CIRCULAR VISIBILITY OF A SIMPLE POLYGON

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ABSTRACT

The point visibility and the edge visibility of a simple polygon are computed. The portion of a simple polygon circularly visible from a given *point* is obtained in $O(n)$ time, where n is the number of vertices. This requires the construction of a circular visibility diagram (CVD) – a classification of arcs emanating from a given point to the edges they hit. This is done in linear time. The portion of a simple polygon circularly visible from a given *edge* is obtained in $O(kn)$ time, where k is the number of CVDs computed. In the worst case, k equals n.

1. Introduction

As visibility can be characterized by lines of sight, a point is (linearly) visible from another point if there exists a line segment connecting them without crossing any obstacle. Using lines to represent the trajectories of visibility facilitates the computation of linear visibility, which subsequently enables the computation of notions such reachability, assemblability, and separability.

One of the fundamental linear visibility problems is the computation of a point visibility polygon, the portion of a polygon that is visible from an internal point. ElGindy and Avis[7] and Lee[12] develop linear time algorithms for constructing a linear visibility polygon inside a simple polygon. O'Rourke[14] shows that $O(n \log n)$ time is required to compute a visibility polygon for a non-simple polygon.

Another fundamental linear visibility problem is the computation of an edge visibility polygon, the portion of a polygon that is visible from an edge, internally. Avis and Toussaint[3] define three levels – complete, strong, and weak – of edge visibility. Lee and Preparata[11] give an algorithm that determines whether a simple polygon is completely or strongly visible from a given edge. Avis and Toussaint[3] present a linear time algorithm for determining whether a polygon is weakly visible. Chazelle and Guibas[5] show that an edge visibility polygon inside a triangulated simple polygon can be constructed in linear time. Suri and O'Rourke[17] develop an algorithm for constructing a weak edge visibility polygon inside a non-simple polygon in $\Omega(n^4)$ time.

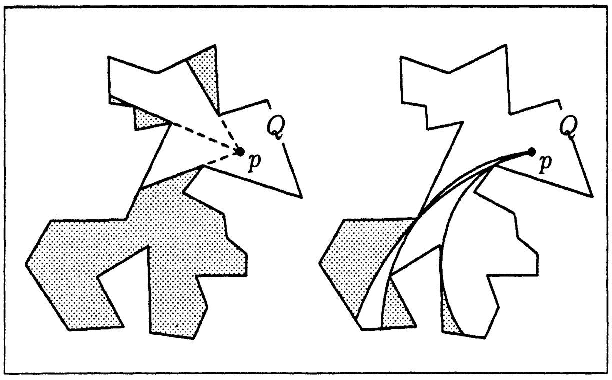

Figure 1: The linear visibility polygon and the circular visibility region of s.

These fundamental algorithms have many extensions. In the *art gallery* problem[14], a minimum number of points inside a polygon is to be positioned such that the union of the point-visibility polygons equals the simple polygon. In the *minimum link path* problem, a path which connects two points inside a polygon and which consist a minimum number of line segments is sought; Suri[16] gives a linear time algorithm which compute such a path by constructing a sequence of visibility polygons. Guibas et al.[8] show that the *shortest path* inside a simple polygon can also be solved in linear time by utilizing the notion of visibility. Not all paths have to be linear. In engineering applications, devices such as machine tools and robots are equipped with circular interpolators, giving rise to the notion of *circular visibility.*

A point q is said to be *circularly visible* from a distinct point p if a circular arc can be drawn from p to q without hitting an obstacle. The set of directed circular arc – clockwise or counterclockwise – drawn from p are called *visibility arcs* emanating from p and can be uniquely defined by their centers, endpoints, and directions. The difference between linear and circular visibility is illustrated in Figure 1. The shaded portions are not visible. Since arcs are involved in circular visibility, the term "region" is used (instead of "polygon"). And since straight lines can be considered as degenerate arcs, the linear visibility polygon is a subset of the circular visibility region.

Research on circular visibility is relatively sparse. Agarwal and Sharir[1] present an $O(n \log n)$ algorithm for computing the circular visibility region of a given point inside a simple polygon. Agarwal and Sharir[2] also show that by preprocessing a simple polygon, with $O(n \log^3 n)$ time, the query to the first intersection point between a circular arc emanating from a given point and the simple polygon can be answered in $O(\log^4 n)$ time. A linear time algorithm is developed by Chou and Woo[6] for classifying the visibility arcs emanating from a given point with respect to the edges of a simple polygon that they hit; such a classification is called the *circular visibility diagram* (or CVD) of a fixed point. In this paper, two types of circular visibility regions are computed by utilizing the CVD: that of a given point and that of a given edge.

The rest of the paper is organized as follows. The structure of the CVD is first

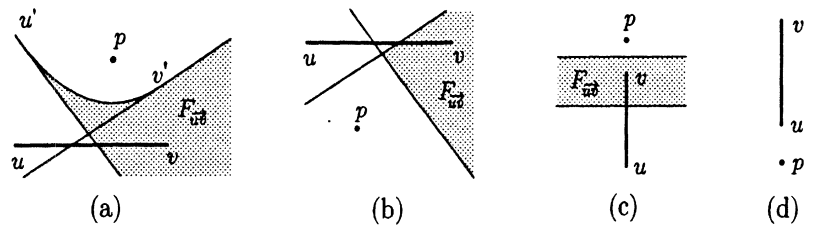

Figure 2: Centers of the CCW visibility arcs that hit the left of $\vec{uv}$.

reviewed; the correspondence between a particular class of circular visibility arcs and their counterparts in the CVD is then established. In Section 3, a linear time algorithm for computing a circular visibility region of a point is described. Finally, in Section 4, an $O(kn)$ time algorithm for computing a circular visibility region of an edge is developed, where n is the total number of vertices in the simple polygon and k is the number of different pairs of supports of the circular arcs that contribute to the boundary of the visibility region.

2. Preliminaries

The representation for a collection of visibility arcs is first established. Whereas there is only one line segment connecting two given points, there are an infinite number of circular arcs connecting them. Additionally, due to the non-linearity, the collection of visibility arcs from a point to an edge can no longer be represented by a one-dimensional direction wedge[7]. The pattern of these visibility arcs is very difficult to visualize as they cross each other along their paths.

It is noted that circles passing through a given point can be uniquely represented by their corresponding centers. Each point in the plane corresponds to the center of exactly two circular arcs – CW or CCW along the same circle – emanating from the given point. By considering CW and CCW arcs separately, there is a one-to-one correspondence between a point in the plane and either a CW or a CCW visibility arc. The visibility arcs from the given point to an edge can therefore be uniquely represented by the collection of the centers of the visibility arcs.

Let the edges of a simple polygon be ordered in the CCW direction. Since p is inside the polygon, all the the visibility arcs that hit an edge $\vec{uv}$ of the polygon will hit the left side of the edge. Thus, only the intersections from the left of the edges are of interest. Figure 2 illustrates the four possible configurations of p and $\vec{uv}$. The loci of the centers of the CCW visibility arcs emanating from p and hitting the left side of $\vec{uv}$ are shaded. Figure 2 (a) depicts the case where p is to the *left* of $\vec{uv}$, whereas Figure 2 (b) depicts the case where p is to the *right* of $\vec{uv}$. Figure 2 (c) and (d) shows the limiting cases, where p is collinear with $\vec{uv}$ and $\vec{uv}$ is directed *towards* and *away from* p, respectively. It is noted that if $\vec{uv}$ is directed away from p, no CCW arc emanating from p will hit the left side of $\vec{uv}$.

The classification of visibility arcs with respect to the edges of the polygon that they hit can now be obtained. Such a classification results in two partitions of the

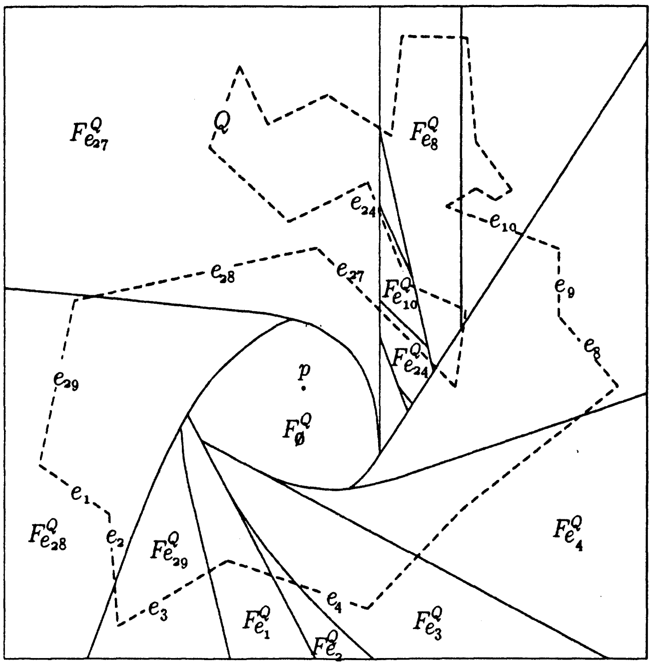

Figure 3: The CCW CVD of a simple polygon Q.

plane, for the centers of CW and CCW arcs, and are called the CW CVD and the CCW CVD, respectively. An example of a CCW CVD is shown in Figure 3, where p is the emanating point, Q in dotted lines the simple polygon of interest, and the CCW CVD of p in solid lines. A CVD can have up to $n+1$ regions. Points in the same region correspond to the centers of all the CCW visibility arcs which emanate from p and hit a particular edge of Q. (The region $F^Q_\emptyset$ contains all the points about which CCW visibility arcs emanating from p and missing all the edges of Q.) The boundaries of the regions of the CVD consist of: line segments or half-lines, each of which is a portion of the perpendicular bisector between p and a vertex of Q, and parabolic curves, each of which is a portion of the parabola[a] defined by p and an edge of Q.

The data structure of the CVD is similar to the dual space data structure used by Chazelle and Guibas[5] for solving a variety of linear visibility problems, in which a line $ax+by+1=0$ in the primal space is represented by a point (a,b) in the dual space[4,13]. Points in the dual space are then grouped into regions according to the edges that their corresponding visibility rays hit in the primal, which results in a planar partition in the dual space.

Prior to the development of the algorithms, some terminologies are introduced to aid the presentation of the ideas. Given an arc $\widehat{pq}$, as shown in Figure 4, let the region bounded by arc $\widehat{pq}$ and line segment $\overline{pq}$ be called the *segment* of $\widehat{pq}$. The *convex side* of a circular arc can subsequently be defined as the side of the arc that

[a] A parabola can be defined as the locus of all the equi-distant points between a point and a line. Every portion of the line corresponds to a portion of such a parabola.

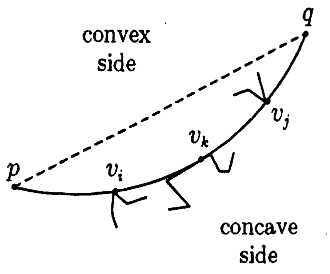

Figure 4: Sides and supports with respect to an arc $\widehat{pq}$.

is the inside of the segment, whereas the *concave side* of a circular arc is the side that is to the outside. Let a point in contact with an arc be called a *support* of the arc. A vertex of a polygon is therefore said to be a *convex support* of a circular arc, if it is in contact with the arc on the convex side. Likewise, a vertex or a non-vertex point[b] on an edge of a polygon in contact with an arc on its concave side is called a *concave support* of this arc. In Figure 4, vertex v_j is a convex support of $\widehat{pq}$, whereas vertex v_i and point v_k are concave supports of $\widehat{pq}$.

The closure of the circular visibility region is first computed. This means that a visibility arc which passes through a vertex of the polygon and stays inside the polygon is not blocked by the vertex. Likewise, a visibility arc tangent to an edge of the polygon is not blocked by the edge. The circular visibility region computed accordingly is a closed region. By taking the open set of this closed region, the desired circular visibility region is obtained. Since the open set of a dangling curve in the plane is empty, the dangling arcs identified are ignored.

3. Circular Visibility Region of a Given Point

In this section, an efficient algorithm that computes the solution to the following visibility problem is developed.

Problem CVR(p,Q): Circular visibility region of a given point
Given: a simple polygon $Q = \{e_1, e_2, ..., e_n\}$, and a point p contained in Q.
Find: the portion of Q that is circularly visible from p.

To construct such a visibility region, the constituents of its boundary are first examined. By establishing the properties for the visibility arcs that contribute to the boundary of the visibility region, and subsequently establishing the correspondence between these visibility arcs and their counterparts in the CVD, the circular visibility region of a point can then be constructed.

Since lines are degenerate circular arcs, and consequently the linear visibility polygon of p is a subset of the circular visibility region of p, only regions not in the linear visibility polygon of p need to be examined. These disjoint regions are called *pockets*, and can be obtained by taking the boolean difference between Q and the

[b]It is noted that if an edge is in contact with an arc on its convex side, the contact must be with the endpoint(s) of the edge.

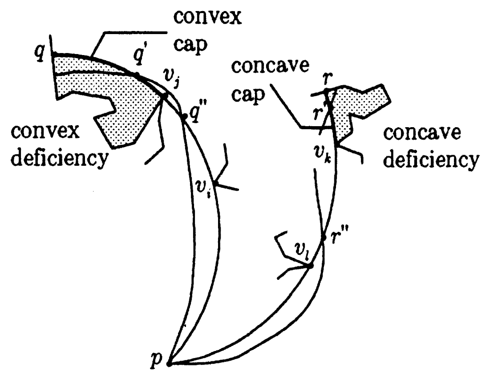

Figure 5: Convex and concave caps of p.

linear visibility polygon of p. Since visibility arcs which do not hit any edge of Q can only travel within the linear visibility polygon of p, visibility arcs entering a pocket always hit the boundary of the pocket. The identification of the boundary of the circular visibility region from p can therefore be reduced to the examination of the visibility arcs hitting the boundary of the pockets.

The boundary of the circular visibility region in a pocket consists of portions of the boundary of Q and portions of visibility arcs. The latter can be classified into two groups. Let $\widehat{v_j q}$ be a portion of a visibility arc emanating from p such that there does not exist any visibility arc that can reach the region to the convex side of $\widehat{v_j q}$, as shown in Figure 5. Arc $\widehat{v_j q}$ is a portion of the boundary of the circular visibility region of p, and is called a *convex cap* of p; the region to the convex side of $\widehat{v_j q}$ is called a *convex deficiency* of p. Similarly, *concave caps* and *concave deficiencies* of p can be defined. Visibility arcs which are supersets of a cap are said to *contain* the cap. To construct the circular visibility regions inside the pockets, all the visibility arcs that contain a convex or a concave cap of p need to be identified.

Whether or not a visibility arc contains a cap can be determined by the configuration of the support(s) on the arc. Suppose that a visibility arc $\widehat{pq}$ hits the boundary of Q at q and does not have any support. Since circular arcs passing through p and q and lying entirely on either side of $\widehat{pq}$ can be obtained, arc $\widehat{pq}$ can not be on the boundary of the circular visibility region of p, and thus does not contain a cap. By a similar argument, visibility arcs that have only one support cannot contain a cap either. The following lemma establishes the properties for a CCW visibility arc having two supports and containing a cap.

Lemma 1 *(1) Let q be a point on the boundary of Q, and let $\widehat{pq}$ be a CCW visibility arc with a concave support, v_i, and a convex support, v_j, ordered as p, v_i, v_j, and q. Then, arc $\widehat{v_j q}$ is a convex cap of p.*
(2) Let r be a point on the boundary of Q, and let $\widehat{pr}$ be a CCW visibility arc with a convex support, v_l, and a concave support, v_k, ordered as p, v_l, v_k, and r. Then, arc $\widehat{v_k r}$ is a concave cap of p.

Proof. (1) Suppose, to the contrary, that there exists a visibility arc which em-

anates from p and crosses $\widehat{v_j q}$ at q' ($q' \neq q$), as shown in Figure 5. Because both of the supports v_i and v_j are on Q, which is a simple polygon, this visibility arc has to cross $\widehat{v_i v_j}$. This means that this arc and $\widehat{pq}$ intersect at three points, p, q'', and q'. As two distinct circles have at most two intersections, an arc emanating from p and crossing $\widehat{v_j q}$ at a point other than v_j cannot exist. As the region that is not circularly visible from p is to the convex side of $\widehat{v_j q}$, arc $\widehat{v_j q}$ is a convex cap of p.
(2) An illustration of such an arc is also given in Figure 5. As the proof for (2) is similar to that of (1), it is omitted. □

It is easy to verify that without the confinement of a concave support on $\widehat{pv_j}$ (as shown in Figure 5), no matter how many more convex and concave supports there are, arc $\widehat{v_j q}$ can not be a convex cap. On the other hand, when $\widehat{pq}$ has support(s) besides v_i and v_j, as established in Lemma 1, the region to the convex side of $\widehat{v_j q}$ is still not circularly visible from p. Arc $\widehat{v_j q}$ therefore remains to be a convex cap and a portion of the boundary of the circular visibility region of p. Likewise, without a convex support on $\widehat{pv_k}$, arc $\widehat{v_k r}$ would not be a concave cap, whereas even if visibility arc $\widehat{v_k r}$ is in contact with supports besides v_l and v_k, arc $\widehat{v_k r}$ will still be a concave cap of p. Additionally, a CW visibility arc contains a convex or a concave cap if its supports satisfy the same configurations as described in Lemma 1 for the supports of a CCW visibility arc. By identifying all the caps of p, and subsequently discarding its corresponding deficiencies, the circular visibility region of p is obtained. In the following, points in the CVD corresponding to visibility arcs that contain caps are identified.

The correspondence between points on the partitioning curves of the CVD and their corresponding visibility arcs is now established. Recall that each partitioning curve in the CVD may be composed of a line segment (or a half-line), a piece of a parabolic curve, or both of them[6]. A partitioning line segment (or a partitioning half-line) is a portion of the perpendicular bisector between p and a vertex of Q, which means that the visibility arc centered at a point on this line segment will hit this vertex of Q. On the other hand, a partitioning parabolic curve, which is a portion of the parabola defined by p and an edge of Q, represents the locus of the centers of the visibility arcs that are tangent to this edge of Q. Let the points joining two or more partitioning curves be called the *nodes* of the CVD.

A node that is joined by exactly two partitioning curves corresponds to the center of a visibility arc in one of the following three configurations: passing through two vertices of Q, passing through a vertex and tangent to an edge of Q, or tangent to two edges of Q. These vertices and/or edges of Q are the supports of this visibility arc. An illustration of the correspondence between a node of the CVD and its corresponding visibility arc is shown in Figure 6. Since this node v_c is at the point where β_i, a portion of the perpendicular bisector of p and v_i, is joined by β_j, a portion of the perpendicular bisector of p and v_j, it corresponds to a visibility arc which has two supports v_i and v_j. Since the visibility arcs containing the convex and concave caps of p are visibility arcs with supports satisfying Lemma 1, they can be identified by examining the nodes of the CVD.

In the degenerate case, a visibility arc may have more than two supports, which

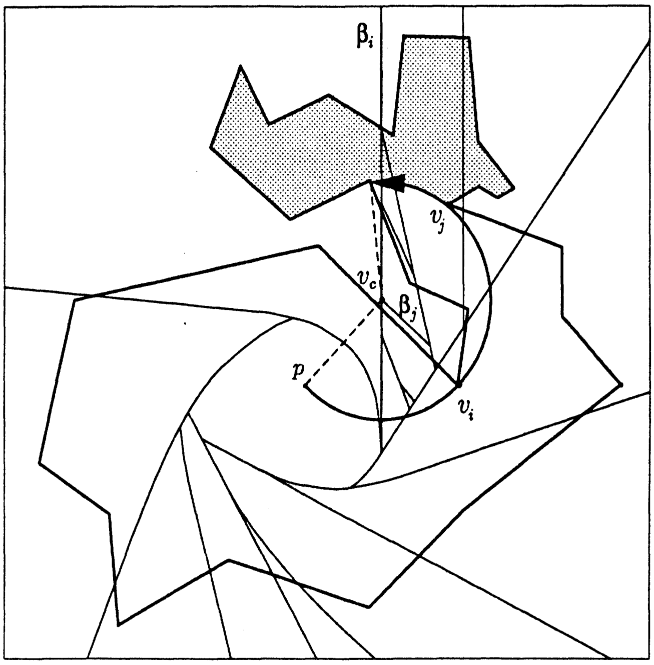

Figure 6: Correspondence between a cap of p and a node in the CVD.

corresponds to a node joined by more than two partitioning curves. In this case, there is still only one cap that needs to be identified. Suppose the sequence of the supports on a visibility arc $\widehat{pq}$ begins with a concave support v_i, as shown in Figure 7 (a), the region to the convex side of the arc between the first convex support v_j and the end of this visibility arc is not circularly visible from p. Arc $\widehat{v_j q}$ is therefore a convex cap of p. On the other hand, suppose the sequence of the supports on a visibility arc $\widehat{pr}$ begins with a convex support v_l, as shown in Figure 7 (b), the region to the concave side of the arc between the first concave support v_k and the end of this visibility arc is not circularly visible from p. Arc $\widehat{v_k r}$ is therefore a concave cap of p. Furthermore, if there exists a concave support v_s on $\widehat{v_j q}$, as shown in Figure

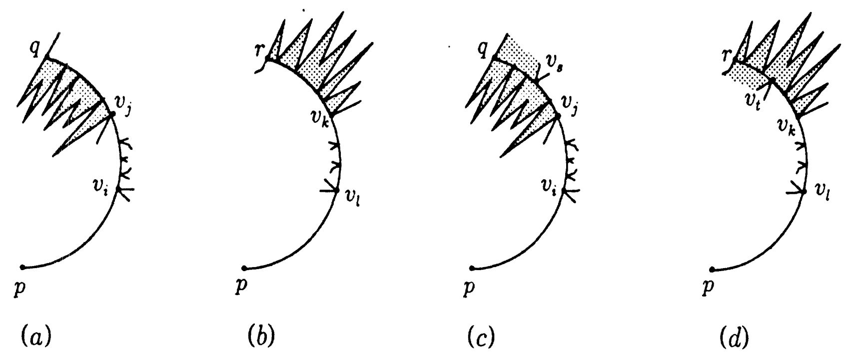

Figure 7: Visibility arcs that have more than two supports.

7 (c). Then, arc $\widehat{v_s q}$ is a dangling arc. As indicated earlier, the open set of the dangling arc $\widehat{v_s q}$ is empty, arc $\widehat{v_s q}$ can therefore be ignored, and arc $\widehat{v_j v_s}$ (rather than arc $\widehat{v_j q}$) becomes the convex cap contained by $\widehat{pq}$. Similarly, if there exists a convex support v_t on $\widehat{v_k r}$, as shown in Figure 7 (d). Then, arc $\widehat{v_t r}$ is a dangling arc, and is therefore ignored. Arc $\widehat{v_k v_t}$ (rather than arc $\widehat{v_k r}$) becomes the concave cap contained by $\widehat{pr}$.

By construction, the number of nodes and the number of partitioning curves of the CVD are of the same order as the vertices of Q, which is $O(n)$. The identification of the caps, which requires the examination of all the nodes of the CVD and the partitioning curves passing through each of the nodes, can therefore be completed in $O(n)$ time. With all the caps of p identified, the boundary of Q is then traversed to sew the caps to the boundary of Q contributing to the boundary of the circular visibility region, which can be achieved trivially in linear time. The total time required for this algorithm to construct the circular visibility region of a given point inside a simple polygon is thus $O(n)$.

Figure 6 illustrates the correspondence between a concave cap of p and its corresponding node in the CVD. The shaded region is the concave deficiency of p associating with the concave cap. Since this cap is the only cap of p, the unshaded portion of Q is thus the circular visibility region of p.

4. Circular Visibility Region of a Given Edge

It is established in this section that the boundary of such a circular visibility region consists of portions of the boundary of the simple polygon and the convex and concave caps computed with respect to the emanating edge. These caps are obtained by sweeping the region of the polygon with various sets of visibility arcs from the emanating edge. The CVDs of some of the vertices of the simple polygon are computed to determine the transition between two sets of sweeping arcs. Due to the lack of amortized behavior in searching of the transitions for the sweeping arcs, the time complexity for constructing the circular visibility region of an edge is substantially higher than that for computing for a point.

Let the emanating edge be denoted by $\overline{uv}$, directed from u to v, and let the rays emanating from u and v and coincident with $\overline{uv}$ hit Q at points u' and v', as shown in Figure 8. It is noted that CCW visibility arcs emanating from the left side of $\overline{uv}$ (except at v) do not cross $\overline{vv'}$. By duplicating $\overline{uv}$ and $\overline{vv'}$, a new polygon which consists of not only all the edges of Q but also edges $\overline{v'v}$, $\overline{vu}$, $\overline{uv}$, and $\overline{vv'}$ is obtained. The emanating edge $\overline{uv}$ becomes an edge on the boundary of this new polygon. Similarly, a new polygon consisting of all the edges of Q and edge $\overline{u'u}$, $\overline{uv}$, $\overline{vu}$, and $\overline{uu'}$ can be constructed, in which CCW arcs emanating from the left of $\overline{vu}$ will not hit $\overline{uu'}$ and $\overline{u'u}$. The circular visibility region of $\overline{uv}$ with CCW visibility arcs can be obtained by merging the circular visibility regions computed individually for the two polygons above. The CW circular visibility region can be established similarly. For simplicity of discussion, it is assumed that the emanating edge is an edge on the boundary of the simple polygon, as defined in the following.

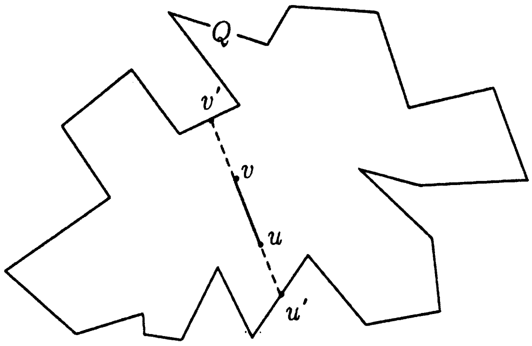

Figure 8: Decomposition of Q into two sub-polygons.

Problem CVR($\overline{uv}$,Q): circular visibility region of an edge
Given: a simple polygon Q with edges $\overline{uv}$, e_1, e_2, ..., and e_n.
Find: the portion of Q that is circularly visible from $\overline{uv}$.

Let a cap computed with respect to $\overline{uv}$ be called a cap of $\overline{uv}$. No other visibility arc emanating from $\overline{uv}$ will cross a cap of $\overline{uv}$. The region separated from the circular visibility region of $\overline{uv}$ by a cap of $\overline{uv}$ is thus called a *deficiency* of $\overline{uv}$. As adopted in Section 3, the deficiency that is to the convex (concave) side of a cap is called a convex (concave) deficiency, and the cap separating it from the circular visibility region is called a convex (concave) cap. In the following lemma, sufficient conditions for a point on $\overline{uv}$ to contribute a CCW visibility arc that contains a cap of $\overline{uv}$ are established.

Lemma 2 *Let p ($p \neq u$ and $p \neq v$) be a point on $\overline{uv}$ and $\widehat{pq}$ be a CCW visibility arc which emanates from p and hits Q at q.*
(1) Let $\widehat{v_j q}$ be a convex cap of p, where v_j is a convex support of $\widehat{pq}$. Then, $\widehat{v_j q}$ is also a convex cap of $\overline{uv}$ if p is the point on $\overline{uv}$ which is circularly visible from v_j and the closest to u, or if $\widehat{pq}$ is tangent to $\overline{uv}$ at p.
(2) Let $\widehat{v_i q}$ be a concave cap of to p, where v_i is a concave support of $\widehat{pq}$. Then, $\widehat{v_i q}$ is also a concave cap of $\overline{uv}$ if p is the point on $\overline{uv}$ which is circularly visible from v_i and the closest to v.

Proof. As visibility arcs only emanate from the left of $\overline{uv}$ and $\widehat{pq}$ is a CCW visibility arc, $\overline{pv}$ must lie on the concave side of $\widehat{pq}$. The two cases in the first part of the lemma are first discussed.

(1) Suppose p is the point on $\overline{uv}$ which is circularly visible from v_j and the closest to u, as shown in Figure 9 (a); points on $\overline{uv}$ which are circularly visible from v_j must lie on $\overline{pv}$. Since Q is a simple polygon, for an arc emanating from $\overline{pv}$ to reach $\widehat{v_j q}$, the arc must first cross $\widehat{v_i v_j}$ from its convex side, where v_i denotes the concave support that confines $\widehat{pq}$ and makes $\widehat{v_j q}$ a convex cap of p. Since $\overline{pv}$ lies on the concave side of $\widehat{pq}$, for an arc emanating from $\overline{pv}$ to reach $\widehat{v_i v_j}$, the arc must first cross $\widehat{pv_i}$ from its concave side. However, as two distinct arcs can have at most two intersections, an arc which emanates from $\overline{pv}$ and crosses $\widehat{pv_i}$ and $\widehat{v_i v_j}$ cannot cross $\widehat{v_j q}$ at a point other than v_j. Therefore, $\widehat{v_j q}$ is a convex cap of $\overline{uv}$.

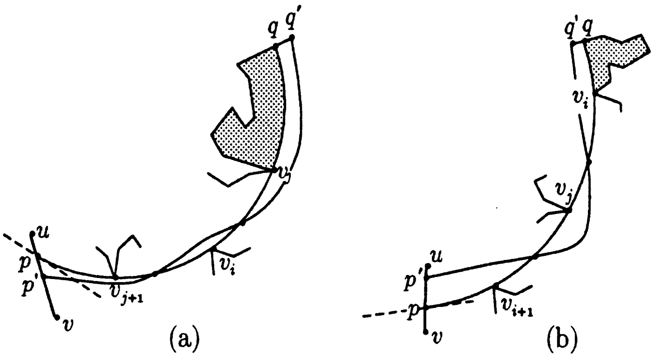

Figure 9: Convex and concave caps of $\overline{uv}$.

On the other hand, suppose p is the tangent point between $\widehat{pq}$ and $\overline{uv}$, which means that except for p, the rest of $\overline{uv}$ lies to the concave side of $\widehat{pq}$. For the same reason as established in the previous case, no other visibility arcs emanating from $\overline{uv}$ will cross $\widehat{v_jq}$ at a point other than v_j. Consequently, in this case, $\widehat{v_jq}$ is also a convex cap of $\overline{uv}$.
(2) The concave cap $\widehat{v_iq}$ of p is shown in Figure 9 (b). Since the proof for showing that this cap is also the concave cap of $\overline{uv}$ is analogous to that for showing the convex cap of $\overline{uv}$ in (1), further analysis is omitted. □

By the above lemma, it follows directly that some caps of an endpoint of $\overline{uv}$ are also the caps of $\overline{uv}$.

Corollary 1 *(1) A CCW convex cap of u is also a convex cap of $\overline{uv}$.*
(2) A CCW concave cap of v is also a concave cap of $\overline{uv}$.

Note that since $\overline{uv}$ is on the boundary of Q, a visibility arc of u computed in the presence of Q is always a visibility arc of $\overline{uv}$. Edge $\overline{uv}$ may be considered as a concave support of a visibility arc emanating from u if this arc is tangent to $\overline{uv}$. While Lemma 2 establishes sufficient conditions for a point on $\overline{uv}$ to emanate visibility arcs that contribute to caps of $\overline{uv}$, the following lemma establishes necessary conditions on the supports for such visibility arcs.

Lemma 3 *Let p ($p \neq u$ and $p \neq v$) be a point on $\overline{uv}$ and $\widehat{pq}$ be a CCW visibility arc which emanates from p and hits Q at q.*
(1) Let $\widehat{v_jq}$ be a convex cap of p, where v_j is a convex support of $\widehat{pq}$ and v_i is the only concave support on $\widehat{pv_j}$. Suppose p is the point on $\overline{uv}$ which is circularly visible from v_j and the closest to u. Then, there exists a convex support, v_{j+1}, on $\widehat{pv_i}$, such that $\widehat{pv_{j+1}}$ is a convex cap of v_j.
(2) Let $\widehat{v_iq}$ be a concave cap of p, where v_i is a concave support of $\widehat{pq}$ and v_j is the only convex support on $\widehat{pv_i}$. Suppose p is the point on $\overline{uv}$ which is circularly visible from v_i and the closest to v. Then, there exists a concave support, v_{i+1}, on $\widehat{pv_j}$, such that $\widehat{pv_{i+1}}$ is a concave cap of v_i.

Proof. The proofs are straightforward. In (1), if there does not exist such a convex support v_{j+1}, there must exist a visibility arc from v_j to $\overline{up}$ with a radius smaller than that of $\widehat{pq}$ as $\overline{up}$ lies to the convex side of $\widehat{pq}$. This contradicts the fact that p is the point on $\overline{uv}$ which is circularly visible from v_j and the closest to u. Since the order of supports on $\widehat{pv_j}$, emanating from v_j, follows the order described in Lemma 1 (1), $\widehat{pv_{j+1}}$ is a convex cap of v_j. Similarly, in (2), if there does not exist such a concave support v_{i+1}, p can not be the point on $\overline{uv}$ which is circularly visible from v_i and the closest to v. The order of supports on $\widehat{v_i p}$ follows the order given in Lemma 1 (2), arc $\widehat{pv_{i+1}}$ is therefore a concave cap of v_i. □

With the properties of the caps of $\overline{uv}$ identified, the details of the algorithm that computes the circular visibility region of $\overline{uv}$ are now described. First, terminologies and the hierarchy of the algorithm are given. It is clear that the portion of the polygon which is linearly visible from $\overline{uv}$ is also circularly visible from $\overline{uv}$. By subtracting this linear visibility polygon from Q, a set of *pockets* with respect to $\overline{uv}$, are obtained. Only circular visibility region inside the pockets need to be computed. A pocket is said to be a *CW pocket* if only CW arcs emanating from $\overline{uv}$ can reach the interior of the pocket, and a *CCW pocket* if only CCW arcs can enter it. (Since only CCW arcs are used for the illustration of the algorithm, CW pockets are ignored for the further consideration.) A line segment that separates a pocket from the linear visibility polygon is called the *lid* of the pocket. Every visibility arc which emanates from $\overline{uv}$ and enters a pocket must cross the lid of the pocket. Let the extension of the lid of a CCW pocket intersect $\overline{uv}$ at a point p, where $p \neq v$. It is noted that only the arcs emanating from points on $\overline{up}$ may reach the inside of this CCW pocket. This means that $\overline{up}$ can replace $\overline{uv}$ as the emanating edge, and therefore only the case where the extension of the lid intersects an endpoint of the emanating edge is analyzed.

The domain in which caps of $\overline{uv}$ need to be identified can be further reduced. In each CCW pocket, the convex and concave caps of v are first computed. This can be achieved by utilizing the CCW CVD of v, as discussed in Section 3. The region in a pocket that is circularly visible from v is also circularly visible from $\overline{uv}$. Furthermore, by Corollary 1, the concave caps of v are also the concave caps of $\overline{uv}$. Consequently, only the convex deficiencies of v may contain regions which are not circular visible from v but are circularly visible from $\overline{uv}$, and requires further examination.

The process for constructing caps of $\overline{uv}$ in the convex deficiencies of v is now described by using the example shown in Figure 10, in which $\widehat{v_j s_1}$ is a convex cap of v, the region to the convex side of $\widehat{v_j s_1}$ is a convex deficiency of v and is denoted as $D_v(\widehat{v_j s_1})$, and v_j is a convex support of $\widehat{vs_1}$. To start, three possible emanating points on $\overline{uv}$ which contribute visibility arcs that pass through v_j and contain convex caps of $\overline{uv}$ are identified. The distinction among these three emanating points and the approach to construct them are given in the following.

Case 1. Suppose that arc $\widehat{p_2 v_{j+1}}$, a CW convex cap of v_j as shown in Figure 11 (a), is identified, where p_2 ($p_2 \neq u$) is a point on $\overline{uv}$ and v_{j+1} is a convex support on $\widehat{p_2 v_j}$. Since arc $\widehat{p_2 v_{j+1}}$ is a convex cap of v_j, there must exist a concave support v_i, on $\widehat{v_j v_{j+1}}$, that makes $\widehat{p_2 v_{j+1}}$ a convex cap of v_j. Suppose that the CCW extension of $\widehat{v_j p_2}$ intersects the boundary of $D_v(\widehat{v_j s_1})$ at a point s_2. This concave support v_i also makes $\widehat{v_j s_2}$ a convex cap of p_2. Again, since arc $\widehat{p_2 v_{j+1}}$ is a convex cap of v_j,

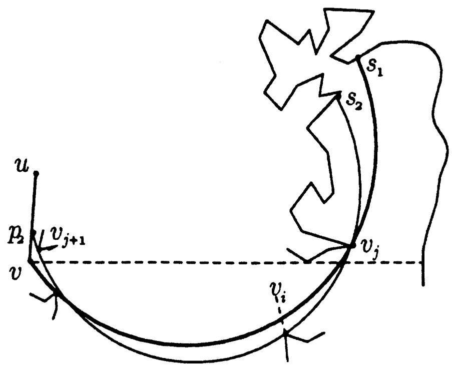

Figure 10: A convex cap $\widehat{v_j s_2}$ of $\overline{uv}$.

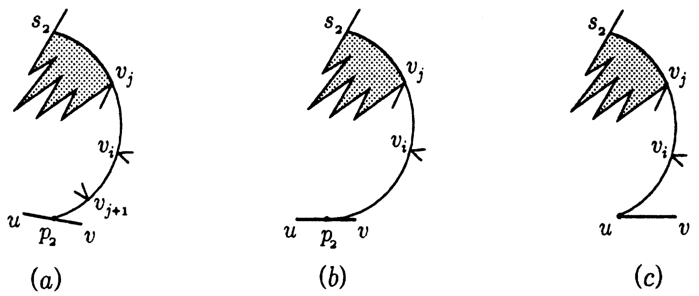

Figure 11: Three emanating points that contribute convex caps of $\overline{uv}$.

p_2 must be the point on $\overline{uv}$ which is circularly visible from v_j and the closest to u. By Lemma 2, arc $\widehat{v_j s_2}$ is a convex cap of $\overline{uv}$.
Case 2. Suppose a CW visibility arc of v_j having a concave support v_i and tangent to $\overline{uv}$ at p_2 is identified, as shown in Figure 11 (b). Due to the configuration of supports v_i and v_j on $\widehat{p_2 s_2}$, arc $\widehat{v_j s_2}$ is a convex cap of p_2, and by Lemma 2, a convex cap of $\overline{uv}$.
Case 3. Suppose that u is circularly visible from v_j, as shown in Figure 11 (c). By Corollary 1, the convex cap $\widehat{v_j s_2}$ of u is also a convex cap of $\overline{uv}$.

As visibility arcs of a point having supports in a certain configuration can be identified in the CVD of that point, which one of these three cases that is encountered can be determined by utilizing the CW CVD of v_j. First, assume that Case 1 is encountered. Let v_{j+1} denote the convex support on $\widehat{p_2 v_i}$.

Then, only the region bounded between $\widehat{v_j s_1}$ and $\widehat{v_j s_2}$ (the region in Q that is to the convex side of $\widehat{v_j s_1}$ and to the concave side of $\widehat{v_j s_2}$), as shown in Figure 10, remains to be decomposed by additional caps. Since whether or not a visibility arc contains a cap can be determined by the configuration of the supports that confine the arc, the pairs of supports that may create a cap of $\overline{uv}$ are identified. By propagating such supports and sweeping with visibility arcs passing through these supports, the caps of $\overline{uv}$ in the region bounded between $\widehat{v_j s_1}$ and $\widehat{v_j s_2}$ can be constructed efficiently.

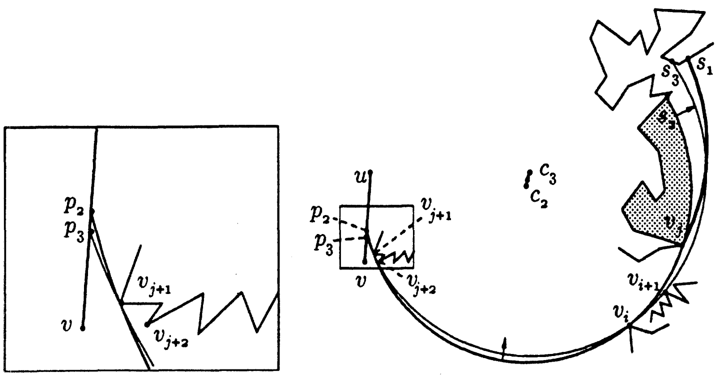

Figure 12: The transition of hinges of the sweeping arc.

The process of *sweeping* for the identification of the convex caps of $\overline{uv}$ is now described. Imagine that $\widehat{p_2s_2}$, as shown in Figure 12, is a physical entity with elasticity which retains circularity when bent. Suppose that this physical arc is bent against its two supports at v_{j+1} and v_i by gradually pulling the center of the arc at c_2 away from $\widehat{p_2s_2}$, along the perpendicular bisector between v_i and v_{j+1}. This process can also be viewed as sweeping with visibility arcs which emanate from $\overline{uv}$ and pass through two fixed points (provided that these arcs can reach $\overline{uv}$). Vertices v_{j+1} and v_i are called the ***hinges*** of this sweep, and the arc that is bent is referred to as a *sweeping arc.*

When the concave support is an edge rather than a vertex, the hinge point moves along the edge as the arc is swept. It is therefore the edge, rather than a fixed point on the edge, that is the hinge. Also, as the sweep proceeds, the center of the sweeping arc will move along a parabolic curve, rather than a straight line, defined by the edge and the other hinge which is a vertex of Q. Suppose instead of Case 1 it is Case 2 that is encountered, the emanating point p_2 and v_i will be used as the hinges, whereas if Case 3 is encountered, u and and v_i will be used as the hinges.

The sweep from $\widehat{p_2s_2}$ with hinges at v_{j+1} and v_i continues until the sweeping arc intersects the boundary of $D_v(\widehat{v_js_1})$ (not including $\widehat{v_js_1}$) at s_1, or comes in contact with either a convex support between v_i and v_{j+1} or a concave support between v_i and the intersection of the sweeping arc with the boundary of $D_v(\widehat{v_js_1})$. The part of the sweeping arc between v_{j+1} and its intersection with $\overline{uv}$ cannot come in contact with any concave support during the sweep because it always lies in the region to the convex side of $\widehat{vv_j}$ and to the concave side of $\widehat{p_2v_j}$. This region does not contain any portion of the boundary of Q because $\widehat{vv_j}$ is confined by a concave support and $\widehat{p_2v_j}$ by a convex support. Therefore, this part of the sweeping arc cannot encounter any support. When the sweeping arc reaches s_1, the sweep is discontinued as the entire region bounded by $\widehat{v_js_1}$ and $\widehat{v_js_2}$ is swept. On the other hand, if the sweeping arc comes in contact with one of the two types of supports before reaching s_1, the sweep is stopped for changing the hinge(s).

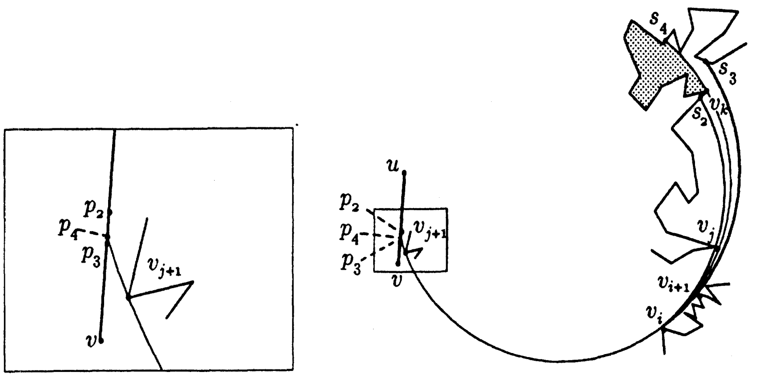

Figure 13: A convex deficiency in the region bounded between $v_i\widehat{s_2}$ and $v_i\widehat{s_3}$.

The case that the sweeping arc comes in contact with a concave support v_{i+1} between v_i and the intersection of the sweeping arc with the boundary of $D_v(v_j\widehat{s_1})$ is presented. Let the sweeping arc at this moment intersect $\overline{p_2v}$ at p_3 and intersect the boundary of $D_v(v_j\widehat{s_1})$ at s_3. All the convex caps in the region bounded between $v_i\widehat{s_2}$ and $v_i\widehat{s_3}$ can be identified during the sweep from $p_2\widehat{s_2}$ to $p_3\widehat{s_3}$ with hinges at v_{j+1} and v_i. Suppose that a convex support, such as v_k shown in Figure 13, lying in $D_v(v_j\widehat{s_1})$ and between v_i and s_4, the intersection of the sweeping arc with the boundary of $D_v(v_j\widehat{s_1})$, is encountered during the sweep. Since the sweeping arc is hinged at v_i and v_{j+1}, arc $p_4\widehat{v_{j+1}}$, where p_4 is the intersection of the sweeping arc at this instance with $\overline{uv}$, is a convex cap of v_k. By Lemma 2, arc $v_k\widehat{s_4}$ is a convex cap of $\overline{uv}$.

While conceptually the immediate successive hinge v_{i+1} and the convex caps of $\overline{uv}$ in the region swept before changing the hinge(s) are identified via the sweeping of visibility arcs, all of them can be identified directly from the CCW CVD of v_{j+1}. Since the sweeping arc is hinged at v_i and v_{j+1}, the centers of the visibility arcs emanating from $\overline{uv}$ which are generated during the sweep must lie on the perpendicular bisector between v_{j+1} and v_i. Recall that a portion of this perpendicular bisector is a partitioning curve to the CVD of v_{j+1}, and the nodes on this partitioning curve correspond to the centers of all the visibility arcs having v_{j+1}, v_i and other vertices or edges as supports. It is therefore sufficient to examine these nodes only. Since the sweeping arc is bent outwards, the radius of the sweeping arc must be getting larger accordingly. The search of the centers of the convex caps of $\overline{uv}$ starts from the node (on this partitioning curve) that defines $p_2\widehat{s_2}$, and moves away from v_{j+1} for visibility arcs with larger radii.

After the engagement of a new concave support at v_{i+1}, as shown in Figure 12, the sweeping arc hinged at supports v_i and v_{j+1} can no longer reach the region bounded by $v_j\widehat{s_1}$ and $v_j\widehat{s_2}$. To avoid the obstruction, the hinge v_i is replaced by the new concave support v_{i+1}. With the hinges v_{j+1} and v_{i+1}, the process of sweeping the region for identifying convex caps of $\overline{uv}$ and updating the pair of hinges is repeated, until the sweeping arc intersects the boundary of $D_v(v_j\widehat{s_1})$ at s_1.

In the degenerate case the sweeping arc may come in contact with more than

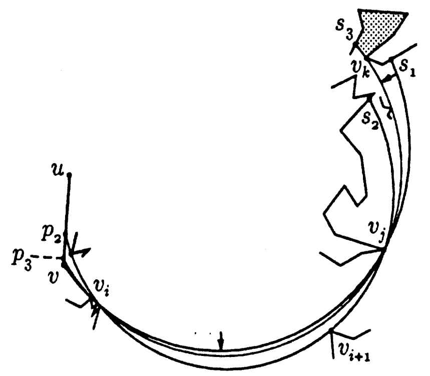

Figure 14: A concave deficiency in the region bounded between $v_j\widehat{s_1}$ and $v_j\widehat{s_2}$.

one support at the same time. Since the sweeping arc is bent outwards, it can be verified easily that when the sweeping arc comes in contact with multiple convex supports between v_i and v_{j+1}, the support that is the closest to v_i should be selected. On the other hand, if the sweeping arc comes in contact with multiple concave supports between v_i and the intersection of the arc with the boundary of $D_v(v_j\widehat{s_1})$, the support that is the farthest from v_i should selected. When there are multiple supports come in contact with the sweeping arc simultaneously between v_i and v_{j+1} and between v_i and the intersection of the sweeping arc with the boundary of $D_v(v_j\widehat{s_1})$, individual rules are applied and both of the hinges are changed.

After the above sweeping procedure, all the convex caps of $\overline{uv}$ in $D_v(v_j\widehat{s_1})$ are identified. Similarly, concave caps in the same region can be identified by bending $\widehat{vs_1}$ inwards against supports v_j and v_i, as shown in Figure 14. The main difference between the identification of convex caps and that of concave caps is in the transition of hinges. When $\widehat{vs_1}$ is bent inwards, it is the portion of the sweeping arc between v_i and v_j that may come in contact with another concave support and the portion of the sweeping arc between v_j and its intersection with the boundary of $D_v(v_j\widehat{s_1})$ that may become in contact with another convex support. Alternating with the changing of the hinges of the sweeping arc, the concave caps of $\overline{uv}$ can be identified by utilizing the CVDs of one of the two current hinges. By this procedure, all the concave caps of $\overline{uv}$ and in $D_v(v_j\widehat{s_1})$ can be identified.

The convex and concave caps of $\overline{uv}$ in all the convex deficiencies of v can be identified by using the above procedure. With the caps of $\overline{uv}$ identified, the boundary of the circular visibility region of $\overline{uv}$ is then completed by connecting these caps with the boundary of Q contributing to the boundary of the circular visibility region.

The time required for constructing the circular visibility region from $\overline{uv}$ is now analyzed. The (linear) edge visibility polygon from $\overline{uv}$ can be computed in linear time with an algorithm developed by Guibas, et al.[8]. The convex and concave caps of v which are computed in the beginning of the algorithm can be achieved in linear time, as illustrated in Section 3. Since the concave deficiencies of v are also the concave deficiencies of $\overline{uv}$ and the region in Q circularly visible from v is also circularly visible from $\overline{uv}$, only the convex deficiencies of v are further examined.

For each convex deficiency of v, the caps of $\overline{uv}$ inside is are computed, which invokes two processes: computing the successive hinges and computing caps of $\overline{uv}$ while sweeping arcs against the two hinges. Since both processes can be achieved in linear time with the aid of the CCW CVD of a hinge, the total time required for the identification of all the caps of $\overline{uv}$ in the convex deficiencies of v is proportional to the number of CVDs computed. A new CVD needs to be constructed only when the hinge with respect to which the CVD is computed is replaced by another support. Let the total number of CVDs computed be denoted as k. As each of the CVDs can be computed in linear time, the total time required for the identification of all the caps inside convex deficiencies of v is $O(kn)$, where n is the total number of vertices in Q.

The connection of the caps of $\overline{uv}$ with the boundary of Q contributing to the boundary of the circular visibility region from $\overline{uv}$ can be achieved in linear time. The total time complexity for constructing the circular visibility region of $\overline{uv}$ is dominated by the identification of the caps of $\overline{uv}$ inside convex deficiencies of v, which is $O(kn)$. It is noted that in the worst case, the number of the hinges with respect to which the CVDs are computed is $O(n)$; the worst-case total time complexity of this algorithm is therefore $O(n^2)$.

5. Conclusion

This paper utilized the structure of circular visibility diagrams in developing algorithms for computing circular visibility regions inside a simple polygon. It is shown that the circular visibility region of a point inside a simple polygon can be constructed in linear time. It is also shown that the circular visibility region of an edge inside a simple polygon can be constructed in $O(nk)$ time, where n is the total number of vertices in Q and k is the total number of different sets of hinges for the caps.

The increase in the time required for computing a circular visibility region from a given edge versus that for computing a linear visibility polygon of a given edge is also observed in the computing of the *width* of a polygon. It has been shown that the minimum distance of a pair of parallel lines which contain a simple polygon can be computed in linear time[9], whereas the minimum distance of a pair of concentric circles which contain the same simple polygon can be computed in $O(n \log n + m)$ time[10], where m is the number of intersections of the farthest point Voronoi diagram and the medial axis. In the worst case, m is equal to $O(n^2)$.

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