at two locations pictured by infinitesimal squares containing an infinitesimal unit of probability. Pulling $\mathrm{p}(\vec{x}')$ back via $W_{2,1}$ and subtracting it from $\mathrm{p}(\vec{x})$ shows that the Lie derivative vanishes.
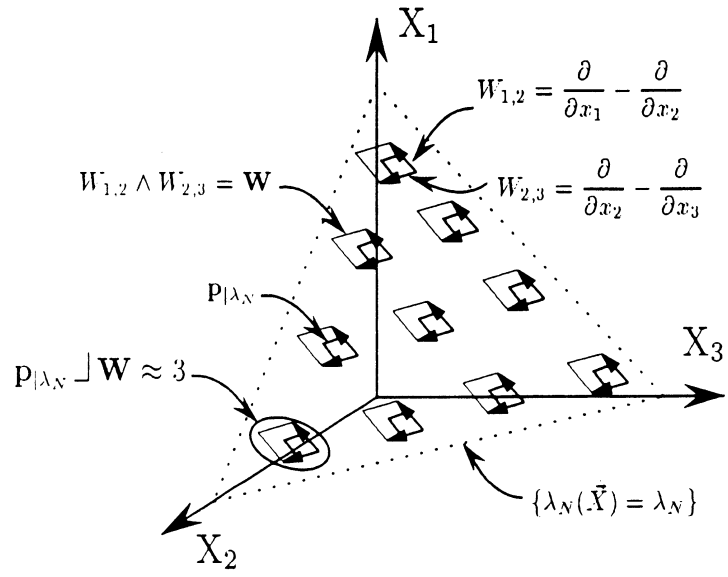
Figure 3: Here $N = 3$. The simplex contains the bivector **W** and the conditional probability 2-forms $p_{|\lambda_N}$ on the simplex. $p_{|\lambda_N} \lrcorner$ **W** is the number of times the 2-form fits into the bivector (in the figure this is approximately 3 times) and this number is constant over the simplex as required.

characterizations.

**1.⇔3.** Assume 3, $\bar{F} = g(\lambda_N)$ for some $g$. Then $\text{Prob}(X_i \geq x_i + h, \vec{X} \geq \vec{x}) =$
$\text{Prob}(X_j \geq x_j + h, \vec{X} \geq \vec{x}) = g(N/(h + \sum_{i=1}^{N} X_i))$. 1 follows after dividing
both sides of the first equality by $\bar{F}(\vec{x})$.

Assume 1, multiply both sides by $\bar{F}(\vec{x})$, collect terms, and substitute $\vec{x} - he_j$ for $\vec{x}$. Thus $\text{Prob}(\vec{X} \geq \vec{x} - he_j + he_i) = \text{Prob}(\vec{X} \geq \vec{x})$ for all $0 < h < x_j$.
Both $\bar{F}$ and $\lambda_N$ are the same at $\vec{x} - he_j + he_i$ and $\vec{x}$. One can similarly
shift lifetime between any pairs $i, j$ while still conserving $\bar{F}$ and $\lambda_N$. As a
consequence, $\bar{F}$ is a function of $\lambda_N$ alone.

**2.⇔3.** Assume 2. Then $(\partial/\partial x_i - \partial/\partial x_j)\bar{F} = 0$, or $\nabla_{W_{i,j}}(\log \bar{F}) = 0$. This implies
that $\log \bar{F}$ and hence $\bar{F}$ is a function of $\lambda_N$ alone. If not, there would be
point where $\nabla_{W_{i,j}} \log \bar{F} \neq 0$.

Assume 3. This implies $\log \bar{F}$ is a function of $\lambda_N$ alone. Then

$$\nabla_{W_{i,j}}(\log \bar{F}) = \frac{d\bar{F}}{d\lambda_N}\left(\frac{\partial \lambda_N}{\partial x_i} - \frac{\partial \lambda_N}{\partial x_j}\right)$$
$$= 0.$$

**4.⇔3.** Because the Lie brackets of the coordinate vectors in the $\vec{X}$ system all
vanish, we have

$$\frac{(-1)^N \partial^N}{\partial x_1 \cdots \partial x_N}\nabla_{W_{i,j}}(\bar{F}) = \nabla_{W_{i,j}}\left(\frac{(-1)^N \partial^N}{\partial x_1 \cdots \partial x_N}\bar{F}\right) = \nabla_{W_{i,j}}(f).$$

If $\bar{F}$ is a function of $\lambda_N$ alone, then $\nabla_{W_{i,j}}(\bar{F}) = 0$. From the above equality
it follows that $\nabla_{W_{i,j}}(f) = 0$, which implies 4. To prove $4 \to 3$ revert this
argument.

**5.⇔3.** Assume 3. Then $\bar{F}(\cdot | \lambda_N)$ is constant on the simplex. Integrate out
$x_{n+1}, \ldots, x_N$. This is a Dirichlet integral and gives the result in 5 (see
also, e.g., Barlow and Mendel [2]). That 5 implies 3 can be read off directly from the expression.

**6.⇔2.** Assume 6. Use the $\sharp$ operator for the Euclidean metric tensor (See e.g.
Abraham and Marsden [1]). In $\vec{X}$ coordinates:

$$\sharp(-d\log \bar{F}) = -\frac{\partial \log \bar{F}}{\partial x_1}\frac{\partial}{\partial x_1} - \cdots - \frac{\partial \log \bar{F}}{\partial x_N}\frac{\partial}{\partial x_N} = -g(\lambda_N)\sharp(d\lambda_N).$$

But the components of $\sharp(d\lambda_N)$ depend only on $\lambda_N$ and 2 follows.

Assume 2. Use the $\flat$ operator of the Euclidean metric tensor.

$$\flat(\nabla(-\log \bar{F})) = -\frac{\partial \log \bar{F}}{\partial x_1}dX_1 - \cdots - \frac{\partial \log \bar{F}}{\partial x_N}dX_N.$$

This is $-d\log \bar{F}$ in $\vec{X}$ coordinates. If the $\frac{\partial \log \bar{F}}{\partial x_i}$'s depend only on $\lambda_N$, 6 follows.

**7.$\Leftrightarrow$4.** Write the density $N$-form in $\vec{X}$ coordinates: $\mathbf{p} = f d\vec{X}$. The Leibniz property of the Lie derivative gives:

$$\mathcal{L}_{W_{i,j}}(f d\vec{X}) = \mathcal{L}_{W_{i,j}}(f) d\vec{X} + f \mathcal{L}_{W_{i,j}}(d\vec{X}).$$

But $\mathcal{L}_{W_{i,j}}(d\vec{X}) = 0$ and so we have $\mathcal{L}_{W_{i,j}}(\mathbf{p}) = 0$ if and only if $\mathcal{L}_{W_{i,j}}(f) = 0$, or $\nabla_{W_{i,j}}(f) = 0$, since for functions the Lie derivative is the directional derivative.

**8.$\Leftrightarrow$7.** Because $\mathcal{L}_{W_{i,j}}(h(\lambda_N)) = 0$ for any function $h$, 8 implies: $\mathcal{L}_{W_{i,j}}(\mathbf{p}_{|\lambda_N} \lrcorner \mathbf{W}) = 0$ for all $i,j$. We have:

$$\mathcal{L}_{W_{i,j}}(\mathbf{p}_{|\lambda_N} \lrcorner \mathbf{W}) = \mathcal{L}_{W_{i,j}}(\mathbf{W}) \lrcorner \mathbf{p}_{|\lambda_N} + \mathbf{W} \lrcorner \mathcal{L}_{W_{i,j}}(\mathbf{p}_{|\lambda_N}).$$

But $\mathcal{L}_{W_{i,j}}(\mathbf{W}) = 0$ and so we have 8 if and only if $\mathcal{L}_{W_{i,j}}(\mathbf{p}_{|\lambda_N}) = 0$ for all $i,j$. Now write $\mathbf{p} = \mathbf{p}_{\lambda_N} \wedge \mathbf{p}_{|\lambda_N}$; $\mathbf{p}_{\lambda_N}$ is a marginal probability 1-form for $\lambda_N$. We have

$$\mathcal{L}_{W_{i,j}}(\mathbf{p}_{\lambda_N} \wedge \mathbf{p}_{|\lambda_N}) = \mathcal{L}_{W_{i,j}}(\mathbf{p}_{\lambda_N}) \wedge \mathbf{p}_{|\lambda_N} + \mathbf{p}_{\lambda_N} \wedge \mathcal{L}_{W_{i,j}}(\mathbf{p}_{|\lambda_N}).$$

But $\mathcal{L}_{W_{i,j}}(\mathbf{p}_{\lambda_N}) = 0$ and so we have 7 also if and only if $\mathcal{L}_{W_{i,j}}(\mathbf{p}_{|\lambda_N}) = 0$ for all $i,j$. Now the $W_{i,j}$ have integral manifolds which are the level sets of $lambda_N$ and the equivalence follows.

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