# A SINGULAR INTEGRAL EQUATION APPROACH TO ELECTROMAGNETIC FIELDS FOR CIRCULAR BOUNDARIES WITH SLOTS *)

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**Summary**

It is shown that for the case of a single cylinder which has any number of axial slots of arbitrary width and infinite length, or for the case of coaxial cylinders where one of the cylindrical boundaries has such slots, the Dirichlet and Neumann problems for the Helmholtz equation (which correspond respectively to E and H waves) can be reduced to that of solving a singular integral equation. It is also shown that the resulting singular integral equation is formally the same for both the Dirichlet and Neumann problems for various kinds of circular boundaries. The exact solution of the integral equation is given and applied to the Dirichlet and Neumann problems. The following three simple cases: (1) a single narrow slot in a cylinder; (2) a single narrow slot in a coaxial cylinder; and (3) narrow circular strips are considered to illustrate the applicability of the method.

§ 1. *Introduction.* A circular cylinder (or a circle) has been a favored geometry of boundary value problems, and has been the subject of many investigations in acoustic or electromagnetic field theory studies. In particular, treatments have been given for fields for the case of a slotted cylinder[1]) and for the solution of the Dirichlet and Neumann problems for the case of a circle with a narrow slit in it[2]). However, in these treatments, it is usual to assume that the distribution of the field components in the slot is known[1]) or that it can be replaced by a known distribution of a

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*) The research reported in this paper was sponsored by the National Aeronautics and Space Administration under Grant NsG-472.

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static field[2]). On the other hand, Lewin[3]) has shown that the discontinuities in a rectangular waveguide may be analyzed with the help of a singular integral equation.

Recently, the problems of electromagnetic fields in domains bounded by a circular cylinder (or coaxial circular cylinders) with a finite number of axial slots of arbitrary width and infinite length, for given axial line sources, have become of interest in connection with some practical applications. Under these circumstances, it is necessary to solve for the fields without any particular assumptions regarding the distribution of the field in the slots. It is the purpose of this paper to show how to solve these problems rigorously and generally. In the following, (i) the solutions of the Dirichlet and Neumann problems (that is, $z$-components of the E- and H-fields respectively) are represented by Fourier-Bessel series with unknown coefficients; and (ii) dual series equations for these coefficients are derived; (iii) the dual series equations are then converted into an integral equation; (iv) then the kernel, after the tangential differentiation, is shown to be a singular one of Cauchy type, (with the help of estimation formulas for the Bessel and Hankel functions due to the author). It is also shown that the singular integral equations for the Dirichlet and Neumann problems are formally the same, and hence (v) both can be solved simultaneously. This process is identical for various kinds of cylindrical boundaries. In this paper, the method is illustrated by some examples, the most detailed of which is for a single cylinder with slots. In particular, the detailed calculation is given for the H-wave case for a narrow slot and for the E-wave case for a narrow circular strip. (If two or more cylinders have slots in them, the method leads to simultaneous singular integral equations. In this paper, for simplicity, we restrict attention to the case where only one cylinder has slots in it.)

The results are valid for all wave numbers $k$ and all values of the radii of the cylinders as well as for any number of slots of arbitrary widths. It should be noted that these results cover not only the problems of slots, but those of circular strips as well.

The theory of a singular integral equation which Lewin[3]) employed is that of the so-called dominant equation[4]) where the kernel is simply $1/(x-y)$, where $x$ and $y$ are points on the path of integration. This theory originally arose during an analysis of static problems[4]) (i.e., of the Laplace equation). Lewin applied this theory neglecting higher order modes, and concluded that his result would be improved if a constant appearing in his equation were replaced by a polynomial. However, in our problem we have to take a kernel the form of which is $(x-y)^{-1}+k(x, y)$, where $k(x, y)$ is a certain rational function, instead of a polynomial. An abbreviated theory for such a case is introduced in Appendix 2.

§ 2. *A cylinder with slots.* As a typical example, we will consider, in detail, the solution for the field when the boundary is a cylinder with arbitrary slots in it.

Suppose that a circular cylinder of perfect conductivity with $\nu$ slots in it is described by cylindrical coordinates $(r, \phi, z)$ as follows:

$$r=a, \quad \alpha_j<\phi<\beta_{j+1}, \quad -\infty<z<\infty, \ (j=1,2, \ldots, \nu),$$

where $\beta_j<\alpha_j<\beta_{j+1}$ and $\beta_{\nu+1}=\beta_1$ (fig. 1). Assume that axial (electric and/or magnetic) line sources are located at

$$Q_{\mathrm{i}}: \ r=r_{\mathrm{i}}, \ \phi=\phi_{\mathrm{i}},$$

$$Q_{\mathrm{e}}: \ r=r_{\mathrm{e}}, \ \phi=\phi_{\mathrm{e}},$$

where $r_{\mathrm{i,e}}$ and $\phi_{\mathrm{i,e}}$ are given arbitrary constants such that

$$0 \leq r_{\mathrm{i}}<a<r_{\mathrm{e}} \ \text{and} \ 0 \leq \phi_{\mathrm{i}}, \phi_{\mathrm{e}} \leq 2\pi .$$

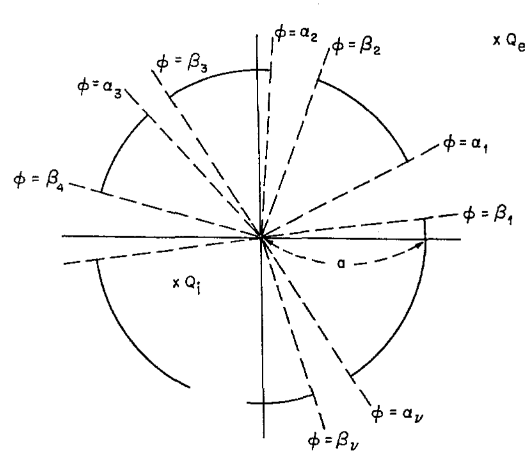


Fig. 1. A circular cylinder with slots.

As is well known, when the time dependence $e^{i\omega t}$ and the $z$-dependence $e^{\gamma z}$ ($i = \sqrt{-1}$, $\omega$ and $\gamma$ are fixed constants) are suppressed, the $z$-components $u = E_z$ and $u = H_z$ of the electric and magnetic fields, respectively, satisfy

$$\Delta u + k^2 u = 0, \tag{1}$$

where $k^2 = \omega^2 \varepsilon \mu + \nu^{\gamma 2}$. ($\varepsilon$ and $\mu$ are given complex constants, representing the complex dielectric constant and magnetic permeability respectively.) Further requirements on $u$ are the radiation condition at infinity, the boundary condition on the walls of the cylinder, the continuity condition of $u$ and $\partial u/\partial r$ throughout the slots, and the edge conditions at the edges of the slots.

First of all, a solution $u$ of (1), which satisfies the radiation condition, can be represented as

$$\begin{aligned} u \equiv u_{\mathrm{e}} &= \sum_{n=-\infty}^{\infty} A_n H_n(kr)\, e^{in\phi} + f_{\mathrm{e}} H_0(kR_{\mathrm{e}}), \quad (a < r) \\ u \equiv u_{\mathrm{i}} &= \sum_{n=-\infty}^{\infty} B_n J_n(kr)\, e^{in\phi} + f_{\mathrm{i}} H_0(kR_{\mathrm{i}}), \quad (0 \leq r < a) \end{aligned} \tag{2}$$

where $A_n$ and $B_n$ are unknown coefficients, and where, respectively, $J_n$ and $H_n$ are the Bessel function and the Hankel function of the second kind, $f_{\mathrm{i}}$ and $f_{\mathrm{e}}$ are given constant amplitudes (including zero) of sources at $Q_{\mathrm{i}}$ and $Q_{\mathrm{e}}$, and $R_{\mathrm{i}}$ and $R_{\mathrm{e}}$ are distances from $Q_{\mathrm{i}}$ and $Q_{\mathrm{e}}$.

Then, the boundary conditions, together with the continuity conditions, are equivalent to

$$u_{\mathrm{e}} = u_{\mathrm{i}};\ r = a,\ 0 \leq \phi \leq 2\pi;\ \frac{\partial u_{\mathrm{e}}}{\partial r} = \frac{\partial u_{\mathrm{i}}}{\partial r}, \tag{3}$$

$$u_{\mathrm{e}} = 0;\ r = a,\ \alpha_j < \phi < \beta_{j+1};\ \frac{\partial u_{\mathrm{e}}}{\partial r} = 0, \tag{4}$$

$$\frac{\partial u_{\mathrm{e}}}{\partial r} = \frac{\partial u_{\mathrm{i}}}{\partial r};\ r = a,\ \beta_j < \phi < \alpha_j;\ u_{\mathrm{e}} = u_{\mathrm{i}} \tag{5}$$

foı $j = 1, 2, \ldots, \nu$. In the first row, $u = E_z$, and in the third row, $u = H_z$. In other words, $u = E_z$ and $u = H_z$ are solutions of the Dirichlet and Neumann problems for (1), respectively. The second row shows the ranges of variables for the expressions in the first and third rows.

On substituting (2) into (3) and making use of the orthogonality of $\{e^{in\phi}\}$ in $(0, 2\pi)$, (3) turns out to be a linear equation in terms of $A_n$ and $B_n$, which, when $u = E_z$ determines $B_n$ in terms of $A_n$ as follows:

$$B_n J_n(a) = A_n H_n(a) + f_{en} J_n(a) H_n(r_e) - f_{in} J_n(r_i) H_n(a), \qquad (6)' \text{ *)}$$

where the abbreviated notations

$$J_n(a) = J_n(ka), \quad J_n(r_i) = J_n(kr_i), \quad H_n(r_e) = H_n(kr_e), \qquad (7)$$
$$f_{en} = f_e \, e^{-in\phi_e}, \quad f_{in} = f_i \, e^{-in\phi_i}$$

have been employed here as well as in the following.

Similarly, when $u = H_z$, we have

$$B_n J_n'(a) = A_n H_n'(a) + f_{en} J_n'(a) H_n(r_e) - f_{in} J_n(r_i) H_n'(a), \qquad (6)''$$

where a prime means the derivative with respect to the argument, e.g.,

$$H_n'(a) = \left. \frac{\partial H_n(kr)}{\partial kr} \right|_{r=a} .$$

On substituting (2) into (4), we obtain

$$\sum_{n=-\infty}^{\infty} \{A_n H_n(a) + f_{en} J_n(a) H_n(r_e)\} \, e^{in\phi} = 0; \quad (\alpha_j < \phi < \beta_{j+1}), \qquad (8)'$$

for $u = E_z$, and

$$\sum_{n=-\infty}^{\infty} \{A_n H_n'(a) + f_{en} J_n'(a) H_n(r_e)\} \, e^{in\phi} = 0; \quad (\alpha_j < \phi < \beta_{j+1}), \qquad (8)''$$

for $u = H_z$.

From (2), (5) and (6), we have

$$\sum_n \frac{-1}{J_n(a)} \{A_n - f_{in} J_n(r_i)\} \, e^{in\phi} = 0, \quad (\beta_j < \phi < \alpha_j) \qquad (9)'$$

for $u = E_z$, and

$$\sum_n \frac{-1}{J_n'(a)} \{A_n - f_{in} J_n(r_i)\} \, e^{in\phi} = 0, \quad (\beta_j < \phi < \alpha_j). \qquad (9)''$$

*) The expressions which specifically refer to the Dirichlet problem ($u = E_z$) have been numbered with a stroke ($'$), and those for the Neumann problem ($u = H_z$) have been numbered with a double stroke ($''$).

These two equations (8) and (9) are the dual series equations for the unknown coefficients $A_n$.

The left hand side of (9)′, which is zero in the slots $\beta_j < \phi < \alpha_j$, will define an unknown function $2\pi\tau(\phi)$ on the wall $\alpha_j < \phi < \beta_{j+1}$. In fact, $\tau(\phi)$ is a quantity proportional to the $z$-component of the surface current on the wall of the cylinder. Hence, by virtue of the orthogonality of $\{e^{in\phi}\}$, (9)′ can be shown to be equivalent to

$$A_n = -J_n(a) \int_L \tau(\theta)\, e^{-in\theta}\, d\theta + f_{in} J_n(r_i), \tag{10$'$}$$

where $L$ stands for the cross section of the wall of the cylinder, i.e.

$$L:\ r = a,\ \alpha_j < \phi < \beta_{j+1},\quad (j = 1, 2, \ldots, \nu).$$

On the other hand, the left hand side of (8)″ will define an unknown function $2\pi\tau(\phi)$ in the slots, which is a quantity proportional to the tangential component of the electric field. Hence, (8)″ is equivalent to

$$A_n = \frac{1}{H_n'(a)} \int_L \tau(\theta)\, e^{-in\theta}\, d\theta - f_{en} \frac{J_n'(a)}{H_n'(a)} H_n(r_e), \tag{10$''$}$$

In this case, $L$ stands for the cross section of the slots, i.e.

$$L:\ r = a,\ \beta_j < \phi < \alpha_j\quad (j = 1, 2, \ldots, \nu).$$

On substituting (10)′ into (8)′, (and similarly, on substituting (10)″ into (9)″), we obtain an integral equation with respect to the unknown function $\tau(\phi)$, which is formally the same for both the Dirichlet and Neumann problems.

$$\sum_{n=-\infty}^{\infty} S_n \int_L \tau(\theta)\, e^{in\Theta}\, d\theta = cf(\phi), \tag{11}$$

where $c$ is a constant (see (17)), $\Theta = \phi - \theta$, and

$$L = \begin{cases} \text{wall};\ r = a,\ \alpha_j < \phi < \beta_{j+1}, & j = 1, 2, \ldots, \nu, \\ \text{slots};\ r = a,\ \beta_j < \phi < \alpha_j, & j = 1, 2, \ldots, \nu, \end{cases} \tag{12}$$

$$-cf(\phi) = \begin{cases} f_e \sum_{n=0}^{\infty} \varepsilon_n J_n(a) H_n(r_e) \cos n(\phi - \phi_e) + \\ \qquad + f_i \sum_{n=0}^{\infty} \varepsilon_n J_n(r_i) H_n(a) \cos n(\phi - \phi_i), \\ f_e \sum_{n=0}^{\infty} \varepsilon_n \dfrac{H_n(r_e)}{H_n'(a)} \cos n(\phi - \phi_e) + \\ \qquad + f_i \sum_{n=0}^{\infty} \varepsilon_n \dfrac{J_n(r_i)}{J_n'(a)} \cos n(\phi - \phi_i), \end{cases} \tag{13}$$

where $\varepsilon_n = 1$ for $n = 1$ and $\varepsilon_n = 2$ for $n > 1$,

$$S_n = \begin{cases} -J_n(a)\, H_n(a), \\ \dfrac{-1}{J_n'(a)\, H_n'(a)}. \end{cases} \tag{14}$$

The upper lines of (12), (13) and (14) are for the Dirichlet problem and the lower lines are for the Neumann problem. Thus these problems are mutually transferable by exchanging dually the meaning of $L$, $c$, $f(\phi)$ and $S_n$.

Because of the edge condition at the edges, the unknown function $\tau(\phi)$ for both the Dirichlet and Neumann problems must have a singularity of order $O(\rho^{-\frac{1}{2}})$ at the edges of the slots, where $\rho$ is the distance from the edge. Hence, we are looking for a solution $\tau(\phi)$ of (11), which has a singularity of $O(\rho^{-\frac{1}{2}})$ at the end points of $L$.

Conversely, if we find a solution $\tau$ of (11) which satisfies this edge condition, and if $A_n$ is determined by (10) in terms of $\tau$, and $B_n$ is determined by (6) in terms of $A_n$, and finally, if $u$ is determined by (2) in terms of $A_n$ and $B_n$, then we can prove that $u = E_z$ and $u = H_z$ are the desired field components, that is they satisfy (1), the radiation condition, the boundary condition, the continuity condition, and the edge condition. (This fact is proved by the uniqueness of the Fourier series for $u$ and $\tau$.)

§ 3. *The solution of the integral equation.* It has been proved, in the preceding section, that the original problems are equivalent to that of solving the fundamental integral equation (11). Now we will proceed to solve it.

With the help of the well known formulas,

$$J_n = (-1)^n J_{-n},\ H_n = (-1)^n H_{-n},\ J_n' = (-1)^n J_{-n}',$$
$$H_n' = (-1)^n H_{-n}',$$

it is easy to see that

$$S_{-n} = S_n. \tag{15}$$

If we write $S_n$ for any integer $n$ ($\neq 0$) as

$$S_n = \frac{c}{|n|}\{1 + s_{|n|}\}, \tag{16}$$

where

$$c = \begin{cases} -i/\pi \\ -i\pi(ka)^2, \end{cases} \tag{17}$$

then, by virtue of (15) and by the application of the formulas in Appendix 1, it can be shown that for any integer $n$

$$s_{|n|} = O((ka)^2/4n).$$

Hence, if we choose a positive integer $N$ such that $(ka)^2/4N$ is sufficiently small, then the $s_{|n|}$ are quantities which are negligibly small when $n > N$ *).

By virtue of (15) and (16), (11) can be rewritten as

$$\int_L \tau(\theta)\left\{S_0 + 2c\sum_{n=1}^{\infty}\left(\frac{1}{n}\cos n\Theta + \frac{s_n}{n}\cos n\Theta\right)\right\} d\theta = cf(\phi),$$

which turns out, with the help of the formula

$$2\sum_{n=1}^{\infty}\frac{1}{n}\cos n\Theta = \log\frac{1}{2-2\cos\Theta}, \quad \Theta \neq 0,$$

to be

$$\int_L \tau(\theta)\, N(\phi, \theta)\, d\theta = f(\phi) + \varepsilon_N, \tag{18}$$

where

$$N(\phi, \theta) = \log\frac{1}{2-2\cos\Theta} + \sum_{n=1}^{N}\frac{2s_n}{n}\cos n\Theta + \frac{S_0}{c},$$

$$\varepsilon_N = -2\sum_{n=N+1}^{\infty}\frac{s_n}{n}\int_L \tau(\theta)\cos n\Theta\, d\theta.$$

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*) We assume that $k$ and $a$ are such that $J_n(a)\, J_n'(a)\, H_n'(a) \neq 0$. Otherwise, additional consideration is necessary for these "resonant" cases where $J_n(a)\, J_n'(a)\, H_n'(a) = 0$. Similar assumptions will be made for other cases in sections 4 and 5.

Because $s_n/n = O(n^{-2})$ for $n \geq N + 1$, and $\sum_{n=1}^{\infty} n^{-2}$ is a convergent series, it can be shown that $|\varepsilon_N| < \varepsilon$ for any given positive number $\varepsilon$, if $n > N$ and $N$ is chosen to be sufficiently large.

Then, on neglecting $\varepsilon_N$ and differentiating both sides of (18) with respect to $\phi$, we have

$$\int_L \tau(\theta)\, N'(\phi, \theta)\, \mathrm{d}\theta = f'(\phi), \tag{19}$$

where $N'(\phi, \theta)$ and $f'(\phi)$ are respectively the derivatives of $N(\phi, \theta)$ and $f(\phi)$ with respect to $\phi$. It is easy to see that (19) is equivalent to

$$\int_L \tau(\theta) \left\{ \frac{i}{2} - \frac{\sin \Theta}{2 - 2\cos \Theta} - k(\phi, \theta) \right\} \mathrm{d}\theta = \tfrac{1}{2} f'(\phi), \tag{20}$$

where

$$k(\phi, \theta) = \frac{i}{2} \{1 - \sum_{n=1}^{N} s_n (\mathrm{e}^{in\Theta} - \mathrm{e}^{-in\Theta})\}. \tag{21}$$

Because of the singularity of the kernel at $\theta = \phi$, the integral in (20) is taken in the sense of Cauchy's principal value. Thus (20) is a singular integral equation derived from (11).

Suppose that $z = r\,\mathrm{e}^{i\theta}$ is a point in a complex plane, then

$$t = a\,\mathrm{e}^{i\theta},\ t_0 = a\,\mathrm{e}^{i\phi}, \tag{22}$$

are points on $L$. By the transformation (22), (20) is transformed into an expression in complex variables as follows:

$$\frac{1}{\pi i} \int_L \tau(t) \left\{ \frac{1}{t - t_0} - k(t_0, t) \right\} \mathrm{d}t = f(t_0), \tag{23}$$

where

$$\tau(t) = i\pi\tau(\theta),$$

$$f(t_0) = \tfrac{1}{2} f'(\phi),$$

and

$$k(t_0, t) = \frac{1}{it} k(\phi, \theta) =$$

$$= \frac{1}{2t} \left\{ 1 - \sum_{n=1}^{N} s_n \left(\frac{t_0}{t}\right)^n + \sum_{n=1}^{N} s_n \left(\frac{t}{t_0}\right)^n \right\} = \sum_{n=-N}^{N} k_n \frac{t_0^n}{t^{n+1}}, \tag{24}$$

where

$$k_n = \begin{cases} \dfrac{1}{2}\left(1 - \dfrac{n}{c} S_n\right), & n \geqq 0, \\ \\ -\dfrac{1}{2}\left(1 + \dfrac{n}{c} S_n\right), & n < 0. \end{cases} \tag{25}$$

The singular integral equation (23) can be solved by the theory described in Appendix 2, and the solution of (23), that is, of (19), which satisfies the edge condition mentioned before, is given by (5) of Appendix 2. Under the transformation (22), this solution is seen to be equivalent to

$$\tau(\phi) = \frac{-iX(\phi)}{2\pi^2} \int_L \frac{1}{1 - e^{i(\phi-\theta)}} \frac{f'(\theta)}{X(\theta)} d\theta - \frac{iX(\phi)}{\pi} \sum_{n=-N}^{N+\nu} p_n a^n e^{in\phi}, \tag{26}$$

where $p_n$ $(n = -N, -N+1, \ldots, N+\nu)$ are constants which are determined by (12) of Appendix 2. Corresponding to (4) of Appendix 2, $X(\phi)$ is defined as

$$X(\phi) = 1/a^\nu \sqrt{\prod_{j=1}^{\nu} (e^{i\phi} - e^{i\alpha_j})(e^{i\phi} - e^{i\beta_j})}, \tag{27}$$

Although (26) satisfies (23), or (19), it does not necessarily satisfy the original integral equation (18), because (19) was derived by the differentiation of (18) with respect to $\phi$. This will be discussed next, and the necessary and sufficient conditions for $\tau(\phi)$ defined by (26) to satisfy (18) too will be given.

On substituting (26) into (19), one has

$$\sum_{n=-N}^{N+\nu} p_n F_n'(\phi) + G'(\phi) = 0, \tag{28}$$

where

$$F_n'(\phi) = \frac{-ia^n}{\pi} \int_L N'(\phi, \theta)\, X(\theta)\, e^{in\theta}\, d\theta$$

$$G'(\phi) = \frac{-i}{2\pi^2} \int_L N'(\phi, \theta)\, X(\theta)\, d\theta \int_L \frac{1}{1 - e^{i(\theta-\psi)}} \frac{f'(\psi)}{X(\psi)} d\psi - f'(\phi).$$

Suppose that $F_n(\phi)$ and $G(\phi)$ are functions defined by

$$F_n(\phi) = \frac{-ia^n}{\pi} \int_L N(\phi, \theta)\, X(\theta)\, e^{in\theta}\, d\theta$$

$$G(\phi) = \frac{-i}{2\pi^2} \int_L N(\phi, \theta)\, X(\theta)\, d\theta \int_L \frac{1}{1 - e^{i(\theta-\psi)}} \frac{f'(\psi)}{X(\psi)}\, d\psi - f(\phi),$$

then, on integrating (28) with respect to $\phi$ from $\phi = \phi_0$ to $\phi = \phi$, where $\phi_0$ is an arbitrary fixed value in $L$, we have,

$$\sum_{n=-N}^{N+\nu} p_n F_n(\phi) + G(\phi) = \sum_{n=-N}^{N+\nu} p_n F_n(\phi_0) + G(\phi_0).$$

Because of (28), the right hand side of the last expression is a constant which is independent of the choice of $\phi_0$. On the other hand, the substitution of (26) into (18) implies that

$$\sum_{n=-N}^{N+\nu} p_n F_n(\phi) + G(\phi) = 0.$$

This means that $\tau$ defined by (26) satisfies (18) if and only if $\{p_n\}$ satisfies the additional condition

$$\sum_{n=-N}^{N+\nu} p_n F_n(\phi_0) + G(\phi_0) = 0. \tag{29}$$

Thus, we conclude that the solution $\tau$ of (18) is given by (26) when $\{p_n\}$ are determined by (12) of Appendix 2 and (29).

This is our principal result. It can be used to determine the fields for any number of arbitrary axial slots (as well as circular strips) and for arbitrary axial line sources. The results are valid for all values of the wave number $k$ and cylinder radius $a$.

In particular, we have obtained the axial component of the surface electric current on the wall of the cylinder for the case of the E-wave, and for the H-wave case we have obtained the tangential component of the electric field in the slots. Although the two solutions are formally the same, the different interpretation to be given to $L$, $cf(\phi)$, and $\{p_n\}$ serves to distinguish them in the two cases. It should also be noted that the determination of $\{p_n\}$ given by (12) of Appendix 2 involves $k_n$ (see (25)), where $k_n$ depends on $S_n$ whose definition itself depends on whether E-waves or H-waves are being considered.

We now consider some simple examples of the theory.

§ 4. *Examples.* A. A single slot in a cylinder. Suppose that there is one slot in a cylinder (fig. 2 and 3), that is, $\nu = 1$, $\beta_1 = -\alpha$ and $\alpha_1 = \alpha$ $(0 \leq \alpha < \pi)$. In fig. 3, it may be a circular

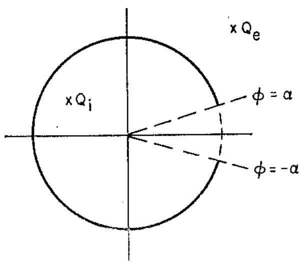


Fig. 2. A slot in a cylinder.

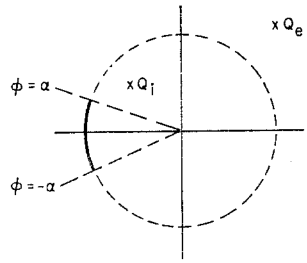


Fig. 3. A circular strip.

strip. Also, for the sake of simplicity, suppose that $|ka| < 1$. Consequently, we can assume that $N = 1$ since for $n > N = 1$, $s_n = 0(1/4n) \doteqdot 0$. Then for the Dirichlet problem, i.e. for E-wave, (26) and (27) can be reduced to

$$\tau(\phi) = \frac{-iX(\phi)}{2\pi^2} \int_{\alpha}^{2\pi-\alpha} \frac{1}{1-e^{i(\phi-\theta)}} \frac{f'(\theta)}{X(\theta)} d\theta - \frac{iX(\phi)}{\pi} \sum_{n=-1}^{2} a^n e^{in\phi},$$

$$X(\phi) = 1/a\sqrt{(e^{i\phi} - e^{i\alpha})(e^{i\phi} - e^{-i\alpha})}, \tag{30}$$

where $p_n$ $(n = -1, 0, 1, 2)$ are determined by

$$\alpha_{-2}p_{-1} + \alpha_{-1}p_0 + \alpha_0 p_1 + (\alpha_1 + \beta_{-1}/k_1)p_2 = \frac{i}{\pi}(\gamma_0 f_{-1} + \gamma_1 f_0).$$

$$\alpha_{-1}p_{-1} + \alpha_0 p_0 + (\alpha_1 + \beta_{-1}/k_0)p_1 + (\alpha_2 + \beta_{-2}/k_0)p_2 = \frac{i}{\pi}\gamma_0 f_0.$$

$$(\alpha_0 + \gamma_0/k_{-1})\,p_{-1} + \alpha_1 p_0 + \alpha_2 p_1 + \alpha_3 p_2 = 0.$$

$$\sum_{n=-1}^{2} p_n F_n(\phi_0) + G(\phi_0) = 0. \tag{31}$$

where $\phi_0$ is an arbitrary value such that $\alpha < \phi_0 < 2\pi - \alpha$, say $\phi_0 = \pi$. For the Neumann problem, i.e., for H-wave, the results are the same as those mentioned above but the integral in (30) should be taken from $\theta = -\alpha$ to $\theta = \alpha$, and $\phi_0$ is $-\alpha < \phi_0 < \alpha$, say $\phi_0 = 0$.

(1) *A narrow slot for H-wave.* Suppose that the slot $-\alpha < \phi < \alpha$ is narrow in the sense that (fig. 2)

$$(2\alpha)^2 \ll 1 \tag{32}$$

then, for the Neumann problem (H-wave), the integral over $(-\alpha, \alpha)$ in (30) can be simplified. In fact, when $\phi \varepsilon L$; $-\alpha < \phi < \alpha$, then

$$|\theta - \alpha|,\ |\phi - \theta| \leq 2\alpha < 1,$$

so if $X(\theta)$ and $1 - e^{i(\phi-\theta)}$ are expanded in series, then we can neglect those terms of $O(\alpha^2)$. Consequently, we have

$$1 - e^{i(\phi-\theta)} \doteqdot -i(\phi - \theta)$$

and

$$X(\phi) \doteqdot e^{-i\phi}/a\sqrt{\alpha^2 - \theta^2}. \tag{33}$$

Hence, we have

$$\tau(\phi) = \frac{1}{2\pi^2\sqrt{\alpha^2 - \phi^2}} \int_{-\alpha}^{\alpha} \left\{\frac{1}{\phi - \theta} - i\right\} f'(\theta)\sqrt{\alpha^2 - \theta^2}\, d\theta - \\ - \frac{i}{\pi} \frac{1}{\sqrt{\alpha^2 - \phi^2}} \sum_{n=-1}^{2} p_n (a\, e^{i\phi})^{n-1}. \tag{34}$$

In this case, (31), which will determine $\{p_n\}$ in (34), will be more precise if we calculate $\alpha_m$ $(m = -2, -1, 0, 1, 2, 3)$, $\beta_m$ $(m = -2, -1)$, $\gamma_m$ $(m = 0, 1)$, $k_m$ $(m = -1, 0, 1)$ and $f_m$ $(m = -1, 0)$, as follows:

To begin with, by its definition (see Appendix 2, (4) and (7)), $X(z)$ is $1/\sqrt{(z - c_1)(z - c_2)}$ in this case, where $c_1 = a\, e^{i\alpha}$ and $c_2 = a\, e^{-i\alpha}$, respectively. Hence,

$$X(z) = \left(\frac{1}{z}\right)\left\{1 - \left(\frac{c_1}{z}\right)\right\}^{-\frac{1}{2}} \left\{\left(\frac{c_2}{z}\right)\right\}^{-\frac{1}{2}} = \sum_{n=-\infty}^{-1} \beta_n z^n,$$

when $|z| > a$, and

$$X(z) = \frac{1}{\sqrt{c_1 c_2}} \left\{1 - \left(\frac{z}{c_1}\right)\right\}^{-\frac{1}{2}} \left\{1 - \left(\frac{z}{c_2}\right)\right\}^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \gamma_n z^n,$$

when $z < a$. By the comparison of the coefficients in the last two expressions, we have $\beta_{-1} = 1$, $\beta_{-2} = a \cos \alpha$, $\gamma_0 = 1/a$, $\gamma_1 = \cos \alpha / a^2$.

According to their definitions (Appendix 2, (7)),

$$\alpha_m = \frac{a^{m-1}}{\pi} \int_{-\alpha}^{\alpha} \frac{e^{i(m-1)\theta}}{\sqrt{\alpha^2 - \theta^2}} d\theta,$$

$$f_m = \frac{a^{m+1}}{2\pi} \int_{-\alpha}^{\alpha} \sqrt{\alpha^2 - \theta^2}\, f'(\theta)\, e^{i(m+1)\theta}\, d\theta.$$

Similarly, $k_1 = \frac{1}{2}\{1 - (S_1/c)\}$, $k_0 = \frac{1}{2}$ and $k_{-1} = -\frac{1}{2}\{1 - (S_1/c)\}$, and

$$N(\phi, \theta) = \log \{\tfrac{1}{2} - 2 \cos (\phi - \theta)\} + 2\left(\frac{S_1}{c} - 1\right) \cos (\phi - \theta) + \frac{S_0}{c},$$

where $S_0/c = 1/i\pi(ka)^2 J_1(a)\, H_1(a)$ and $S_1/c = 1/i\pi(ka)^2 J_1'(a)\, H_1'(a)$, respectively (see (25), (14), (16), (17) and (18)). Hence, (see (29)),

$$F_n(0) = \frac{-ia^{n-1}}{\pi} \int_{-\alpha}^{\alpha} \left[\log \frac{1}{2 - 2\cos\theta} + 2\left(\frac{S_1}{c} - 1\right) \cos \theta + \frac{S_0}{c}\right] \frac{e^{i(n-1)\theta}}{\sqrt{\alpha^2 - \theta^2}} d\theta,$$

$$G(0) = \frac{1}{2\pi^2} \int_{-\alpha}^{\alpha} \left[\log \frac{1}{2 - 2\cos\theta} + 2\left(\frac{S_1}{c} - 1\right) \cos \theta + \frac{S_0}{c}\right] \frac{d\theta}{\sqrt{\alpha^2 - \theta^2}} \times \int_{-\alpha}^{\alpha} \left\{\frac{1}{\theta - \psi} - i\right\} \sqrt{\alpha^2 - \psi^2}\, f'(\psi)\, d\psi - f(0),$$

where we have assumed that $\phi_0 = 0$. With the help of these values of $\alpha_m$, $\beta_m$, $\gamma_m$, $k_m$, $f_m$, $F_m(0)$ and $G(0)$, the detailed expressions for $\{p_n\}$, $(n = -1, 0, 1, 2)$ are obtained from (31).

If there is a source in the „interior" of the cylinder at $r = r_i$, $\phi = \phi_i$ $(r_i < a)$ and no source in the „exterior" $(r > a)$, (*i*, *e*,

$f_e = 0$), then from the lower line of (13) and a formula in Appendix 1, we have

$$f'(\theta) = \frac{2i}{\pi ka} f_i \sum_{n=1}^{\infty} \left(\frac{r_i}{a}\right)^n \sin n(\phi - \phi_i),$$

which is approximately

$$\frac{2i}{\pi ka} f_i \frac{r_i}{a} \sin(\phi - \phi_i), \tag{35}$$

if $r_i < a$. If $r_i = 0$, that is, if the source is located at the center of the cylinder, then,

$$f(\theta) = \frac{+i}{\pi (ka)^2 J_1(a)} f_i = \text{constant},$$

$$f'(\theta) = 0. \tag{36}$$

Hence, (34) along with (35) and (36) give the field for each case.

(2) *A narrow circular strip for E-wave.* In a way dual to that of the previous example, simple expressions are possible for the E-wave for the case of a narrow strip (fig. 3). The strip, which is represented by $r = a$, $\alpha < \phi < 2\pi - \alpha$, is assumed to be narrow in the sense that

$$\{2(\pi - \alpha)\}^2 \ll 1,$$

then, in the integral in (30) which is taken over $(\alpha, 2\pi - \alpha)$, we have

$$|\theta - \alpha|, \ |\phi - \theta| < 1.$$

Hence (33) is again true for this case, and the field distribution $\tau(\phi)$ is given by (34), where the integral, however, is over $(\alpha, 2\pi - \alpha)$. These values $\alpha_m$, $\beta_m$, $\gamma_m$, $k_m$ and $f_m$ are the same as those in A. (1) if all integrals involved in them are taken over $(\alpha, 2\pi - \alpha)$ and if $S_n$ and $f(\phi)$ take the values for E-wave.

If there is a source at $Q_e(r = r_e, \phi = \phi_e)$ and if there is no source in the „interior", $r < a$ (i.e., $f_i = 0$), then, from the upper line of (13) and formula in Appendix 1, we have

$$f'(\theta) = \pi ika f_e \{H_1(r_e) \sin(\theta - \phi_e) + \tfrac{1}{2} ka\, H_2(r_e) \sin 2(\theta - \phi_e) + \dots\} =$$
$$= \pi ika f_e\, H_1(r_e) \sin(\theta - \phi_e).$$

The substitution of the last expression in (34), where the integral is taken over $(\alpha, 2\pi - \alpha)$ gives the distribution of electric current on the circular strip.

B. Two or more slots. If there are two slots in a cylinder (fig. 4), that is, $\nu = 2$, we may assume that $\beta_1 = -\alpha$ and $\alpha_1 = \alpha$, then (26) and (27) give the field, where

$$X(\phi) = \frac{1}{a^2\sqrt{(e^{i\phi} - e^{i\alpha})(e^{i\phi} - e^{-i\alpha})(e^{i\phi} - e^{i\beta_2})(e^{i\phi} - e^{i\alpha_2})}}, \quad (37)$$

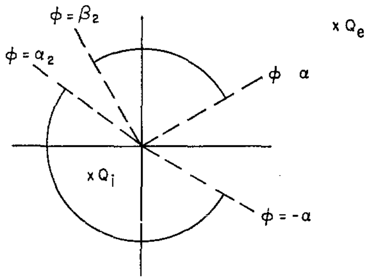


Fig. 4. Two slots in a cylinder.

For this case, the integral in (26) should read:

$$\int_L d\theta = \int_{\alpha}^{\beta_2} d\theta + \int_{\alpha_2}^{2\pi-\alpha} d\theta \quad (38)$$

for the E-wave, and

$$\int_L d\theta = \int_{-\alpha}^{\alpha} d\theta + \int_{\beta_2}^{\alpha_2} d\theta \quad (39)$$

for the H-wave.

(1) *Two narrow slots for H-wave.* If the slots are narrow in the sense that

$$(2\alpha)^2 \ll 1 \text{ and } \{2(\alpha_2 - \beta_2)\}^2 \ll 1,$$

then, the approximation (33) is applicable to (39). In fact, when the point of observation $\phi$ is in the slot $(-\alpha < \phi < \alpha)$, (26) will be

$$\tau(\phi) = \frac{1}{2\pi^2\sqrt{\alpha^2-\phi^2}}\,\frac{1}{\sqrt{(\mathrm{e}^{i\phi}-\mathrm{e}^{i\beta_2})(\mathrm{e}^{i\phi}-\mathrm{e}^{i\alpha_2})}} \times$$

$$\times \left\{ \int_{-\alpha}^{\alpha} \left(\frac{1}{\phi-\theta} - i\right) \sqrt{\alpha^2-\theta^2}\,\sqrt{(\mathrm{e}^{i\theta}-\mathrm{e}^{i\beta_2})(\mathrm{e}^{i\theta}-\mathrm{e}^{i\alpha_2})}\; f'(\theta)\,\mathrm{d}\theta - \right.$$

$$- i \int_{\beta_2}^{\alpha_2} \frac{\mathrm{e}^{-i(\phi-\theta)} f'(\theta)}{1-\mathrm{e}^{i(\phi-\theta)}} \sqrt{(\alpha_2-\theta)(\theta-\beta_2)(\mathrm{e}^{i\theta}-\mathrm{e}^{i\alpha})(\mathrm{e}^{i\theta}-\mathrm{e}^{-i\alpha})}\,\mathrm{d}\theta -$$

$$\left. - i\,\frac{2\pi}{a} \sum_{n=-1}^{3} p_n (a\,\mathrm{e}^{i\phi})^{n-1} \right\}. \tag{40}$$

The comparison of (40) and (34) gives us the effect of the existence of the second slot ($\beta_2 < \phi < \alpha_2$) on the first slot ($-\alpha < \phi < \alpha$).

(2) *Symmetric strips for E-wave.* A similar result can be obtained for two arbitrary circular strips (fig. 4). However, we will consider the more particular case of a symmetric pair of circular strips (fig. 5).

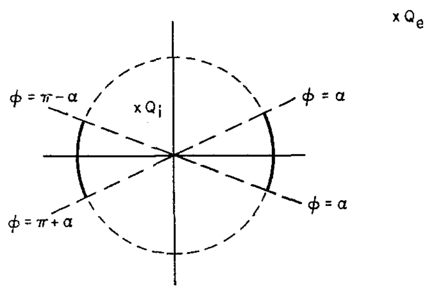


Fig. 5. Symmetric pair of circular strips.

Suppose that two strips are located symmetrically with respect to the $y$-axis, that is,

$$\nu = 2,\ \beta_1 = -\alpha,\ \alpha_1 = \alpha,\ \beta_2 = \pi - \alpha \text{ and } \alpha_2 = \pi + \alpha.$$

(This involves no restriction regarding the sources.) Then (27) becomes

$$X(\phi) = \frac{1}{a^2\sqrt{(\mathrm{e}^{2i\phi}-\mathrm{e}^{-2i\alpha})(\mathrm{e}^{2i\phi}-\mathrm{e}^{2i\alpha})}}$$

and (26) becomes

$$\tau(\phi) = \frac{-iX(\phi)}{2\pi^2} \int_{-\alpha}^{\alpha} \left\{ \frac{1}{1 - e^{i(\phi-\theta)}} \frac{f'(\theta)}{X(\theta)} + \right.$$

$$\left. + \frac{1}{1 + e^{(\phi-\theta)}} \frac{f'(\pi+\theta)}{X(\pi+\theta)} \right\} d\theta - \frac{iX(\phi)}{\pi} \sum_{n=-1}^{3} p_n a^n e^{in\phi}. \qquad (41)$$

(3) *Symmetric narrow strips for E-wave.* If the strips mentioned above are narrow, i.e., if $(2\alpha)^2 \ll 1$, then

$$X(\phi) = \frac{e^{-2i\phi}}{2a^2} \frac{1}{\sqrt{\alpha^2 - \phi^2}}$$

and (41) reduces to

$$\tau(\phi) = \frac{1}{2\pi^2\sqrt{\alpha^2 - \phi^2}} \left[ \int_{-\alpha}^{\alpha} \left\{ \left( \frac{1}{\phi - \theta} - 2i \right) \sqrt{\alpha^2 - \theta^2} \times \right. \right.$$

$$\left. \times f'(\theta) - \frac{i\, e^{-2i(\phi-\theta)}}{2 + i(\phi - \theta)} \sqrt{\alpha^2 - (\pi+\theta)^2}\, f'(\pi+\theta) \right\} d\theta -$$

$$\left. - i\pi \sum_{n=-1}^{3} p_n (a\, e^{i\phi})^{n-2} \right]. \qquad (42)$$

The comparison of (42) and (34) shows what effect the existence of the second strip has on the first strip.

In a way similar to this, three or more slots or strips can be analysed.

§ 5. *A coaxial circular cylinder with slots.* In this section, an analysis very similar to that in § 2 for a single cylinder with slots will be given for a coaxial cylinder with a finite number of arbitrary slots in the outer cylinder.

Suppose that a coaxial circular cylinder of outer radius $a$ and inner radius $b$ with slots is described as:

$$r = b, \; 0 \leq \phi \leq 2\pi, \quad -\infty < z < \infty,$$

$$r = a, \; \alpha_j < \phi < \beta_{j+1}, \; -\infty < z < \infty, \quad (j = 1, 2, \ldots, \nu),$$

where $0 < b < a$, $\beta_j < \alpha_j < \beta_{j+1}$ and $\beta_{\nu+1} = \beta_1$ (fig. 6). This implies that there are $\nu$ slots:

$$r = a, \ \beta_j < \phi < \alpha_j, \quad -\infty < z < \infty, \quad (j = 1, 2, \ldots, \nu).$$

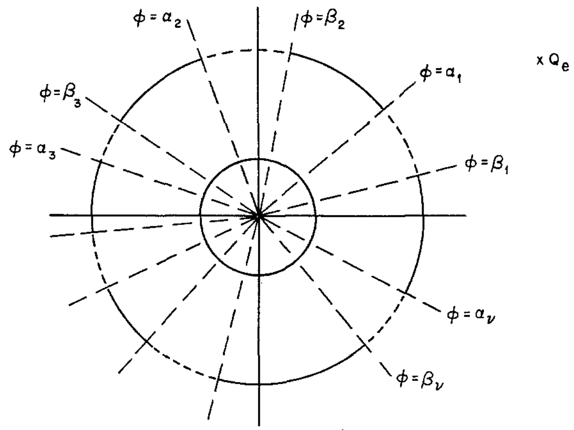


Fig. 6. A coaxial circular cylinder with slots.

Let us assume that there are axial (electric and/or magnetic) line sources at $Q_i(r = r_i, \phi = \phi_i)$ and $Q_e(r = r_e, Q = Q_e)$ where $b \leq r_i < a$ and $a < r_e$.

Then the solution is represented as

$$u = u_e = \sum_{n=-\infty}^{\infty} A_n H_n(kr)\, e^{in\phi} + f_e H_0(kR_e)\,; \quad (a < r),$$

$$u = u_i = \sum_{n=-\infty}^{\infty} \{B_n J_n(kr) + C_n H_n(kr)\}\, e^{in\phi} + f_i H_0(kR_i)\,;$$
$$(b \leq r < a), \qquad (43)$$

where the notations are the same as those in § 2. Corresponding to (3) to (5), we have,

$$u_i = 0;\ r = b,\ 0 \leq \phi \leq 2\pi;\ \frac{\partial u_i}{\partial r} = 0, \qquad (44a)$$

$$u_e = u_i;\ r = a,\ 0 \leq \phi \leq 2\pi;\ \frac{\partial u_e}{\partial r} = \frac{\partial u_i}{\partial r}, \qquad (45a)$$

$$u_e = 0;\ r = a,\ \alpha_j < \phi < \beta_{j+1},\ \frac{\partial u_e}{\partial r} = 0, \tag{46a}$$

$$\frac{\partial u_e}{\partial r} = \frac{\partial u_i}{\partial r};\ r = a,\ \beta < \phi < \alpha_j;\ u_e = u_i, \tag{47a}$$

where $j = 1, 2, \ldots, \nu$.

By the substitution of (43) into (44a) and (45a), we obtain simultaneous equations in terms of $A_n$, $B_n$, and $C_n$. Hence, $B_n$ and $C_n$ are determined in terms of $A_n$, as follows:

$$B_n = \frac{-H_n(a)}{D_n(a, b)} \left\{ H_n(b)A_n + f_{in}D_n(b, r_i) + f_{en} \frac{H_n(b)}{H_n(a)} J_n(a)\, H_n(r_e) \right\};$$

$$C_n = \frac{J_n(b)}{D_n(a, b)} \{H_n(a)A_n + f_{in}D_n(a, r_i) + f_{en}J_n(a)\, H_n(r_e)\}, \tag{44b}$$

where

$$D_n(r, \rho) = J_n(k\rho)\, H_n(kr) - J_n(kr)\, H_n(k\rho)$$

for the E-wave, while for the H-wave

$$B_n = \frac{-H_n'(a)}{T_n(a, b)} \times$$
$$\times \left\{ H_n'(b)A_n - f_{in}L_n(r_i, b) + f_{en}J_n'(a)\, H_n(r_e) \frac{H_n'(b)}{H_n'(a)} - \alpha \right\},$$

$$C_n = \frac{J_n'(b)}{T_n(a, b)} \times$$
$$\times \left\{ H_n'(a)A_n - f_{in}L_n(r_i, a) + f_{en}J_n'(a)\, H_n(r_e) - \alpha \frac{J_n'(a)}{J_n'(b)} \right\}, \tag{45b}$$

where

$$\alpha = \begin{cases} \dfrac{i}{\pi k b} f_{in} & \text{when } r_i = b \\ 0 & \text{when } r_i > b \end{cases}$$

$$T_n(r, \rho) = J_n'(k\rho)\, H_n'(kr) - J_n'(kr)\, H_n'(k\rho),$$

$$L_n(r, \rho) = J_n(k\rho)\, H_n'(kr) - J_n'(kr)\, H_n(k\rho).$$

By a method similar to that in § 2, we have, from (43), (46a) and (47a), the dual series equations for $A_n$:

$$\sum_n \{A_n H_n(a) + f_{en} J_n(a)\, H_n(r_e)\}\, e^{in\phi} = 0 \qquad (\alpha_j < \phi < \beta_{j+1}) \tag{46b}$$

$$\sum_n \frac{e^{in\phi}}{D_n(a, b)} \{A_n H_n(b) - f_{in} D_n(r_i, b) + f_{en} J_n(b)\, H_n(r_e)\} = \begin{cases} 0, & (\beta_j < \phi < \alpha_j), \\ 2\pi\tau(\phi), & (\alpha_j < \phi < \beta_{j+1}), \end{cases} \tag{47b}$$

for the E-wave, and

$$\sum_n \{A_n H_n'(a) + f_{en} J_n'(a)\, H_n(r_e)\}\, e^{in\phi} = \begin{cases} 0, & (\alpha_j < \phi < \beta_{j+1}), \\ 2\pi\tau(\phi), & (\beta_j < \phi < \alpha_j), \end{cases} \tag{48}$$

$$\sum_n \frac{e^{in\phi}}{T_n(a, b)} \{A_n H_n'(b) - f_{in} L_n(r_i, b) + f_{en} J_n'(b)\, H_n(r_e) - \alpha\} = 0, \qquad (\beta_j < \phi < \alpha_j), \tag{49}$$

for the H-wave.

From (47b) and (49) respectively, we have

$$A_n = \frac{D_n(a, b)}{H_n(b)} \int_L \tau(\theta)\, e^{-in\theta}\, d\theta + \frac{1}{Hn(b)} \{f_{in} D_n(r_i, b) - f_{en} J_n(b)\, H_n(r_e)\} \tag{50}$$

for the E-wave, and

$$A_n = \frac{1}{H_n'(a)} \int_L \tau(\theta)\, e^{-in\theta}\, d\theta - \frac{J_n'(a)}{H_n'(a)} H_n(r_e) f_{en} \tag{51}$$

for the H-wave. The substitution of (50) and (51) into (46b) and (48) respectively then leads to

$$\sum_n S_n \int_L \tau(\theta)\, e^{in\Theta}\, d\theta = c f(\phi), \tag{52}$$

where

$$S_n = \begin{cases} \dfrac{H_n(a)}{H_n(b)} D_n(a, b), \\[2ex] \dfrac{H_n'(b)}{H_n'(a)\, T_n(a, b)}, \end{cases} \tag{53}$$

$$cf(\phi) = \begin{cases} f_e \sum_{n=0}^{\infty} \varepsilon_n \dfrac{H_n(r_e)}{H_n(b)} D_n(a, b) \cos n(\phi - \phi_e) - \\ \qquad - f_i \sum_{n=0}^{\infty} \varepsilon_n \dfrac{H_n(a)}{H_n(b)} D_n(r_i, b) \cos n(\phi - \phi_i), \\ f_i \sum_{n=0}^{\infty} \varepsilon_n \left\{ \dfrac{L_n(r_i, b)}{T_n(a, b)} + \dfrac{i\delta(r_i)}{\pi kb\, T_n(a, b)} \right\} \cos n(\phi - \phi_i) - \\ \qquad - f_e \sum_{n=0}^{\infty} \varepsilon_n \dfrac{H_n(r_e)}{H_n'(a)} \cos n(\phi - \phi_e), \end{cases} \tag{54}$$

where $\varepsilon_0 = 1$; $\varepsilon_n = 2$, $n > 0$, and $\delta(r_i) = 1$ when $r_i = b$, and $\delta(r_i) = 0$ when $r_i > b$.

The upper lines are for the E-wave, and the lower lines are for the H-wave. $L$ and $c$ are the same as those defined by (12) and (17) respectively.

Equation (52) is our fundamental integral equation for $\tau$. We can prove that if a solution $\tau$ of (52) which satisfies the edge condition (i.e. a solution which has singularity of $O(\rho^{-\frac{1}{3}})$ at every edge point of $L$.) is found, and if $A_n$ is determined by (50) ((51)) in terms of $\tau$, and if $B_n$ and $C_n$ are determined by (44b) ((45b)) in terms of $A_n$, and finally, if $u$ is determined by (43), in terms of $A_n$, $B_n$ and $C_n$, then $u = E_z$ ($u = H_z$) is the desired solution of the original Dirichlet (Neumann) problem for (1), which satisfies the boundary condition, the continuity condition, the radiation condition and the edge condition. Note that (52) is formally the same as (11) which is the fundamental integral equation for the case of a single cylinder with slots. Furthermore, $S_n$ given by (53) is shown, with the help of Appendix 1, to be

$$S_n = \left(\frac{c}{|n|}\right)\{1 + s_{|n|}\}, \tag{16}$$

where $s_n = O((ka)^2/4n)$. Also, it is easy to see that (15) $S_{-n} = S_n$. Hence, (53) is essentially the same as (11). Therefore, (52) is solved by the method mentioned in § 3 and the solution of it which satisfies the edge condition is given by (26), if $S_n$ and $f(\phi)$ are understood to be given by (53) and (54), respectively. Thus, we have solved the problems for a coaxial cylinder completely.

It is apparent that the solution $\tau$ is applicable in the case of a single cylinder in § 2 as well as of coaxial cylinders in § 5. The solution represents the distribution of the axial component $J_z$ of surface current on the wall of cylinder for the case of E-wave and the tangential component $E_\phi$ of electric field in the slots for the case of H-wave and $S_n$, $f(\phi)$ and $k_n$ (see (25)) must be determined appropriately for each case. Therefore the results obtained in § 4 are true for the case of a coaxial cylinder case, too.

§ 6. *Conclusion.* In this paper, the Dirichlet and Neumann problems for the Helmholtz equation (E- and H-waves, respectively, in the electromagnetic case) and for a circular cylinder and coaxial circular cylinder with arbitrary number of arbitrary slots in them, have been shown to be equivalent to that of a singular integral equation. The integral equation has been solved exactly and the solution of it gives the distributions of the field components on the walls of the cylinder or in the slots. Thanks to the estimation formulas for the Bessel and Hankel functions, the results are true for any wave number and for any radius of cylinder. Detailed calculation has been given especially for the H-wave case when a cylinder has a narrow slot in it. We have restricted ourselves to the cases where only one cylinder has slots in it. If more than two cylinders have slots in them, we will have simultaneous singular integral equations, which will be solved by the way similar to that employed in this paper. The method may be extended to other boundaries, e.g. a plane boundary with slots.

## APPENDIX 1

*Estimation formulas for the Bessel and Hankel functions.* In the text, the following formulas played an essential role, especially when we picked up the singularity of the kernel $\sum S_n e^{in\Theta}$ of the fundamental integral equation (11).

*Theorem.* For any positive integer $n$ and any complex number $\rho$, we have,

$$J_n(\rho) = \frac{1}{n!}\left(\frac{\rho}{2}\right)^n \{1 + j_n(\rho)\},$$
$$J_n'(\rho) = \frac{1}{2(n-1)!}\left(\frac{\rho}{2}\right)^{n-1} \{1 + j_n'(\rho)\}, \tag{1-1}$$

$$H_n(\rho) = \frac{i(n-1)!}{\pi}\left(\frac{2}{\rho}\right)^n \{1 + h_n(\rho)\},$$
$$H'_n(\rho) = \frac{(-i)n!}{2\pi}\left(\frac{2}{\rho}\right)^{n+1} \{1 + h'_n(\rho)\}, \tag{1-1}$$

where $j_n$, $j'_n$, $h_n$, and $h'_n$ are estimated as follows:

$$|j_n(\rho)| \leqq \exp\{|\rho|^2/4(n+1)\} - 1,$$
$$|j'_n(\rho)| \leqq \exp\{|\rho|^2/4n\} - 1 + (|\rho|/2n)^2 \exp\{|\rho|^2/4(n+2)\},$$
$$|h_n(\rho)| \leqq \{|\rho|^2/4(n-1)\}[1 - \{|\rho|^2/4(n-1)\}^{n-1}] \times$$
$$\times [1 - \{|\rho|^2/4(n-1)\}]^{-1} + \left[\frac{1}{(n-1)!}\left(\frac{|\rho|}{2}\right)^n\right]^2 \times \tag{1-2}$$
$$\times \left[i\pi + 2\log\left(\frac{\lambda|\rho|}{2n}\right) + \exp\{|\rho|^2/4n\}\right],$$
$$|h'_n(\rho)| \leqq h_{n+1}(\rho) - \left(\frac{\rho}{2n}\right)^2 \{1 + h_{n-1}(\rho)\},$$

where $\lambda$ is the Euler constant. Equation (1-2) can be replaced by more rough estimation as follows:

$$j_n(\rho)|,\ |j'_n(\rho)|,\ |h_n(\rho)|,\ |h'_n(\rho)| = O(|\rho|^2/4n). \tag{1-3}$$

This theorem is proved with the help of the series expansions of $J_n(\rho)$ and $H_n(\rho)$. The details are supposed to be published in another mathematical periodical and are not described here.

To the best knowledge of the author, these results have not been published yet, except[5])

$$|J_n(\rho)| \leqq \frac{1}{n!}\left|\frac{\rho}{2}\right|^n \exp(|\rho|^2/4), \tag{1-4}$$

$$\leqq \frac{1}{n!}\left|\frac{\rho}{2}\right|^n \exp(|\rho|^2 \tag{1-5}$$

and

$$|J_n(\rho)| \leqq \frac{1}{n!}\left(\frac{\rho}{2}\right)^n (1+\theta),$$

where

$$\theta \leqq \{\exp(\rho^2/4) - 1\}/(n+1). \tag{1-6}$$

However, the first formula in (1-1), together with the first one in (1-2), is better than formulas (1-4), (1-5) and (1-6).

In some cases, formulas such as

$$J_n(\rho) \sim \frac{1}{n!}\left(\frac{\rho}{2}\right)^n, \quad H_n(\rho) \sim \frac{i(n-1)!}{\pi}\left(\frac{2}{\rho}\right)^n, \tag{1-7}$$

are employed. However, (1-7) for any $n$ implies $|\rho| < 1$, or in our case, $|ka| < 1$. Therefore, results obtained with the help of (1-7) are valid only for this restricted case. On the other hand, thanks to the formulas (1-1) and (1-2), our results are valid for all values of $ka$.

## APPENDIX 2

*On the solution of a singular integral equation.* In this appendix, it is shown briefly how to solve a singular integral equation

$$\frac{1}{\pi i}\int_L \left\{\frac{1}{t-t_0} - k(t_0, t)\right\} \tau(t)\,\mathrm{d}t = f(t_0), \tag{2-1}$$

defined on $L$ in the complex $z$-plane. $k(t_0, t)$ is defined by the function

$$k(z, t) = \sum_{n=-N_1}^{N_2} k_n z^n t^{-(n+1)} \tag{2-2}$$

when $z = t_0$ on $L$, where $k_n$ ($n = -N_1, \dots, N_2$) are arbitrary given complex constants. (In the text, $N_1 = N_2 = N$.)

In this appendix, $L$ is taken to be a union of a finite number of arbitrary smooth, non-intersecting, open arcs $L_j$ ($j = 1, 2, \dots, \nu$) in $z$-plane; $L = \sum_{j=1}^{\nu} L_j$. (In the text, $L_j$ was restricted to be a circular arc.)

Suppose that $\tau(t)$ is a solution of (2-1) which has singularities of $o(\rho^{-\frac{1}{2}})$ at every end point $c_l$ ($l = 1, 2, \dots, 2\nu$) of $L = \sum L_j$, where $\rho$ is the distance from $c_l$, then the function

$$\Phi(z) = \frac{1}{2\pi i}\int_L \left\{\frac{1}{t-z} - k(z, t)\right\} \tau(t)\,\mathrm{d}t$$

is easily proved to have the following properties[6]);

(i) $\Phi(z)$ is holomorphic everywhere except on $L$, $z = 0$ and $z = \infty$,

(ii) $\Phi(0) = o(z^{-N_1})$,

(iii) $\Phi(\infty) = o(z^{N_2})$,

(iv) $\lim_{z \to c_l} \Phi(z) = o(1/\sqrt{z - c_l})$, $(l = 1, 2, \ldots, 2\nu)$.

On the other hand, with the help of the Plemelj theorem, we have

$$\Phi^{\pm}(t_0) = \pm \tfrac{1}{2}\tau(t_0) + \frac{1}{2\pi i}\int_L \left\{\frac{1}{t - t_0} - k(t_0, t)\right\} \tau(t)\, \mathrm{d}t,$$

where $\Phi^+(t_0)$ $(\Phi^-(t_0))$ is the limiting value of $\Phi(z)$ when $z$ tends to a point $t_0$ on $L$ $(\neq c_l)$ from the left (right) with respect to the positive direction of $L$ in which $L$ is measured. Hence we have

$$\text{(v)} \qquad \Phi^+(t_0) + \Phi^-(t_0) = \frac{1}{\pi i}\int_L \left\{\frac{1}{t - t_0} - k(t_0, t)\right\} \tau(t)\, \mathrm{d}t = f(t_0),$$

and

$$\text{(vi)} \qquad \Phi^+(t_0) - \Phi^-(t_0) = \tau(t_0).$$

Conversely, we can see, from (vi), that if we find a function $\Psi(z)$ which satisfies (i) to (v), then the solution $\tau$ of (2-1) will be found among these functions obtained by $\Psi^+ - \Psi^-$.

It is not difficult to see that the general expression for $\Psi(z)$ is given by

$$\Psi(z) = \frac{X(z)}{2\pi i}\int_L \frac{1}{t - z}\,\frac{f(t)}{X(t)}\, \mathrm{d}t + X(z)\sum_{n=-N_1}^{N_2+\nu} p_n z^n, \qquad (2\text{-}3)$$

where

$$X(z) = 1/\{\prod_{j=1}^{2\nu} (z - c_j)\}^{\frac{1}{2}} \qquad (2\text{-}4)$$

and $p_n$ $(n = -N_1, -N_1 + 1, \ldots, N_2)$ are complex constants which are not determined yet. Hence $\tau(t)$ should be expressed as

$$\tau(t_0) = \Psi^+(t_0) - \Psi^-(t_0) =$$

$$= \frac{X(t_0)}{\pi i}\int_L \frac{1}{t - t_0}\,\frac{f(t)}{X(t)}\, \mathrm{d}t + X(t_0)\sum_{n=-N_1}^{N_2+\nu} p_n t_0^n. \qquad (2\text{-}5)$$

However, since $\tau(t)$ obtained by (2-5) does not necessarily satisfy (2-1), we have to give pertinent values to $\{p_n\}$ so that (2-5) satisfies (2-1).

On substituting (2-5) into (2-1), we can see, with the help of the Poincaré-Bertrand theorem, that the necessary and sufficient condition for $\tau(t_0)$ defined by (2-5) to be the solution of (2-1) is

$$\frac{-i}{\pi}\int_L \frac{f(\zeta)}{X(\zeta)}\,d\zeta \int_L \frac{X(t)}{\zeta - t}\left\{\frac{1}{t-t_0} - k(t_0, t)\right\} dt + \\ + \sum_{n=-N_1}^{N_2+\nu} p_n \int_L \left\{\frac{1}{t-t_0} - k(t_0, t)\right\} X(t)\, t^n\, dt = 0. \qquad (2\text{-}6)$$

In order to reduce (2-6) to simultaneous linear equations with respect to $\{p_n\}$, the following definitions and lemmas are introduced.

*Definitions.*

$$H_n(\xi) = \frac{1}{\pi i}\int_L \frac{X(t)t^n}{t-\xi}\,dt,$$

$$\alpha_n = \frac{1}{\pi i}\int_L X(t)t^{n-1}\,dt,$$

$$f_n = \frac{1}{\pi i}\int_L \frac{f(t)t^{n-1}}{X(t)}\,dt,$$

$$\delta_{N\nu} = \begin{cases} 1, & N > \nu \\ 0, & N \leqq \nu \end{cases}$$

$$X(z) = \begin{cases} \sum_{n=-\infty}^{-\nu} \beta_n z^n, & |z| > \max |c_l|, \\ \sum_{n=0}^{\infty} \gamma_n z^n, & |z| < \min |c_l|, \end{cases}$$

(i.e., $\beta_n$ and $\gamma_n$ are defined as the coefficients of the Laurent (Taylor) expansions of $X(z)$ with respect to $z = \infty$ and $z = 0$, respectively). Then we can prove

*Lemma*

$$H_n(\xi) = \begin{cases} -\sum_{m=n}^{-1} \gamma_{m-n}\xi^m, & n \leqq -1 \\ 0, & 0 \leqq n \leqq \nu - 1, \\ -\sum_{m=0}^{n-\nu} \beta_{m-n}\zeta^m, & \nu \leqq n \end{cases} \tag{2-8}$$

where $\xi$ is a point on $L$.

*Proof of lemma.* By virtue of Cauchy's integral theorem, we have

$$\frac{1}{\pi i}\int_L \frac{X(t)t^n}{t-z}\,\mathrm{d}t = X(z)z^n + \phi_n^0(z) - \phi_n^\infty(z), \tag{2-9}$$

where

$$\phi_n^0(z) = \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{X(t)t^n}{t-z}\,\mathrm{d}t,$$

$$\phi_n^\infty(z) = \frac{1}{2\pi i}\int_{|z|=R} \frac{X(t)t^n}{t-z}\,\mathrm{d}t$$

and $\varepsilon$ and $R$ are arbitrary constants such that $\varepsilon < \min|c_l|$ and $R > \max|c_l|$, respectively. On applying the Plemelj theorem to (2-9), one has

$$H_n(\xi) = \phi_n^0(\xi) - \phi_n^\infty(\xi). \tag{2-10}$$

On the other hand, (2-9) tells us that the right hand side of (2-9) is a quantity of $o(z^{-1})$ when $|z| \to \infty$ and of $o(1)$ when $|z| \to 0$. Thus, the Laurent coefficients of $\phi_n^0(z)$ and $\phi_n^\infty(z)$ are determined by the comparisons of them with those of $X(z)z^n$ in (2-9), and then (2-10) gives (2-8).

Now in terms of these notations, (2-6) is reduced, with the help of (2-8), to

$$\sum_{n=0}^{N_2} k_n t_0^n \sum_{m=0}^{n} \gamma_m f_{m-n} + \delta_{N_1\nu} \sum_{n=-N_1}^{\nu-1} k_n t_0^n \sum_{m=n+1}^{-\nu} \beta_m f_{m-n} +$$
$$+ i\pi\Big[\sum_{n=-1}^{-N_1} t_0^n \sum_{m=n}^{-N_1} \gamma_{n-m} p_m +$$
$$+ \sum_{n=0}^{N_2} t_0^n \sum_{m=n+\nu}^{N_2+\nu} \beta_{n-m} p_m + \sum_{n=-N_1}^{N_2} k_n t_0^n \sum_{m=-N_1}^{N_2+\nu} \alpha_{m-n} p_m\Big] = 0. \tag{2-11}$$

Because (2-11) must be true for any $t_0$ on $L$, it is equivalent to the following simultaneous linear equations with respect to $\{p_n\}$;

$$\sum_{m=-N_1}^{N_2+\nu} \alpha_{m-n} p_m + \frac{1}{k_n} \sum_{m=n+\nu}^{N_2+\nu} \beta_{n-m} p_m = \frac{i}{\pi} \sum_{m=0}^{n} \gamma_m f_{m-n}, \quad (0 \leqq n \leqq N_2)$$

$$\sum_{m=-N_1}^{N_2+\nu} \alpha_{m-n} p_m + \frac{1}{k_n} \sum_{m=n}^{-N_1} \gamma_{n-m} p_m = 0, \qquad (-\nu \leqq n \leqq -1) \quad \text{(2-12)}$$

$$\sum_{m=-N_1}^{N_2+\nu} \alpha_{m-n} p_m + \frac{1}{k_n} \sum_{m=n}^{-N_1} \gamma_{n-m} p_m = \frac{i}{\pi} \delta_{N_1\nu} \sum_{m=n+1}^{-\nu} \beta_m f_{m-n}.$$
$$(-N_1 \leqq n \leqq -\nu - 1).$$

If $N_1 \leqq \nu$, then the third expression of (2-12) is reduced to the second, where, in this case, the range of $n$ is $-N_1 \leqq n \leqq -1$ instead of $-\nu \leqq n \leqq -1$.

Equation (2-12) is the necessary and sufficient conditions for $p_n$ with which (2-5) is a solution of (2-1). Thus, we have solved (2-1) completely. Note that $\tau(t_0)$ defined by (2-5) has singularities of $o(1/\sqrt{z - c_l})$ at every edge point $c_l$ because of the factor $X(t_0)$.

Received 9th September, 1965.

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