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THE GREATEST COMMON QUOTIENT OF BOREL–SERRE AND THE TOROIDAL COMPACTIFICATIONS OF LOCALLY SYMMETRIC SPACES

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Abstract

In this paper, we identify the greatest common quotient (GCQ) of the Borel–Serre compactification and the toroidal compactifications of Hermitian locally symmetric spaces with a new compactification. Using this compactification, we completely settle a conjecture of Harris–Zucker that this GCQ is equal to the Baily–Borel compactification. We also show that the GCQ of the reductive Borel–Serre compactification and the toroidal compactifications is the Baily–Borel compactification. There are two key ingredients in the proof: ergodicity of certain adjoint action on nilmanifolds and incompatibility between the ambient linear structure and the intrinsic Riemannian structure of homothety sections of symmetric cones.

1 Statement of Results

1.1 Introduction. Let X be a bounded symmetric domain in a complex Euclidean space and $\Gamma \subset \mathrm{Aut}(X)$ a cofinite volume subgroup. Then $\Gamma\backslash X$ is a Hermitian locally symmetric space of finite volume, and any such Hermitian locally symmetric space is of this form. In the following, we assume that $\Gamma\backslash X$ is noncompact.

Hermitian locally symmetric spaces play an important role in automorphic forms [BBo], [S], cohomology theory of Γ [BoSe], [HaZ1,2] and arithmetic algebraic geometry [AMuRT], [N], [CF]. Useful for these applications, $\Gamma\backslash X$ admits respectively the Baily–Borel compactification $\overline{\Gamma\backslash X}^{BB}$ [BBo] (§3), the Borel–Serre compactification $\overline{\Gamma\backslash X}^{BS}$ [BoSe] (§2), and the toroidal compactifications $\overline{\Gamma\backslash X}^{Tor}_{\Sigma}$ [AMuRT] (§5), where Σ is a Γ-admissible

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polyhedral cone decomposition (§5.2). In higher rank, there are, in general, infinitely many Γ-admissible polyhedral cone decompositions and hence infinitely many toroidal compactifications.

The compactification $\overline{\Gamma\backslash X}^{BB}$ is a singular projective variety, $\overline{\Gamma\backslash X}^{BS}$ is a manifold with corners (In fact, it is a real analytic manifold with corners as shown recently in [BoJ].), and $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ are varieties with toric singularities, many of which are smooth projective varieties. A natural problem is to understand relations between these compactifications.

In the following, we say that one compactification $\overline{\Gamma\backslash X}^{1}$ of $\Gamma\backslash X$ dominates another compactification $\overline{\Gamma\backslash X}^{2}$ if the identity map on $\Gamma\backslash X$ extends to a continuous surjective map from $\overline{\Gamma\backslash X}^{1}$ to $\overline{\Gamma\backslash X}^{2}$ and that $\overline{\Gamma\backslash X}^{1}$ strictly dominates $\overline{\Gamma\backslash X}^{2}$ if the extended map is not a homeomorphism.

It was shown by Zucker [Z2, Theorem 3.11] that $\overline{\Gamma\backslash X}^{BS}$ dominates $\overline{\Gamma\backslash X}^{BB}$ (see 3.4.2 below); and it was shown in [AMuRT, Proposition on p. 254] that all toroidal compactifications $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ dominate $\overline{\Gamma\backslash X}^{BB}$, and hence smooth toroidal compactifications resolve the singularities of $\overline{\Gamma\backslash X}^{BB}$, which was the motivation for introducing the toroidal compactifications in [AMuRT].

If $\Gamma\backslash X$ is a Riemann surface, then $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ is unique and equal to $\overline{\Gamma\backslash X}^{BB}$, which is obtained by adding one point to every cusp of $\Gamma\backslash X$, while $\overline{\Gamma\backslash X}^{BS}$ is obtained by adding a circle to each cusp. Therefore, $\overline{\Gamma\backslash X}^{BS}$ strictly dominates $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ by collapsing each boundary circle in $\overline{\Gamma\backslash X}^{BS}$ to a boundary point in $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$.

For other spaces $\Gamma\backslash X$, it has been recognized for a long time that $\overline{\Gamma\backslash X}^{BS}$ and $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ are not compatible. One way to measure the incompatibility of these two compactifications is to study their greatest common quotient (GCQ), denoted by $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ (see §6.1 for its definition and existence). Since $\overline{\Gamma\backslash X}^{BB}$ is a common quotient of $\overline{\Gamma\backslash X}^{BS}$ and $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ as mentioned earlier, $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ dominates $\overline{\Gamma\backslash X}^{BB}$. In [HaZ1, Conjecture 1.5.8], Harris and Zucker made the following conjecture.

CONJECTURE 1.1.1. *For every toroidal compactification $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$, the GCQ $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ of $\overline{\Gamma\backslash X}^{BS}$ and $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ is equal to $\overline{\Gamma\backslash X}^{BB}$.*

1.2 Statements of results. In this paper, we identify $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ with a new compactification and determine the cases when Conjecture 1.1.1 holds.

For simplicity, we assume in the following that there exists a semisimple algebraic group $\mathbf{G}$ which is $\mathbb{Q}$-simple such that $X = \mathbf{G}(\mathbb{R})/K$, where K is a maximal compact subgroup of the real locus $\mathbf{G}(\mathbb{R})$, and $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is a neat arithmetic subgroup [Bo3, §17.1].

Theorem 1.2.1 (§7)**.** *For every toroidal compactification $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$, the GCQ $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ is equal to the intermediate compactification $\overline{\Gamma\backslash X}^{Int}$ defined in Proposition 5.5.1 below and hence is independent of the choice of Γ-admissible polyhedral cone decomposition Σ in the toroidal compactification.*

The fibers of $\overline{\Gamma\backslash X}^{Int}$ over $\overline{\Gamma\backslash X}^{BB}$ are described explicitly in Proposition 5.5.1. As a corollary of Theorem 1.2.1 and the description of $\overline{\Gamma\backslash X}^{Int}$, we get the following results.

Theorem 1.2.2 (§8)**.** *If $\mathbf{G}$ is $\mathbb{Q}$-simple but not absolutely simple, then for every toroidal compactification $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$, the GCQ $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ is equal to $\overline{\Gamma\backslash X}^{BB}$ and hence Conjecture 1.1.1 is true.*

Hilbert modular varieties satisfy the condition in Theorem 1.2.2, and hence Conjecture 1.1.1 holds for them.

Theorem 1.2.3 (§8)**.** *If $\mathbf{G}$ is $\mathbb{Q}$-simple and $\mathbb{Q}$-split but not equal to* SL(2), *then for every toroidal compactification $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$, the GCQ $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ strictly dominates $\overline{\Gamma\backslash X}^{BB}$, and hence Conjecture 1.1.1 is not true.*

Siegel modular varieties and Picard modular varieties satisfy the conditions in Theorem 1.2.3, and hence Conjecture 1.1.1 fails for them. The conditions that $\mathbf{G}$ is $\mathbb{Q}$-simple and $\mathbb{Q}$-split imply that $\mathbf{G}$ is absolutely simple. For other absolutely simple cases, see Theorem 8.2.1 below. Together with Theorem 1.2.2 above, they cover all cases.

There is another compactification of $\Gamma\backslash X$, called the reductive Borel–Serre compactification $\overline{\Gamma\backslash X}^{RBS}$, which lies between $\overline{\Gamma\backslash X}^{BS}$ and $\overline{\Gamma\backslash X}^{BB}$ and which also plays an important role in cohomology of Γ [Z1, p. 190], [HaZ1, 1.3(b)], [GHM]. Then the first step of the proof of Theorem 1.2.1 in §7 shows the following result.

Theorem 1.2.4 (§8)**.** *For every toroidal compactification $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$, the GCQ $\overline{\Gamma\backslash X}^{RBS} \wedge \overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ of the reductive Borel–Serre compactification*

$\overline{\Gamma\backslash X}^{RBS}$ and $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ is equal to $\overline{\Gamma\backslash X}^{BB}$.

In [GT], Goresky and Tai study the least common refinement of $\overline{\Gamma\backslash X}^{RBS}$ and $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ (see §6.1).

1.3 Idea of proof. If $\Gamma\backslash X$ is a Hilbert modular surface, then the boundary $\partial\overline{\Gamma\backslash X}^{BS}$ is a union of rank two torus bundles over a circle, one bundle for each end of $\Gamma\backslash X$; while $\partial\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ is a union of cycles of rational curves $\mathbb{C}P^1$, one cycle for each end of $\Gamma\backslash X$, whose length depends on Σ. Then it is conceivable that $\overline{\Gamma\backslash X}^{BS}$ is completely incompatible with $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ at each end, and hence $\overline{\Gamma\backslash X}^{BS}\wedge\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ is the compactification obtained by adding one point to every end of $\Gamma\backslash X$, which is exactly $\overline{\Gamma\backslash X}^{BB}$. Therefore, Conjecture 1.1.1 is true in this case.

On the other hand, for a Picard modular surface $\Gamma\backslash X = \Gamma\backslash B^2$ [L], where B^2 is the unit ball in $\mathbb{C}^2$, $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ is unique and $\partial\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ is a union of elliptic curves, one for each end of $\Gamma\backslash X$, while $\partial\overline{\Gamma\backslash X}^{BS}$ is a union of circle bundles over the elliptic curves which appear in $\partial\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$. Since $\overline{\Gamma\backslash X}^{BB}$ is obtained by adding one point to each end of $\Gamma\backslash X$, these three compactifications fit into a tower

$$\overline{\Gamma\backslash X}^{BS} \underset{\neq}{\longrightarrow} \overline{\Gamma\backslash X}^{Tor} \underset{\neq}{\longrightarrow} \overline{\Gamma\backslash X}^{BB}.$$

So the GCQ $\overline{\Gamma\backslash X}^{BS}\wedge\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ is equal to $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$, which strictly dominates $\overline{\Gamma\backslash X}^{BB}$, and Conjecture 1.1.1 is false in this case.

In every case where Conjecture 1.1.1 fails, this phenomenon in the Picard modular surface is present, i.e., $\overline{\Gamma\backslash X}^{BS}$ dominates $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ near some boundary components at infinity in $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$, but $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ is strictly bigger than $\overline{\Gamma\backslash X}^{BB}$.

Besides this dominance and the fact that they both are bigger than $\overline{\Gamma\backslash X}^{BB}$, the compactifications $\overline{\Gamma\backslash X}^{BS}$ and $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ are incompatible.

To prove this incompatibility, we proceed in two steps. The fiber of $\overline{\Gamma\backslash X}^{BS}$ over $\overline{\Gamma\backslash X}^{BB}$ is a family of nilmanifolds over a lower dimensional locally symmetric space. Assume that the base has positive dimension. We use the incompatibility between the geodesic action in $\overline{\Gamma\backslash X}^{BS}$ and the torus action in $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ to show that every horizontal section of this bundle collapses to a point in any common quotient of $\overline{\Gamma\backslash X}^{BS}$ and $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$, using Lemma 6.2.1. Then we show that the fundamental group of the base manifold acts ergodically on the fibers of the bundle, and hence the fibers have to collapse also in any common quotient because of the Hausdorff property. The possibility of carrying out this two step approach has been suggested in [HaZ1, Note 1 added in proof, p. 308].

1.4 Organization. The rest of the paper is organized as follows. In §2, we recall the Borel–Serre and reductive Borel–Serre compactifications, emphasizing the point of view of the horospherical decomposition. In §3, we recall the Baily–Borel compactification, in particular its connection with the Borel–Serre compactification through the refined horospherical decomposition. In §4, we recall the toroidal embedding, emphasizing the topology as a partial compactification. In §5, we recall the toroidal compactifications and a new compactification lying between the toroidal compactifications and the Baily–Borel compactification. In §6, we define GCQ of two compactifications and propose a general method to determine GCQ. In §7, we prove Theorem 1.2.1 using the collapsing methods in §6. In §8, we prove Theorems 1.2.2, 1.2.3 and 1.2.4, and a more general version of Theorem 1.2.3.

In our presentation of the compactifications $\overline{\Gamma\backslash X}^{BS}$, $\overline{\Gamma\backslash X}^{RBS}$, $\overline{\Gamma\backslash X}^{BB}$ and $\overline{\Gamma\backslash X}_{\Sigma}^{tor}$, we emphasize the geometric point of view, in particular the horospherical decomposition. Such geometric descriptions are convenient for studying relations between the compactifications. In fact, these descriptions have also been used in [J] to study metric properties of compactifications of locally symmetric spaces.

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2 Borel–Serre Compactification

In this section, we recall the Langlands decomposition of a parabolic subgroup (2.1) and the induced horospherical decomposition of X (2.2). Then we recall the Borel–Serre compactification $\overline{\Gamma\backslash X}^{BS}$ (2.3) and point out a simple but important property of convergence of geodesics in the compactification $\overline{\Gamma\backslash X}^{BS}$ (2.3.2). In (2.4), we recall the reductive Borel–Serre compactification $\overline{\Gamma\backslash X}^{RBS}$.

2.1 Langlands decomposition. Let $x_0 = K$ be the basepoint in $X = G/K$ corresponding to the maximal compact subgroup K in $G = \mathbf{G}(\mathbb{R})$ chosen in §1.2. For any rational parabolic subgroup $\mathbf{P}$, let $\mathbf{N_P}$ be its unipotent radical. Then the Levi quotient $\mathbf{L_P} = \mathbf{P}/\mathbf{N_P}$ has a unique lift $\mathbf{L_P}(x_0)$ in $\mathbf{P}$ which is stable under the Cartan involution associated with x_0 [BoSe, §1.6, §1.9], and $\mathbf{P} = \mathbf{N_P}\mathbf{L_P}(x_0)$.

Let $\mathbf{S_P}$ be the maximal $\mathbb{Q}$–split torus in the center of $\mathbf{L_P}$. Then $\mathbf{L_P} = \mathbf{M_P}\mathbf{S_P}$, where $\mathbf{M_P} = \bigcap_{\chi\in X_{\mathbb{Q}}(\mathbf{L_P})} \mathrm{Ker}(\chi^2)$. Denote the lifts of $\mathbf{M_P}$ and $\mathbf{S_P}$ in $\mathbf{L_P}(x_0)$ still by $\mathbf{M_P}$ and $\mathbf{S_P}$.

Let $N_P = \mathbf{N_P}(\mathbb{R})$, $M_P = \mathbf{M_P}(\mathbb{R})$, and A_P be the connected component of $\mathbf{S_P}(\mathbb{R})$. These subgroups induce the Langlands decomposition of $P = \mathbf{P}(\mathbb{R})$:

$$P = N_P M_P A_P\,, \quad g = n(g)m(g)a(g)\,, \tag{1}$$

where $n(g) \in N_P$, $m(g) \in M_P$, $a(g) \in A_P$ are uniquely determined by g.

Let $\mathfrak{a}_P$ be the Lie algebra of A_P, and $\mathfrak{n}_P$ be the Lie algebra of N_P. Then $\mathfrak{a}_P$ acts on $\mathfrak{n}_P$, and the set of roots is denoted by $\Phi^+(P, A_P)$. Define a positive cone $\mathfrak{a}_P^+$ by

$$\mathfrak{a}_P^+ = \left\{H \in \mathfrak{a}_P \mid \beta(H) > 0\,,\ \text{for all } \beta \in \Phi^+(P, A_P)\right\}.$$

2.2 Horospherical Decomposition Through the identification $X = G/K$ and the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G, the Killing form on the Lie algebra $\mathfrak{g}$ defines a G-invariant Riemannian metric on X. In the following, we fix this metric on X.

The Langlands decomposition of a parabolic subgroup P above induces a horospherical decomposition of X. Define $K_P = M_P \cap K$. Then K_P is a maximal compact subgroup of M_P, and the quotient $X_P = M_P/K_P$ is the product of a symmetric space of noncompact type and a possible Euclidean space, and hence is called the boundary symmetric space associated with P.

Since P acts transitively on X, the Langlands decomposition induces the following horospherical decomposition of X:

$$X = N_P \times X_P \times A_P, \quad x = n(x)a(x)z(x), \tag{1}$$

where $n(x) \in N_P, z(x) \in X_P$, and $a(x) \in A_P$ are uniquely determined by x.

If G has $\mathbb{R}$-rank one, then X_P consists of one point, and the orbits under N_P are horospheres associated with the point at infinity determined by P. Because of this, the above decomposition is called the horospherical decomposition of X determined by P. This decomposition also play an important role in [K] and [GuJTa].

For convenience, we identify $(n, z, a) \in N_P \times X_P \times A_P$ with the point $x = naz \in X$.

LEMMA 2.2.1 [Bo2, Proposition 1.6]. *For any $n \in N_P$ and $z \in X_P$, nA_Pz is a totally geodesic flat submanifold; for any $n \in N_P$ and $a \in A_P$, naX_P is a totally geodesic submanifold; and nA_Pz is perpendicular to naX_P and N_Paz at (n, z, a). In particular, for any $n \in N_P$, $z \in X_P$, $H \in \mathfrak{a}_P^+$, and $|H| = 1$, the curve $c(t) = (n, z, \exp(tH))$, $t \in \mathbb{R}$, is a unit speed geodesic in X.*

The horospherical decomposition and Lemma 2.2.1 are illustrated in Figure 2.2.

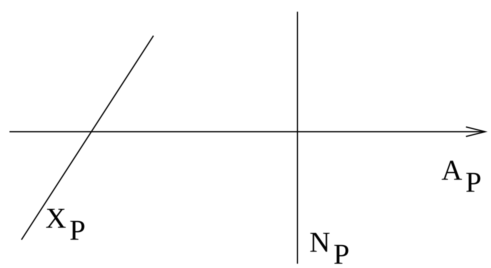

Figure 2.2

2.3 Boundary components and $\overline{\Gamma\backslash X}^{BS}$ The Borel–Serre compactification $\overline{\Gamma\backslash X}^{BS}$ [BoSe] is constructed in three steps:

1. Define a boundary component for every proper rational parabolic subgroup of $\mathbf{G}$.
2. Define a partial compactification $\overline{X}^{BS}$ of X by adding these (rational) boundary components.
3. Show that Γ acts continuously on $\overline{X}^{BS}$ with a compact Hausdorff quotient, which is the compactification.

For any rational parabolic subgroup $\mathbf{P}$, its boundary component $e(\mathbf{P})$ is defined by $e(\mathbf{P}) = N_P \times X_P$. The boundary component $e(\mathbf{P})$ is added at infinity of X through the horospherical decomposition (Eq. (1) in §2.2). More precisely, for any sequence $x_j = (n_j, z_j, \exp H_j)$ in X, x_j converges to $(n_\infty, z_\infty) \in e(\mathbf{P})$ if and only if $n_\infty = \lim_{j\to\infty} n_j$, $z_\infty = \lim_{j\to\infty} z_j$, and for any $\beta \in \Phi^+(P, A_P)$, $\beta(H_j) \to +\infty$ as $j \to +\infty$.

This gluing of $e(\mathbf{P})$ to X at infinity can be represented by Figure 2.3.

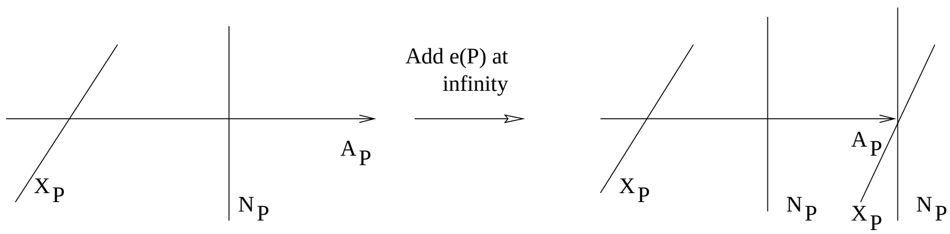

Figure 2.3

By adding all the boundary components $e(\mathbf{P})$ to X as above, we get a partial compactification $\overline{X}^{BS} = X \cup \coprod_{\mathbf{P}} e(\mathbf{P})$ of X, where the union over $\mathbf{P}$ is over all proper rational parabolic subgroups of $\mathbf{G}$.

Proposition 2.3.1 [BoSe, Theorem 9.3]. *The arithmetic subgroup Γ acts continuously on $\overline{X}^{BS}$ with a compact quotient. The quotient $\Gamma\backslash\overline{X}^{BS}$ is called the Borel–Serre compactification and denoted by $\overline{\Gamma\backslash X}^{BS}$.*

Let $\mathbf{P}_1, \dots, \mathbf{P}_l$ be representatives of Γ-conjugacy classes of proper rational parabolic subgroups of $\mathbf{G}$. Then

$$\overline{\Gamma\backslash X}^{BS} = \Gamma\backslash X \cup \coprod_{i=1}^{l} \Gamma_{P_i}\backslash e(\mathbf{P}_i)\,, \tag{1}$$

where $\Gamma_{P_i} = \Gamma \cap P_i$.

Lemma 2.3.2. *For any $\mathbf{P}_i$, let $c_j(t)$ be the projection in $\Gamma\backslash X$ of the geodesic $\tilde{c}_j(t) = (n_j, z_j, \exp(tH_j))$ in X, where $n_j \in N_P$, $z_j \in X_P$, $H_j \in \mathfrak{a}_P^+$, $|H_j| = 1$, and $j = 1, 2$. Then $c_1(t)$ and $c_2(t)$ converge to the same boundary point in $\overline{\Gamma\backslash X}^{BS}$ as $t \to +\infty$ if and only if (n_1, z_1) and (n_2, z_2) project to the same point in $\Gamma_{P_i}\backslash e(\mathbf{P}_i)$.*

Proof. From the definition, it is clear that $\tilde{c}_i(t)$ converges to $(n_i, z_i) \in e(\mathbf{P}) \subset \overline{X}^{BS}$. Then the lemma is clear.

2.4 Reductive Borel–Serre compactification. The reductive Borel–Serre compactification $\overline{\Gamma\backslash X}^{RBS}$ is constructed similarly as $\overline{\Gamma\backslash X}^{BS}$ [Z1, p. 190], [HaZ1, 1.3(b)], [GHM, §8].

For any rational parabolic subgroup $\mathbf{P}$, its boundary component $\hat{e}(\mathbf{P})$ is defined by $\hat{e}(\mathbf{P}) = X_P$. The boundary component $\hat{e}(\mathbf{P})$ is added at infinity of X also through the horospherical decomposition (Eq. (1) in §2.2). More precisely, for any sequence $x_j = (n_j, z_j, \exp H_j)$ in X, x_j converges to $z_\infty \in \hat{e}(\mathbf{P})$ if and only if $z_\infty = \lim_{j\to\infty} z_j$, and for any $\beta \in \Phi^+(P, A_P)$, $\beta(H_j) \to +\infty$ as $j \to +\infty$. Intuitively, the nilpotent factor N_P shrinks to a point, and only X_P is added at infinity as illustrated in Figure 2.4.

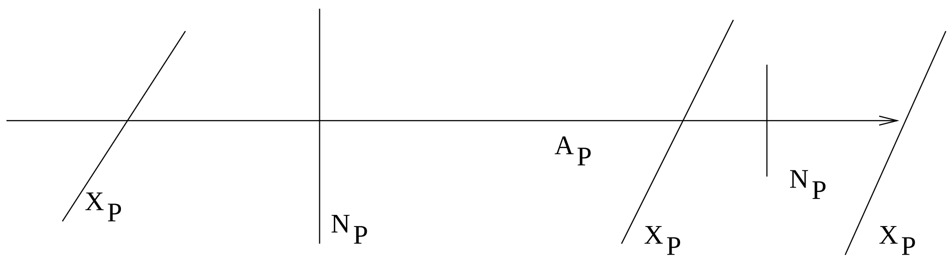

Figure 2.4

Adding all the (rational) boundary components $\hat{e}(\mathbf{P})$ to X, we get a partial compactification $\overline{X}^{RBS} = X \cup \coprod_{\mathbf{P}} \hat{e}(\mathbf{P})$ of X.

LEMMA 2.4.1 [Z1, Proposition 4.2]. *The arithmetic subgroup Γ acts continuously on $\overline{X}^{RBS}$ with a compact Hausdorff quotient, which is the reductive Borel–Serre compactification $\overline{\Gamma\backslash X}^{RBS}$.*

3 Baily–Borel Compactification

In this section, we recall the root structure of Hermitian symmetric spaces (3.1) and a refined horospherical decomposition of X (3.2). Then we recall the Baily–Borel compactification $\overline{\Gamma\backslash X}^{BB}$ (3.3) and its connection with $\overline{\Gamma\backslash X}^{BS}$ and $\overline{\Gamma\backslash X}^{RBS}$ (3.4), following the philosophy of [Z2].

3.1 Root structure of Hermitian symmetric spaces. Let $\mathbf{S} \subset \mathbf{G}$ be a maximal $\mathbb{Q}$-split torus, and $\Phi(\mathbf{G}, \mathbf{S})$ the set of $\mathbb{Q}$-roots of $\mathbf{G}$ with respect to $\mathbf{S}$. Choose an order on $\mathbf{S}$, and denote the set of positive roots by $\Phi^+(\mathbf{G}, \mathbf{S})$ and the set of simple roots by $\Delta(\mathbf{G}, \mathbf{S})$. Since $\mathbf{G}$ is $\mathbb{Q}$-simple, $\Phi(\mathbf{G}, \mathbf{S})$ is irreducible. By [BBo, Proposition 2.9], the root system $\Phi(\mathbf{G}, \mathbf{S})$ is either of type BC_r or of type C_r, where $r = \mathrm{rk}_{\mathbb{Q}}(\mathbf{G})$. Denote the simple roots by $\beta_1, \dots, \beta_r$ such that β_i is not orthogonal to β_{i+1}, and β_r is the short root in case of BC_r type and the long root in case of C_r type. The root β_r is called the distinguished root of $\Delta(\mathbf{G}, \mathbf{S})$.

3.2 Refined horospherical decomposition. The set of positive roots $\Phi^+(\mathbf{G}, \mathbf{S})$ determines a unique minimal rational parabolic subgroup $\mathbf{P}$ whose radical $\mathbf{N_P}$ is spanned by the root spaces of the positive roots. Any rational parabolic subgroup $\mathbf{Q}$ containing $\mathbf{P}$ is called a standard rational parabolic subgroup and is of the form $\mathbf{P}_I$, where I is a proper subset of $\Delta(\mathbf{G}, \mathbf{S})$, and $\mathbf{P}_I$ is generated by the centralizer of $\mathbf{S}_I = \{s \in \mathbf{S} \mid s^\beta = 1, \beta \in I\}$ and $\mathbf{N_P}$.

A subset of $\Delta(\mathbf{G}, \mathbf{S})$ is called connected if it is not a union of two orthogonal disjoint subsets with respect to the Killing form. For every standard parabolic subgroup $\mathbf{P}_I$, let $\Delta_{I,h}$ be the connected component of I containing the distinguished root β_r. If I does not contain β_r, $\Delta_{I,h}$ is defined to be empty.

Then $\Delta_{I,h}$ spans a subroot system in $\Phi(\mathbf{G}, \mathbf{S})$, whose root spaces generate a semisimple algebraic subgroup $\mathbf{G}_{I,h}$ of $\mathbf{G}$. Let $\mathbf{G}_{I,l}$ be the normal $\mathbb{Q}$-subgroup in the Levi group $\mathbf{L}_{\mathbf{P}_I}(x_0)$ complementary to $\mathbf{G}_{I,h}$, i.e., $\mathbf{L}_{\mathbf{P}_I}(x_0) = \mathbf{G}_{I,h}\mathbf{G}_{I,l}$ is an almost direct product. Define $K_{I,h} = K \cap G_{I,h}$, and $K_{I,l} = K \cap G_{I,l}$, where $G_{I,h} = \mathbf{G}_{I,h}(\mathbb{R})$, $G_{I,l} = \mathbf{G}_{P,l}(\mathbb{R})$.

Lemma 3.2.1. *The space $X_{I,h} = G_{I,h}/K_{I,h}$ is a Hermitian symmetric space of noncompact type. And $X_{I,l} = G_{I,l}/K_{I,l}A_{P_I}$ is a symmetric space of noncompact type. If $\mathbf{P}_I$ is a maximal rational parabolic subgroup, then $C_{\mathbf{P}_I} = G_{I,l}/K_{I,l}$ is a symmetric cone in the center of the nilpotent radical of P_I.*

Proof. The first statement follows from the fact that $\Delta_{I,h}$ spans a root system of type either BC or C. If $\mathbf{P}_I$ is maximal, the second and the third statements follows from [AMuRT, Theorem 1, p. 227]; and the nonmaximal cases are similar.

Lemma 3.2.2. *The boundary symmetric space X_I associated with $\mathbf{P}_I$ in §2.2 can be decomposed as a Riemannian product $X_I = X_{I,h} \times X_{I,l}$.*

Proof. Since $M_{P_I} = G_{I,h}G_{I,l}/A_{P_I}$ and $G_{I,h}$ commutes with $G_{I,l}$, the lemma follows from the definition of $X_{I,h}$ and $X_{I,l}$.

Since every rational parabolic subgroup $\mathbf{Q}$ is conjugate to a standard parabolic subgroup $\mathbf{P}_I$, we also get subgroups $G_{Q,h}, G_{Q,l}$ of M_Q, and the boundary spaces $X_{Q,h}, X_{Q,l}$.

Lemma 3.2.3. *With the above notation, the space X has the following refined horospherical decomposition with respect to the parabolic subgroup $\mathbf{Q}$: $X = N_Q \times X_{Q,h} \times X_{Q,l} \times A_Q$.*

Proof. It follows from the horospherical decomposition $X = N_Q \times X_Q \times A_Q$ (Eq. (1) in §2.2) and the decomposition $X_Q = X_{Q,h} \times X_{Q,l}$ in Lemma 3.2.2.

This refined horospherical decomposition of X is illustrated in Figure 3.2.

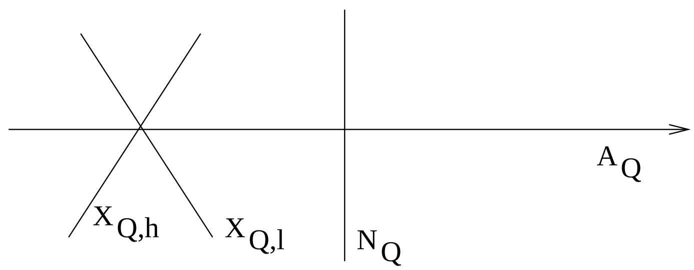

Figure 3.2

To compare $\overline{\Gamma\backslash X}^{BS}$ and $\overline{\Gamma\backslash X}^{BB}$, we need the following lemma.

LEMMA 3.2.4. *For any proper rational parabolic subgroup* $\mathbf{Q}$, *there exists a unique maximal rational parabolic subgroup* $\mathbf{Q}'$ *containing* $\mathbf{Q}$ *such that* $X_{Q,h} = X_{Q',h}$.

Proof. Without loss of generality, assume that $\mathbf{Q} = \mathbf{P}_I$. If I does not contain the distinguished root β_r, then $X_{Q,h}$ consists of one point, and $\mathbf{Q}' = \mathbf{P}_{\Delta - \{\beta_r\}}$ is the unique maximal standard parabolic subgroup containing $\mathbf{Q}$ such that $X_{Q',h}$ is trivial. Otherwise, I contains β_r. Let $\{\beta_i, \dots, \beta_r\}$ be the connected component of I containing β_r. Then $\mathbf{P}_{\Delta - \{\beta_{i-1}\}}$ is the unique maximal standard parabolic subgroup containing $\mathbf{Q}$ such that $X_{Q',h} = X_{Q,h}$.

3.3 Boundary components and $\overline{\Gamma\backslash X}^{BB}$. The construction of the Baily–Borel compactification $\overline{\Gamma\backslash X}^{BB}$ [BBo] is similar to $\overline{\Gamma\backslash X}^{BS}$ in §2.3 except that only proper maximal rational parabolic subgroups contribute to the boundary $\partial\overline{\Gamma\backslash X}^{BB}$, i.e., the boundary components of the partial compactification $\overline{X}^{BB}$ of X is parametrized by maximal rational parabolic subgroups of $\mathbf{G}$.

For each maximal rational parabolic subgroup $\mathbf{Q}$, define its boundary component $\tilde{e}(\mathbf{Q})$ by $\tilde{e}(\mathbf{Q}) = X_{Q,h}$. This boundary component can be realized as a maximal analytic subset in the boundary of the closure of X in the complex Euclidean space when X is realized canonically as a bounded symmetric domain [BBo].

These rational boundary components $\tilde{e}(\mathbf{Q})$ can be added to X according to the refined horospherical decomposition in Lemma 3.2.3 to form the partial compactification $\overline{X}^{BB} = X \cup \coprod_{\mathbf{Q}} \tilde{e}(\mathbf{Q})$, where the union over $\mathbf{Q}$

is over all maximal rational parabolic subgroups. More precisely, for any sequence $x_j = (n_j, z_{j,h}, z_{j,l}, \exp H_j) \in N_Q \times X_{Q,h} \times X_{Q,l} \times A_Q$, the sequence x_j converges to $z_{\infty,h} \in \tilde{e}(\mathbf{Q})$ if and only if $z_{\infty,h} = \lim_{j\to\infty} z_{j,h}$ in $X_{Q,h}$, and for any $\beta \in \Phi^+(Q, A_Q)$, $\lim_{j\to+\infty} \beta(H_j) = +\infty$.

This gluing procedure is illustrated in Figure 3.3, where $X_{Q,l}$ and N_Q shrink to one point at infinity, and only the Hermitian factor is added at infinity.

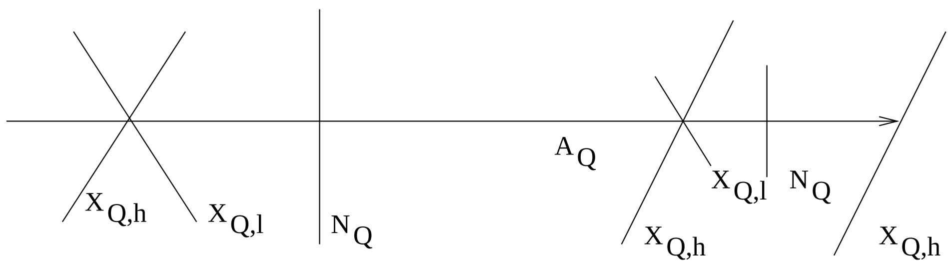

Figure 3.3

LEMMA 3.3.1 [BBo, Corollary 4.11]. *The arithmetic group* Γ *acts continuously on* $\overline{X}^{BB}$ *with a compact Hausdorff quotient. This quotient is called the Baily–Borel compactification of* $\Gamma\backslash X$ *and denoted by* $\overline{\Gamma\backslash X}^{BB}$.

REMARK 3.3.2. By Lemma 3.2.4, non-maximal rational parabolic subgroups do not give rise to new boundary components for $\overline{X}^{BB}$. This is the reason that the boundary components of $\overline{X}^{BB}$ are parametrized by maximal rational parabolic subgroups.

3.4 The fibers of $\overline{\Gamma\backslash X}^{BS}$ over $\overline{\Gamma\backslash X}^{BB}$. To determine the GCQ $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}_\Sigma^{Tor}$ and understand its relation with $\overline{\Gamma\backslash X}^{BB}$, we need to describe the fibers of $\overline{\Gamma\backslash X}^{BS}$ over $\overline{\Gamma\backslash X}^{BB}$ explicitly.

LEMMA 3.4.1. *The identity map on* X *extends to a continuous* Γ*-equivariant map from* $\overline{X}^{BS}$ *to* $\overline{X}^{BB}$*; i.e., the partial compactification* $\overline{X}^{BS}$ *dominates* $\overline{X}^{BB}$.

Proof. From the description in §2.3 and §3.3 above, it is clear that a convergent sequence in $\overline{X}^{BS}$ with limit $(n_\infty, z_\infty) \in N_Q \times X_Q$ is also convergent in $\overline{X}^{BB}$ with limit $z_{\infty,h} \in X_{Q',h}$, where $\mathbf{Q}'$ is the unique maximal rational parabolic subgroup containing $\mathbf{Q}$ such that $X_{Q',h} = X_{Q,h}$ (Lemma 3.2.4), and $z_\infty = (z_{\infty,h}, z_{\infty,l}) \in X_{Q,h} \times X_{Q,l} = X_Q$. This implies that the identity map on X extends to a well-defined continuous map from $\overline{X}^{BS}$ to $\overline{X}^{BB}$.

Since Γ acts on the set of rational parabolic subgroups by conjugation and preserves the horospherical decomposition and the refined horospherical decomposition, the extended map is Γ equivariant.

LEMMA 3.4.2 [Z2, Theorem 3.11]. *The Borel–Serre compactification* $\overline{\Gamma\backslash X}^{BS}$ *dominates the Baily–Borel compactification* $\overline{\Gamma\backslash X}^{BB}$.

Proof. Since the map from $\overline{X}^{BS}$ to $\overline{X}^{BB}$ is Γ-equivariant, we get a continuous map from $\overline{\Gamma\backslash X}^{BS}$ to $\overline{\Gamma\backslash X}^{BB}$ which restricts to the identity map on X. Therefore, $\overline{\Gamma\backslash X}^{BS}$ dominates $\overline{\Gamma\backslash X}^{BB}$.

Let $\mathbf{P}_1, \dots, \mathbf{P}_n$ be representatives of Γ-conjugacy classes of maximal rational parabolic subgroups. Then

$$\overline{\Gamma\backslash X}^{BB} = \Gamma\backslash X \cup \coprod_{i=1}^{n} \Gamma_{P_i,h}\backslash \tilde{e}(P_i) = \Gamma\backslash X \cup \coprod_{i=1}^{n} \Gamma_{P_i,h}\backslash X_{P_i,h} ,$$

where $\Gamma_{P_i,h}$ is the image of $\Gamma_{P_i} = \Gamma \cap P_i$ under the projection $P_i = N_{P_i} G_{P_i,h} G_{P_i,l} A_{P_i} \to G_{P_i,h}$ and defines a lattice subgroup in $G_{P_i,h}$.

For each $\mathbf{P}_i$, let $\mathbf{P}_{i,j}$, $j = 1, \dots, n_i$, be representatives of Γ-conjugacy classes of rational parabolic subgroups which are contained in $\mathbf{P}_i$ and which satisfy $X_{P_{i,j},h} = X_{P_i,h}$. Then by Lemma 3.2.4, $\mathbf{P}_{i,j}$, $i = 1, \dots, n$, $j = 1, \dots, n_i$, are representatives of Γ-conjugacy classes of all proper rational parabolic subgroups of $\mathbf{G}$, and

$$\overline{\Gamma\backslash X}^{BS} = \Gamma\backslash X \cup \coprod_{i=1}^{n} \coprod_{j=1}^{n_i} \Gamma_{P_{i,j}}\backslash e(\mathbf{P}_{i,j}) = \Gamma\backslash X \cup \coprod_{i=1}^{n} \coprod_{j=1}^{n_i} \Gamma_{P_{i,j}}\backslash N_{P_{i,j}} \times X_{P_{i,j}} .$$

LEMMA 3.4.3 [Z3, pp. 350–351]. *For any* $z \in \Gamma_{P_i,h}\backslash X_{P_i,h} \subset \partial\overline{\Gamma\backslash X}^{BB}$, *the fiber in* $\overline{\Gamma\backslash X}^{BS}$ *over* z *is equal to*

$$\coprod_{j=1}^{n_i} \Gamma''_{P_{i,j}}\backslash N_{P_{i,j}} \times X_{P_{i,j},l} ,$$

where $\Gamma''_{P_{i,j}}$ *is the kernel in* $\Gamma_{P_{i,j}}$ *under the projection* $P_{i,j} = N_{P_{i,j}} G_{P_{i,j},l} G_{P_{i,j},h} A_{P_{i,j}} \to G_{P_{i,j},h}$.

Proof. It follows from the proof of Lemma 3.4.1 that the fiber in $\overline{X}^{BS}$ over a boundary component $\tilde{e}(\mathbf{P}_i)$ is equal to $\coprod_{j=1}^{n_i} N_{P_{i,j}} \times X_{P_{i,j},l}$. By passing to the quotient, we get Lemma 3.4.3.

LEMMA 3.4.4. *Let* $\mathbf{P}_{i,1} = \mathbf{P}_i$ *be the unique maximal parabolic subgroup in the collection* $\mathbf{P}_{i,1}, \dots, \mathbf{P}_{i,n_i}$. *Then* $\Gamma''_{P_i}\backslash N_{P_i} \times X_{P_i,l}$ *is a dense subset of the total fiber* $\coprod_{j=1}^{n_i} \Gamma''_{P_{i,j}}\backslash N_{P_{i,j}} \times X_{P_{i,j},l}$ *over* z.

Proof. We note that $\overline{X}^{BS}$ has the structure of a manifold with corners, $\coprod_{j=1}^{n_i} e(\mathbf{P}_{i,j}) = \coprod_{j=1}^{n_i}(N_{P_{i,j}} \times X_{P_{i,j},l}) \times X_{P_i,h}$, and $e(\mathbf{P}_i)$ is a maximal boundary face and other boundary components $e(\mathbf{P}_{i,j})$ are contained in the closure of $e(\mathbf{P}_i)$ [BoSe, §7]. This implies that $N_{P_i} \times X_{P_i,l}$ is dense in $\coprod_{j=1}^{n_i}(N_{P_{i,j}} \times X_{P_{i,j},l})$. Passing to the quotient, we get Lemma 3.4.4.

Similarly, we get the following description of the fibers of $\overline{\Gamma\backslash X}^{RBS}$ over $\overline{\Gamma\backslash X}^{BB}$.

LEMMA 3.4.5. *The reductive Borel-Serre compactification $\overline{\Gamma\backslash X}^{RBS}$ dominates $\overline{\Gamma\backslash X}^{BB}$. For any $z \in \Gamma_{P_i,h}\backslash X_{P_i,h} \subset \partial\overline{\Gamma\backslash X}^{BB}$, the fiber in $\overline{\Gamma\backslash X}^{RBS}$ over z is equal to $\coprod_{j=1}^{n_i} \Gamma_{P_{i,j},l}\backslash X_{P_{i,j},l}$, where $\Gamma_{P_{i,j},l}$ is the image of $\Gamma_{P_{i,j}}$ in $G_{P_{i,j},l}$ under the projection $P_{i,j} = N_{P_{i,j}} G_{P_{i,j},l} G_{P_{i,j},h} A_{P_{i,j}} \to G_{P_{i,j},l}$.*

4 Toroidal Embeddings

In this section, we recall the toroidal embeddings. For details, see [AMuRT, §1.1], [O], [N, §6]. In the following, we emphasize the topological aspect of the toroidal embeddings and ignore their algebraic structure.

4.1 Rational partial polyhedral decomposition. Let N be a lattice in $\mathbb{R}^n$. Then $T = \mathbb{C}^n/N$ is equal to $(\mathbb{C}^\times)^n$, a complex torus. The lattice N defines a rational structure on $N \otimes \mathbb{R} \cong \mathbb{R}^n$.

A subset σ of $N \otimes \mathbb{R}$ is called a strongly convex rational polyhedral cone if there exist a finite number of elements $n_1, \dots, n_s$ in N such that

$$\sigma = \{a_1 n_1 + \cdots + a_s n_s \mid a_i \geq 0, i = 1, \dots, s\}.$$

Then $\mathrm{Span}(\sigma) = \sigma + (-\sigma)$ is a linear subspace of $N \otimes \mathbb{R}$, and the interior of σ considered as a subset of $\mathrm{Span}(\sigma)$ is an open cone and denoted by $\overset{o}{\sigma}$.

For any such cone σ, there exist finitely many linear functionals $l_1, \dots, l_k$ on $N \otimes \mathbb{R}$ and $p \leq k$ such that

$$\sigma = \{n \in N \otimes \mathbb{R} \mid l_1(n) \geq 0, \dots, l_p(n) \geq 0; l_{p+1}(n) = 0, \dots, l_k(n) = 0\},$$

and

$$\overset{o}{\sigma} = \{n \in N \otimes \mathbb{R} \mid l_1(n) > 0, \dots, l_p(n) > 0; l_{p+1}(n) = 0, \dots, l_k(n) = 0\}. \tag{1}$$

A rational partial polyhedral cone decomposition of $N \otimes \mathbb{R}$ is a collection Σ of strongly convex rational polyhedral cones in $N \otimes \mathbb{R}$ satisfying the following conditions:

1. Every face of any $\sigma \in \Sigma$ is also a cone in Σ.
2. For any $\sigma, \sigma' \in \Sigma$, the intersection $\sigma \cap \sigma'$ is a face of both σ and σ'.

4.2 Boundary components of cones. A rational partial polyhedral cone decomposition Σ in $N \otimes \mathbb{R}$ defines a partial compactification of the torus T, called the toroidal embedding of T associated with Σ and denoted by $\overline{T}_\Sigma$.

For any $\sigma \in \Sigma$, the complex subspace $\mathrm{Span}_{\mathbb{C}}(\sigma) = \mathrm{Span}(\sigma) \otimes \mathbb{C}$ acts on T by translation. Define the boundary component $O(\sigma)$ associated with σ to be the quotient $T/\mathrm{Span}_{\mathbb{C}}(\sigma)$, which is equal to $(\mathbb{C}^r/\mathrm{Span}_{\mathbb{C}}(\sigma))/N$, a complex torus of smaller dimension.

Then the toroidal embedding $\overline{T}_\Sigma$ is defined by

$$\overline{T}_\Sigma = T \cup \coprod_{\sigma\in\Sigma,\sigma\neq\{0\}} O(\sigma)\,, \tag{1}$$

with the following topology: A sequence $z_j = x_j + iy_j$ in T converges to a point $z_\infty \in O(\sigma)$ for some $\sigma \in \Sigma$ if and only if for the defining linear functionals $l_1, \dots, l_k$ of σ in Eq. (1) in §4.1, the following conditions are satisfied:

1. $l_1(x_j) \to +\infty, \dots, l_p(x_j) \to +\infty$ as $j \to +\infty$, while $l_{p+1}(x_j), \dots, l_k(x_j)$ are bounded.
2. The projection of z_j in $O(\sigma)$ converges to the point z_∞.

Lemma 4.2.1. *Write $T = N \otimes \mathbb{R}/N + iN \otimes \mathbb{R}$. For any point $x \in N \otimes \mathbb{R}/N$ and a ray $c(t)$ in $N \otimes \mathbb{R}$ which starts from the origin and is contained in the interior $\overset{o}{\sigma}$ of a cone $\sigma \in \Sigma$, then the ray $x + ic(t)$ in T converges to a boundary point in $O(\sigma) \subset \overline{T}_\Sigma$ as $t \to +\infty$. If σ has codimension zero in $N \otimes \mathbb{R}$, then any two such rays whose imaginary parts are contained in the interior $\overset{o}{\sigma}$ converge to the same boundary point.*

Proof. It is clear that both conditions above are satisfied and hence the ray $x + ic(t)$ converges to a point in $O(\sigma)$ as $t \to +\infty$. If σ has codimension zero in $N \otimes \mathbb{R}$, then $O(\sigma)$ consists of one point, and the second statement follows immediately.

Remark 4.2.2. The decomposition in Eq. (1) above is the T-orbit decomposition of $\overline{T}_\Sigma$. The reason why we call $O(\sigma)$ the boundary component is to emphasize $\overline{T}_\Sigma$ as a partial compactification in analogue with the partial compactification $\overline{X}^{BS}$.

4.3 Examples. If $\dim N \otimes \mathbb{R} = 1$ and $\Sigma = \{\{0\}$, the positive half line$\}$, then $\overline{T}_\Sigma$ is represented by Figure 4.3.1. In this figure, the circle $N \otimes \mathbb{R}/N$ shrinks to one point at infinity, and $\overline{T}_\Sigma$ is the complex plane. If $\dim N \otimes \mathbb{R} = 2$ and Σ is the product of the polyhedral cone decomposition in dimension 1 above, then Σ consists of four elements and $\overline{T}_\Sigma$ is represented geometrically in Figure 4.3.2. In this figure, the two dimensional compact torus $N \otimes \mathbb{R}/N$ shrinks to either a circle or a point depending on the direction at infinity. Any two rays in T whose imaginary parts are rays in the positive quadrant converge to the distinguished corner point.

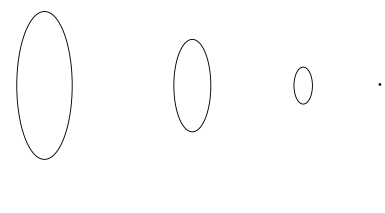

Figure 4.3.1

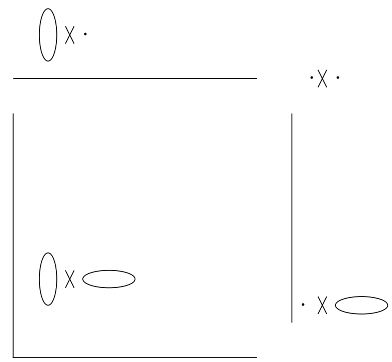

Figure 4.3.2

5 Toroidal and Intermediate Compactifications

In this section, we recall the toroidal compactifications $\overline{\Gamma\backslash X}^{Tor}_{\Sigma}$ of $\Gamma\backslash X$ (5.4) and a new compactification $\overline{\Gamma\backslash X}^{Int}$, which lies between $\overline{\Gamma\backslash X}^{Tor}_{\Sigma}$ and $\overline{\Gamma\backslash X}^{BB}$ and is hence called the intermediate compactification (5.5).

A key point in understanding the toroidal compactifications is to visualize the complex torus needed to define the toroidal embeddings (5.3). The realization of X as a Siegel domain of the third kind and its connection with the horospherical decomposition in (5.1) show clearly how the complex torus arises.

5.1 Siegel domains of the third kind. For every maximal rational parabolic subgroup $\mathbf{Q}$, let U_Q be the center of the nilpotent radical N_Q of Q, and let $V_Q = N_Q/U_Q$. Then V_Q is a vector group, i.e., abelian and diffeomorphic to its Lie algebra $\mathfrak{v}_Q$. Since N_Q is a U_Q-principal bundle over V_Q, we get that, as differential manifolds, $N_Q = U_Q \times V_Q$.

LEMMA 5.1.1. *The Lie algebra $\mathfrak{v}_Q$ of the quotient group $V_Q = N_Q/U_Q$ can be identified with a subspace of the algebra $\mathfrak{n}_Q$ of N_Q which is complementary to the Lie algebra $\mathfrak{u}_Q$ of U_Q. The adjoint action of $G_{Q,l}$ on $\mathfrak{n}_Q$*

preserves this subspace, which is denoted also by $\mathfrak{v}_Q$.

Proof. By [BBo, Corollary 2.10], Q is also a maximal real parabolic subgroup. For simplicity, we assume that Q is the normalizer of a standard boundary component F_s in the notation of [AMuRT, §4.1]. Then the third equation on [AMuRT, p. 224] shows that the Lie algebra $\mathfrak{u}_Q$ is the direct sum of some of the root spaces which appear in $\mathfrak{n}_Q$, and hence $\mathfrak{v}_Q$ can naturally identified with the direct sum of the other root spaces in $\mathfrak{n}_Q$, given by the second equation on [AMuRT, p. 224]. This equation also shows that this complementary subspace of $\mathfrak{u}_Q$ in $\mathfrak{n}_Q$ is an abelian subalgebra. The root space decomposition of the Lie algebra $\mathfrak{g}_{Q,l}$ of $G_{Q,l}$ on [AMuRT, p. 226] shows that the adjoint action of $G_{Q,l}$ on $\mathfrak{n}_Q$ leaves both subspaces $\mathfrak{u}_Q$ and $\mathfrak{v}_Q$ invariant. This completes the proof.

This lemma shows that the adjoint action of $G_{Q,l}$ on the quotient group V_Q and its Lie algebra $\mathfrak{v}_Q$ can be studied by the restriction of the adjoint action of $G_{Q,l}$ on $\mathfrak{n}_Q$ to the invariant subspace $\mathfrak{v}_Q$ (see Lemma 7.5.1). In §7.4 and §7.5, we will show that the adjoint (or holonomy) action of $\Gamma_{Q,l}$ on the compact nilmanifold $\Gamma_{V_Q}\backslash V_Q$ is ergodic, where $\Gamma_{V_Q} = \Gamma_{N_Q}/\Gamma_{U_Q}$, $\Gamma_{N_Q} = \Gamma \cap N_Q$, $\Gamma_{U_Q} = \Gamma \cap U_Q$. This ergodicity result plays an important role in this paper.

Recall from Lemma 3.2.1 that $X_{Q,h}$ is the boundary Hermitian symmetric space and C_Q is the symmetric cone in U_Q with $X_{Q,h}$ as a homothety section. Then we have the following realization of X as a Siegel domain of the third kind over $X_{Q,h}$

Proposition 5.1.2 [WoKo, Theorem 7.7], [AMuRT, §3.4, pp. 238-239], [N, §5]. *With the above notation, there exists an injective holomorphic map* $\pi : X \to X_{Q,h} \times \mathbb{C}^n \times (U_Q \otimes \mathbb{C})$ *such that*

$$\pi(X) = \left\{(z, v, u_1 + iu_2) \mid z \in X_{Q,h}, v \in \mathbb{C}^n, u_1 \in U_Q, u_2 \in h_z(v,v) + C_Q\right\},$$

where $n = \frac{1}{2}\dim V_Q$ *and* $h_z(v,v) \in C_Q$ *is a quadratic form in* v *depending holomorphically on* z.

This realization represents X as a family of tube domains $U_Q + iC_Q$ over $X_{Q,h} \times \mathbb{C}^n$ and is closely related to the refined horospherical decomposition in Lemma 3.2.3.

Using the decomposition $N_Q = U_Q \times V_Q$ explained above, we can write the refined horospherical decomposition of X in Lemma 3.2.3 as follows:

$$X = U_Q \times V_Q \times X_{Q,h} \times X_{Q,l} \times A_Q\,. \tag{1}$$

Then the relation between the horospherical decomposition and the realization as a Siegel domain of the third kind is as follows.

Lemma 5.1.3. *For any* $x = (u, v, z, x_l, a) \in U_Q \times V_Q \times X_{Q,h} \times X_{Q,l} \times A_Q = X$, *denote the image* $\pi(x)$ *of* x *under the map* π *in Proposition 5.1.2 by* $(z', v', u'_1 + iu'_2) \in X_{Q,h} \times \mathbb{C}^n \times (U_Q \otimes \mathbb{C})$. *Then* $z' = z$, *the map* $v \to v'$ *defines a* $\mathbb{R}$*-linear isomorphism from* V_Q *to* $\mathbb{C}^n$, *the map* $u \to u'_1$ *is a* $\mathbb{R}$*-linear transformation on* U_Q, *and* $u'_2 \in h_z(v', v') + C_Q$. *Furthermore, for any* $u \in U_Q$, $v \in V_Q$, *and* $z \in X_{Q,h}$, *the image of* $\{u\} \times \{v\} \times \{z\} \times X_{Q,l} \times A_Q$ *is exactly the shifted cone* $u'_1 + i(h_z(v', v') + C_Q)$ *over the point* $(z, v') \in X_{Q,h} \times \mathbb{C}^n$.

Proof. It follows from the discussions in [AMuRT, pp. 235-238]; in particular, the linear isomorphisms $v \to v'$ and $u \to u'$ come from trivialization of the two principal bundles.

This relation in Lemma 5.1.3 can be represented by Figure 5.1.1. The left-hand side represents the fivefold decomposition of X in Eq. (1) above, while the right-hand side represents the structure of X as a family of tube domains over $X_{Q,h} \times \mathbb{C}^n \cong X_{Q,h} \times V_Q$ (we only draw the fibers of the fibrations).

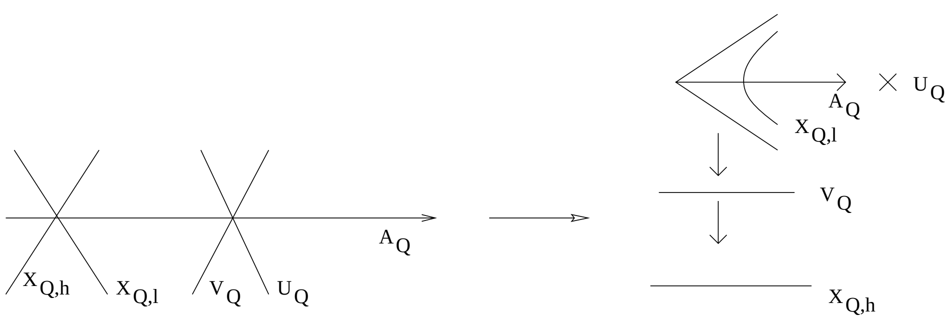

Figure 5.1.1

Since it is easier to visualize the horospherical decomposition, this figure is helpful in understanding the realization of X as a Siegel domain of the third kind.

Lemma 5.1.4. *For any* $u \in U_Q$, $v \in V_Q$, $z \in X_{Q,h}$, $x_l \in X_{Q,l}$, *and* $H \in \mathfrak{a}_Q^+$, $|H| = 1$, *the curve* $c(t) = (u, v, z, x_l, \exp(tH))$, $t \in \mathbb{R}$, *is a geodesic in* X. *In the realization of* X *as a Siegel domain of the third kind,* $c(t)$ *becomes a ray in the tube domain* $U_Q + i(h_z(v', v') + C_Q)$ *whose imaginary part is a ray starting from the vertex* $h_z(v', v')$.

Proof. By Lemma 2.2.1, $c(t)$ is a geodesic in X. By Lemma 5.1.3, the geodesic $c(t)$, $t \in \mathbb{R}$, is mapped into the cone $u'_1 + i(h_z(v', v') + C_Q)$ over the point $(z, v') \in X_{Q,h} \times \mathbb{C}^n$. Since $X_{Q,h}$ is a section of the symmetric cone C_Q, any geodesic in the cone $u_1' + i(h_z(v', v') + C_Q)$ with respect to the invariant metric is a ray from the vertex.

This lemma shows that when x_l varies in $X_{Q,l}$, the family of parallel geodesics $(u, v, z, x_l, \exp(tH))$ in X are mapped to a family of rays in the cone $h_z(v', v') + C_Q$ issued from the vertex, as illustrated in Figure 5.1.2

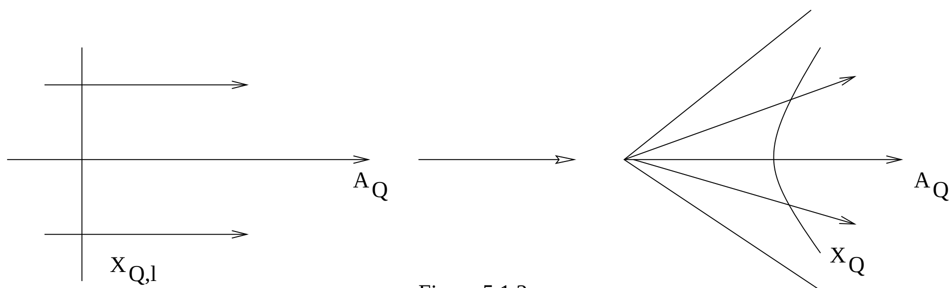

Figure 5.1.2

5.2 Γ-admissible family of polyhedral cone decomposition. As in the case of toroidal embeddings, the toroidal compactifications $\overline{\Gamma\backslash X}^{Tor}_{\Sigma}$ of $\Gamma\backslash X$ depend on polyhedral decompositions Σ.

For any maximal rational parabolic subgroup $\mathbf{Q}$, let $\Gamma_{Q,l}$ be the image of Γ_Q in $G_{Q,l}$ under the projection $Q = N_Q G_{Q,h} G_{Q,l} A_Q \to G_{Q,l}$. Then $\Gamma_{Q,l}$ is a torsion free lattice subgroup acting on $X_{Q,l}$. Denote the intersection $\Gamma_Q \cap U_Q$ by Γ_{U_Q}. Then Γ_{U_Q} is a torsion free lattice in the vector group U_Q.

Recall from [AMuRT, p. 117, 252], [N, p. 59-60] that a $\Gamma_{Q,l}$-admissible polyhedral decomposition of C_Q is a collection Σ_Q of polyhedral cones satisfying the following conditions:

1. Each cone in Σ_Q is a strongly convex rational polyhedral cone in $\overline{C_Q} \subset U_Q$ with respect to the rational structure on U_Q induced by the lattice Γ_{U_Q}.
2. Every face of any $\sigma \in \Sigma_Q$ is also an element in Σ_Q.
3. For any $\sigma, \sigma' \in \Sigma_Q$, the intersection $\sigma \cap \sigma'$ is a face of both σ and σ'.
4. For any $\gamma \in \Gamma_{Q,l}$ and $\sigma \in \Sigma_Q$, $\gamma\sigma$ is also a cone in Σ_Q.
5. There are only finitely many classes of cones in Σ_Q modulo $\Gamma_{Q,l}$.
6. $C_Q \subset \cup_{\sigma \in \Sigma_Q} \sigma$, and hence $C_Q = \cup_{\sigma \in \Sigma_Q} C_Q \cap \sigma$.

A Γ-admissible family of polyhedral cone decomposition $\Sigma = \{\Sigma_Q\}$ is a union of $\Gamma_{Q,l}$-admissible polyhedral cone decompositions Σ_Q of C_Q over all maximal rational parabolic subgroups satisfying the following compatibility conditions:

1. If $Q_1 = \gamma Q_2 \gamma^{-1}$, then $\gamma \Sigma_{Q_1} = \Sigma_{Q_2}$.

2. If C_{Q_1} is contained in the boundary of C_{Q_2}, then $\Sigma_{Q_1} = \{\sigma \cap \overline{C_{Q_1}} \mid \sigma \in \Sigma_{Q_2}\}$.

5.3 Partial toroidal compactifications. For any maximal rational parabolic subgroup Q, Γ_{U_Q} is a lattice in U_Q, and $\Gamma_{U_Q}\backslash U_Q \otimes \mathbb{C}$ is a complex torus. Using Proposition 5.1.2, identify X with the subset $\pi(X)$ in $X_{Q,h} \times \mathbb{C}^n \times (U_Q \otimes \mathbb{C})$. Then $\Gamma_{U_Q}\backslash X$ is contained in a bundle $\Gamma_{U_Q}\backslash X_{Q,h} \times \mathbb{C}^n \times (U_Q \otimes \mathbb{C})$ over $X_{Q,h} \times \mathbb{C}^n$ with fiber $\Gamma_{U_Q}\backslash U_Q \otimes \mathbb{C}$, which is denoted by $\Gamma_{U_Q}\backslash X(Q)$.

A $\Gamma_{Q,l}$-admissible polyhedral cone decomposition Σ_Q of $C_Q \subset U_Q$ defines a partial compactification (a toroidal embedding) $\overline{\Gamma_{U_Q}\backslash U_Q \otimes \mathbb{C}}_{\Sigma_Q}$ of every fiber $\Gamma_{U_Q}\backslash U_Q \otimes \mathbb{C}$ in $\Gamma_{U_Q}\backslash X(Q)$ (§4.2). Putting all these partial compactifications together, we get a partial compactification $\overline{\Gamma_{U_Q}\backslash X(Q)}_{\Sigma_Q}$ of the bundle $\Gamma_{U_Q}\backslash X(Q)$. The interior of the closure of $\Gamma_{U_Q}\backslash X$ in $\overline{\Gamma_{U_Q}\backslash X(Q)}_{\Sigma_Q}$ defines a partial compactification $\overline{\Gamma_{U_Q}\backslash X}_{\Sigma_Q}$ of $\Gamma_{U_Q}\backslash X$, which is a bundle over $X_{Q,h} \times \mathbb{C}^n$ [AMuRT, pp. 249–250].

This partial compactification $\overline{\Gamma_{U_Q}\backslash X}_{\Sigma_Q}$ can be represented by Figure 5.3.

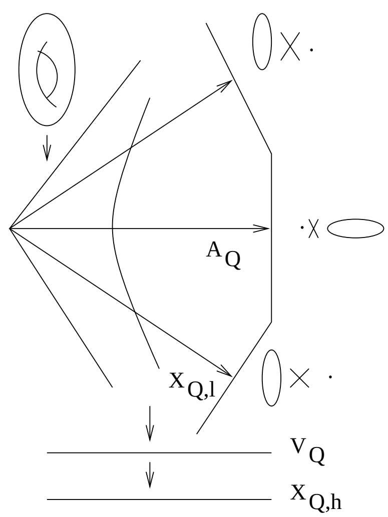

Figure 5.3

5.4 Toroidal compactification $\overline{\Gamma\backslash X}^{Tor}_{\Sigma}$. For every Γ-admissible family of polyhedral cone decomposition $\Sigma = \{\Sigma_Q\}$ in §5.2, we get a family of partially compactified spaces $\overline{\Gamma_{U_Q}\backslash X}_{\Sigma_Q}$.

PROPOSITION 5.4.1 [AMuRT, Main Theorem I, p. 252] [N, Main Theorem 7.10]. *For any Γ-admissible family of polyhedral cone decomposition Σ, there exists a unique compact Hausdorff analytic compactification $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ satisfying the following conditions:*

1. *For every maximal rational parabolic subgroup $\mathbf{Q}$, the projection map $\pi_Q : \Gamma_{U_Q}\backslash X \to \Gamma\backslash X$ extends to an open holomorphic map $\pi_Q : \overline{\Gamma_{U_Q}\backslash X}_{\Sigma_Q} \to \overline{\Gamma\backslash X}_{\Sigma}^{Tor}$.*
2. *The images $\pi_Q(\overline{\Gamma_{U_Q}\backslash X}_{\Sigma_Q})$ for all maximal rational parabolic subgroups $\mathbf{Q}$ cover $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$.*

This compactification $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ is called the toroidal compactification of $\Gamma\backslash X$ associated with the polyhedral cone decomposition Σ. For any such Σ, there always exists a refinement of Σ such that the corresponding toroidal compactification is a smooth projective variety. This is the motivation for defining the toroidal compactifications in [AMuRT].

To describe the boundary of $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$, we first construct essential boundary components in $(\Gamma_Q/\Gamma_{U_Q})\backslash\overline{\Gamma_{U_Q}\backslash X}_{\Sigma_Q}$. Let $\sigma_1, \dots, \sigma_m$ be representatives of $\Gamma_{Q,l}$-classes of cones in Σ_Q whose interiors are contained in the cone C_Q. These classes of cones in Σ_Q are called the essential cones for Q. For each such cone σ_j, its boundary component $O(\sigma_j)$ in the toroidal embedding $\overline{T}_{\Sigma_Q}$ of the torus $T = \Gamma_{U_Q}\backslash U_Q \otimes \mathbb{C}$ (4.2) defines a subbundle of $\overline{\Gamma_{U_Q}\backslash X}_{\Sigma_Q}$ whose fiber over $X_{Q,h} \times \mathbb{C}^n$ is $O(\sigma_j)$. Denote this bundle by $O(\sigma_j)$ also.

LEMMA 5.4.2 [AMuRT, p. 274] [N, p. 66]. *Let Γ'_Q be the kernel of the projection $\Gamma_Q \to \Gamma_{Q,l}$. Then Γ'_Q/Γ_{U_Q} acts properly discontinuously on $O(\sigma_j)$ and the quotient $(\Gamma'_Q/\Gamma_{U_Q})\backslash O(\sigma_j)$ is a fiber bundle over $\Gamma_{Q,h}\backslash X_{Q,h}$. The union $\coprod_{j=1}^{m}(\Gamma'_Q/\Gamma_{U_Q})\backslash O(\sigma_j)$ is mapped injectively into the boundary of $(\Gamma_Q/\Gamma_{U_Q})\backslash\overline{\Gamma_{U_Q}\backslash X}_{\Sigma_Q}$, and is called the essential boundary from Q.*

LEMMA 5.4.3 [N, Corollary 7.13]. *The map $\pi_Q : \overline{\Gamma_{U_Q}\backslash X}_{\Sigma_Q} \to \overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ factors through the map*

$$\overline{\pi_Q} : (\Gamma_Q/\Gamma_{U_Q})\backslash\overline{\Gamma_{U_Q}\backslash X}_{\Sigma_Q} \to \overline{\Gamma\backslash X}_{\Sigma}^{Tor},$$

and the map $\overline{\pi_Q}$ is injective on the essential boundary of $(\Gamma_Q/\Gamma_{U_Q})\backslash\overline{\Gamma_{U_Q}\backslash X}_{\Sigma_Q}$.

Proof. Since Σ is Γ-admissible, Γ_Q acts on $\overline{\Gamma_{U_Q}\backslash X}_{\Sigma_Q}$. Clearly Γ_{U_Q} acts trivially on $\overline{\Gamma_{U_Q}\backslash X}_{\Sigma_Q}$. Therefore, Γ_Q/Γ_{U_Q} acts on $\overline{\Gamma_{U_Q}\backslash X}_{\Sigma_Q}$. Since the map $\Gamma_{U_Q}\backslash X \to \Gamma\backslash X$ factors through the map $(\Gamma_Q/\Gamma_{U_Q})\backslash\Gamma_{U_Q}\backslash X \to \Gamma\backslash X$ and $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ is Hausdorff, the map $\pi_Q : \overline{\Gamma_{U_Q}\backslash X}_{\Sigma_Q} \to \overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ also factors through the map $\overline{\pi_Q}$.

On the other hand, by the reduction theory, two points on the essential boundary of $\overline{\Gamma_{U_Q}\backslash X}_{\Sigma_Q}$ are mapped to the same point in $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ if and only if they are identified under Γ_Q/Γ_{U_Q}. This implies that $\overline{\pi_Q}$ is injective on the boundary of $(\Gamma_Q/\Gamma_{U_Q})\backslash\overline{\Gamma_{U_Q}\backslash X}_{\Sigma_Q}$.

Let $\mathbf{P}_1, \dots, \mathbf{P}_n$ be representatives of Γ-conjugacy classes of maximal rational parabolic subgroups of $\mathbf{G}$. For each $\mathbf{P}_i$, let $\sigma_{i,1}, \dots, \sigma_{i,m_i}$ be representatives of $\Gamma_{\mathbf{P}_i,l}$-classes of essential cones in Σ_{P_i}. By the definition of Σ in §5.2, every cone in Σ is essential with respect to some P_i. Therefore, we get from Proposition 5.4.1(2) the following description of the boundary $\partial\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ of the toroidal compactification $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$:

$$\partial\overline{\Gamma\backslash X}_{\Sigma}^{Tor} = \coprod_{i=1}^{n}\coprod_{j=1}^{m_i}(\Gamma'_{P_i}/\Gamma_{U_{P_i}})\backslash O(\sigma_{i,j})\,. \tag{1}$$

Lemma 5.4.4 [AMuRT, Proposition, p. 254], [N, pp. 63-64]. *Every toroidal compactification* $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ *dominates* $\overline{\Gamma\backslash X}^{BB}$. *For any boundary point* $z \in \Gamma_{P_i,h}\backslash X_{P_i,h} \subset \partial\overline{\Gamma\backslash X}^{BB}$, *the fiber in* $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ *over* z *is a union* $\coprod_{j=1}^{m_i}\pi_{\sigma_{i,j}}^{-1}(z)$, *where* $\pi_{\sigma_{i,j}}$ *is the projection map from the fiber bundle* $(\Gamma'_{P_i}/\Gamma_{U_{P_i}})\backslash O(\sigma_j)$ *to* $\Gamma_{P_i,h}\backslash X_{P_i,h}$.

Proof. From the proof of [AMuRT, Lemma 2, p. 255] and the definition of the map $\overline{\Gamma\backslash X}_{\Sigma}^{Tor} \to \overline{\Gamma\backslash X}^{BB}$, we get that among all the boundary components in Eq. (1), only the essential boundary components of P_i are mapped to $\Gamma_{P_i,h}\backslash X_{P_i,h}$. Then the lemma follows easily.

5.5 Intermediate compactification $\overline{\Gamma\backslash X}^{Int}$. Using the above description of the fibers of $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ over $\overline{\Gamma\backslash X}^{BB}$, we introduce a new compactification $\overline{\Gamma\backslash X}^{Int}$, which will be shown to be equal to the GCQ $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ in §7.

Proposition 5.5.1. *For every toroidal compactification* $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$, *there exists a compactification of* $\Gamma\backslash X$ *satisfying the following properties:*

1. *It dominates* $\overline{\Gamma\backslash X}^{BB}$ *and is dominated by* $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$.
2. *Let* $\Gamma_{P,h}\backslash X_{P,h}$ *be a boundary component of* $\overline{\Gamma\backslash X}^{BB}$. *For any* $z \in \Gamma_{P,h}\backslash X_{P,h} \subset \partial\overline{\Gamma\backslash X}^{BB}$, *if* $\dim X_{P,l} \geq 1$, *then the fiber over* z *in*

$\overline{\Gamma\backslash X}^{Int}$ consists of only one point.

3. *If $\dim X_{P,l} = 0$, then the fiber over z in $\overline{\Gamma\backslash X}^{Int}$ is $\Gamma_{V_P}\backslash V_P$, a compact torus, which is also the fiber in $\overline{\Gamma\backslash X}^{Tor}_{\Sigma}$ over z, where $\Gamma_{V_P} = \Gamma_{N_P}/\Gamma_{U_P}$ is a lattice in V_P.*

This compactification is called the intermediate compactification of $\Gamma\backslash X$ and denoted by $\overline{\Gamma\backslash X}^{Int}$.

Proof. In $\overline{\Gamma\backslash X}^{BB}$, let $\coprod' \Gamma_{P_i,h}\backslash X_{P_i,h}$ be the union of those boundary components such that $\dim X_{P_i,l} \geq 1$. Since the boundary components $\Gamma_{P_i,h}\backslash X_{P_i,h}$ with $\dim X_{P_i,l} = 0$ are of higher dimension than the components in the union above and the closure of each boundary component in $\overline{\Gamma\backslash X}^{BB}$ is a union of the boundary itself and some other components of lower dimension, this union $\coprod' \Gamma_{P_i,h}\backslash X_{P_i,h}$ is a compact subset of $\partial\overline{\Gamma\backslash X}^{BB}$. The fiber in $\overline{\Gamma\backslash X}^{Tor}_{\Sigma}$ over z in $\coprod' \Gamma_{P_i,h}\backslash X_{P_i,h}$ is compact. Then it follows that the following relation is closed: Two boundary points in $\overline{\Gamma\backslash X}^{Tor}_{\Sigma}$ are equivalent if and only if they belong to the fiber over the same point in $\coprod' \Gamma_{P_i,h}\backslash X_{P_i,h}$. Therefore, the quotient of $\overline{\Gamma\backslash X}^{Tor}_{\Sigma}$ by this relation defines a Hausdorff compactification of $\Gamma\backslash X$.

This compactification clearly dominates $\overline{\Gamma\backslash X}^{BB}$, and the fiber over a point $z \in \Gamma_{P_i,h}\backslash X_{P_i,h} \subset \overline{\Gamma\backslash X}^{BB}$ is trivial if $\dim X_{P_i,l} \geq 1$. On the other hand, if $\dim X_{P_i,l} = 0$, the fiber is the same as the fiber in $\overline{\Gamma\backslash X}^{Tor}_{\Sigma}$, which is equal to $\Gamma_{V_{P_i}}\backslash V_{P_i}$ by Lemma 5.4.4.

Proposition 5.5.2. *The intermediate compactification $\overline{\Gamma\backslash X}^{Int}$ is independent of the choice of Γ-admissible polyhedral cone decomposition Σ in $\overline{\Gamma\backslash X}^{Tor}_{\Sigma}$, and is dominated by $\overline{\Gamma\backslash X}^{BS}$; in particular, $\overline{\Gamma\backslash X}^{Int}$ is a common quotient of $\overline{\Gamma\backslash X}^{BS}$ and every toroidal compactification $\overline{\Gamma\backslash X}^{Tor}_{\Sigma}$.*

Proof. From its definition, $\partial\overline{\Gamma\backslash X}^{Int} = \coprod' \Gamma_{P_i,h}\backslash X_{P_i,h} \cup \coprod'' (\Gamma'_{P_i}/\Gamma_{U_{P_i}})\backslash X_{P_i,h} \times V_{P_i}$, where the union $\coprod'$ is over P_i with $\dim X_{P_i,l} \geq 1$, and $\coprod''$ is over P_i with $\dim X_{P_i,l} = 0$. So as a set, $\overline{\Gamma\backslash X}^{Int}$ is independent of Σ.

We need to show that the topology of $\overline{\Gamma\backslash X}^{Int}$ is also independent of Σ. Notice that $\coprod' \Gamma_{P_i,h}\backslash X_{P_i,h}$ is a part of the boundary $\partial\overline{\Gamma\backslash X}^{BB}$, and $\coprod'' (\Gamma'_{P_i}/\Gamma_{U_{P_i}})\backslash X_{P_i,h} \times V_{P_i}$ is a part of the boundary $\partial\overline{\Gamma\backslash X}^{Tor}_{\Sigma}$. Then the topology of $\overline{\Gamma\backslash X}^{Int}$ can be described as follows:

1. A sequence x_j in $\Gamma\backslash X$ converges to a point $x_\infty \in \coprod' \Gamma_{P_i,h}\backslash X_{P_i,h}$ in the compactification $\overline{\Gamma\backslash X}^{Int}$ if and only if x_j converges to x_∞ in $\overline{\Gamma\backslash X}^{BB}$.
2. A sequence x_n in $\Gamma\backslash X$ converges to a point $x_\infty \in \coprod''(\Gamma'_{P_i}/\Gamma_{U_{P_i}})\backslash X_{P_i,h}\times V_{P_i}$ in $\overline{\Gamma\backslash X}^{Int}$ if and only if x_n converges to x_∞ in $\overline{\Gamma\backslash X}^{Tor}_{\Sigma}$.

The convergence of a sequence to a point in $\coprod'$ clearly does not depend on Σ. We show that it is also the case with the convergence to a point in $\coprod''$.

It follows from the construction of $\overline{\Gamma\backslash X}^{Tor}_{\Sigma}$, in particular, the reduction theory (see [AMuRT, p. 247, last paragraph] and [N, p. 69, first paragraph]) that the following important fact holds: If $\dim X_{P_i,l} = 0$, then a small neighborhood of the boundary $(\Gamma'_{P_i}/\Gamma_{U_{P_i}})\backslash X_{P_i,h}\times V_{P_i}$ in $(\Gamma_{P_i}/\Gamma_{U_{P_i}})\backslash(\overline{\Gamma_{U_{P_i}}\backslash X}_{\Sigma_{P_i}})$ is also a small neighborhood of the boundary component $(\Gamma'_{P_i}/\Gamma_{U_{P_i}})\backslash X_{P_i,h}\times V_{P_i}$ in $\overline{\Gamma\backslash X}^{Tor}_{\Sigma}$.

Since $\dim C_{P_i} = 1$, the $\Gamma_{P_i,l}$-admissible polyhedral cone decomposition Σ_{P_i} is unique (consisting of the origin and $\overline{C_{P_i}}$), the convergence to boundary points in $(\Gamma_{P_i}/\Gamma_{U_{P_i}})\backslash(\overline{\Gamma_{U_{P_i}}\backslash X}_{\Sigma_{P_i}})$ is independent of Σ. Then the above fact implies that the convergence to boundary points in $\coprod''$ in the compactification $\overline{\Gamma\backslash X}^{Tor}_{\Sigma}$ does not depend on Σ either. This proves that the compactification $\overline{\Gamma\backslash X}^{Int}$ is independent of Σ.

To prove the second statement, we need to show that any convergent sequence x_j in $\overline{\Gamma\backslash X}^{BS}$ also converges in $\overline{\Gamma\backslash X}^{Int}$. Use the notation in §3.4. Let $\mathbf{P}_{i,j}$ be the unique representative such that the limit x_∞ of x_j belongs to $\Gamma_{P_{i,j}}\backslash e(\mathbf{P}_{i,j})$.

If $\dim X_{P_i,l} \geq 1$, then Lemmas 3.4.2 and 3.4.3 show that x_j converges to the image of x_∞ in $\Gamma_{P_i,h}\backslash X_{P_i,h}$ in the compactification $\overline{\Gamma\backslash X}^{BB}$. The above discussion shows that x_j also converges in the compactification $\overline{\Gamma\backslash X}^{Int}$.

On the other hand, if $\dim X_{P_i,l} = 0$, we claim that any rational parabolic subgroup $\mathbf{Q}$ contained in $\mathbf{P}_i$ with $X_{Q,h} = X_{P_i,h}$ is equal to $\mathbf{P}_i$. In other words, there is only one boundary component $\Gamma_{P_i}\backslash e(\mathbf{P}_i)$ in $\overline{\Gamma\backslash X}^{BS}$ lying above the boundary component $\Gamma_{P_i,h}\backslash X_{P_i,h}$ in $\overline{\Gamma\backslash X}^{BB}$.

In fact, by the root structure of $\mathbf{G}$ in §3.1, $\mathbf{P}_i$ is conjugate to the standard parabolic subgroup $\mathbf{P}_{\Delta-\{\beta_1\}}$. Since $\Delta - \{\beta_1\}$ is connected, the condition $X_{Q,h} = X_{P_i,h}$ forces $\mathbf{Q}$ to be equal to $\mathbf{P}_i$ (see Lemma 3.2.4).

Using the realization of X as a Siegel domain of the third kind in Prop. 5.1.2, $\Gamma_{U_{P_i}}\backslash X$ can be realized as a subset of $X_{P_i,h} \times \mathbb{C}^n \times (\Gamma_{U_{P_i}}\backslash U_{P_i}) \times U_{P_i}$, where $n = \frac{1}{2}\dim_{\mathbb{R}} V_{P_i}$. For any sequence x_j in $\Gamma_{U_{P_i}}\backslash X$, write $x_j = (z_j, v_j, u_{1,j}, u_{2,j})$ according to this decomposition. Since $\dim X_{P_i,l} = 0$, $U_{P_i} \cong A_{P_i}$, $\dim U_{P_i} = \dim A_{P_i} = 1$; and a sequence x_j in $\Gamma_{U_{P_i}}\backslash X$ converges to a boundary point in $\overline{\Gamma_{U_{P_i}}\backslash X}_{\Sigma_{P_i}}$ if and only if the following conditions are satisfied: $\lim_{j\to\infty} z_j$ exists, $\lim_{j\to\infty} v_j$ exists, and $\lim_{j\to\infty} u_{2,j} = +\infty$ (i.e., under the identification $U_{P_i} \cong A_{P_i}$, for any $\alpha \in \Phi^+(P_i, A_{P_i})$, $\lim_{j\to+\infty} u_{2,j}^\alpha = +\infty$).

From the relation between the horospherical decomposition and the realization as a Siegel domain in Lemma 5.1.3, we get that if a sequence x_j in X converges to a point in the boundary component $e(\mathbf{P}_i)$ in $\overline{X}^{BS}$, then in the realization of X as a Siegel domain of the third kind, $x_j = (z_j, v_j, u_{1,j}+iu_{2,j}) \in X_{Q,h} \times \mathbb{C}^n \times (U_Q \otimes \mathbb{C})$, the limits $\lim_{j\to+\infty} z_j$, $\lim_{j\to+\infty} v_j$, $\lim_{j\to+\infty} u_{1,j}$ all exist, and $\lim_{j\to+\infty} u_{2,j} = +\infty$. It follows from the previous paragraph that the projection of x_j in $\Gamma_{U_{P_i}}\backslash X$ also converges to a boundary point in $\overline{\Gamma_{U_{P_i}}\backslash X}_{\Sigma_{P_i}}$ in §5.3. We note that every sequence x_j in $\Gamma\backslash X$ converging to a point in $\Gamma_{U_{P_i}}\backslash e(\mathbf{P}_i)$ in $\overline{\Gamma\backslash X}^{BS}$ has a lift in X which converges to a point in $e(\mathbf{P}_i) \subset \overline{X}^{BS}$. Then the above discussions on relation between $\overline{\Gamma_{U_{P_i}}\backslash X}_{\Sigma_{P_i}}$ and $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ imply that if a sequence x_j in $\Gamma\backslash X$ converges to a boundary point in $\Gamma_{U_{P_i}}\backslash e(\mathbf{P}_i)$ in the compactification $\overline{\Gamma\backslash X}^{BS}$, then x_j also converges to point in $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ and hence in $\overline{\Gamma\backslash X}^{Int}$. This completes the proof the second statement.

REMARK 5.5.3. If $\mathrm{rk}_{\mathbb{R}}(\mathbf{G}) \geq 2$, then $\overline{\Gamma\backslash X}^{Int}$ is strictly dominated by the eccentric Borel–Serre compactification $\overline{\Gamma\backslash X}^{exc}$ defined in [HaZ1, 1.4(b)], which lies between $\overline{\Gamma\backslash X}^{BS}$ and $\overline{\Gamma\backslash X}^{RBS}$. If $\mathrm{rk}_{\mathbb{R}}(\mathbf{G}) = 1$, then $\overline{\Gamma\backslash X}^{Int}$ is equal to $\overline{\Gamma\backslash X}^{exc}$.

6 GCQ of Two Compactifications of a Topological Space

In this section, we define the GCQ of two compactifications of a topological space and explain two useful techniques for determining the GCQ. In the following, all the topological spaces are Hausdorff.

6.1 Existence of GCQ. In this subsection, we follow closely §1.1 of [HaZ1]. Let Y be a noncompact topological space. Recall that a compactification Y_1 of Y dominates another compactification Y_2 if the identity map on Y extends to a continuous map from Y_1 to Y_2.

DEFINITION 6.1.1. Let Y_1, Y_2 be two compactifications of Y.

1. A compactification of Y is called a common quotient of Y_1, Y_2 if it is dominated by both Y_1 and Y_2. A common quotient of Y_1, Y_2 is called the greatest common quotient (GCQ) if it dominates any other common quotient of Y_1, Y_2. The GCQ of Y_1, Y_2 is unique and is denoted by $Y_1 \wedge Y_2$.
2. A compactification of Y is called a common refinement of Y_1, Y_2 if it dominates both Y_1, Y_2. A common refinement of Y_1, Y_2 is called the least common refinement (LCR) if it is dominated by any other common refinement of Y_1, Y_2. The LCR of Y_1, Y_2 is unique and denoted by $Y_1 \vee Y_2$.

The LCR $Y_1 \vee Y_2$ can be constructed easily. In fact, let $d : X \to Y_1 \times Y_2$ be the diagonal embedding. Then the closure $\overline{d(X)}$ in $Y_1 \times Y_2$ is $Y_1 \vee Y_2$.

On the other hand, there does not seem to be an explicit method to construct $Y_1 \wedge Y_2$.

LEMMA 6.1.2. *For any two compactifications Y_1, Y_2 of Y, the GCQ $Y_1 \wedge Y_2$ exists and is unique.*

Proof. The uniqueness follows from the definition. To prove the existence, consider the set $\mathcal{C}$ of all compactifications of Y which are dominated by both Y_1 and Y_2. This set is clearly nonempty since the one point compactification is dominated by both Y_1 and Y_2. We claim that any ordered set in $\mathcal{C}$ has an upper bound in $\mathcal{C}$. In fact, for any ordered set Y_α, $\alpha \in I$, the closure of the diagonal embedding of Y in the product $\prod_{\alpha \in I} Y_\alpha$ is a compactification which is dominated by both Y_1 and Y_2, and dominates every compactification Y_α, $\alpha \in I$. By Zorn's lemma, there exists a maximal element Z in $\mathcal{C}$.

We claim that $Z = Y_1 \wedge Y_2$. If not, there exists a common quotient Y' of Y_1 and Y_2 which is not dominated by Z. Since Y' belongs to $\mathcal{C}$, this contradicts the fact that Z is maximal.

6.2 Determination of GCQ. By Lemma 6.1.2, $Y_1 \wedge Y_2$ always exists. A natural question is how to construct $Y_1 \wedge Y_2$ from Y_1 and Y_2. One approach is as follows:

1. Find a common quotient Z of Y_1 and Y_2.
2. Show that for every boundary point in Z, its fiber in Y_1 (or Y_2) has to collapse to one point in any common quotient of Y_1 and Y_2.

Then Z is equal to $Y_1 \wedge Y_2$.

By Proposition 5.5.2, $\overline{\Gamma\backslash X}^{Int}$ is a candidate of $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}_{\Sigma}^{Tor}$. The problem is to use some incompatibility between $\overline{\Gamma\backslash X}^{BS}$ and $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ to collapse their fibers over $\overline{\Gamma\backslash X}^{Int}$.

In the following, we introduce two collapsing methods, each being responsible for half of the collapsing that is needed to prove that $\overline{\Gamma\backslash X}^{Int}$ is the GCQ $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}_{\Sigma}^{Tor}$.

Lemma 6.2.1. *Let Y_1, Y_2 be two compactifications of Y. Let $A \subset \partial Y_1$ be a boundary subset. Suppose every point $a \in A$ is the limit of a sequence in Y which is also convergent in Y_2, and the limit of this sequence in Y_2 is the same for all choices of a in A. Then the closure $\overline{A}$ of A in Y_1 collapses to one point in any common quotient of Y_1 and Y_2.*

Proof. Let $a, a' \in A$ be any two points, and x_j, x'_j be sequences in Y which converge to a, a' in Y_1 respectively and converge to the same point in Y_2. Define a new sequence y_j as follows: $y_j = x_{j/2}$ if j is even, $y_j = x'_{j+1/2}$ if j is odd. Then y_j is convergent in Y_2 and hence in any common quotient Z in Y_1 and Y_2. This implies that a, a' are mapped to the same point in Z. Since the map from Y_1 to Z is continuous, $\overline{A}$ is mapped to one point in Z.

This lemma can be illustrated by the following example. Take $Y = \mathbb{R}^2$. Let Y_1 be the compactification obtained by adding the unit circle S^1 at infinity with the following topology: a sequence x_j converges to $v \in S^1$ if and only if $x_j/|x_j| \to v$. The circle S^1 can also be interpreted as the set of equivalence classes of geodesics in $\mathbb{R}^2$. Let Y_2 be the closure of the imaginary part $\mathbb{R}^2$ of the complex torus $\mathbb{C}^2/\mathbb{Z}^2$ in the toroidal compactification $\overline{\mathbb{C}^2/\mathbb{Z}^2}_{\Sigma}$, where Σ is determined by the coordinate quadrants and their sides (see §4). These two compactifications are illustrated in Figure 6.2.

Let $A = \{(\cos\theta, \sin\theta) \mid 0 < \theta < \pi/2\} \subset \partial Y_1$. For any $v \in A$, the ray tv, $t \geq 0$, converges to v in Y_1 as $t \to +\infty$. On the other hand, tv converges to the distinguished right upper corner point in Y_2 as $t \to +\infty$, which clearly does not depend on v. Lemma 6.2.1 implies that $\overline{A} \subset Y_1$ is mapped to one point in $Y_1 \wedge Y_2$. Similarly, other parts of the boundary ∂Y_1 are also mapped to a single point in $Y_1 \wedge Y_2$. Since the closures of these boundary parts of ∂Y_1 are connected, the whole boundary ∂Y_1 is mapped to one point. Therefore, $Y_1 \wedge Y_2$ is the one point compactification. We can also apply Lemma 6.2.1 to show that every line in Y_2 is mapped to one point in $Y_1 \wedge Y_2$, and hence $Y_1 \wedge Y_2$ is the one point compactification.

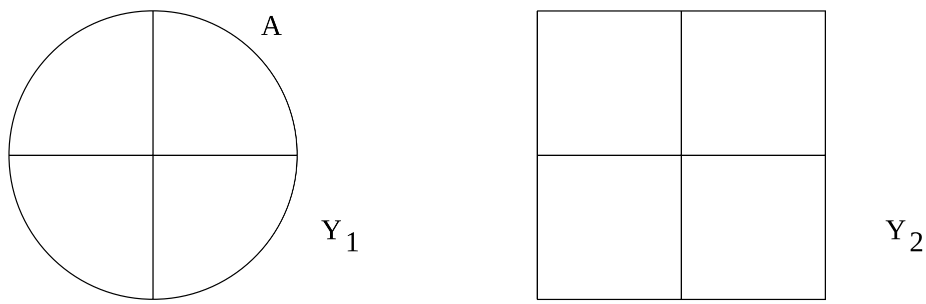

Figure 6.2

This example lies behind the incompatibility of $\overline{\Gamma\backslash X}^{BS}$ and $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$. Briefly, $\overline{\Gamma\backslash X}^{BS}$ is a geodesic compactification, resembling Y_1; while $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ is similar to Y_2.

Lemma 6.2.2. *Let Y_1, Y_2 be two compactifications of Y. Suppose ∂Y_1 contains a fiber bundle B over a base manifold Z with fiber V. Assume that the bundle B has a flat connection. If every horizontal section in B is mapped to one point in $Y_1 \wedge Y_2$ and the holonomy action of $\pi_1(Z)$ on V is ergodic, then the whole fiber bundle B is mapped to one point in $Y_1 \wedge Y_2$.*

Proof. Let $\tilde{Z}$ be the universal covering space of Z. Then $B = \pi_1(Z)\backslash\tilde{Z} \times V$. By assumption, for any $v \in V$, the image of $\tilde{Z} \times \{v\}$ in B is mapped to one point $Y_1 \wedge Y_2$. This implies that the map from B to $Y_1 \wedge Y_2$ factors through the quotient $\pi_1(Z)\backslash V$. Since $\pi_1(Z)$ acts ergodically on V, $\pi_1(Z)$ has dense orbits in V [H, Lemma, p. 26]. The only Hausdorff space which is an image of $\pi_1(Z)\backslash V$ is the one point space. This implies that B is mapped to one point in $Y_1 \wedge Y_2$.

The fiber in $\overline{\Gamma\backslash X}^{BS}$ over $\overline{\Gamma\backslash X}^{Int}$ contains nilpotent bundles over locally symmetric spaces of lower dimension (see §7.1). This is the example in this paper to which we apply this lemma.

The collapsing of the horizontal sections is achieved by Lemma 6.2.1. To get the ergodicity of the holonomy action, we use the following classical result.

Lemma 6.2.3 [H, Automorphism Theorem, p. 53]. *Let Λ be a lattice in a real vector space V. An element $g \in \mathrm{SL}(V, \mathbb{R})$ preserving Λ induces an action on $\Lambda\backslash V$. This action of g on $\Lambda\backslash V$ is ergodic if and only if g has no eigenvalue which is a root of unity.*

7 GCQ of the Borel–Serre and the Toroidal Compactifications

In this section, we identify $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}^{Tor}_{\Sigma}$ with $\overline{\Gamma\backslash X}^{Int}$ and hence prove Theorem 1.2.1. The idea is to show the fibers in $\overline{\Gamma\backslash X}^{BS}$ over $\overline{\Gamma\backslash X}^{Int}$ have to collapse to a point in $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}^{Tor}_{\Sigma}$. In §7.1, we reduce the proof of Theorem 1.2.1 to Proposition 7.1.1. In §7.2–§7.5, we prove Proposition 7.1.1 in four steps.

7.1 Reduction of the proof of Theorem 1.2.1. Let $\mathbf{P}_1, \dots, \mathbf{P}_n$ be representatives of Γ-conjugacy classes of maximal rational parabolic subgroups of $\mathbf{G}$. Recall from Lemma 3.4.3 that for any $z \in \Gamma_{P_i,h}\backslash X_{P_i,h} \subset \partial\overline{\Gamma\backslash X}^{BB}$, the fiber in $\overline{\Gamma\backslash X}^{BS}$ over z contains $\Gamma''_{P_i}\backslash N_{P_i} \times X_{P_i,l}$.

Proposition 7.1.1. *If $\dim X_{P_i,l} \geq 1$, then for any $z \in \Gamma_{P_i,h}\backslash X_{P_i,h} \subset \partial\overline{\Gamma\backslash X}^{BB}$, the subset $\Gamma''_{P_i}\backslash N_{P_i} \times X_{P_i,l}$ in the fiber in $\overline{\Gamma\backslash X}^{BS}$ over z is mapped to one point in $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}^{Tor}_{\Sigma}$.*

Corollary 7.1.2. *If $\dim X_{P_i,l} \geq 1$, then for any $z \in \Gamma_{P_i,h}\backslash X_{P_i,h} \subset \partial\overline{\Gamma\backslash X}^{Int}$, the fiber over z in $\overline{\Gamma\backslash X}^{BS}$ is mapped to one point in $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}^{Tor}_{\Sigma}$.*

Proof. According to its definition in Proposition 5.5.1, if $\dim X_{P_i,l} \geq 1$, $\Gamma_{P_i,h}\backslash X_{P_i,h}$ is contained in $\partial\overline{\Gamma\backslash X}^{Int}$. Since the topologies of $\overline{\Gamma\backslash X}^{BB}$ and $\overline{\Gamma\backslash X}^{Int}$ are the same near $\Gamma_{P_i,h}\backslash X_{P_i,h}$, the fibers of the maps $\overline{\Gamma\backslash X}^{BS} \to \overline{\Gamma\backslash X}^{BB}$ and $\overline{\Gamma\backslash X}^{BS} \to \overline{\Gamma\backslash X}^{Int}$ over z are the same. By Lemma 3.4.4, $\Gamma''_{P_i}\backslash N_{P_i} \times X_{P_i,l}$ is dense in the fiber over z. Therefore, Proposition 7.1.1 implies that this fiber is mapped to one point in $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}^{Tor}_{\Sigma}$.

Proof of Theorem 1.2.1. Assuming Proposition 7.1.1 first, we prove Theorem 1.2.1. Let $\mathbf{P}_i$ be any maximal rational parabolic subgroup. If $\dim X_{P_i,l} \geq 1$, $\Gamma_{P_i,h}\backslash X_{P_i,h} \subset \partial\overline{\Gamma\backslash X}^{Int}$. Corollary 7.1.2 shows that any sequence x_j in $\Gamma\backslash X$ which converges to a point in $\Gamma_{P_i,h}\backslash X_{P_i,h} \subset \overline{\Gamma\backslash X}^{Int}$ also converges to a point in $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}^{Tor}_{\Sigma}$.

On the other hand, if $\dim X_{P_i,l} = 0$, by Proposition 5.5.1 and its proof, the boundary component in $\overline{\Gamma\backslash X}^{Int}$ associated with $\mathbf{P}_i$ is the same as the boundary component of $\mathbf{P}_i$ in $\overline{\Gamma\backslash X}^{Tor}_{\Sigma}$, and the topologies of $\overline{\Gamma\backslash X}^{Int}$ and $\overline{\Gamma\backslash X}^{Tor}_{\Sigma}$ are the same near this boundary component; in particular, if a sequence x_j in $\Gamma\backslash X$ converges to a point in this boundary component in $\overline{\Gamma\backslash X}^{Int}$, then x_j also converges in $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ and hence in $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}_{\Sigma}^{Tor}$. Together with the previous paragraph, this shows that $\overline{\Gamma\backslash X}^{Int}$ dominates $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}_{\Sigma}^{Tor}$.

According to Proposition 5.5.2, $\overline{\Gamma\backslash X}^{Int}$ is a common quotient of $\overline{\Gamma\backslash X}^{BS}$ and $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$. Therefore, $\overline{\Gamma\backslash X}^{Int} = \overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}_{\Sigma}^{Tor}$, and Theorem 1.2.1 is proved.

Outline of the proof of Proposition 7.1.1. In the rest of this section, we prove Proposition 7.1.1. We want to explore the fibration structure of $\Gamma''_{P_i}\backslash N_{P_i} \times X_{P_i,l}$ over $\Gamma_{P_i,l}\backslash X_{P_i,l}$, in particular, the flat connection, to show that it collapses to a point in $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}_{\Sigma}^{Tor}$.

For simplicity, we drop the subindex i. The subset $\Gamma''_P\backslash N_P \times X_{P,l}$ is a bundle over $\Gamma_{P,l}\backslash X_{P,l}$ with fiber $\Gamma_{N_P}\backslash N_P$. This bundle has a flat connection, and the horizontal sections are images of $\{n\} \times X_{P,l}$, $n \in N_P$. In fact, the trivial fibration $N_P \times X_{P,l} \to X_{P,l}$ has a flat connection whose horizontal sections are $\{n\} \times X_{P,l}$. Since the action of Γ''_P preserves the distribution of the horizontal subspaces of the tangent space of $N_P \times X_{P,l}$ defined by the submanifolds $\{n\} \times X_{P,l}$, this connection induces a flat connection on the nilmanifold fibration $\Gamma''_P\backslash N_P \times X_{P,l} \to \Gamma_{P,l}\backslash X_{P,l}$. (For more detailed discussions about the flat connection on nilmanifold fibrations, see [GHM, §7].) The fiber $\Gamma_{N_P}\backslash N_P$ is itself a bundle whose base is $\Gamma_{V_P}\backslash V_P$ and whose fiber is $\Gamma_{U_P}\backslash U_P$.

We prove Proposition 7.1.1 in the following four steps:

1. Collapsing the horizontal sections $X_{P,l}$ in $\Gamma''_P\backslash N_P \times X_{P,l}$.
2. Collapsing the fiber $\Gamma_{U_P}\backslash U_P$ in $\Gamma_{N_P}\backslash N_P$.
3. Collapsing the fiber $\Gamma_{V_P}\backslash V_P$ in the reduced fiber bundle $\Gamma''_P U_P\backslash N_P \times X_{P,l}$ when $\mathbf{G}$ is not absolutely simple.
4. Collapsing $\Gamma_{V_P}\backslash V_P$ when $\mathbf{G}$ is absolutely simple.

The collapsing in (1) and (2) comes from Lemma 6.2.1, and the collapsing in (3) and (4) comes from Lemma 6.2.2.

The above collapsing can be viewed using Figure 5.1.1: The factors $X_{P,l}, U_P, V_P$ shrink to a point at infinity, only the factor $X_{P,h}$ remains. We can also show similarly that the fibers in $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ over $\overline{\Gamma\backslash X}^{Int}$ collapse to a point in $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}_{\Sigma}^{Tor}$.

A good example to keep in mind while reading the proof below is the Hilbert modular surface since all the collapsing phenomena can be seen clearly here. In this example, only Steps 1 and 2 are required, but the collapsing in Step 2 can also be achieved by the method in Step 3.

7.2 Step 1. Collapsing the horizontal sections $\boldsymbol{X_{P,l}}$. Recall the refined horospherical decomposition $X = U_P \times V_P \times X_{P,h} \times X_{P,l} \times A_P$ (Eq. (1) in §5.1). Then $\dim A_P = 1$. Let $H \in \mathfrak{a}_P^+$ be the unique unit vector. Fix any $u \in U_P$, $v \in V_P$, $z \in X_{P,h}$. Then for any two points $x_{l,1}, x_{l,2} \in X_{P,l}$, the two geodesics $\tilde{c}_j(t) = (u, v, z, x_{l,j}, \exp(tH))$ converge to $(u, v, z, x_{l,j}) \in e(\mathbf{P})$ in $\overline{X}^{BS}$ respectively as $t \to +\infty$, $j = 1, 2$. By Lemma 2.3.2, their projections $c_j(t)$ in $\Gamma\backslash X$ converge to the same point in $\overline{\Gamma\backslash X}^{BS}$ if and only if $(u, v, z, x_{l,j})$ project to the same point in $\Gamma_P\backslash e(\mathbf{P})$. Since Γ_P acts properly discontinuously on $e(\mathbf{P})$, $c_1(t)$ and $c_2(t)$ converge to different points as $t \to +\infty$ if $x_{l,1}$ and $x_{l,2}$ are close but different.

On the other hand, in the realization of X as a Siegel domain of the third kind, the set $\{u\} \times \{v\} \times \{z\} \times X_{P,l} \times A_P$ is mapped to the shifted cone $h_z(v, v) + C_P$ in U_P (Lemma 5.1.3). The two geodesics $\tilde{c}_1(t)$ and $\tilde{c}_2(t)$ become two rays starting from the vertex, i.e., $\lim_{t\to-\infty} \tilde{c}_j(t) = h_z(v, v)$ (see Lemma 5.1.4 and Figure 5.1.2).

If $x_{l,1}, x_{l,2}$ are generic and close, then the two rays belong to the interior of one polyhedral cone in Σ_P of codimension zero in U_P. By Lemma 4.2.1, in particular, the examples in §4.3, the two rays $\tilde{c}_1(t)$ and $\tilde{c}_2(t)$ converge to the same point in the partial compactification $\overline{\Gamma_{U_p}\backslash X}_{\Sigma_P}$ in §5.3 as $t \to +\infty$. Then it follows from the definition of $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ that $c_1(t), c_2(t)$ converge to the same point in $\overline{\Gamma\backslash X}_{\Sigma}^{Tor}$ as $t \to +\infty$.

By Lemma 6.2.1, the points $(u, v, z, x_{l,j}) \in \Gamma_P\backslash e(\mathbf{P})$ are mapped to the same point in $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}_{\Sigma}^{Tor}$. This implies that for $n = uv$, the horizontal section in $\Gamma'_P\backslash N_P \times X_{P,l}$ determined by $\{n\} \times X_{P,l}$ is mapped to one point in $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}_{\Sigma}^{Tor}$.

7.3 Step 2. Collapsing the fiber $\boldsymbol{\Gamma_{U_P}\backslash U_P}$ in $\boldsymbol{\Gamma_{N_P}\backslash N_P}$. For any $v \in V_P, z \in X_{P,h}, x_l \in X_{P,l}$, as u varies in U_P, $\tilde{c}_u(t) = (u, v, z, x_l, \exp(tH))$ defines a family of geodesics in X. As $t \to +\infty$, their limits in $\overline{X}^{BS}$ cover the subset $U_P \times \{v\} \times \{z\} \times \{x_l\} \subset e(\mathbf{P})$, and the limits of their projections $c_u(t)$ in $\Gamma\backslash X$ cover the fiber $\Gamma_{U_P}\backslash U_P$ in $\Gamma''_P\backslash N_P \times X_{P,l}$ over the points $x_l \in \Gamma_{P,l}\backslash X_{P,l}$, $v \in \Gamma_{V_P}\backslash V_P$. (As explained in §7.1, $\Gamma_{U_P}\backslash U_P$ is the top fiber in a two step fibration.)

On the other hand, as in §7.2, in the realization of X as a Siegel domain of the third kind, this family of geodesics $\tilde{c}_u(t)$ becomes a family of rays in the tube domain $U_P + i(h_z(v,v) + C_P)$ whose imaginary parts are the same ray from the vertex.

If x_l is generic, i.e., the ray in the cone $h_z(v,v) + C_P$ is contained in the interior of a polyhedral cone of codimension zero, we can show as in §7.2 that this family of geodesics $c_u(t)$ converge to the same point in $\overline{\Gamma\backslash X}_\Sigma^{Tor}$ as $t \to +\infty$. Then Lemma 6.2.1 shows that this fiber $\Gamma_{U_P}\backslash U_P$ is mapped to one point in $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}_\Sigma^{Tor}$.

Since the fibers $\Gamma_{U_P}\backslash U_P$ over any x_l can be approximated by the fibers over generic x_l, it follows from §7.2 that any fiber $\Gamma_{U_P}\backslash U_P$ is mapped to one point in $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}_\Sigma^{Tor}$.

7.4 Step 3. Collapsing $\Gamma_{V_P}\backslash V_P$ when G is not absolutely simple. By §7.3, any fiber $\Gamma_{U_P}\backslash U_P$ in the bundle $\Gamma_{N_p}\backslash N_P$ is mapped to one point in $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}_\Sigma^{Tor}$, and hence the map from the bundle $\Gamma''_P\backslash N_P \times X_{P,l}$ to $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}_\Sigma^{Tor}$ factors through the reduced bundle $\Gamma''_P U_P\backslash N_P \times X_{P,l}$, which is a bundle over $\Gamma_{P,l}\backslash X_{P,l}$ with fiber $\Gamma_{V_P}\backslash V_P$.

By §7.2, every horizontal section of this reduced bundle collapses to a point in $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}_\Sigma^{Tor}$. Therefore, by Lemma 6.2.2, to prove the collapsing of the fibers $\Gamma_{V_P}\backslash V_P$, we need to show that $\Gamma_{P,l}$ acts ergodically on $\Gamma_{V_P}\backslash V_P$.

Recall from §5.1 that V_P is a vector group, and $\Gamma_{P,l}$ acts on V_P by conjugation. If we identify V_P with its Lie algebra $\mathfrak{v}_P$, then Γ_{V_P} becomes a lattice in $\mathfrak{v}_P$, still denoted by Γ_{V_P}, and $\Gamma_{V_P}\backslash V_P = \Gamma_{V_P}\backslash\mathfrak{v}_P$. By Lemma 6.2.3, to prove the ergodicity of the action of $\Gamma_{P,l}$ on $\Gamma_{V_P}\backslash\mathfrak{v}_P$, it suffices to show the following result.

LEMMA 7.4.1. *With the above notation and the assumption in Proposition 7.1.1, there exists an element $\gamma \in \Gamma_{P,l}$ such that the restriction of the adjoint action $\mathrm{Ad}(\gamma)$ to $\mathfrak{v}_P$ has no eigenvalue which is a root of unity.*

The above discussions show that Lemma 7.4.1 finishes the proof of Theorem 1.2.1. The rest of this section is devoted to the proof of Lemma 7.4.1.

We first prove Lemma 7.4.1 when $\mathbf{G}$ is not absolutely simple. In the next subsection, we prove the absolutely simple case. Since $\mathbf{G}$ is not absolutely simple, i.e., $\mathbf{G}(\mathbb{C})$ is not simple, there exists a totally real number field $k \neq \mathbb{Q}$ and an absolutely simple k group $\mathbf{G}'$ such that $\mathbf{G} = R_{k/\mathbb{Q}}\mathbf{G}'$, where $R_{k/\mathbb{Q}}$ is the functor of restriction of the ground fields (see [BBo, Lemma 3.2, p. 469]).

Let $\mathcal{V}$ be the set of Archimedean valuations of k. Then

$$\mathbf{G}(\mathbb{R}) = \prod_{v\in\mathcal{V}} \mathbf{G}'(k_v) = \prod_{v\in\mathcal{V}} \mathbf{G}'(\mathbb{R})\,,$$

and $P = \mathbf{P}(\mathbb{R})$ is a product $\prod_{v\in\mathcal{V}} P_v$, where P_v is a proper maximal parabolic subgroup of $\mathbf{G}'(\mathbb{R})$. Let A_{P_v} be the split component of P_v. Then $A_P \subset \prod_{v\in\mathcal{V}} A_{P_v}$. Denote the orthogonal complement of A_P in $\prod_{v\in\mathcal{V}} A_{P_v}$ by $A_P^\perp$. Since $k \neq \mathbb{Q}$, $\dim A_P^\perp \geq 1$. Then $A_P^\perp$ can be identified with a subset of $X_{P,l}$. Since the image of $A_P^\perp$ in $\Gamma_{P,l}\backslash X_{P,l}$ is compact, $\Gamma_{P,l}$ induces a lattice Λ in $A_P^\perp$.

We claim that there exists an element λ in Λ such that none of the eigenvalues of $\mathrm{Ad}(\lambda)$ acting on the Lie algebra $\mathfrak{n}_P$ of N_P has absolute value 1. This claim implies Lemma 7.4.1, since, by Lemma 5.1.1, $\mathfrak{v}_P$ can be identified with a subspace of $\mathfrak{n}_P$ and the adjoint action on $\mathfrak{v}_P$ is the restriction of the adjoint action on $\mathfrak{n}_P$.

In fact, $N_P = \prod_{v\in\mathcal{V}} N_{P_v}$, $\mathfrak{n}_P = \sum_{v\in\mathcal{V}} \mathfrak{n}_{P_v}$. For any $a \in A_{P_v}$, $a \neq \mathrm{Id}$, none of the eigenvalues of $\mathrm{Ad}(a)$ on $\mathfrak{n}_{P_v}$ has absolute value equal to 1, since $\dim A_{P_v} = 1$ and the eigenvalue of $\mathrm{Ad}(a)$ on a root space $\mathfrak{g}_\alpha$ in $\mathfrak{n}_{P_v}$ is $a^{-\alpha}$; while for $v' \neq v$, $\mathrm{Ad}(a)$ acts trivially on $\mathfrak{n}_{P_{v'}}$. Since Λ is a lattice in $A_P^\perp$ and none of the factors A_{P_v} is equal to A_P, there exists an element $\lambda \in \Lambda \in A_P^\perp$ such that none of its components in $\prod_{v\in\mathcal{V}} A_{P_v}$ is equal to the identity. For such a λ, $\mathrm{Ad}(\lambda)$ has no eigenvalue on $\mathfrak{n}_P$ which has absolute value equal to 1. This proves the claim, and hence Lemma 7.4.1 in this non-simple case.

7.5 Step 4. Collapsing the reduced bundle $\Gamma''_P U_P\backslash N_P \times X_{P,l}$ when G is absolutely simple. In this subsection, we prove Lemma 7.4.1 when $\mathbf{G}$ is absolutely simple. Then by the discussions at the beginning of §7.4, this will finish the proof that the reduced bundle $\Gamma''_P U_P\backslash N_P \times X_{P,l}$ collapses to a point in $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}_\Sigma^{Tor}$, and hence the proof of Theorem 1.2.1.

Since $\mathbf{G}$ is absolutely simple, $\mathbf{G}(\mathbb{R})$ is simple. Let $\mathbf{T}$ be a maximal $\mathbb{R}$-split torus of $\mathbf{G}$ containing the maximal $\mathbb{Q}$-split torus $\mathbf{S}$. Let ${}_{\mathbb{R}}\Phi(\mathbf{G},\mathbf{T})$ be the set of $\mathbb{R}$-roots of $\mathbf{G}$ with respect to $\mathbf{T}$. Then ${}_{\mathbb{R}}\Phi(\mathbf{G},\mathbf{T})$ is of either type BC_s or type C_s, where $s = \mathrm{rk}_{\mathbb{R}}(\mathbf{G})$. Choose an order on $\mathbf{T}$ which is compatible with the order on $\mathbf{S}$, i.e., a positive root in ${}_{\mathbb{R}}\Phi(\mathbf{G},\mathbf{T})$ restricts to either a positive root in $\Phi(\mathbf{G},\mathbf{S})$ or zero. Denote the simple roots in ${}_{\mathbb{R}}\Phi(\mathbf{G},\mathbf{T})$ by ${}_{\mathbb{R}}\Delta(\mathbf{G},\mathbf{T}) = \{\alpha_1,\dots,\alpha_s\}$ such that α_i is not orthogonal to α_{i+1}, and α_s is the short root in case of type BC_s or the long root in case of type C_s. Then each $\mathbb{Q}$-simple root β_i in $\Phi(\mathbf{G},\mathbf{S})$ is the restriction of a unique simple $\mathbb{R}$-root $\alpha_{c(i)}$ for some $1 \leq c(i) \leq s$ (see [BBo, Prop. 2.9]).

Assume the maximal rational parabolic subgroup $\mathbf{P}$ is standard and hence of the form $\mathbf{P}_{\Delta-\{\beta_i\}}$. Since $\mathbf{P}$ is also a maximal real parabolic subgroup [BBo, Corollary 2.10], the real locus $P = \mathbf{P}(\mathbb{R})$ is standard with respect to the set of $\mathbb{R}$-simple roots ${}_{\mathbb{R}}\Phi(\mathbf{G},\mathbf{T})$ and hence of the form $P_{{}_{\mathbb{R}}\Delta-\{\alpha_{c(i)}\}}$, where $\alpha_{c(i)}$ restricts to β_i as above. From the assumption in Proposition 7.1.1 that $\dim X_{P,l} \geq 1$, we get $c(i) \geq 2$.

For each $\alpha \in {}_{\mathbb{R}}\Phi(\mathbf{G},\mathbf{T})$, let H_α denote its root vector defined by $\alpha(H) = \langle H, H_\alpha\rangle$ for all $H \in \mathfrak{t}$, where $\langle\cdot,\cdot\rangle$ is the Killing form. Then $H_{\alpha_1},\dots,H_{\alpha_{c(i)-1}}$ generate a subalgebra of $\mathfrak{t}$, the Lie algebra of $T = \mathbf{T}(\mathbb{R})$. Denote this subalgebra by $\mathfrak{t}_i$.

Then Lemma 7.4.1 follows from the following two lemmas.

LEMMA 7.5.1. *Assume that* $\mathbf{G}$ *is absolutely simple and* $\dim X_{P,l} \geq 1$. *For any nonzero vector* $V \in \mathfrak{v}_P$, *there exists* $H \in \mathfrak{t}_i$ *such that* $[H, V] \neq 0$.

Proof. Since $P = P_{{}_{\mathbb{R}}\Delta-\{\alpha_{c(i)}\}}$, the Lie algebra of $\mathfrak{n}_P$ of N_P is generated by the $\mathbb{R}$-root spaces of the positive roots which are not linear combinations of the simple roots in ${}_{\mathbb{R}}\Delta - \{\alpha_{c(i)}\}$. This implies that any root α appearing in $\mathfrak{n}_P$ contains a positive multiple of $\alpha_{c(i)}$, i.e., $\alpha = \sum_{j=1}^{s} m_j\alpha_j$, where $m_j \geq 0$, and $m_{c(i)} > 0$. Since $\alpha_j(H_{\alpha_{j-1}}) = \langle\alpha_j,\alpha_{j-1}\rangle \neq 0$ and $\alpha_j(H_{\alpha_j}) = \langle\alpha_j,\alpha_j\rangle \neq 0$, it follows that $\alpha(H_{\alpha_{j-1}}) \neq 0$ and $\alpha(H_{\alpha_j}) \neq 0$, where $j \leq c(i)$ is the first index such that $m_j > 0$. Because $m_{c(i)} > 0$ and $c(i) \geq 2$ (see above), there exists $j \leq c(i)$ such that $m_j > 0$. Since $H_{\alpha_{j-1}} \in \mathfrak{t}_i$ if $j > 1$ and $H_{\alpha_j} \in \mathfrak{t}_i$ if $j = 1$, the above discussions imply that the root α does not vanish on $\mathfrak{t}_i$. By Lemma 5.1.1, $\mathfrak{v}_P$ can identified with a subspace of $\mathfrak{n}_P$ and the action of $\mathfrak{t}_i \subset \mathfrak{g}_{P,l}$ on $\mathfrak{v}_P$ is equal to the restriction to $\mathfrak{v}_P$ of the adjoint action on $\mathfrak{n}_P$. Then Lemma 7.5.1 follows immediately.

LEMMA 7.5.2. *Assume that* $\mathbf{G}$ *is absolutely simple and* $\dim X_{P,l} \geq 1$. *If Lemma 7.4.1 is not true, then there exists a nonzero vector* $V \in \mathfrak{v}_P$ *such that for any* $H \in \mathfrak{t}_i$, $[H, V] = 0$.

Proof. Recall that $X_{P,l} = {}^oG_{P,l}/K_{P,l}$, where ${}^oG_{P,l} = G_{P,l}/A_P$. Let ${}^o\mathfrak{g}_{P,l}$ be the Lie algebra of ${}^oG_{P,l}$.

By assumption, for every $\gamma \in \Gamma$, $\mathrm{Ad}(\gamma)$ acting on $\mathfrak{v}_P$ has an eigenvalue which is a root of unit $\exp i\frac{2\pi p}{q}$, where p, q are coprime integers. The degree of the characteristic polynomial of $\mathrm{Ad}(\gamma)$ is bounded by $\dim \mathfrak{v}_P$. By a basic fact of cyclotomic fields [W, Theorem 2.5] that the degree of $\mathbb{Q}(\exp i\frac{2\pi p}{q})$ over $\mathbb{Q}$ goes to infinity as $q \to +\infty$, it follows that the denominator q is bounded. Therefore, there exists an integer n independent of γ such that $(\exp i\frac{2\pi p}{q})^n = 1$.

For any $g \in {}^oG_{P,l}$, let $\lambda_1(g), \dots, \lambda_d(g)$ be the eigenvalues of $\mathrm{Ad}(g)$ acting on $\mathfrak{v}_P$, where $d = \dim \mathfrak{v}_P$. Let S_d be the symmetry group on d elements, and $\mathbb{C}^d/S_d$ be the quotient space, which is an algebraic variety. Then the map $\Lambda : g \in {}^oG_{P,l} \to (\lambda_1(g), \dots, \lambda_d(g)) \in \mathbb{C}^d/S_d$ is a well-defined algebraic map. In fact, let $\chi : \mathrm{GL}(\mathfrak{v}_P) \to \mathbb{C}^d/S_d$ be the adjoint quotient map as defined in [Sp, §2, pp. 178–179], then the map Λ is the composition of the map $\mathrm{Ad} : {}^oG_{P,l} \to \mathrm{GL}(\mathfrak{v}_P)$ with χ.

The above discussion shows that for every $\gamma \in \Gamma_{P,l}$, $\Lambda(\gamma)$ belongs to the proper subvariety

$$\big\{(\lambda_1, \dots, \lambda_d) \in \mathbb{C}^d/S_d \;\big|\; (\lambda_1^n - 1) \dots (\lambda_d^n - 1) = 0\big\} \subset \mathbb{C}^d/S_d\,.$$

By the Borel density theorem [Bo1], the image $\Lambda({}^oG_{P,l})$ is also contained in this subvariety. This means that for any $g \in {}^oG_{P,l}$, $\mathrm{Ad}(g)$ has an eigenvalue on $\mathfrak{v}_P$ whose nth power is equal to one. So for any $X \in {}^o\mathfrak{g}_{P,l}$, $\mathrm{Ad}(e^{nX})$ has an eigenvalue equal to 1. Differentiating the eigen-equation, we get that for any $X \in {}^o\mathfrak{g}_{P,l}$, there exists a nonzero vector in $V \in \mathfrak{v}_P$ such that $[X, V] = 0$. Since $\mathfrak{t}_i$ is a subalgebra of $\mathfrak{t}$, the action of $\mathfrak{t}_i$ on $\mathfrak{v}_P$ can be diagonalized, and hence there exists a nonzero vector $V \in \mathfrak{v}_P$ such that for any $H \in \mathfrak{t}_i$, $[H, V] = 0$. This proves Lemma 7.5.2.

Proof of Lemma 7.4.1 in the absolutely simple case. If Lemma 7.4.1 is not true, then by Lemma 7.5.2, there exists a nonzero $V \in \mathfrak{v}_P$ such that $[H, V] = 0$ for all $H \in \mathfrak{t}_i$. But this contradicts Lemma 7.5.1. Therefore, Lemma 7.4.1 holds, and the proof of Theorem 1.2.1 is now complete.

REMARK 7.5.3. As pointed out by the referee, the result of Parry in [P] shows that the ergodicity of the $\Gamma_{P,l}$-action on Γ_{V_P} implies the holonomy action of $\Gamma_{P,l}$ on the full nilmanifold $\Gamma_{N_P}\backslash N_P$ is also ergodic. Since $\Gamma_{N_P}\backslash N_P$ is a fiber bundle over $\Gamma_{V_P}\backslash V_P$ with fiber $\Gamma_{U_P}\backslash U_P$, the collapsing of the fiber $\Gamma_{U_P}\backslash U_P$ in §7.3 could be combined with the collapsing of the space $\Gamma_{V_P}\backslash V_P$ in §7.4 and §7.5.

8 Proof of Theorems 1.2.2, 1.2.3 and 1.2.4

In this section, we prove Theorems 1.2.2, 1.2.3 and 1.2.4 and determine when Conjecture 1.1.1 is true if $\mathbf{G}$ is absolutely simple (8.2.1) and hence cover all cases.

8.1 Proofs of the theorems. First we reformulate Theorem 1.2.1 as follows.

LEMMA 8.1.1. *Let* $\mathbf{G}$ *be a simple* $\mathbb{Q}$*-group, and* $\mathbf{P}_1, \dots, \mathbf{P}_n$ *be representa-*

tives of Γ*-conjugacy classes of proper maximal rational parabolic subgroups of* $\mathbf{G}$*. Then* $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}_{\Sigma}^{Tor} = \overline{\Gamma\backslash X}^{BB}$ *if and only if for every* $\mathbf{P}_i$*, either* $\dim X_{P_i,l} \geq 1$*, or* $\dim X_{P_i,l} = 0$ *and* $\dim V_{P_i} = 0$.

Proof. It follows from Propositions 5.5.1 and 5.5.2 that $\overline{\Gamma\backslash X}^{Int} = \overline{\Gamma\backslash X}^{BB}$ if and only if for every $\mathbf{P}_i$, either $\dim X_{P_i,l} \geq 1$, or $\dim X_{P_i,l} = 0$ and $\dim V_{P_i} = 0$. Then Lemma 8.1.1 follows from Theorem 1.2.1.

Proof of Theorem 1.2.2. By assumption, $\mathbf{G}$ is not absolutely simple. The proof in §7.4 shows that for any rational maximal parabolic subgroup $\mathbf{P}$, $\dim X_{P,l} \geq \dim A_P^{\perp} \geq 1$. Then Theorem 1.2.2 follows from Lemma 8.1.1.

Proof of Theorem 1.2.3. Since $\mathbf{G}$ is $\mathbb{Q}$-split, $\mathbf{S}$ is also a maximal $\mathbb{R}$-split torus, and hence $\beta_1, \dots, \beta_r$ are all the $\mathbb{R}$-simple roots. Then the maximal rational parabolic subgroup $\mathbf{P} = \mathbf{P}_{\Delta - \{\beta_1\}}$ satisfies $\dim X_{P,l} = 0$. If $\mathbf{G} \neq \mathrm{SL}(2)$, then $\dim V_P \geq 1$, and Lemma 8.1.1 implies that $\overline{\Gamma\backslash X}^{BS} \wedge \overline{\Gamma\backslash X}_{\Sigma}^{Tor} \neq \overline{\Gamma\backslash X}^{BB}$.

Proof of Theorem 1.2.4. It follows from the description of the fibers in $\overline{\Gamma\backslash X}^{RBS}$ over $\overline{\Gamma\backslash X}^{BB}$ in Lemma 3.4.5 and the collapsing in §7.2 that all these fibers in $\overline{\Gamma\backslash X}^{RBS}$ collapse to one point in $\overline{\Gamma\backslash X}^{RBS} \wedge \overline{\Gamma\backslash X}_{\Sigma}^{Tor}$. Therefore $\overline{\Gamma\backslash X}^{RBS} \wedge \overline{\Gamma\backslash X}_{\Sigma}^{Tor} = \overline{\Gamma\backslash X}^{BB}$.

8.2 The absolutely simple case. Theorem 1.2.2 shows that Conjecture 1.1.1 always holds in the case that $\mathbf{G}$ is not absolutely simple. On the other hand, Theorem 1.2.3 shows that Conjecture 1.1.1 fails in that special absolutely simple case. We now deal with the general absolutely simple case.

Assume that $\mathbf{G}$ is absolutely simple. Let $\alpha_1, \dots, \alpha_s$ be the set of $\mathbb{R}$-simple roots ordered as in §7.5. Then each $\mathbb{Q}$-simple root β_i is the restriction of a unique $\mathbb{R}$-root $\alpha_{c(i)}$, and the sequence $c(i)$ is strictly increasing (see §7.5 above and [BBo, p. 471]).

Theorem 8.2.1. *Suppose that* $\mathbf{G}$ *is an absolutely simple* $\mathbb{Q}$*-group and not equal to* $\mathrm{SL}(2)$*. Then in the above notation, Conjecture 1.1.1 holds if and only if* $c(1) \geq 2$.

Proof. In the notation of §7.5, the noncompact part of ${}^0\mathfrak{g}_{P_i,l}$ is generated by the root spaces of roots which are linear combinations of $\alpha_1, \dots, \alpha_{c(i)-1}$. Then $\dim X_{P_i,l} \geq 1$ if and only if $c(i) \geq 2$. On the other hand, the assumption that $\mathbf{G} \neq \mathrm{SL}(2)$ implies that if $\dim X_{P_i,l} = 0$, $\dim V_{P_i} \geq 1$. Since $c(i)$

is strictly increasing, Lemma 8.1.1 implies that Conjecture 1.1.1 holds if and only if $c(1) \geq 2$.

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