THE UNIVERSITY OF MICHIGAN

INDUSTRY PROGRAM OF THE COLLEGE OF ENGINEERING

A STUDY OF NONLINEAR SYSTEMS WITH RANDOM INPUTS

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I. INTRODUCTION

The response of a nonlinear system to a random input usually cannot be obtained using the phase space or the describing function methods of analysis, since these methods are applicable only to a nonlinear system which contains only prescribed functions for its input. On the other hand, when a system contains a random input, this input can be characterized only by a set of statistical properties. To overcome this difficulty, Booton first introduced a linearization technique[(1)], which was based upon the assumption that the probability distribution of the response of a nonlinear closed-loop system containing a zero memory nonlinearity when excited by a Gaussian random input, is approximately Gaussian. This statistical linearization method has been checked experimentally and the results agree with the theoretical calculations within the limits of experimental measurement.

This paper determines the theoretical probability distributions of Booton's type of nonlinear system when some special conditions are imposed on the random input function. It simultaneously explains, therefore, the validity of Booton's assumption under these special conditions.

II. STATISTICAL IDEA FROM PHASE SPACE POINT OF VIEW

It is well known by most engineers that the response of any dynamical system can be represented by a point in phase space. For a second order system the phase space reduces to a phase plane. In this plane a point represents the position and the velocity of the dynamical system at a certain instant of time. With increasing time the representative point, which is continuously moving in this plane, describes a phase trajectory. Its path is analogous to the stream line of flow in hydrodynamics, and it represents the history of the dynamical system. If the input to the system is a random function of time which has a wide uniform power spectrum, then under this condition the representative point of the system, when described in the phase space, undergoes a random motion. This random motion if observed on an oscilloscope reminds one of the Brownian movement of a particle on the surface of liquid. It is this analogy that led the authors to consider a Markoff random process technique for this motion. It is interesting to point out that the analogy between the phase space flow and the nonlinear differential equation was first considered by Kaplan.[2] The present idea of the random motion of a representative point in phase space could be considered as an extension of the ideas of phase space flow. A moment's reflection would have revealed that both of these ideas could have been derived from Boltzmann's equation.[3]

III. CONDITIONS IMPOSED ON THE FUNCTIONS AND EFFECT OF FEEDBACK

In order to apply the Markoff random process technique to a dynamical system, some conditions must be imposed on the random input, F(t). These conditions[4] are:

$$\langle F(t) \rangle_{AV} = 0 \qquad (1)$$

$$\langle F(t)\, F(t+\tau) \rangle_{AV} = 2D\delta(\tau) \qquad (2)$$

$$F(t) \text{ is Gaussian distributed} \qquad (3)$$

Condition (2) implies that the input function F(t) has a white spectrum.

By imposing conditions (1) and (2) on the random input function, one might assume that the Markoff random process technique is not useful in analyzing a feedback control system. This impression is based upon the fact that the power spectrum of the random input to a feedback control system usually is limited to an extremely low frequency region, while condition (2) requires the random input to have a white power spectrum. However, in the following discussion it is shown that this impression is not true. In reality the feedback action of the system creates an effective input, which is a linear combination of the true random input and its higher order derivatives, rather than a true random input.

Consider, for example, the first order servomechanism which contains an integrator and a nonlinear element $K(\epsilon)$ in its forward path. The following two equations define the dynamic behavior of the system:

$$\epsilon = \theta_i - \theta_o, \tag{4}$$

and

$$\frac{d\theta_o}{dt} = K(\epsilon), \tag{5}$$

where θ_o, θ_i and ϵ are respectively the output, input and error. Substitution of Equation (4) into Equation (5) results in an equation relating the error, ϵ, and input, θ_i.

$$\frac{d\epsilon}{dt} + K(\epsilon) = \frac{d\theta_i}{dt} = F(t) \tag{6}$$

The random function $F(t)$ is called an effective random input and is assumed to have the properties prescribed by Equations (1), (2), and (3). Since $F(t)$ is a linear function of the true random input $\theta_i(t)$ which is a Gaussian random function, it follows immediately that the conditions (1) and (3) are satisfied by $F(t)$. According to power spectrum analysis, the power spectrum $\varphi_F(\omega)$ of the effective random input

$F(t)$ and the power spectrum $\varphi_{\Theta_i}(\omega)$ of the true random input $\Theta_i(t)$ are related by

$$\varphi_F(\omega) = |j\omega|^2 \varphi_{\Theta_i}(\omega). \tag{7}$$

If the power spectrum $\varphi_{\Theta_i}(\omega)$ is of the following form:

$$\varphi_{\Theta_i}(\omega) = \frac{D}{(\omega^2 + \alpha^2)(\omega^2 + \beta^2)} \quad , \tag{8}$$

where α, β, D are constants, then from Equation (7) the power spectrum $\varphi_F(\omega)$ of the effective random input, $F(t)$, will have the following form:

$$\varphi_F(\omega) = \frac{\omega^2 D}{(\omega^2 + \alpha^2)(\omega^2 + \beta^2)} \quad . \tag{9}$$

If in the frequency range of interset the value of the constant, α^2, in the expression (9) is very much smaller than ω^2, and if the value of the constant β^2 is very much larger than ω^2, then under these conditions the power spectrum $\varphi_F(\omega)$ can be approximated by the following equations:

$$\varphi_F(\omega) \cong \frac{D\omega^2}{\alpha^2\beta^2} \quad \text{for } \omega^2 < \alpha^2 \tag{10.a}$$

$$\varphi_F(\omega) \cong \frac{D}{\beta^2} \quad \text{for } \alpha^2 < \omega^2 < \beta^2 \tag{10.b}$$

$$\varphi_F(\omega) \cong \frac{D}{\omega^2} \quad \text{for } \omega^2 > \beta^2 \tag{10.c}$$

In Figure 1 will be found the power spectra $\varphi_{\Theta_i}(\omega)$ and $\varphi_F(\omega)$ as function of ω.

Inspection of Figure 1 reveals the power spectrum of the effective random input, $F(t)$, to be almost flat over the frequency range $\alpha < \omega < \beta$. If the frequency range $\alpha < \omega < \beta$ is much larger than the system bandwidth, and, if in addition, the value α is very small, then the

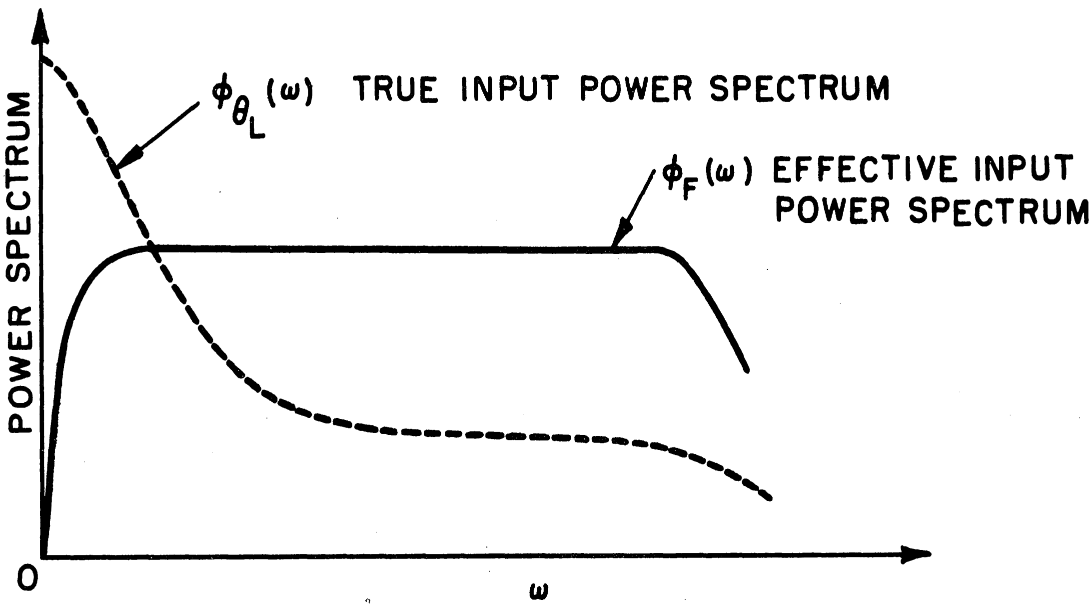

Figure 1. Power Spectrum of True and Effective Inputs

power spectrum $\varphi_F(\omega)$ of the effective random input can be considered as a white spectrum.* Under these conditions the effective random input has approximately the properties described by Equation (2). The Markoff random process technique can be applied therefore to the first order system provided that the random input θ_i has the power spectrum $\varphi_{\theta_i}(\omega)$ of the form described by Equation (9).

In general, if the error of a servomechanism or of a feedback system can be described by the following equation:

$$\sum_{n=0}^{N} a_n \frac{d^n\epsilon}{dt^n} + K(\epsilon) = \sum_{n=0}^{N} a_n \frac{d^n\theta_i}{dt^n} \tag{11}$$

where a_n's are constants, as previously defined, then the right-hand side of Equation (11) can be lumped together to form an effective input function F(t) which is given by the following equation:

$$F(t) = \sum_{n=0}^{N} a_n \frac{d^n\theta_i}{dt^n} \tag{12}$$

Accordingly, the power spectrum $\varphi_F(\omega)$ of the effective input F(t) and the power spectrum $\varphi_{\theta_i}(\omega)$ of the true input $\theta_i(t)$ are related as follows:

$$\varphi_F(\omega) = \left|\sum_{n=0}^{N} (j\omega)^n a_n\right|^2 \varphi_{\theta_i}(\omega) \tag{13}$$

By properly choosing the form of the power spectrum $\varphi_{\theta_i}(\omega)$, it

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* In an actual case, if a power spectrum is uniform over a frequency range which is greater than the frequency range of the system, then the power spectrum can be considered as a white spectrum.

is always possible to obtain a power spectrum $\varphi_F(\omega)$ which is approximately a white spectrum from a physical point of view.

The block diagram representation of Equation (11) can be given in two equivalent forms: (I) As a closed-loop representation as shown in Figure 2; (II) as an open-loop representation as shown in Figure 3.

From Figure 3 it is readily seen that the effective input to the system is essentially white noise provided that the input power spectrum is restricted to a special form determined by Equation (13). Because of the above power spectrum's changing property in feedback system, it is possible to use Markoff random process technique to analyze several commonly encountered nonlinear servomechanisms.

IV. MARKOFF RANDOM PROCESS AND FOKKER-PLANCK EQUATION

A discrete Markoff process can be interpreted as a process in which the occurrence of an event depends only on the occurrence of an event immediately preceding it. For a continuous case it is usually defined in terms of its conditional probability density function by an important relation given by Equation (14) which is known as the Chapmann-Kolmogoroff equation:

$$\begin{aligned} & f(t_2;\, y_1,\, y_2,\, \ldots\, y_n \,|\, S;\, \xi_1,\, \xi_2,\, \ldots\, \xi_n) \\ & = \int f(t_1;\, x_1,\, x_2,\, \ldots\, x_n \,|\, S;\, \xi_1,\, \xi_2,\, \ldots\, \xi_n) \\ & \quad f(t_2;\, y_1,\, y_2,\, \ldots\, y_n \,|\, t_1;\, x_1,\, x_2,\, \ldots\, x_n) \\ & \quad dx_1,\, dx_2\, \ldots\, dx_n \end{aligned} \tag{14}$$

The function $f(t_1;\, x_1,\, x_2\, \ldots\, x_n \,|\, S;\, \xi_1,\, \xi_2,\, \ldots\, \xi_n)$ stands for the probability density function which states mathematically that at time t_1 the random variables are at $x_1,\, x_2,\, \ldots\, x_n$ on the assumption that at time S they were at $\xi_1,\, \xi_2,\, \ldots\, \xi_n$. For simplification in writing,

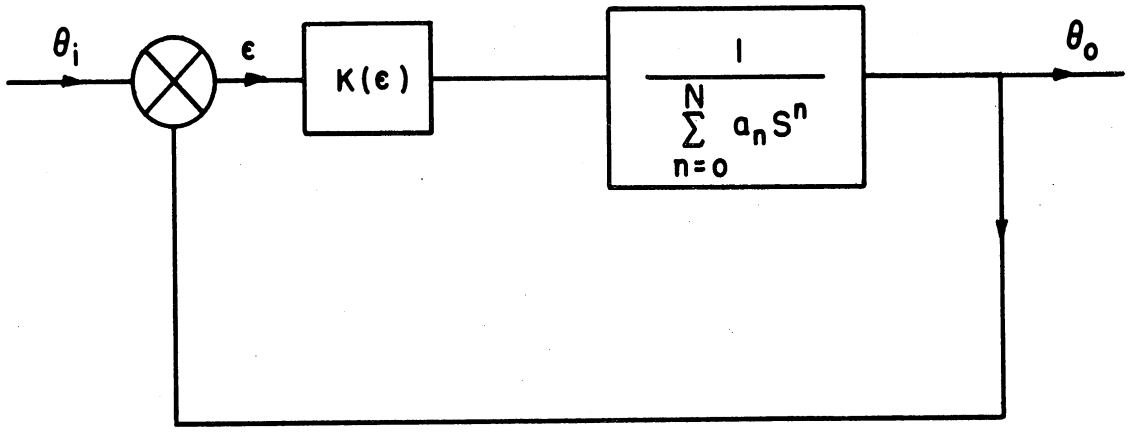

Figure 2. Closed Loop Representation of an nth Order Nonlinear Servomechanism

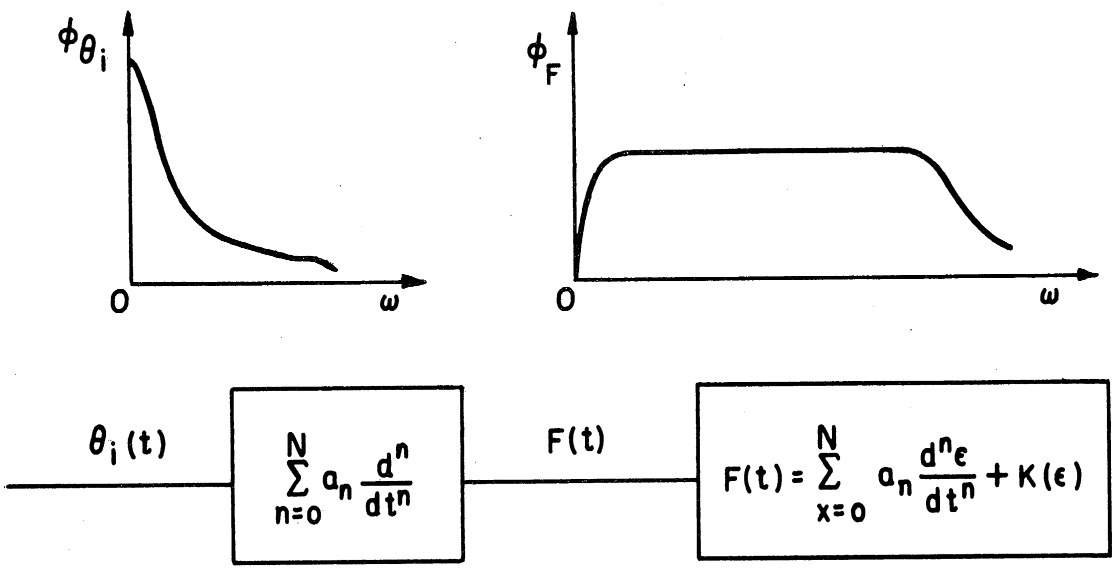

Figure 3. Open Loop Representation of an nth Order Nonlinear Servomechanism

vector notation is often used, Equation (14) when written in vector form becomes:

$$f(t_2; \vec{y} \,|S; \vec{\xi}) = \int f(t_1; \vec{x} \,|S; \vec{\xi})\, f(t_2; \vec{y} \,|t_1; \vec{x})\, d\vec{x} . \qquad (15)$$

Based upon some conditions imposed on the conditional moments of those random variables, Equation (15) can be reduced to a linear partial differential equation known as Fokker-Planck equation[4]:

$$\frac{\partial f}{\partial t} = - \sum_{i=1}^{N} \frac{\partial}{\partial y_i} [\beta_i \, f] + \frac{1}{2} \sum_{i,j}^{N} \frac{\partial}{\partial y_i} \frac{\partial}{\partial y_j} [\beta_{ij} f] \qquad (16)$$

where

$$\beta_i = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int (\Delta y_i)\, f(t + \Delta t; \vec{y} + \Delta\vec{y} \,|\vec{t}; \vec{y})\, d\,(\Delta\vec{y})$$

$$\beta_{ij} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int (\Delta y_i)(\Delta y_j) f(t + \Delta t; \vec{y} + \Delta\vec{y} \,|t; \vec{y})\, d(\Delta\vec{y})\, d\,(\Delta\vec{y})$$

$\beta_{i,j,k..l.} = 0$ for any β whose subscripts have three or more letters.

V. A SECOND ORDER NONLINEAR SERVO SYSTEM AND ITS RELATION WITH FOKKER-PLANCK EQUATION

The dynamical behavior of a second order servo system can be described by the following second-order differential equation:

$$\frac{d^2\theta_o}{dt} + a\,\frac{d\theta_o}{dt} + b\theta_o = K(\epsilon) \qquad (17)$$

where $K(\epsilon)$ is a function ϵ, a and b are constant, and θ_i and θ_o are as previously defined. By substituting $\epsilon = \theta_i - \theta_o$, Equation (17) can be written into the following form:

$$\frac{d^2\epsilon}{dt^2} + a\,\frac{d\epsilon}{dt} + b\epsilon + K(\epsilon) = \frac{d^2\theta_i}{dt^2} + a\,\frac{d\theta_i}{dt} + b\theta_i \qquad (18)$$

If the left side of Equation (18) is written as a single function $F(t)$, Equation (18) will reduce to the following form:

$$\frac{d^2\epsilon}{dt^2} + a\frac{d\epsilon}{dt} + b\epsilon + K(\epsilon) = F(t) \tag{19}$$

Since the function $F(t)$ is a linear combination of a Gaussian input function Θ_i, it follows that $F(t)$ is a Gaussian random function. Furthermore by properly choosing the power spectrum of Θ_i as shown in Section III, $F(t)$ can be approximately treated as Gaussian white noise, and the Markoff random process technique will apply under this condition.

If we now let $\frac{d\epsilon}{dt} = y_1$, $\epsilon = y_2$ then the above equation becomes two first order simultaneous differential equations in phase space $y_1 - y_2$:

$$\frac{dy_1}{dt} + ay_1 + by_2 + K(y_2) = F(t) \tag{20.a}$$

$$\frac{dy_2}{dt} = y_1 \tag{20.b}$$

Let t be increased to $t + \Delta t$ where Δt is very small compared with the response time of the system, but is not necessarily small compared with the function $F(t)$. Putting the above two simultaneous equations into the incremental form they become:

$$\Delta y_1 + ay_1\Delta t + [by_2 + K(y_2)]\Delta t = \int_t^{t+\Delta t} F(t)\, dt \tag{21.a}$$

and

$$\Delta y_2 = y_1\Delta t \tag{21.b}$$

where $\int_t^{t+\Delta t} F(t)\, dt$ indicates the fact that $F(t)$, in this small time interval Δt, is a rapidly changing function of time. By using Equations (21.a) and (21.b), it is possible to calculate the functions β_i, $\beta_{i,j}$, β_{ijk}, ..., $\beta_{ij\ldots m}$ of the Fokker-Planck equation given in Section IV.

For the case of the second order servomechanism expressed by Equation (19), the Fokker-Planck equation has the following form:

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial y_2}(y_1 f) + \frac{\partial}{\partial y_1}\left\{[ay_1 + by_2 + K(y_2)]f\right\} + D\frac{\partial^2 f}{2y_1{}^2} \qquad (22)$$

Here we use the assumption that $\langle F(t)\ F(t+\tau)\rangle_{AV} = 2D\delta(\tau)$ where $\delta(\tau)$ is a delta function and D a constant.

The solution of Equation (22) gives the conditional probability density function of the nonlinear servo system. This solution, in general, is a function of y_1, y_2, t and initial conditions y_{10}, y_{20}, t_0. However, as $t \rightarrow \infty$ the solution becomes independent of y_{10}, y_{20} and t_0. This is what one would expect, since any random variable after a long time interval becomes independent of its initial conditions. Consequently, as $t \rightarrow \infty$ the conditional probability function becomes a first probability function.(5)

Since we are interested in the first probability density function, this is equivalent to finding the steady state solution or having $\frac{\partial f}{\partial t} = 0$ in Equation (22). The Fokker-Planck equation (22) reduces, in this case, to the following form:

$$0 = -\frac{\partial}{\partial y_2}(y_1 f) + \frac{\partial}{\partial y_1}\left\{[ay_1 + by_1 + K(y_2)]f\right\} + D\frac{\partial^2 f}{dy_1{}^2} \qquad (23)$$

The additional conditions that the stationary solution must satisfy are listed below:

1. $f(y_2, y_1, t = \infty) \rightarrow 0$ as $|y_1| \rightarrow \infty$ or as $|y_2| \rightarrow \infty$ or both $|y_1|$ and $|y_2| \rightarrow \infty$
2. $f(y_2, y_1, t = \infty)$ should always be a positive function
3. $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(y_2, y_1\ t = \infty)\ dy_1\ dy_2 = 1$

If we assume a stationary solution of the form $f[my_1^2 + n \int [by_2+K(y_2)]dy_2]$, and use the above three conditions, it can be shown that

$$f(y_2, y_1\ t = \infty) = A \exp [\frac{-a}{2D} y_1^2 - \frac{a}{D} \int [by_2 + K(y_2)]\ dy_2], \qquad (24)$$

where A is determined by the normalized condition 3.

The steady-state solution or the first probability density function in phase-space is the most general type solution that can be obtained from a second order system which has an arbitrary zero memory nonlinearity in its forward path, when it has been subjected to a Gaussian random input. From this solution it is readily seen that in spite of a nonlinearity in this type of second order system, the rate of the response of the system, namely y_1, still retains the Gaussian property. However, the response y_2 itself is changed to a random process that is of a non-Gaussian type.

For the case when the nonlinear function $K(y_2)$ is approximately linear in the region around y_2, such as in the case of saturated amplifier approximated by $K(y_2) = \tanh\ (cy_2)$, the first probability function $f(y_2,\ y_1,\ t = \infty)$ can almost be considered as approximating a Gaussian probability density function.

This can be seen from the expression:

$$f(y_2,\ y_1,\ t = \infty) = A \exp [\frac{-a^2}{2D} y_1^2] \exp [\ \frac{a}{D} \int \tanh\ (cy_2)\ dy_2]$$

$$= A \exp [\frac{-a^2}{2D} Y_1^2] \exp [\frac{-a}{D} \ln \cosh\ (cy_2)]$$

when $|cy_2| < 1\quad \ln \cosh\ (cy_2) \approx (cy_2)^2$

then

$$f(y_2,\ y_1,\ t = \infty) = A \exp [\frac{-a^2}{2D} y_1^2] \exp [\frac{-a}{D} (cy_2)^2]$$

which is evidently the same as a Gaussian probability density function.

VI. HIGHER-ORDER SYSTEM

The above technique which was applied to a class of second order systems in Section V can be extended to higher-order systems. However, in the case of higher order systems, only under very special conditions could the steady state solution be obtained by the authors. To be specific, only those systems that can be decomposed into the following form can the steady state solution of the Fokker-Planck equation be obtained:

$$\frac{d^2x_i}{dt^2} + a_i \frac{dx_i}{dt} + K_i(x_i) + \sum_{j=1}^{N} g_{ij}(x_j) = F_i(t), \; i = 1, 2, \ldots, N. \quad (25)$$

where a_i's are constants, $g_{ij}(x_j)$ and $g_{ji}(x_j)$ are of the same functional form, and $F_i(t)$'s are Gaussian white noise. The first probability density function usually has the following form:

$$f = A \exp \left\{ \sum_{i=1}^{N} b_i V_i^2 + \sum_{i=1}^{N} C_i \int K_i(x_i) dx_i + \sum_{i=1}^{N} d_i \sum_{i=1}^{N} \int g_{ij}(x_j) dx_i \right\}$$

where $V_i = \frac{dx_i}{dt}$ and b_i's, C_i's, d_i's are constants. The constant, A, is the normalization factor.

It should be pointed out that when a higher-order system is decomposed into a system of equations (25) then it is no longer of the form of Equation (11), namely:

$$\sum_{n=0}^{N} a_n \frac{d^n \theta_o}{dt^n} + K(\epsilon) = \sum_{n=0}^{N} a_n \frac{d^n \theta_i}{dt^n} . \quad (11)$$

Consequently, the argument used in Section IV is no longer true, because in this case one cannot write the effective input, F(t), as a linear combination of the true input θ_i. However, if the input is non-Gaussian distributed and the overall effect of the nonlinear combination of the true input Θ_i still gives an effective input, F(t) which is Gaussian distributed and has white spectrum, then it is possible to obtain the first probability density function of this system.

VII. COMPUTER STUDY

In order to verify the result of the analysis presented in the previous sections, an analog computer study was made on a second order nonlinear servomechanism. Before the results of the computer study are discussed, the measuring equipment or the single channel amplitude distribution analyzer will be described. The functional diagram of the analyzer is given in Figure 4. This analyzer measures the probability distribution function P(x) of a random variable x, the relation between the probability distribution function P(x) and the probability density function f(x) which is of the form:

$$P(x) = \int_{-\infty}^{x} f(x)\,dx. \tag{26}$$

The resolution of the analyzer was checked against a known function of time such as a triangular wave. It can be shown the probability distribution function P(x) vs. the amplitude, x, of the triangular wave theoretically is a straight line. The agreement between the theoretical straight line and the experimental points was found to be excellent. The noise generator which was used, was constructed in the Servomechanism Laboratory. It had a flat spectrum from approximately $\frac{1}{37.5}$ cps to 25 cps and was Gaussian distributed.

The equation of the second order servomechanisms under study is of the form:

$$\frac{d^2\epsilon}{dt^2} + \beta\frac{d\epsilon}{dt} + K(\epsilon) = \frac{d^2\Theta_i}{dt^2} + \beta\frac{d\Theta_i}{dt} = F(t). \tag{27}$$

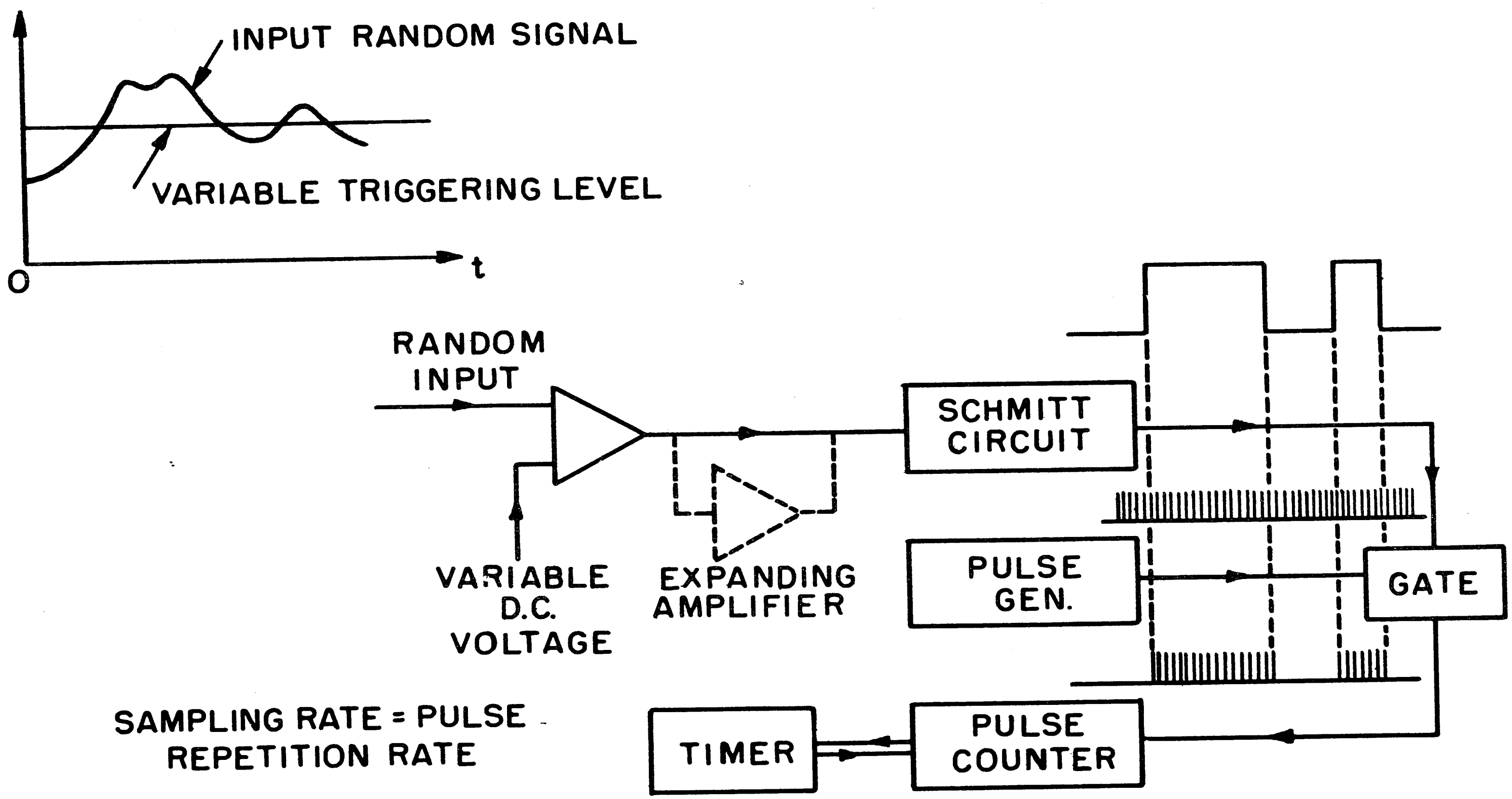

Figure 4. Functional Diagram of Amplitude Distribution Analyzer

Two different kinds of function $K(\epsilon)$ were used, these are listed below. For the case of the saturated amplifier:

$$(1)\quad K(\epsilon) = \begin{cases} 100 & \epsilon > 1 \\ 100\epsilon & |\epsilon| < 1 \\ -100 & \epsilon < 1 \end{cases} \quad \text{and } \beta = 10 \qquad (28)$$

For the case of the relay amplifier:

$$(2)\quad K(\epsilon) = \begin{cases} K, & \epsilon > 0 \\ -K, & \epsilon < 0 \end{cases} \quad \text{where } K = 100 \text{ and } \beta = 20. \qquad (29)$$

In the first case, the measurements were intended to show the departure of the distribution of the error, ϵ, from normality in the nonlinear region; therefore, under these conditions only, is the test of normality of the error important. In order to confirm the belief that the departure from normality was not due to statistical fluctuation of the data, a confidence interval test was performed. The results of the experimental data were plotted on normal probability papers and are given in Figure 5 and Figure 6. In the second case, a detailed study was made because of the nonlinearity occurring for small values of error as well as for large values of error. It presents, therefore, a severe test of the validity of the analysis given in previous sections.

For Equation (27) with $K(\epsilon)$ and β as given in (29), the first probability density function $f(\epsilon, \dot{\epsilon})$, according to Equation (24), is of the form:

$$f(\dot{\epsilon}, \epsilon) = A \exp \left[\frac{-\beta}{2D} \dot{\epsilon}^2 + \frac{K\beta}{D} \epsilon\right] \text{ for } e < 0 \qquad (30.a)$$

$$= A \exp \left[\frac{-\beta}{2D} \dot{\epsilon}^2 - \frac{K\beta}{D} \epsilon\right] \text{ for } \epsilon > 0, \qquad (30.b)$$

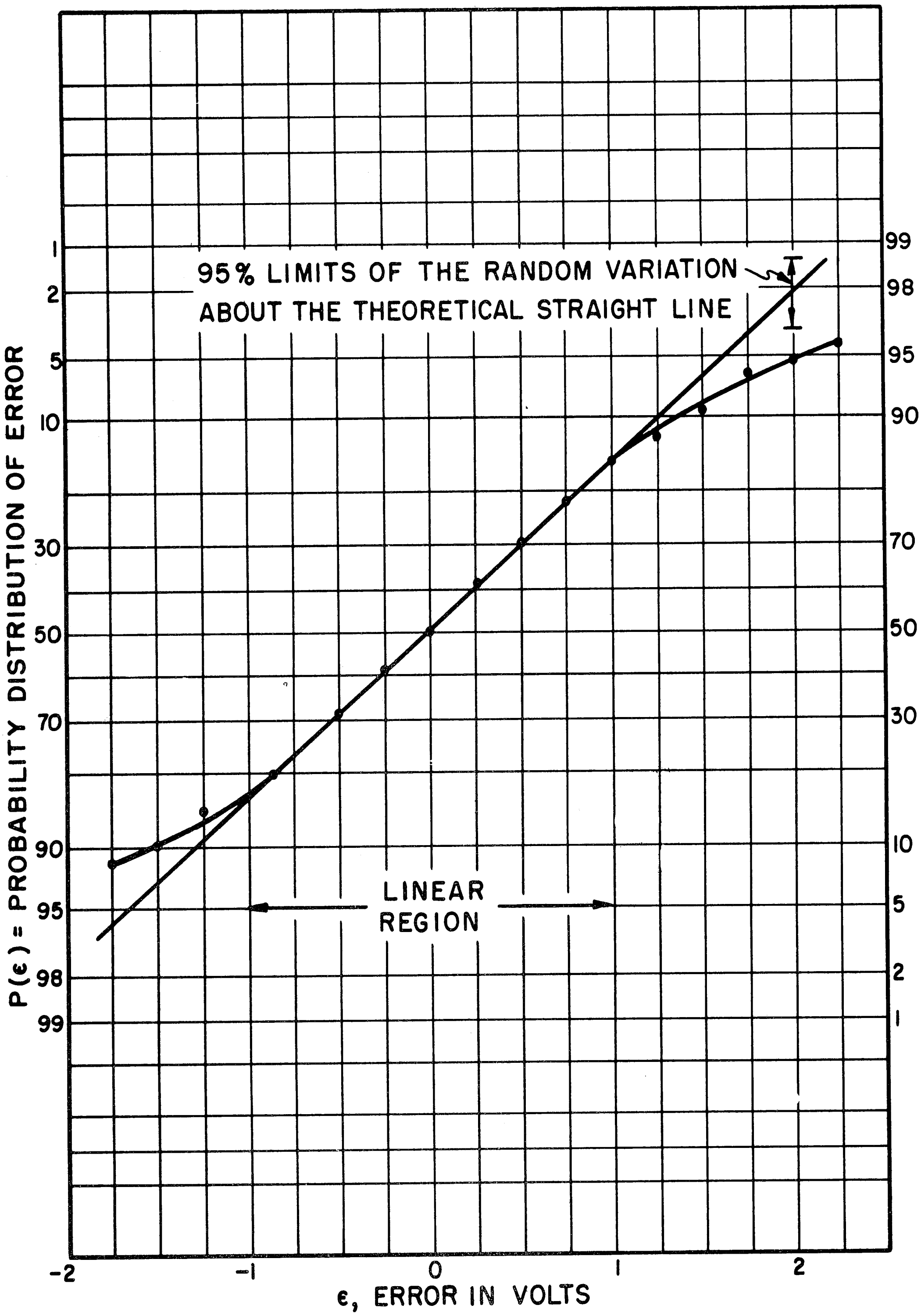

Figure 5. Error Probability Distribution of a Second Order Servomechanism with Saturation

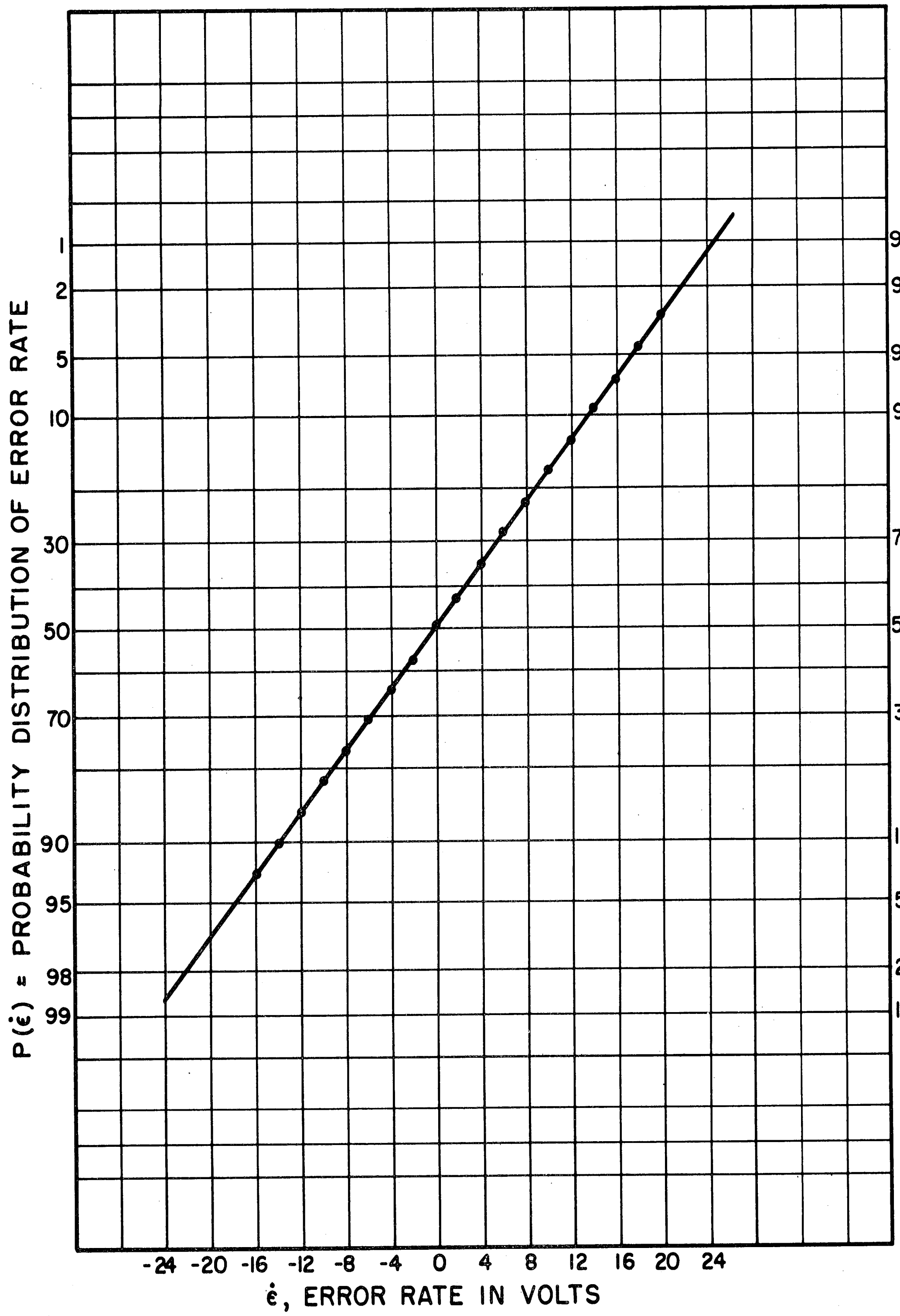

Figure 6. Error Rate Probability Distribution of a Second Order Servomechanism with Saturation

where A is a constant. By using the normalization condition

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\dot{\epsilon}, \epsilon)\, d\dot{\epsilon}\, d\epsilon = 1 ,$$

it is possible to determine the constant A. The first probability density function of ϵ alone is determined from the relation:

$$f(\epsilon) = \int_{-\infty}^{+\infty} f(\dot{\epsilon},\epsilon)\, d\dot{\epsilon} = \frac{K}{2D} \exp \left[\frac{K\beta}{D} \epsilon\right] \text{ for } \epsilon < 0 \quad (31.a)$$

$$= \frac{K}{2D} \exp \left[\frac{-K\beta}{D} \epsilon\right] \text{ for } \epsilon > 0 \quad (31.b)$$

The probability distribution function $P(\epsilon)$, which is defined below, is related to the probability density function $f(\epsilon)$ by the following relations:

$$P(\epsilon) = \int_{-\infty}^{\epsilon} f(t)\, dt = \frac{1}{2} \exp \left[\frac{K\beta}{D} \epsilon\right] \text{ for } \epsilon < 0 \quad (32.a)$$

$$P(\epsilon) = 1 - \frac{1}{2} \exp \left[\frac{-K\beta}{D} \epsilon\right] \text{ for } \epsilon > 0 \quad (32.b)$$

Since the functions $P(\epsilon)$ and $1 - P(\epsilon)$ are equal to some exponential functions of ϵ, it is evident that the graph of the data plotted on semi-logarithmic paper should be a straight line. Furthermore, if $\epsilon = \frac{-D}{\beta K}$ is substituted in Equation (32.a), $P(\epsilon)$ is found to equal 0.1834. Based upon this fact, by finding the point on the graph of $P(\epsilon)$ vs. ϵ at which the function $P(\epsilon)$ is equal to 18.34 per cent, it is possible to determine the value of ϵ which corresponds to the quantity $\frac{D}{\beta K}$. The results of the experimental data were plotted on semi-logarithmic paper and normal probability paper. They are given in Figure 7, Figure 8 and Figure 9. The effects of the parameters on the probability distribution were checked by comparing the change with an arbitrarily chosen standard system. The theoretical calculation and the measured data are listed below, the results were amazingly close:

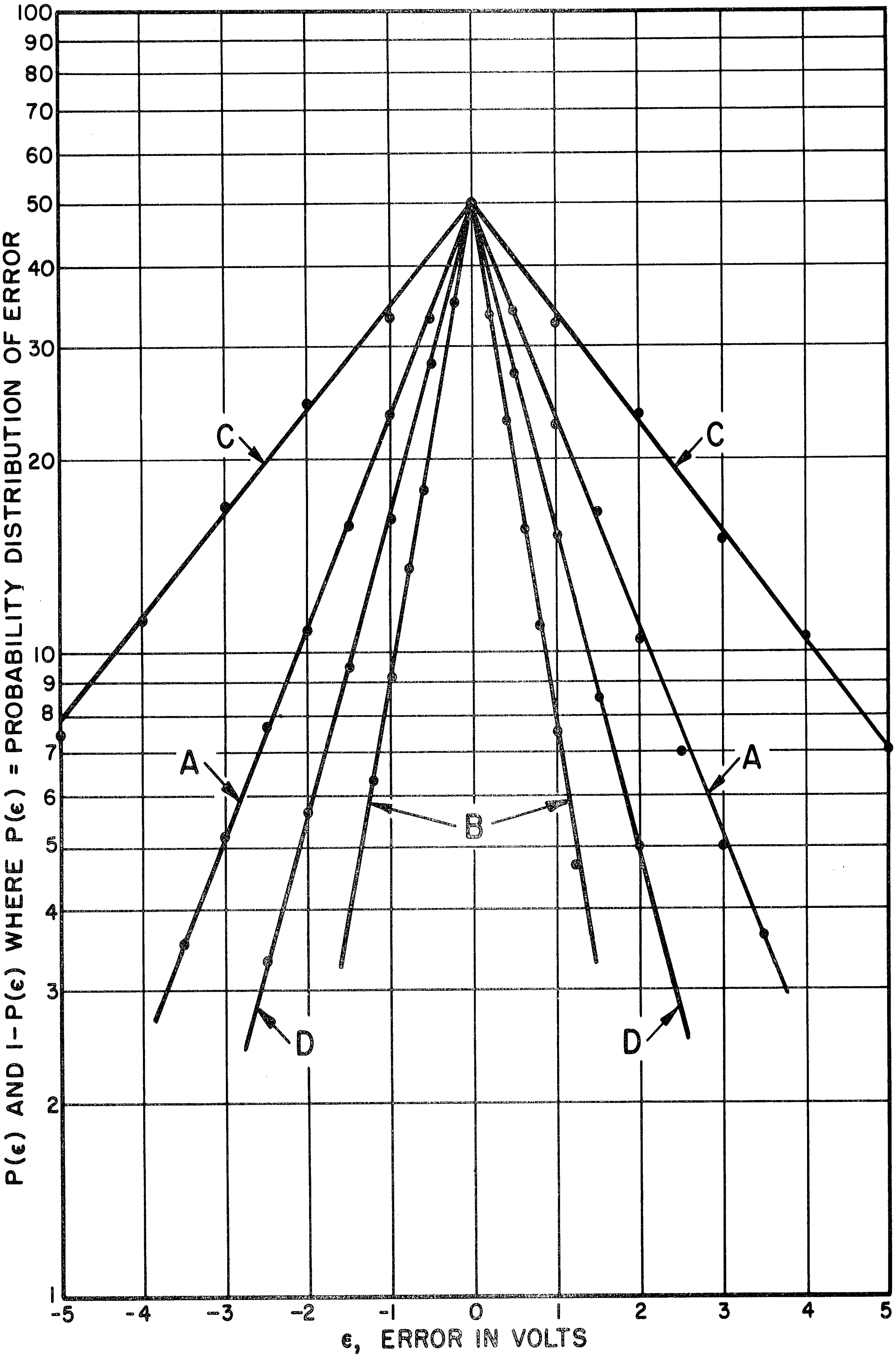

Figure 7. Error Probability Distribution of a Second Order Relay Servomechanism

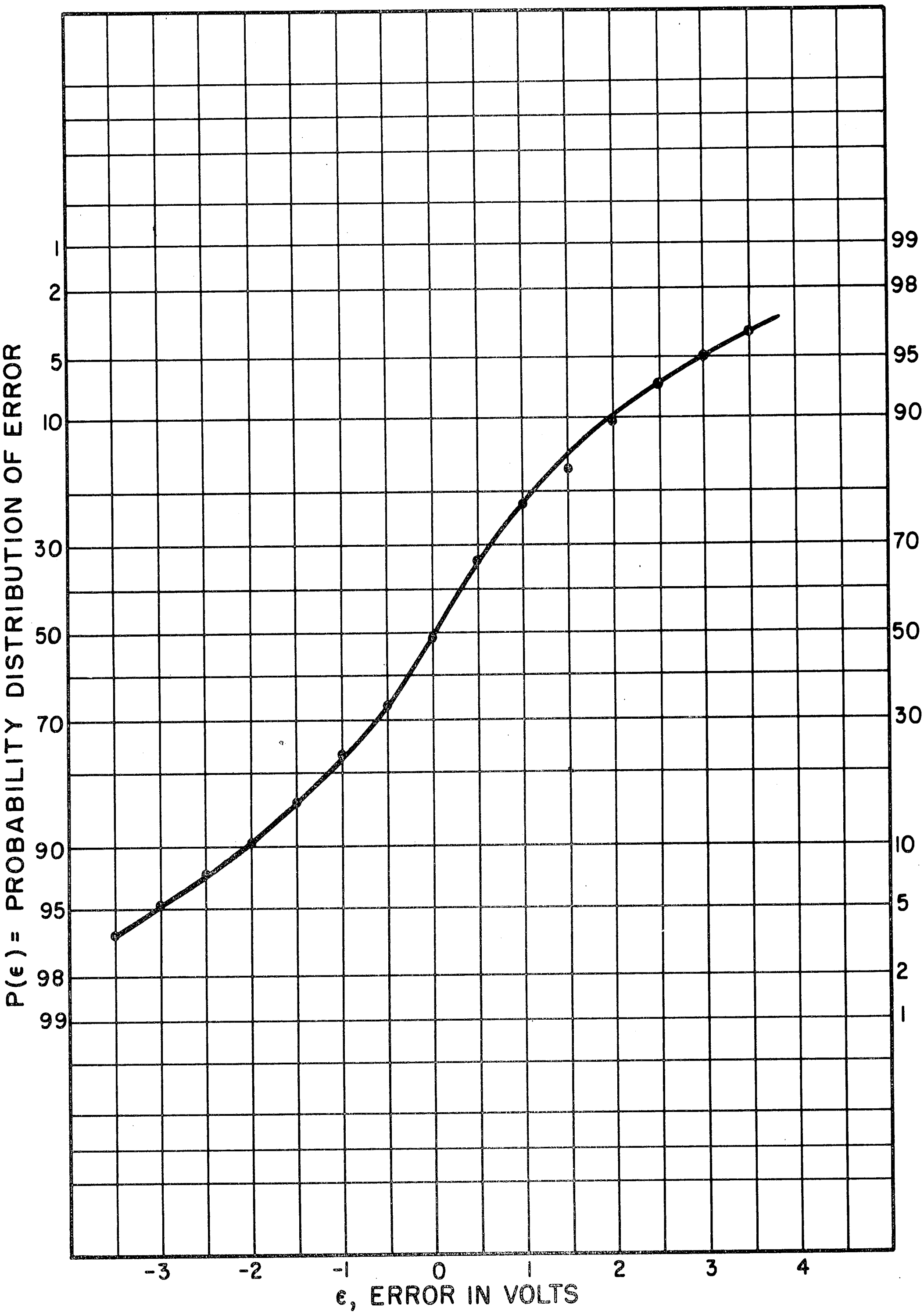

Figure 8. Error Probability Distributions of a Second Order Relay Servomechanism Plotted on Gaussian Paper

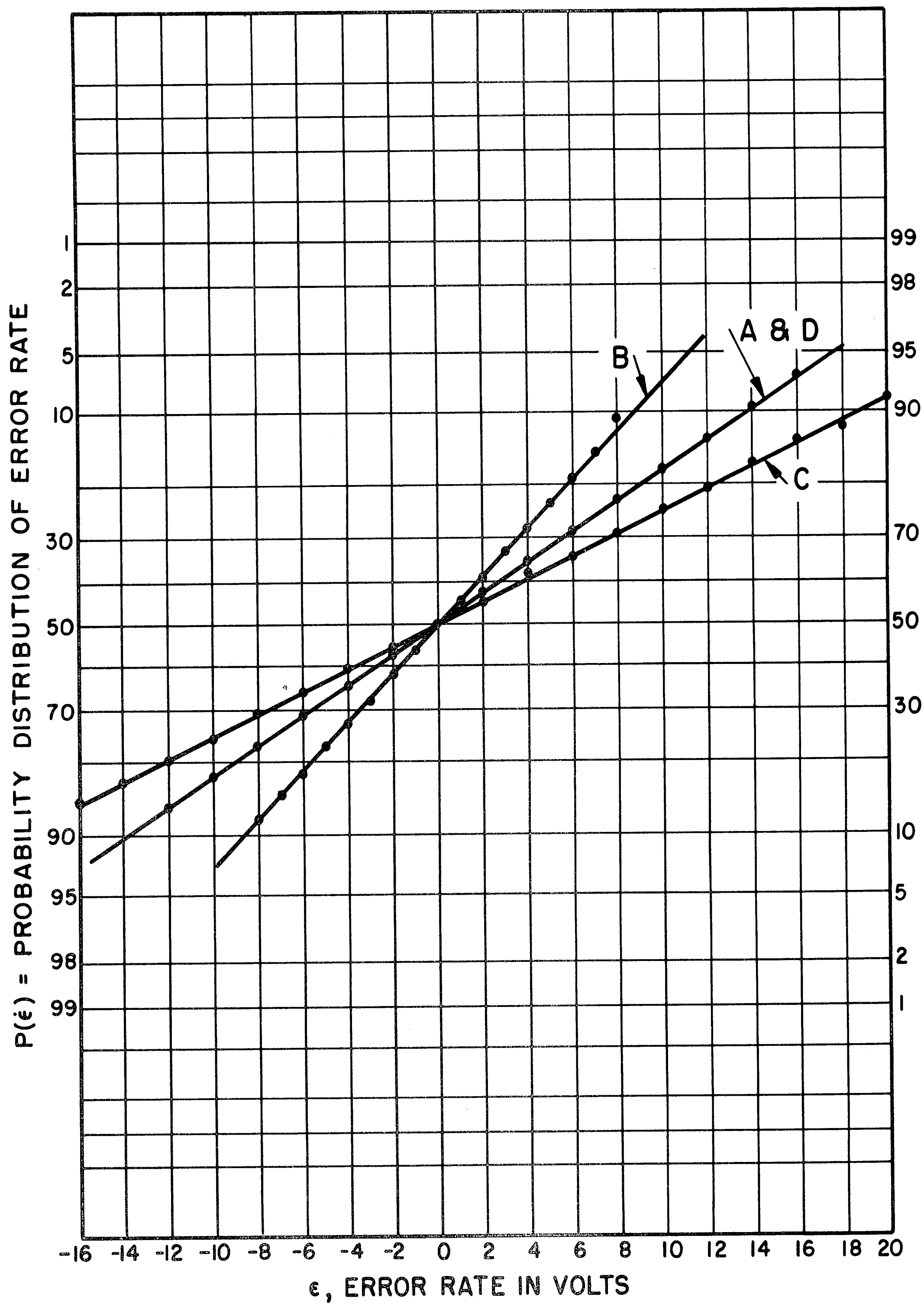

Figure 9. Error Rate Probability Distributions of a Second Order Relay Servomechanism

Curve	Changed Parameter	Theoretical Calculation	Measured Data	Remarks
B	$D_N = \frac{1}{1.5^2} D_s$	$\frac{D_s}{D_N} = 2.25$	$\frac{D_s}{D_N} = 2.22$	D_N = changed effective input power spectrum
C	$\beta_N = \frac{1}{2} \beta_s$	$\frac{\beta_s}{\beta_N} = 2.0$	$\frac{\beta_s}{\beta_N} = 2.0$	β_N = changed damping
D	$K_N = 1.5 K_s$	$\frac{K_s}{K_N} = 0.666$	$\frac{K_s}{K_N} = 0.690$	K_N = changed force
A	STANDARD SYSTEM			D_s = effective input power spectrum of the standard system β_s = damping of the standard system K_s = restarting force of the standard system

SUMMARY

The purpose of this paper is to present a method of determining the first probability density function of a nonlinear system subjected to a Gaussian random input. The Markoff random process technique was used. The results from the computer study agree with the theoretical calculations within the limits of experimental error.

Although this study is limited to a Gaussian input with some definite power spectrum, this analysis can be applied to a large class of nonlinear systems.

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