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On the Approximate Determination of Natural Frequencies and Modes of Cantilever Beams

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With 6 Figures

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Summary — Zusammenfassung

On the Approximate Determination of Natural Frequencies and Modes of Cantilever Beams. The free oscillations of cantilever beams of variable cross-section are considered. By lumping the mass properties of the beam at discrete points, approximate modes and upper and lower bounds to approximate natural frequencies are obtained essentially by a simple iteration scheme. Both Euler-Bernoulli and Timoshenko beams are considered. Example problems are exhibited and compared to known results.

Zur näherungsweisen Ermittlung der Eigenfrequenzen und Schwingungsformen von Kragträgern. Die freien Schwingungen von einseitig eingespannten Trägern mit variablen Querschnitt werden betrachtet. Durch Zusammenziehung der Masse auf diskrete Punkte werden — im wesentlichen durch ein einfaches Iterationsschema — Näherungen für die Schwingungsformen und für die oberen und unteren Grenzen der Eigenfrequenzen gefunden. Sowohl Euler-Bernoulli als auch Timoshenko-Träger werden betrachtet. Anhand von Beispielen wird der Vergleich mit bekannten Resultaten gezogen.

Introduction

The idea of determining the natural frequencies of beams by lumping the beam properties at various points is hardly new. What we propose here is a procedure for the analysis of the free vibrations of beams based upon the complementary variational principle which yields an algorithm particularly suited to modern digital computation. Such a complementary formulation of the problem has been suggested by Prager [1] although from a different point view. Gaines and Volterra [2, 3] have worked on the continuous problem along lines similar to those followed by Prager, but in addition they have obtained lower bounds to beam frequencies through use of a technique due to Tricomi [4]. The algorithm which we propose utilizes ideas from all these previous papers.

Euler-Bernoulli Beams

We begin by considering sections of the beam, Fig. 1, of equal mass, m. Let us suppose there are n such sections. Then we consider the following positions along the beam: 0, x_1', x_2', . . ., x_n'. The positions are defined such that the mass from x'_{i-1} to x_i' is m. Thus

$$x_i' : \int_0^{x_i'} (\gamma\, A/g)\, dx = i\, m \qquad i = 1, 2, \ldots, n, \qquad x_n' = L. \tag{1}$$

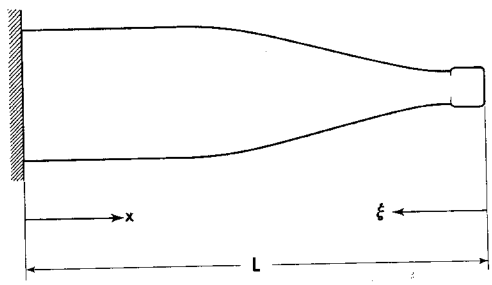

Fig. 1. Beam Configuration

We now represent the entire mass of a section by a point mass. We will have complete freedom in our algorithm concerning the position of the mass. In this paper we locate it at the center of mass of the section the position of which we denote by x_i:

$$x_i = \left[\int_{x_{i-1}'}^{x_i'} (\gamma\, A/g)\, x\, dx \right] / \, m \qquad i = 1, 2, \ldots, n. \tag{2}$$

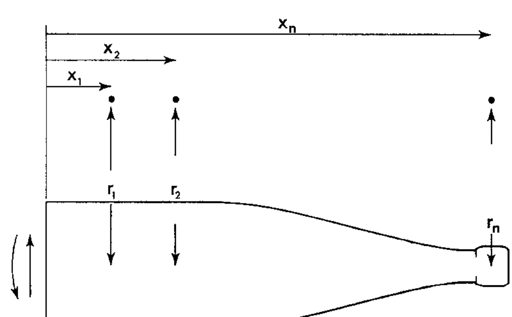

Fig. 2. Euler-Bernoulli Beam, Free Body Diagram

We denote the impulse between the beam and the mass m at x_i by r_i (see Fig. 2). The kinetic energy of the system, T, is then [5]:

$$T = \frac{1}{2}\,(1/m) \sum_{i=1}^{n} r_i^2. \tag{3}$$

The potential coenergy of the system, V^*, is obtained from

$$V^* = \frac{1}{2} \int_0^L [M^2/EI]\, dx, \tag{4}$$

where $M(x)$ is the bending moment distribution in the beam. In the present problem it is easier to write first $M(\xi)$ where ξ is defined in Fig. 1.

$$\begin{aligned} M(\xi) &= \dot{r}_n (\xi + x_n - L)\, U(\xi + x_n - L) \\ &\quad + \dot{r}_{n-1} (\xi + x_{n-1} - L)\, U(\xi + x_{n-1} - L) \ldots . \\ &\quad + \dot{r}_1 (\xi + x_1 - L)\, U(\xi + x_1 - L), \end{aligned}$$

where U is Heaviside's unit function:

$$\begin{aligned} U(x) &= 0 \; x < 0, \\ &= 1 \; x \geqslant 0. \end{aligned}$$

But since $\xi + x = L$, we have

$$M(x) = \sum_{i=1}^{n} \dot{r}_i (x_i - x)\, U(x_i - x). \tag{5}$$

(Note that $U(x_i - x)$ is 1 from $x = 0$ to $x = x_i$, and is 0 for $x > x_i$.) From (4) and (5) we obtain

$$V^* = \frac{1}{2} \int_0^L (1/EI) \left[\sum_{i=1}^{n} \dot{r}_i (x_i - x)\, U(x_i - x) \right]^2 dx. \tag{6}$$

From (6) it is clear that V^* is a quadratic form in the variables $\dot{r}_i$. Thus we examine

$$\frac{\partial^2 V^*}{\partial \dot{r}_i\, \partial \dot{r}_j} = \int_0^L (1/EI)(x_i - x)(x_j - x)\, U(x_i - x)\, U(x_j - x)\, dx,$$

or setting this equal to $A_{ij} = A_{ji}$,

$$A_{ij} = \int_0^{x_k} (1/EI)(x_i - x)(x_j - x)\, dx, \qquad x_k = \min(x_i, x_j). \tag{7}$$

And we have then

$$V^* = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}\, \dot{r}_i\, \dot{r}_j. \tag{8}$$

Equations of Motion

The equations of motion, or more properly, the equations of compatibility, are obtained from the complementary equations [5]:

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \dot{r}_i}\right) - \frac{\partial L^*}{\partial r_i} = 0 \qquad i = 1, 2, \ldots, n, \qquad L^* = T - V^*. \tag{9}$$

Thus from (3), (8), and (9) we obtain the system of equations:

$$\sum_{j=1}^{n} A_{ij}\, \ddot{r}_j + (1/m)\, r_i = 0, \qquad i = 1, 2, \ldots, n. \tag{10}$$

or in matrix notation:

$$[A]\, \underline{\ddot{r}} + (1/m)\, \underline{r} = 0.$$

For the case of free oscillations, we set

$$r_i = R_i \sin(p\, t).$$

And thus (10) becomes

$$\sum_{j=1}^{n} A_{ij}\, R_j = (1/m\, p^2)\, R_i, \tag{11}$$

or

$$[A]\, \underline{R} = (1/m\, p^2)\, \underline{R}.$$

We note at this point that in formulating our problem in the above manner, we will always obtain an ordinary eigenvalue problem rather than a generalized eigenvalue problem

$$[A]\, \underline{R} = (1/p^2)\, [B]\, \underline{R}$$

which one would expect. We will also arrange for this when we come to include TIMOSHENKO effects in our analysis.

Natural Frequencies and Modes

We have reduced the problem of determining the lowest natural frequencies of the beam to that of finding the eigenvalues of the matrix $[A]$. To obtain upper bounds on the frequencies represented by $[A]$ (from this point on we shall distinguish between the frequencies of $[A]$ and the beam frequencies), we form a complementary RAYLEIGH quotient, Q^*, [6]. In the present case we have

$$Q^* = \left[\sum_{i=1}^{n} R_i^2/m \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}\, R_i\, R_j\right]. \tag{12}$$

In (12) if $\underline{R}$ is the k-th eigenvector of $[A]$, then Q^* will equal p_k^2. If $\underline{R}$ is an approximation to the lowest eigenvector, Q^* will give an upper bound on p_1^2, the square of the lowest natural frequency.

The usual procedure one follows once (12) is obtained is to apply the RAYLEIGH-RITZ technique. However, the eigenvalues of $[A]$ are essentially $1/p^2$. Thus the largest eigenvalue of $[A]$ corresponds to the lowest natural frequency. Thus rather than employ a RAYLEIGH-RITZ procedure to obtain the eigenvalues, we establish an iteration scheme:

$$[A]\,\underline{R}^q = \underline{R}^{q+1}. \tag{13}$$

It is well known that (13) will converge to the mode which corresponds to the highest eigenvalue (hence the lowest natural frequency). With (13), we write (12)

$$Q^* = \left[\sum_{i=1}^{n} (R_i^q)^2/m \sum_{i=1}^{n} (R_i^q\, R_i^{q+1})\right]. \tag{14}$$

Once the lowest mode is determined to a desired accuracy (the simplest method of which is to set some tolerance on the difference of Q^* from one iteration to the next) we sweep out the lowest mode from consideration of the $\underline{R}$ space and proceed to determine the second mode, etc. Suppose that the first modes determined are $\underline{S}_1, \underline{S}_2, \ldots, \underline{S}_h$ $(h < n)$. Then our revised iteration scheme is

$$\begin{gathered}[A]\,\underline{R}^q = \underline{T}^{q+1},\\ R^{q+1} = \underline{T}^{q+1} - \sum_{i=1}^{h} (\underline{T}^{q+1}\cdot \underline{S}_i/\underline{S}_i\cdot \underline{S}_i)\,\underline{S}_i,\end{gathered} \tag{15}$$

where $\cdot$ denotes the scalar or inner product. The iteration (15) will now converge to $\underline{S}_{h+1}$.

Another advantage of formulating our problem in terms of the matrix $[A]$ is that we can easily determine lower bounds to the computed frequencies (if not the actual beam frequencies). Suppose that the eigenvalues of $[A]$ are τ_k and the corresponding normalized eigenvectors are φ_i^k (the i-th component of the k-th vector). Then

$$\sum_{j=1}^{n} A_{ij}\,\varphi_j^k = \tau_k\,\varphi_i^k, \quad \sum_{i=1}^{n} \varphi_i^k\,\varphi_i^h = \delta_{kh}.$$

One can easily show that since the vectors φ^k are complete ($[A]$ is real and symmetric) A_{ij} has the expansion

$$A_{ij} = \sum_{k=1}^{n} \tau_k\,\varphi_i^k\,\varphi_j^k$$

and further

$$\sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}^2 = \sum_{i=1}^{n} \tau_k^2. \tag{16}$$

In our particular case, we have by (11)

$$m^2 \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}^2 = \sum_{i=1}^{n} (1/p_i)^4. \tag{17}$$

Suppose now that we have computed the following upper bounds to the frequencies of $[A]$:

$$\hat{p}_1 \geqslant p_1,\ \hat{p}_2 \geqslant p_2, \ldots, \hat{p}_h \geqslant p_h,\ h \leqslant n.$$

Then if we set

$$(1/\bar{p}_j)^4 = m^2 \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}^2 - \sum_{\substack{i=1\\ i \neq j}}^{h} (1/\hat{p}_i)^4 \tag{18}$$

and compare to (17), we see that

$$\bar{p}_j \leqslant p_j. \tag{19}$$

Timoshenko Effects

We have dealt with the EULER-BERNOULLI beam in great detail in the above. We will now see that the TIMOSHENKO effects, that is the effects of rotary inertia and shear deformation, can be included in the above scheme with comparative ease.

In addition to locating discrete mass elements on the beam as shown in Fig. 2, we now consider sections of equal inertia. Thus we define positions y_i' and y_i by

$$\int_0^{y_i'} (\gamma\, I/g)\, dx = (i/n) \int_0^{L} (\gamma\, I/g)\, dx = i\, \eta, \tag{20}$$

where $I = I\,(x)$ is the moment of inertia of the section and η is a parameter, defined by (20), which plays a role analogous to that played by m in (1). Thus

$$y_i = \left[\int_{y_{i-1}'}^{y_i'} (\gamma\, I/g)\, x\, dx/\eta \right]. \tag{21}$$

The position y_i might be termed the center of inertia of the section between y'_{i-1} and y_i'.

At the positions y_i along the beam we consider to be placed discrete (pure) rotary inertia elements. We think of these as ideal elements which are defined only by an angular impulse—angular velocity relation without any associated (linear) impulse—velocity relation.

We consider now a free body diagram of our system, Fig. 3. At each point x_i we place a mass, m. The reaction between m and the beam is $\dot{r}_i$. At each point y_i we place an inertia η. The reaction between η and the beam is denoted by $\dot{\sigma}_i$, a pure couple. Thus we can write the moment and shear distributions in the beam:

$$\begin{aligned} M(x) &= \sum_{i=1}^{n} [\dot{\sigma}_i \, U(y_i - x) - \dot{r}_i \,(x_i - x)\, U(x_i - x)], \\ Q(x) &= \sum_{i=1}^{n} [\dot{r}_i \, U(x_i - x)]. \end{aligned} \tag{22}$$

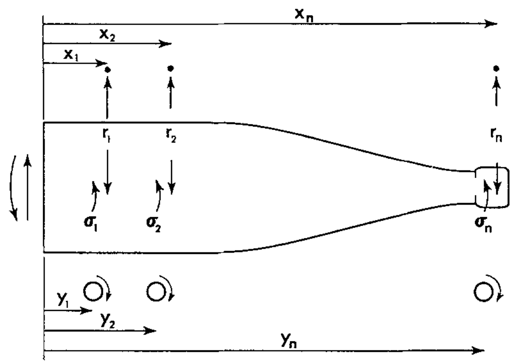

Fig. 3. Timoshenko Beam, Free Body Diagram

The functions T and V^* [7] are now given by

$$\begin{aligned} T &= \frac{1}{2}\left[(1/m) \sum_{i=1}^{n} r_i^2 + (1/\eta) \sum_{i=1}^{n} \sigma_i^2\right], \\ V^* &= \frac{1}{2}\left[\int_0^L (1/E\,I)\, M^2\, dx + \int_0^L (1/k\,A\,G)\, Q^2\, dx\right]. \end{aligned} \tag{23}$$

For the sake of future convenience, we scale and rename the variables σ_i and the distances y_i:

$$\sigma_i = a\,L\,r_{i+n} \qquad y_i = x_{i+n}, \tag{24}$$

where a is chosen so that

$$\frac{\sigma_i^2}{\eta} = \frac{r_{i+n}^2}{m} \text{ thus } a = (\eta/m)^{\frac{1}{2}}/L.$$

Thus we now have (22) as

$$M(x) = -\sum_{i=1}^{n} \dot{r}_i\,(x_i - x)\, U(x_i - x) + a\,L \sum_{i=n+1}^{2n} \dot{r}_i\, U(x_i - x),$$

$$Q(x) = \sum_{i=1}^{n} \dot{r}_i \, U(x_i - x)$$

and thus we obtain (23)

$$T = \frac{1}{2}(1/m) \sum_{i=1}^{2n} r_i^2,$$
$$V^* = \sum_{i=1}^{2n} \sum_{j=1}^{2n} A_{ij} \, \dot{r}_i \, \dot{r}_j, \tag{25}$$

where

$$
\begin{aligned}
A_{ij} &= \int_0^{x_k} [(x_j - x)(x_i - x) / EI + 1/kAG] \, dx && i \leqslant n \quad j \leqslant n \\
&= -aL \int_0^{x_k} [(x_i - x) / EI] \, dx && i \leqslant n \quad j > n \\
&= -aL \int_0^{x_k} [(x_j - x) / EI] \, dx && i > n \quad j \leqslant n \\
&= (aL)^2 \int_0^{x_k} (1 / EI) \, dx && i > n \quad j > n
\end{aligned} \tag{26}
$$

where $x_k = \min(x_i, x_j)$.

By (9), where n is replaced by $2n$, we now obtain the system equations for free oscillations

$$[A] \, \underline{R} = (1/m \, p^2) \, \underline{R}.$$

The matrix $[A]$ is the $2n$ dimensional matrix from (26) which now includes corrections for rotary inertia and shear deformation. We can now apply the same computational scheme as developed for the EULER-BERNOULLI beam. We note that in doing so, the first n elements of $\underline{R}$ represent the point impulses r_i and the last n the angular impulses σ_i.

In the determination of the modes of the TIMOSHENKO beam, we will obtain, directly, approximations to the shear distribution $Q(x)$ and the moment distribution $M(x)$ from (22). In order to obtain the deflection distribution $Y(x)$ and the bending slope distribution $\psi(x)$, we note (see [7])

$$\dot{y} = \left(\frac{g}{\gamma A}\right) r,$$
$$\dot{\psi} = \left(\frac{g}{\gamma I}\right) \sigma. \tag{27}$$

Thus from a knowledge of the r_i and the σ_i, the deflection and bending slope modes can be plotted directly.

Examples

We consider now two examples to illustrate the use of the above theory: a uniform TIMOSHENKO beam and a truncated wedge. In the first case we compare our eigenvalues and eigenfunctions to the exact results obtained by HUANG [8, 9]. In the second case, we compare our frequencies to those obtained by VOLTERRA and GAINES [2].

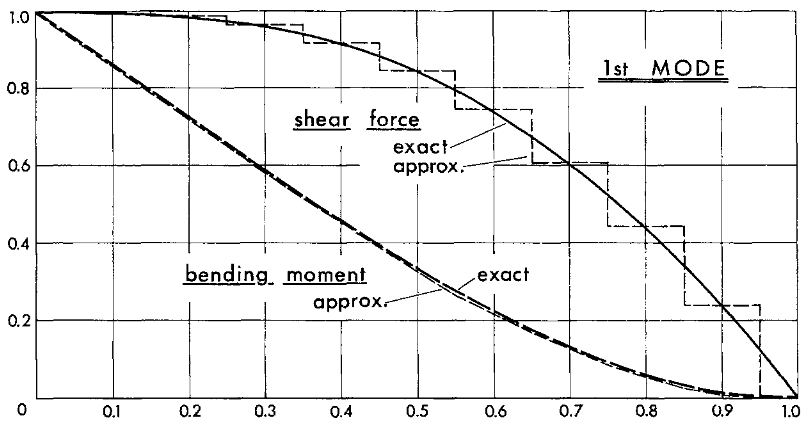

Fig. 4. Bending Moment and Shear Distributions, Mode 1

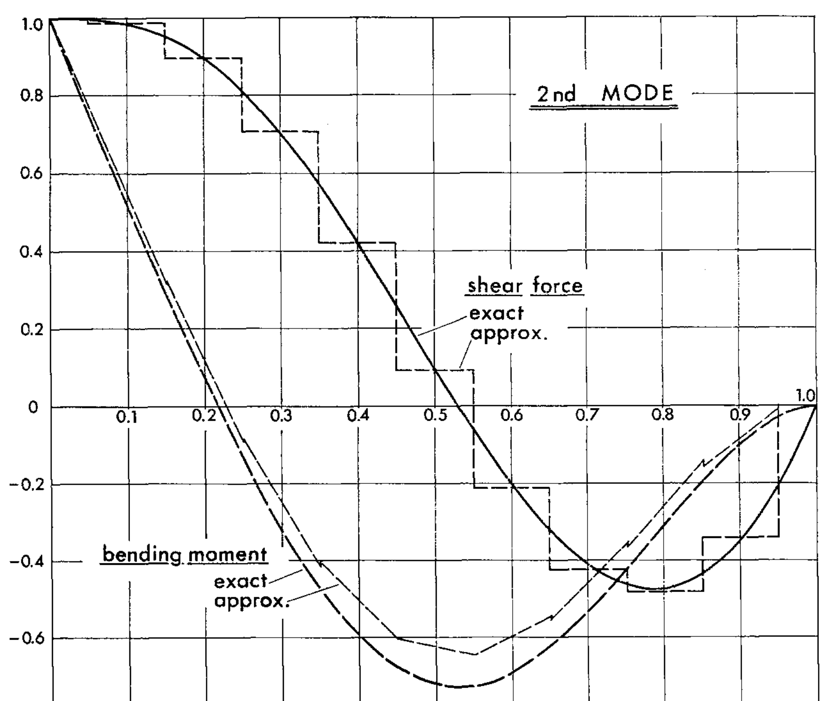

Fig. 5. Bending Moment and Shear Distributions, Mode 2

Example 1: Uniform Beam. We have chosen a beam with the following characteristics:

$$(I/A\,L^2)^{\frac{1}{2}} = (E\,I/kA\,G\,L^2)^{\frac{1}{2}} = 0.03.$$

In the approximate analysis, we have used ten subdivisions. A comparison of the eigenvalues obtained by the approximate method and the exact values is shown below:

	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5
Approximate (b)	3.509	21.580	58.471	109.326	170.596
Exact	3.501	21.421	57.874	108.224	169.647
% Error	0.23	0.78	1.03	1.02	0.56

where $b^2 = (\gamma\,A\,L^4\,p^2/E\,I\,g)$.

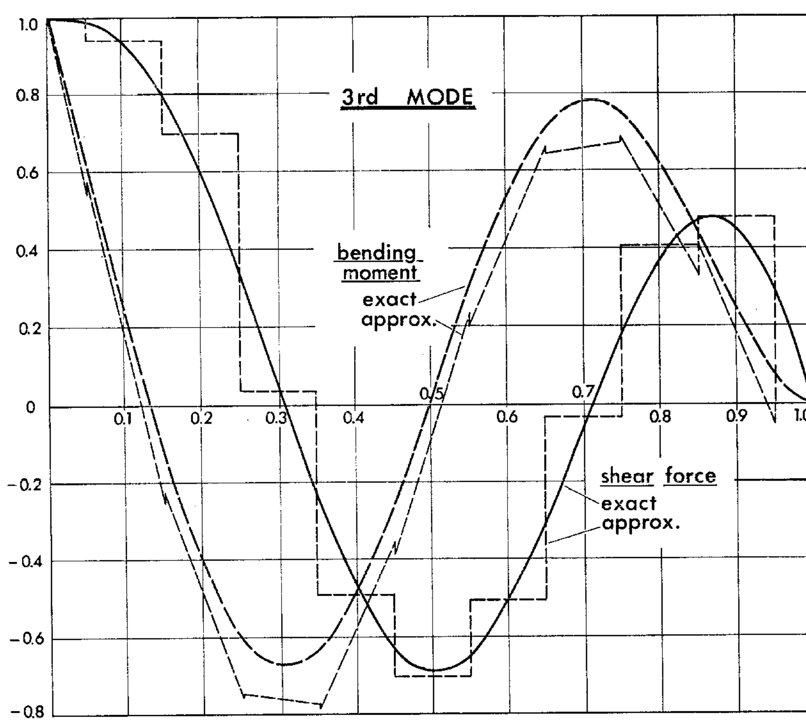

Fig. 6. Bending Moment and Shear Distributions, Mode 3

The moment and shear diagrams are compared for the first three modes in Figs. 4, 5, and 6.

Example 2: Truncated Wedge. We consider now a beam of rectangular cross section with a constant width. The height of the beam varies uniformly from H_0 at the fixed end to H_1 at the free end. In order to compare to the results obtained by Volterra and Gaines, we have chosen the following values to describe the beam:

$$H_1/H_0 = 0.5, \qquad L/H_0 = 3.0, \qquad E/G = 2.6, \qquad k = 0.833.$$

We use ten descrete masses and ten discrete inertia elements to describe the beam. The natural frequencies of Volterra and Gaines are compared to the approximate values for the first three modes:

	Mode 1	Mode 2	Mode 3
Approximate (c)	3.588	14.112	29.623
Volterra and Gaines (c)	3.568	13.538	26.300

where $c^2 = (\gamma A_0 L^4 p^2 / E I_0 g)$, and A_0 and I_0 are the area and moment of inertia at the fixed end.

In the above, we have reported only the approximate upper bounds. The reason for this is that our lower bounds are a measure only of the accuracy of the approximate computation. And since the approximate frequencies may be different from the actual frequencies, the approximate lower bounds may, for example, be higher than the actual frequencies.

Conclusions

The virtues of the above method are twofold. First it can be applied to arbitrarily shaped beams. In practice, one may well encounter beams which are non-uniform in shape. Thus tables of natural frequencies of beams are of limited utility. Second, the method we propose can be changed from n subdivisions to any other number of subdivisions with a minimum of difficulty in the cases in which the A_{ij} can be evaluated in terms of anti-derivatives.

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