Math. Ann. 311, 305–333 (1998)

Mathematische Annalen

Hyperbolic maps on $\mathbb{P}^2$

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Received: 21 February 1997 / Revised version: 8 October 1997

Mathematics Subject Classification (1991): 32H50, 32L30

1 Introduction

In this paper we discuss hyperbolic holomorphic maps on $\mathbb{P}^2$. Our aim is to introduce invariant currents and measures which describe the dynamics.

The simplest model is the example (z^2, w^2) on $\mathbb{C}^2$ which on $\mathbb{P}^2$ can be written as $[z^2 : w^2 : t^2]$.

Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a holomorphic self-map of degree d. In homogeneous coordinates, $f = [P : Q : R]$ where P, Q, R are homogenous polynomials of degree d with no common zeros except for the origin, so f is a well defined holomorphic map. The space of such maps is denoted by $\mathscr{H}_d$.

Some results on the dynamics of such maps have been obtained in ([FS1]), ([FS2]), ([FS3]). We recall a few notions.

Definition 1.1 *The Fatou set $F = F_f$ of $f \in \mathscr{H}_d$ is the largest open set on which the iterates (f^n) is a normal family. The Julia set $J = J_f$ is the complement of F_f.*

Definition 1.2 *Let $g : M \to M$ be a continuous self map on a manifold. The non wandering set $\Omega = \Omega_g$ is the set of points $x \in M$ such that for every neighborhood U of x there is an $n \geq 1$ with $f^n(U) \cap U \neq \emptyset$.*

We want to study the dynamics of $f \in \mathscr{H}_d$, under the assumption that f is prehyperbolic on Ω_f, see ([Ru]). The examples we have in mind are perturbations of maps of the following type:

$$f([z : w : t]) = [P(z, t) : Q(w, t) : t^d]$$

First author supported by an NSF grant.

where P, Q define polynomial maps on $\mathbb{P}^1$ which are hyperbolic on their non-wandering set. It is well known that on $\mathbb{P}^1$ this is equivalent to the fact that all critical points are in the basin of attraction of attracting cycles.

We study in this paper the class of maps $f \in \mathscr{H}_d$, which are strongly hyperbolic on Ω_f (s-hyperbolic), see definition 3.2. In particular we assume that periodic points are dense in Ω_f. If f is prehyperbolic on Ω_f, then $\Omega_f = S_0 \cup S_1 \cup S_2$ where the unstable dimension of S_j is j. In this case we show that each S_i is nonempty (Theorem 3.4) and that $J \setminus S_2 = \cup_{x \in S_1} W^s(x)$ where $W^s(x)$ is the stable manifold of x.

The set S_2 is the support of a probability measure μ which is mixing and of maximal entropy. It is a basic set ([FS2]).

The set S_1 decomposes into a finite number of basic sets: $S_1 = \cup_j S_1^j$. There is an ordering among the basic sets: $S_1^j < S_1^k$ if one can go from S_1^k to S_1^j see the paragraph after Theorem 2.5 for a precise definition. We are interested in describing dynamical objects related to S_1^j assuming S_1^j is minimal for this order.

More precisely, in order to describe the "foliated" structure of J it is natural to consider positive closed currents. It is shown in ([FS2]) that J is the support of a positive closed current T which is obtained as the limit of $\frac{(f^n)^*\omega}{d^n}$, where ω is the standard Kahler form on $\mathbb{P}^2$.

We show that if S_1^j is minimal and f is s-hyperbolic then the restriction of the current T to $W^s(S_1^j)$ is laminar, i.e. near S_1^j it is an integral of currents of integration of local stable manifolds. Indeed $W^s(S_1^j)$ is open in J. We prove that the stable manifold of a point $x \in S_1^j$ is dense in $W^s(S_1^j)$. The approach is to obtain the result by proving a convergence result for the currents $\frac{(f^n)^*}{d^n}[W^s_R(x)]$, where $x \in S_1^j$ and $[W^s_R(x)]$ is the current of integration on the local stable manifold at x. The approach is similar to the one used by Bedford and Smillie ([BS2]) in the case of Hénon maps in $\mathbb{C}^2$, which are biholomorphisms.

The set S_2 is the repelling part of the dynamics. By analogy with Hénon maps, paragraph 4, we introduce an open set

$$U^- = \{z; \{f^{-n}(z)\} \text{ converge locally uniformly near z to}; S_2\}$$

and $K^- := \mathbb{P}^2 \setminus U^-$. The set $K^- \setminus S_0$ is the union of unstable manifolds $W^u(\tilde{x})$ with $x_0 \in S_1$ (Proposition 4.2). Recall that for endomorphisms the unstable manifold depends on the prehistory of a point.

We prove the existence of positive closed $(1,1)$ currents σ, supported on K^- and satisfying the functional equation $f_*\sigma = d\sigma$, where d is the algebraic degree of f. When S_1^j is minimal we describe the structure of a current σ satisfying the previous equation and whose support is $\overline{W^u(S_1^j)}$.

Indeed when S_1^j is minimal the closure of $W^u(S_1^j)$ is obtained by adding a finite number of attracting cycles.

For this current we show that $\nu := T \wedge \sigma$ is an invariant probability measure with support equal to S_1^j and f is mixing with respect to ν (see Theorems 6.2 and 6.3).

2 Hyperbolicity for endomorphisms

We first recall a few definitions and results on hyperbolic sets for smooth maps on a compact Riemannian manifold M. We refer to ([Ru]) for the general theory.

Let $f : M \to M$ a smooth map on M and suppose K is a compact forward invariant set for f, i.e. $f(K) = K$.

The first problem is to generalize the notion of hyperbolic set for diffeomorphisms to general smooth maps (endomorphisms).

We could say that (but as explained below we will not) K is hyperbolic if there is a continuous splitting of the tangent bundle $T_x M = V_x^s + V_x^u$ for $x \in K$ with V^s contracted and V^u expanded under f'. More precisely, there exist constants $C > 1, \Theta > 1$ such that for every $x \in K$, $f'(V_x^u) \subset V_{f(x)}^u$, $f'(V_x^s) \subset V_{f(x)}^s$ and $|Df^k(x)\xi| \leq \frac{C}{\Theta^k}|\xi| \ \forall\ k = 1, 2, ..., \xi \in V_x^s$ and $|Df^k(x)\xi| \geq \frac{\Theta^k}{C}|\xi| \ \forall\ k = 1, 2, ..., \xi \in V_x^u$. If K is hyperbolic in this sense, there are associated local stable manifolds W_x^s to each point $x \in K$. These are smooth manifolds of the same dimension as V_x^s and $f(W_x^s) \subset W_{f(x)}^s$ as germs. Moreover $T_x W_x^s = V_x^s$.

However, if one tries to construct unstable manifolds, they are not necessarily unique because preimages are not unique. Rather, an unstable manifold at $x \in K$ depends on the prehistory chosen for x, i.e. one fixes any sequence $\tilde{x} = (x_k)_{k \leq 0}, x_0 = x, f(x_k) = x_{k+1}$. Note that $\tilde{x}$ is not necessarily unique since points can have several preimages. There is an unstable manifold $W_{\tilde{x}}^u$ through x of the same dimension as V_x^u, and these manifolds have the following invariance property: $f(W^u_{(\dots,x_{-1})}) \subset W^u_{(\dots x_0)}$ as germs. Moreover, $T_x W_{\tilde{x}}^u = V_x^u$.

So all the possible unstable manifolds are tangent to each other, but depend otherwise on the prehistory chosen.

The condition that all the unstable manifolds are tangent is too restrictive, because the tangent space should depend on the prehistory.

Hence we introduce

$$\tilde{K} := \{\tilde{x} = (x_k) \in \Pi_{k \leq 0} K; f(x_{k-1}) = x_k\}$$

and similarly $\tilde{M}$.

The map f induces a map $\tilde{f} : \tilde{K} \to \tilde{K}$, $\tilde{f}(\tilde{x}) := (f(x_k))$.

The tangent bundle of $\tilde{K}$ is the set of $(\tilde{x}, \xi)$ with $\xi \in T_{x_0} M$. The set K is said to be prehyperbolic for f if there is a continuous splitting of the tangent bundle for $\tilde{K}$, $T_{\tilde{x}} = V_{x_0}^s + V_{\tilde{x}}^u$ with $V_{x_0}^s$ contracted and $V_{\tilde{x}}^u$ expanded under f'. More precisely there are constants $C > 1$, $\Theta > 1$ such that for every $\tilde{x} = (x_k) \in \tilde{K}$ there is a splitting of $T_{x_0} M = V_{x_0}^s + V_{\tilde{x}}^u$ and

$$|Df^k(x_0)\xi'| \leq C\Theta^{-k}|\xi'|, \ \ \xi' \in V_{x_0}^s; \ |Df^{-k}(x_0)\xi''| \leq C\Theta^{-k}|\xi''|, \ \ \xi'' \in V_{\tilde{x}}^u.$$

We also assume that $Df(V_{x_0}^s) \subset V_{f(x_0)}^s$ and $(Df)(V_{\tilde{x}}^u) = V_{f(\tilde{x})}^u$. Under these assumptions the dimensions of $V_{x_0}^s$ and $V_{\tilde{x}}^u$ are locally constant. We will call the dimension of $V^u_{\tilde{x}}$, the unstable dimension. Given $x \in K$, for R sufficiently small we define the local stable manifold

$$W^s_{x,R} := \{y \in M; d(f^n(y), f^n(x)) < R \text{ for } n \geq 0\}$$

Given $x_0 \in K$ let $\tilde{x} = (x_k) \in \tilde{K}$ be a prehistory for x_0. We define the local unstable manifold for that prehistory as:

$$W^u_{\tilde{x},R} := \{y_0 \in M; \exists (y_k)_{k\leq 0} \in \tilde{M} \text{ and } d(y_k, x_k) < R\}$$

Similarly we define the global stable and unstable sets:

$$W^s_x := \{y \in M; \lim_{n\to\infty} d(f^n(y), f^n(x)) = 0\}$$

$$W^u_{\tilde{x}} := \{y_0 \in M; \exists (y_k)_{k\leq 0} \in \tilde{M}\ d(y_k, x_k) \to 0\}.$$

These are tangent at $\tilde{x}$ to $V^s_{x_0}$ and $V^u_{\tilde{x}}$ respectively. The maps $x \to W^s_{x,R}$ and $\tilde{x} \to W^u_{\tilde{x},R}$ are continuous for the $\mathscr{C}^r$ topology on the space of parametrized discs. Moreover there are constants $L > 0,\ \lambda > 1$ such that if $y, z \in W^s_{x,R}$ then

$$d(f^n(y), f^n(z)) \leq L\lambda^{-n} d(y, z).$$

If $\tilde{y} = (y_n),\ \tilde{z} = (z_n)$ are prehistories as in the definition of $W^u_{\tilde{x},R}$, then $d(y_n, z_n) \leq L\lambda^{-|n|} d(y_0, z_0)$.

Definition 2.1 *The set K has local product structure if R can be chosen such that for all $x \in K,\ y \in \tilde{K}$*

$$W^s_{x,R} \cap W^u_{\tilde{y},R} \subset K$$

and the intersection consists of at most one point.

In this case there is an $\epsilon > 0$ such that if $d(x, y) < 2\epsilon$ then $W^s_{x,R} \cap W^u_{\tilde{y},R}$ consists of exactly one point $[x, y]$. Moreover there is an $L > 0$ such that

$$d(x, [x, y]) \leq Ld(x, y),$$
$$d(y, [x, y] \leq Ld(x, y).$$

Definition 2.2 *Let $f : M \to M$ be as above and let Ω be the non wandering set for f. We say that f satisfies Axiom A if*
i) Ω is a compact prehyperbolic set.
ii) Periodic points are dense in Ω.

Theorem 2.3 *([Ru p.160]) Let $f : M \to M$ be a smooth map with non wandering set Ω. Assume f satisfies Axiom A. Then*
i) Ω has local product structure.
ii) $\Omega = \Omega_1 \cup \cdots \cup \Omega_N$ is a finite union of pairwise disjoint closed forward invariant sets such that f is topologically transitive on each Ω_i (i.e. f has a dense forward orbit on each Ω_i). The above decomposition is unique, the sets Ω_i are called the basic sets.
iii) Each basic set Ω_i is a finite union $\Omega_i = \Omega_{i,1} \cup \cdots \cup \Omega_{i,N_i}$ of smallest pairwise disjoint closed sets which are permuted cyclically by f.

Remark 2.4 Each Ω_i is isolated in the nonwandering set, more precisely, there exists a compact neighborhood $\overline{U}$ of Ω_i such that if $\{x_n\}_{n\in\mathbb{Z}}$, $f(x_n) = x_{n+1}$ $\forall n$ and $x_n \in \overline{U}$ $\forall n$, then $x_n \in \Omega_i$ $\forall n$ ([Ru p. 160], [Pz],[Mo]).

Recall the ordering among basic sets. We say $\Omega_i > \Omega_j$ if and only if $W^u(\Omega_i)\cap W^s(\Omega_j) \neq \emptyset$, which means that it is possible to go from Ω_i to Ω_j. The No-cycle condition means that one cannot find basic sets $\Omega_{i_1}, \dots, \Omega_{i_p}$, $p > 1$ such that $\Omega_{i_1} > \Omega_{i_2} > \dots \Omega_{i_p} > \Omega_{i_1}$.

The interest of hyperbolic maps on their nonwandering set Ω is their stability. We have the following result ([Ru, p.168]), ([Pz]) and ([Mo, p. 182]).

Theorem 2.5 *The set of $\mathscr{C}^r$ maps on M, satisfying Axiom A and the No-cycle condition is open. In particular the non-wandering sets of such maps are close if the maps are close.*

If f is such a map, there exists a neighborhood $\mathscr{U}(f) \subset \mathscr{C}^r(M, M)$ such that for $g \in \mathscr{U}(f)$, $\tilde{f}$ is conjugate to $\tilde{g}$ on their prehyperbolic cover. More precisely, there is a homeomorphism $h : \tilde{\Omega}(f) \to \tilde{\Omega}(g)$ such that the following diagram is commutative.

$$\begin{array}{ccc} \tilde{\Omega}(f) & \xrightarrow{\tilde{f}} & \tilde{\Omega}(f) \\ h\downarrow & & h\downarrow \\ \tilde{\Omega}(g) & \xrightarrow{\tilde{g}} & \tilde{\Omega}(g) \end{array}$$

We consider $\tilde{\Omega}(f)$ and $\tilde{\Omega}(g)$ as subsets of $\tilde{M} \subset (\mathbb{P}^2)^{\mathbb{N}}$ with the product topology, so the fact that the conjugating homeomorphism is close to the identity makes sense.

3 Holomorphic maps in $\mathbb{P}^2$

Let $f : \mathbb{P}^2 \to \mathbb{P}^2$, $f \in \mathscr{H}_d$. Let ω be the Kähler form in $\mathbb{P}^2$ normalized such that $\int \omega \wedge \omega = 1$. It was shown in ([FS2]) that the sequence of positive closed $(1,1)$ forms $\frac{(f^n)^*\omega}{d^n}$ converges in the sense of currents to a positive closed current T, whose support is equal to the Julia set J_f and which satisfies the following functional equation

$$f^*T = dT$$

where d denotes the algebraic degree of f.

It was shown in [FS2] that $\mu := T \wedge T$ is a probability measure satisfying $f^*\mu = d^2\mu$ and hence is a measure of maximal entropy, $\ln d^2$, see ([Gr]). Observe that by Bezout's Theorem f is a d^2 to 1 map.

Let $\Omega = \Omega_f$ be the non wandering set of f and assume Ω is prehyperbolic. Then Ω can be decomposed into disjoint sets $\Omega =: S_0 \cup S_1 \cup S_2$ where S_j is of unstable dimension j. A priori some of the S_j could be empty. However since $f(\Omega) = \Omega$ it is clear that $f(S_j) = S_j$ $j = 0, 1, 2$. The following result is proved in ([FS2]).

Theorem 3.1 *Assume $f \in \mathscr{H}_d$ is prehyperbolic on Ω_f. Then*
i) $\Omega_f \neq J_f$, $J_f \neq S_2$.
ii) S_0 is the union of a finite number of attracting periodic orbits and the Fatou components are preperiodic to attracting basins.
iii) $S_2 \supset S_\mu :=$ support of μ, $f^{-1}(S_\mu) = S_\mu$.

Definition 3.2 *We will say that $f \in \mathscr{H}_d$ is $s-$ hyperbolic if*
i) f is prehyperbolic on Ω_f
ii) $f^{-1}(S_2) = S_2$
iii) There is an algebraic variety A of dimension 1 such that $A \cap S_1 = \emptyset$.
iv) Periodic points are dense in Ω_f.
v) There exists a neighborhood U of S_1 such that $f^{-1}(S_1) \cap U = S_1$.

Because of i) and iv), $s-$ hyperbolic maps satisfy Axiom A.

Remark 3.3 Observe that if Df is injective at every point of S_1, then condition iii) is satisfied with $A = C$, the critical set.

Condition v) allows us to prove a shadowing Lemma and the following result, see also ([Ru p. 103]) and ([Pz]).

Theorem 3.4 *If K is prehyperbolic, satisfies v) above and has a local product structure, there is a neighborhood U of K such that if $f^m(y) \in U$ for all $m \geq 0$, then $y \in W^s_x$ for some $x \in K$, in fact y is in the local stable manifold of x. If $\tilde{y} = (y_k)$ is a prehistory such that all $y_k \in U$ then $y_0 \in W^u_{\tilde{x}}$ for some $\tilde{x} = (x_k) \in \tilde{K}$.*

If K satisfies only conditions i) to iv), then $W^s(S^1_1) = \cup_{x \in S^1_1} W^s(x)$.

Remark 3.5 The last statement is proved in a standard way ([Ru]) by using a similar statement in the hyperbolic cover. We need condition v) only to prove that a neighborhood of S^1_1 in $W^s(S^1_1)$ is contained in the union of local stable manifolds centered on S^1_1.

Theorem 3.6 *Assume $f \in \mathscr{H}_d$ is $s-$ hyperbolic. Then each of the S_j is nonempty. Moreover $J \setminus S_2 = W^s(S_1) := \{y; \lim_{n\to\infty} d(f^n(y), S_1) = 0\}$, S_1 has local product structure. Furthermore, $W^s(S_1) = \cup_{x \in S_1} W^s_x$, and more precisely, there is a $1 >> \epsilon > 0$ and a $1 >> R > 0$ so that if an orbit $\{f^n(y)\}_{n \geq 0}$ remains at distance $< \epsilon$ from S_1, then $y \in W^s_R(x)$ for some $x \in S_1$.*

Proof. We know by Theorem 3.1 that S_2 is nonempty. Assume S_1 is empty. Then $\Omega = S_0 \cup S_2$. Let $x \in J \setminus S_2$. The Julia set J is totally invariant and $\{f^n(x)\}$ has to cluster somewhere on Ω. Hence $\{f^n(x)\}$ has to cluster on S_2. But this is impossible since $f^{-1}(S_2) = S_2$ and S_2 is repelling. Hence S_1 is nonempty.

Assume S_0 is empty. Then $\Omega = S_1 \cup S_2$ and by Theorem 3.1, ii), $J = \mathbb{P}^2$. There is a neighborhood $U \supset S_2$ such that $f^{-1}(U) \subset\subset U$. So orbits of points in $\mathbb{P}^2 \setminus S_2$ cluster only at S_1. Let $S_1 = \cup S^j_1$ be the decomposition of S_1 in basic sets given by Theorem 2.3. For R small enough, let

$$W^j_R = \{z; d(\{f^n(z)\}, S^j_1) \leq R\}.$$

Each W_R^j is closed and $\cup_{n,j} f^{-n}(W_R^j) = \mathbb{P}^2 \setminus S_2$. It follows that for some j_0, $W_R^{j_0}$ has nonempty interior. Moreover $W_R^{j_0} = \cup_{x \in S_1^{j_0}} (W^s(x) \cap W_R^{j_0})$ is a union of local stable manifolds. Fix $x \in S_1^{j_0}$ such that the local stable manifold $W^s(x) \cap W_R^{j_0}$ intersects the interior of $W_R^{j_0}$. We can assume that x is a periodic point. Let $\tilde{x}$ be the prehistory of x in $\tilde{S}_1$ consisting of the periodic orbit of x. Consider $W^u_{\tilde{x},R}$. Using the local product structure one sees that $W_R^{j_0} \cap W^u_{\tilde{x},R} \subset S_1^{j_0}$ and contains a disc from $W^u_{\tilde{x},R}$ about x, because one can just follow a transverse disc to $W^s(x)$ contained in the interior of $W_R^{j_0}$. Hence there is a ball $B(x,r)$ such that $B(x,r) \cap W^u_{\tilde{x},R} \subset S_1^{j_0}$. Since f maps $S_1^{j_0}$ to itself, it follows that

$$W^u_{\tilde{x}} \subset S_1^{j_0}.$$

We can assume, using the result of Siu-Yeung ([SY]) that $\mathbb{P}^2 \setminus A$ is Kobayashi hyperbolic. Indeed, we can suppose that A is of large degree and contained in a Zariski open dense set of varieties X such that $\mathbb{P}^2 \setminus X$ is Kobayashi hyperbolic. This would imply that f^n restricted to $B(x,r) \cap W^u_{\tilde{x},R}$ is equicontinuous, which is impossible. So S_0 is non empty.

Let $y \in J \setminus S_2$. Let U be a neighborhood of S_2 such that $f^{-1}(U) \subset\subset U$. We can assume $y \notin U$. We want to show that $f^n(y) \to S_1$. If not, $f^n(y)$ will also cluster on S_2, let m be the smallest integer such that $f^m(y) \in U$. This implies that $f^{m-1}(y) \in U$, a contradiction. So $J \setminus S_2 \subset W^s(S_1)$. If $y \in W^s(S_1)$ it cannot be in the Fatou set nor in S_2, the other inclusion follows.

That S_1 has local product structure follows from the density of periodic points and Theorem D.2, in ([Ru] p.155). One deduces from the local product structure that $W^s(S_1) = \cup_{x \in S_1} W^s(x)$ and that more precisely, there is an $0 < \epsilon << 1$ and an $0 < R << 1$ so that if an orbit $\{f^n(y)\}_{n \geq 0}$ remains closer to S_1 than ϵ, then $y \in W^s_R(x)$ for some $x \in S_1$. □

Corollary 3.7 *Assume $f \in \mathscr{H}_d$ is $s-$ hyperbolic on Ω. Then the Julia set has empty interior.*

Proof. Assume first that the interior of S_2 is non empty. Then $(f^n)(\mathrm{int}(S_2)) \to \mathrm{int}(S_2)$ and $S_2 \cap [\cup_{n=0}^N f^{-n} C] = \emptyset$ where C denotes the critical set, and this is still valid for any algebraic variety close to C. Since $\mathrm{int}(S_2)$ is Kobayashi hyperbolic, the family (f^n) is normal, which contradicts the uniform expansion of f on S_2.

As a consequence, if $\mathrm{int}(J)$ is non empty, then the interior of $W^s(S_1)$ is not empty. Let Δ^2 be a polydisc in $W^s(S_1)$. Fix $0 < \epsilon << 1$ small enough. Define

$$F_m := \{z \in \Delta^2; \mathrm{dist}(f^n(z), S_1) \leq \epsilon \ \forall n \geq m.\}$$

There is an m such that F_m has nonempty interior. By the last part of Theorem 3.4, $f^m(F^m)$ is contained in the union of local stable manifolds of S_1.

Let $W^s(p)$ be such a manifold. We can assume that p is periodic. We show that $W^u_R(\tilde{p})$ is contained in S_1 for the prehistory $\tilde{p}$ consisting of the periodic orbit of p.

Any $W^s_R(x)$ close enough to $W^s_R(p)$ is going to intersect $W^u_R(\tilde{p})$. So a disc in $W^u(\tilde{p})$ about p is contained in S_1, by the local product structure. Since S_1 is forward invariant it follows that $W^u(\tilde{p}) \subset S_1$. Let k be the period of p. Then f^k maps $W^u(\tilde{p})$ to itself. Clearly the sequence $\{(f^k_{|W^u(\tilde{p})})^m\}_m$ is not a normal family. Since S_1 is contained in the Kobayashi hyperbolic complement of a perturbation of A, ([SY]) this leads to a contradiction. □

We discuss now the decomposition of $\Omega = S_0 \cup S_1 \cup S_2$ into basic sets, i.e. closed disjoint sets with dense orbit as given in the abstract setting by Theorem 2.3.

We study the special case of $f \in \mathscr{H}_d$ which is assumed to be $s-$ hyperbolic on Ω. We have already seen that the basic sets for S_0 are just finitely many attracting periodic orbits. The corresponding stable sets are the basins of attraction. We next show that S_2 is a basic set.

Theorem 3.8 *Suppose that $f \in \mathscr{H}_d$ is $s-$ hyperbolic. The set S_2 of unstable dimension* 2 *is a basic set and $S_2 = S_\mu$. The unstable set of S_2 is open with locally pluripolar complement.*

Proof. We know from Theorem 3.1 that S_2 contains the support S_μ of μ. Let $\sigma_2 := S_2 \setminus S_\mu$. We claim that σ_2 is closed. Let V be a neighborhood of S_μ such that $f^{-k}(V) \subset\subset V$ for some $k \geq 1$, recall that $f^{-1}(S_\mu) = S_\mu$. Observe that points in $V \setminus f^{-nk}(V)$ are wandering. Hence $S_2 \setminus S_\mu$ cannot intersect that set. So $V \cap \sigma_2$ is empty, hence σ_2 is closed.

Let C be the critical set of f. Clearly $C \cap S_2 = \emptyset$. Define $\mathscr{C} := \overline{\cup_{n\geq 0} f^n(C)}$. Since f is $s-$ hyperbolic, $f^{-1}(S_2) = S_2$. Using that $f^{-\ell}(W) \subset\subset W$ for some small neighborhood W of S_2, and some $\ell \geq 1$, it follows that $\mathscr{C} \cap S_2 = \emptyset$. Locally in $\mathbb{P}^2 \setminus \mathscr{C}$ one can define holomorphic local branches of inverses of f^n f_i^{-n}. A theorem of Ueda ([U]) asserts that they are equicontinuous.

For any continuous function ϕ on $\mathbb{P}^2$, define $A^n_\phi(x) = \frac{1}{d^{2n}} \sum_{i=1}^{d^{2n}} \phi(f_i^{-n}(x))$.

By the above result it is clear that for any given ϕ, the sequence of functions (A^n_ϕ) is locally equicontinuous in $\mathbb{P}^2 \setminus \mathscr{C}$. On the other hand it is shown in ([FS3]) that there exists a locally pluripolar set $\mathscr{E}$ (independent of ϕ) such that for $x \in \mathbb{P}^2 \setminus \mathscr{E}$, $A^n_\phi(x) \to \mu(\phi)$, in particular the limit does not depend on x. As a consequence $A^n_\phi(x) \to \mu(\phi)$ uniformly on compact sets of the nonempty open set $\mathbb{P}^2 \setminus \mathscr{C}$.

Observe that σ_2 is also totally invariant. Let $\phi = 1$ in a neighborhood of S_μ and 0 in a neighborhood of σ_2. Since $A^n_\phi(x) \to 1$, for $x \in \sigma_2$, this implies that σ_2 is empty.

It is shown in ([FS2]) that the measure μ is ergodic. Hence μ almost every point has a dense orbit. As a consequence, $S_2 = S_\mu$ is a basic set.

Recall that

$$W^u(S_2) = \{y; \exists\, \tilde{y} = (y_n)\; y_0 = y,\; f(y_{n-1}) = y_n \text{ and } d(y_n, S_2) \to 0\}.$$

We have already mentioned that except on a pluripolar set $\mathscr{E}$, $\frac{1}{d^{2n}} \sum_{i=1}^{d^{2n}} \epsilon_{x_i^n} \to \mu$.

So $\mathbb{P}^2 \setminus \mathscr{E} \subset W^u(S_2)$. Since there is an open set $U \supset S_2$ such that $f^{-1}(U) \subset\subset U$, then $W^u(S_2)$ is open with locally pluripolar complement. $\square$

4 The inverse Julia set J^-

We recall first a few facts about invariant currents for Hénon maps.

Let f be a Hénon map in $\mathbb{C}^2$, for example $f(z,w) = (z^2 + c + aw, z)$, $c \in \mathbb{C}$, $a \neq 0$. Consider $\tilde{f}[z : w : t] = [z^2 + ct^2 + awt : zt : t^2]$, the rational extension of f to $\mathbb{P}^2$. Let ω denote the standard Kähler form of $\mathbb{P}^2$.

It is shown in ([FS5]) that $\frac{(\tilde{f}^n)^*\omega}{2^n}$ converges to a positive current $\tilde{\mu}^+$ on $\mathbb{P}^2$. The restriction μ^+ of $\tilde{\mu}^+$ to $\mathbb{C}^2$ has been considered in ([BS1]), ([BS2]) where it was extensively studied.

It is proved in ([FS3]) that the current $\tilde{\mu}^+ \wedge \tilde{\mu}^+$ is well defined and that this measure is equal to the Dirac mass δ_{p_-} at the point of indeterminacy of $\tilde{f}$ i.e. $p_- = [0 : 1 : 0]$. The point p_- appears as the only "repelling" point for $\tilde{f}$. (For any small neighborhood $U(p_)$ and any point $q \in U(p_)$, $q \neq p_,$ there is an integer $m(q)$ so that $\tilde{f}^n(q) \notin U(p_)\ \forall n \geq m(q)$.)

Similarly it is possible to define $\tilde{\mu}^-$ for the map $\tilde{f}^{-1}$ (the extension to $\mathbb{P}^2$ of the automorphism f^{-1} of $\mathbb{C}^2$) and to consider the probability measure $\tilde{\mu}^- \wedge \tilde{\mu}^+ =: \nu$, which is an invariant measure of maximal entropy supported on a compact subset of $\mathbb{C}^2$. The properties of the measure ν are studied in ([BS1]), ([BS2]), ([BLS]).

When f is a holomorphic map on $\mathbb{P}^2$ of degree $d \geq 2$, the analogue of the current $\tilde{\mu}^+$ is what we have called T. So $T = \lim \frac{(f^n)^*\omega}{d^n}$. The probability measure $\mu := T \wedge T$ is well defined. If f is prehyperbolic on its nonwandering set Ω, then $S_\mu :=$ supp μ is contained in the repelling part S_2, see Theorem 3.1.

For any holomorphic map $f : \mathbb{P}^2 \to \mathbb{P}^2$ of degree $d \geq 2$ the probability measure μ is mixing and of maximal entropy ([FS2]).

In the case of Hénon maps, K^- is defined as the set of points z such that $\{f^{-n}(z)\}$ is bounded. Equivalently we can consider

$$U^- = \{z; \{f^{-n}(z)\} \text{ converges to } p_, \text{ the support of } \delta_{p_-}\}$$

and define $K^- := \mathbb{C}^2 \setminus U^-$. The convergence to p_- is always uniform in a small neighborhood , so U^- is open. By analogy with the previous definition we define U_f^- for $f \in \mathscr{H}_d$ as follows:

For any set V, set $V_n := \{f^{-n}(z); z \in V\}$.

$$U_f^- := \{z_0; \exists \text{ a neighbh } V \text{ of } z_0 \text{ with dist}\{V_n, \text{Supp } \mu\} \to 0 \text{ when } n \to \infty\}$$

Then $K_f^- := \mathbb{P}^2 \setminus U_f^-$.

Definition 4.1 *The backward Julia set $J^- = J_f^-$ is defined to be $\partial K^- = \partial K_f^-$.*

It is clear that $f(K^-) = K^-$, $f^{-1}(K^-) \supset K^-$, $f(J^-) \supset J^-$, $f(U^-) \supset U^-$ and $f^{-1}(U^-) \subset U^-$.

Assume that f is $s-$ hyperbolic. Let (S_1^k), $1 \le k \le \ell$, be the basic sets in S_1. We will consider the corresponding unstable sets

$$W^u(S_1^k) = \{y; \exists \tilde{y} = (y_n)_{n \le 0},\ \text{a prehistory of } y = y_0,\ \text{such that } d(y_n, S_1^k) \to 0\}.$$

Proposition 4.2 *Suppose f is $s-$ hyperbolic. Then $K^- = \cup_{\tilde{x} \in \tilde{S}_1} W^u(\tilde{x}) \cup S_0 = W^u(S_1) \cup S_0$. More precisely, for arbitarily small $R > 0$, $\cup_{\tilde{x} \in \tilde{S}_1} W^u_R(\tilde{x})$ contains a neighborhood of S_1 in $W^u(S_1)$. In particular, $K^- \cap S_2 = \emptyset$.*

Proof. Let $x \notin W^u(S_1) \cup S_0$. Then no prehistory of x converges to S_1. So all prehistories cluster only on $S_2 \cup S_0$. Therefore since S_2 is attracting for f^{-1}, they converge to S_2. And the same holds in a neighborhood of x. Hence K^- is contained in $W^u(S_1) \cup S_0$. The inclusion $W^u(S_1) \cup S_0 \subset K^-$ is clear.

Since S_1 has the local product structure, see Theorem 3.5, Theorem 2.5 implies that $W^u(S_1) = \cup_{\tilde{x} \in \tilde{S}_1} W^u(\tilde{x})$. Indeed S_2 is completely invariant so $K^- \cap S_2 = \emptyset$. □

Corollary 4.3 *Assume f is $s-$ hyperbolic. Then the complement of K^- in $\mathbb{P}^2$ is a domain of holomorphy. So K^- is connected.*

Proof. Except for a finite set of points in S_0, there is a possibly singular holomorphic disc (a piece of some $W^u(\tilde{x})$) through any point in K^-, which is contained in K^-. It follows from the solution of the Levi Problem in $\mathbb{P}^2$ that $\mathbb{P}^2 \setminus K^-$ is a domain of holomorphy. Hence K^- is connected. □

We want to describe more precisely the sets $W^s(S_1^j)$. Recall that we have an ordering among the (S_1^j).

Theorem 4.4 *Assume f is an $s-$ hyperbolic map. Then $W^s(S_1^j) \cap W^u(S_1^j) = S_1^j$. If S_1^j is minimal, then $W^u(S_1^j) \setminus S_1^j$ is contained in the region of attraction of S_0, $\overline{W^u(S_1^j)} \setminus W^u(S_1^j) \subset S_0$. The set $W^s(S_1^j)$ is relatively open in J. Also $\cup_{x \in S_1^j} \overline{W^s_R(x)} =: J_R$, the union of local stable manifolds, $R << 1$ contains a neighborhood of S_1^j in J.*

Proof. We know by Theorem 3.4 and Proposition 4.2 that $W^s(S_1) = \cup_{x \in S_1} W^s(x)$ and $W^u(S_1) = \cup_{\tilde{x} \in \tilde{S}_1} W^u(\tilde{x})$. Hence for each j, $W^s(S_1^j) = \cup_{p \in S_1^j} W^s(p)$ and $W^u(S_1^j) = \cup_{\tilde{x} \in \tilde{S}_1^j} W^u(\tilde{x})$. Let $p \in S_1^j$ and $\tilde{q} \in \tilde{S}_1^j$. Assume $x \in W^s(p) \cap W^u(\tilde{q})$. Notice that we can replace x by a forward iterate of x. Hence we can assume that $x \in W^s_R(p)$, the local stable manifold of p. We want to show first that x is recurrent.

Let B be a ball containing x. Then B will contain a disc Δ intersecting $W^s_R(p)$ transversally. Since periodic orbits are dense in S_1^j, (see Definition 3.2) we have that Δ will also intersect $W^s_R(p')$ transversally, where p' is a periodic point in S_1^j. By topological transitivity and expansion, we can assume that $f^m(\Delta)$

is close to $W^u_{\tilde{q}_k,R}$ for arbitrarily large m for any k. On the other hand, we have a prehistory of x, $\tilde{x}$ for which the distance between x_k and q_k approaches 0. By the contraction in the stable direction it then follows that for some k, $f^{k+m}(\Delta)$ will intersect B. So x is recurrent. This proves that $W^s(S_1^j) \cap W^u(S_1^j) = S_1^j$.

Suppose S_1^j is minimal for the ordering $>$, and $p \in W^u(S_1^j) \setminus S_1^j$. We want to show that p is in the domain of attraction of S_0. Let C denote the set of cluster points of the forward orbit of p. Then C is contained in the nonwandering set, C cannot intersect S_2. If C intersects two separate S_1^k, C must contain points that are not in $S_1 \cup S_0$. Hence C can only intersect one S_1^k. If $k \neq j$, this contradicts that S_1^j is minimal, if $k = j$, this contradicts that $W^s(S_1^j) \cap W^u(S_1^j) = S_1^j$. So necessarily $C \subset S_0$. Hence p is in the region of attraction for S_0.

Next, suppose $p_0 \in \overline{W^u(S_1^j)} \setminus (S_0 \cup W^u(S_1^j))$. Note that any such point must have a preimage in the same set. Hence we can find a prehistory $\{p_n\}_{n \leq 0}$ of p_0 in this set. The cluster points must be nonwandering and cannot intersect S_0, so must be in some S_1^k.

If $k \neq j$, then $W^u(S_1^j)$ clusters at S_1^k. This contradicts that for small R, an arbitrarily small neighborhood U of S_1^j, $\cup_{\tilde{x} \in \tilde{S}_1^j} \overline{W_R^u(\tilde{x})} \setminus U$ is a compact subset of the region of attraction of S_0 and that $\cup_{\tilde{x} \in \tilde{S}_1^j} W^u(\tilde{x})$ contains a neighborhood of S_1^j in $W^u(S_1^j)$. (See Proposition 4.2.) Hence $k = j$, so $p_0 \in W^u(S_1^j)$, a contradiction.

By Proposition 4.2. $p_0 \in W^u_{\tilde{x}}$ for some $\tilde{x} \in \tilde{S}_1^k$.

We prove next that if S_1^j is minimal, then $W^s(S_1^j)$ is open in J. Consider $\cup_{x \in S_1^j} \overline{W_R^s(x)} = F$, the union of local stable manifolds $R << 1$. Then F is closed and we want to show it contains a neighborhood of S_j^1 in J. If not there is $x \in J$ arbitrarily close to S_1^j, not in F, say $x \in W^s(p)$, $p \in S_1^k$. By Theorem 3.5, the orbit $\{f^n(x)\}$ cannot remain close to S_1^j. The distance from $f^n(x)$ to S_j^1 can increase only if the orbit follows an unstable manifold of a point in S_1^j. But we have seen that $W^u(S_1^j)$ enters immediately into a basin of attraction. This is impossible for $\{f^n(x)\}$. □

Proposition 4.5 *Assume f is $s-$ hyperbolic. The set K^- is an attractor for f in $\mathbb{P}^2 \setminus S_2$, i.e. $\{f^n\}$ converges uniformly on compact set in $\mathbb{P}^2 \setminus S_2$ to K^-.*

Proof. Fix a neighborhood $U(K^-)$ and a neighborhood $V(S_2)$. There exists an integer $N > 1$ so that is $x \in \mathbb{P}^2 \setminus (U(K^-))$ and $n \geq N$, then all preimages $f^{-n}(x)$ are in $V(S_2)$. Hence for any $x \in \mathbb{P}^2 \setminus V(S_2)$, $n \geq N$, $f^n(x) \in U(K^-)$. □

Remark 4.6 The Julia set J is also an attractor for f^{-1} in $\mathbb{P}^2 \setminus S_0$, in the following sense. If $\overline{B(x,r)}$ is a ball disjoint from S_0 then $(f^{-n})(B(x,r))$ converge uniformly to J. The analogy with Hénon maps can be continued. The attracting point at infinity, $p = [1:0:0]$ for the Hénon map $h = [z^2 + ct^2 + awt : zt : t^2]$, plays the role of S_0.

5 Positive closed currents on J and K^-.

5.1 Invariant currents on K^-

We first introduce a class of invariant currents supported on K^- and study their intersection with T.

We recall first the definition of the direct image or push forward of a current. Let R be a current on a smooth manifold M. Assume $g : M \to N$ is a smooth proper map, N a smooth manifold. The direct image g_*R of R is defined by

$$< g_*R, \phi >:=< R, g^*\phi > \quad (1)$$

for any test form ϕ. Observe that g_* preserves the dimension of the current, i.e. the type of the forms $\phi,\ g^*\phi$.

Similarly we would like to to define the pull-back of the current R on N by the equation

$$< g^*R, \phi >=< R, g_*\phi > \quad (2)$$

where ϕ is a smooth test form and $g_*\phi$ is the current defined in (1). This works if g is a diffeomorphism, in fact $g_*\phi = (g^{-1})^*\phi$, so is a smooth test form also.

This works also well if g is a finitely sheeted unbranched covering. In this case

$$g_*\phi = \sum_{j=1}^{k}(g_j^{-1})^*\phi$$

where $g_1^{-1}, \ldots, g_k^{-1}$ are the local inverses of g, so is also a smooth test form which can hence be paired with R.

Unfortunately, we will use maps $g : \mathbb{P}^2 \to \mathbb{P}^2$ which are holomorphic of degree $d \geq 2$ and are always branched.

If R is a closed, positive $(1,1)$ current on $\mathbb{P}^2$, then $\pi^*R = dd^c u$ where u is a plurisubharmonic function on $\mathbb{C}^3$. We define f^*R by the equation $\pi^* f^* R = dd^c(u \circ F)$ ([FS3]). Let $Z = f(C)$ be the branch locus of f, here C denotes the critical set. Then

$$f : \mathbb{P}^2 \setminus f^{-1}(Z) \to \mathbb{P}^2 \setminus (Z)$$

is a finitely sheeted covering map and f^*R is classically defined as a current on $\mathbb{P}^2 \setminus f^{-1}(Z)$ using the relation

$$< f^*R, \phi >=< R, f_*R > .$$

The two definitions coincide on $\mathbb{P}^2 \setminus f^{-1}(Z)$. When R gives no mass to Z then f^*R as defined in $\mathbb{P}^2$ is just the trivial extension to $\mathbb{P}^2 \setminus f^{-1}(Z)$.

For a positive current R on $\mathbb{P}^2$ we have $\mathrm{supp} f_*R = f(\mathrm{supp} R)$: The fact that $\mathrm{supp} f_*(R) \subset f(\mathrm{supp} R)$ is valid for all currents. Suppose $x = f(y) \in f(\mathrm{supp}\ R)$

and let ϕ be a positive form supported in a neighborhood of x. Then $f^*\phi > 0$ near y and $< f_*R, \phi >=< R, f^*\phi >> 0$. So x is in the support of f_*R.

Recall also that for a positive closed $(1,1)$ current R on $\mathbb{P}^2$ the mass $\|R\|$ of R is given by

$$\|R\| = \int R \wedge \omega.$$

It is possible to choose a plurisubharmonic function u in $\mathbb{C}^3$ satisfying $u(\lambda z) = c \ln|\lambda| + u(z)$, with $c = \|R\|$ and $dd^c u = \pi^* R$. See [FS3].

Claim 5.1 *A potential v for $\pi^*(f_*R)$ is given by*
$v(z) = \frac{1}{d}\sum_{F(z_i)=z} u(z_i)$ *counted with multiplicity.*

Proof. Obvious. □

Observe that for a positive current R on $\mathbb{P}^2$ we have the formula

$$f_* f^* R = d^2 R. \quad (3)$$

Given $f \in \mathscr{H}_d$. We assume f is $s-$ hyperbolic on Ω. We consider the set $\mathscr{S}$ of positive closed $(1,1)$ currents S on $\mathbb{P}^2$ with $\|S\| = 1$ such that $f_*S = dS$ and support of S is disjoint from the support of μ which is equal to S_2 (since f is $s-$ hyperbolic).

Theorem 5.2 *Assume that f is $s-$ hyperbolic. The set $\mathscr{S}$ is a nonempty convex weakly compact set. The currents in $\mathscr{S}$ are supported in K^-.*

Proof. Let R be any positive closed current, $\|R\| = 1$, with support disjoint from S_2. Such currents exist. For example $R = \frac{[C]}{\|C\|}$ where $[C]$ is the current of integration on the critical set of f. We can assume that supp$R$ is disjoint from an open set $U \supset S_2$ such that $f^{-k}(U) \subset\subset U$ for some positive integer k. Consider $\sigma_N := \frac{1}{N}\sum_{n=0}^{N-1}\frac{f_*^n R}{d^n}$. Let σ be any limit point of the sequence σ_N. Since $\sigma_N - \frac{f_*\sigma_N}{d} = O(\frac{1}{N})$ it follows that $f_*\sigma = d\sigma$. Since $f(\mathbb{P}^2 \setminus U) \subset \mathbb{P}^2 \setminus U$ the support of σ is disjoint from U. If R is a positive closed $(1,1)$ current, then the mass of f_*R is the same as $d*$ the mass of R:

$$\int f_*R \wedge \omega = \int R \wedge f^*\omega$$

and $f^*\omega =$ (degree of f)$\omega + dd^c u$ where u is a $\mathscr{C}^\infty$ function on $\mathbb{P}^2$. Since R is closed, $\int R \wedge dd^c u = 0$.

We prove next that the support of any element S in $\mathscr{S}$ is contained in K^-.

Fix $x_0 \notin K^- \cup S_2$. All prehistories of x_0 cluster only on S_2. Define $\phi_n(x) := \max_{g \in f_i^{-n}(x)} \mathrm{dist}(S_2, y)$. Given $\epsilon > 0, \mathrm{dist}(x_0, S_2) > 2\epsilon$, there is an n_0 such that for $n \geq n_0$, $\phi_n(x_0) < \epsilon$. By continuity there is an $\epsilon > r > 0$ such that for $x \in B(x_0, r)$, $\phi_n(x) < \epsilon$. If Θ is a test form with support in $B(x_0, r)$, S has no mass on supp$(f^n)^*(\Theta)$ when $n \geq n_0$, hence $< (f^n)_*S, \Theta >=< S, (f^n)^*\Theta >= 0$ so $< S, \theta >= 0$ since $(f^n)_*S = d^n S$. It follows that supp$S \subset K^-$. Consequently $\mathscr{S}$ is compact.

The convexity is clear. □

Example 5.1 $f_0[z : w : t] = [z^d : w^d : t^d]$. Then $K^- = \{zwt = 0\}$.

The only positive closed $(1,1)$ currents supported by K^- are of the form $\alpha[z = 0] + \beta[w = 0] + \gamma[t = 0]$ ([Fe]). So the elements of $\mathscr{S}$ are of the form above, with $\alpha \geq 0, \beta \geq 0, \gamma \geq 0, \alpha + \beta + \gamma = 1$. We will need some estimates for the mass of the push forward of a current.

Proposition 5.3 *Let τ be a positive closed $(1,1)$ current in an open set U of $\mathbb{P}^2$. Let $\chi \in \mathscr{C}_0^\infty(U)$. Then there is a constant A (depending on χ) and a constant C depending on f such that*

$$\|\frac{(f^n)_*\chi\tau}{d^n}\| \leq \|\chi\tau\| + AC\|\tau\|.$$

Moreover $\|\frac{f_*^n\chi\tau}{d^n}\| \to \int \chi\tau\wedge T$. If τ is a relatively compact region in a Riemann surface R and $T_{|R}$ has no mass on $\partial\tau$, then $\|\frac{f_*^n\tau}{d^n}\| \to \int \tau \wedge T$.

Proof. Let F be a pull back of f to $\mathbb{C}^3$. There is a constant C such that

$$\log|z| - C \leq \frac{1}{d^n}\log|F^n| \leq \log|z| + C.$$

So

$$\frac{(f^n)^*\omega}{d^n} = \omega + dd^c u_n$$

with $-C \leq u_n \leq C$. Hence

$$\begin{aligned} < \frac{f_*^n(\chi\tau)}{d^n}, \omega > &= < \chi\tau, \frac{(f^n)^*\omega}{d^n} > \\ &= < \chi\tau, \omega > + < \chi\tau, dd^c u_n > \\ &= < \chi\tau, \omega > + < dd^c\chi\tau, u_n > . \end{aligned}$$

The estimate follows.

The second statment follows from

$$\begin{aligned} \|\frac{f_*^n(\chi\tau)}{d^n}\| &= < \frac{f_*^n(\chi\tau)}{d^n}, \omega > \\ &= < \chi\tau, \frac{(f^n)^*\omega}{d^n} > \\ &= < \chi\tau, dd^c G_n > \\ &= < dd^c(\chi\tau), G_n > \end{aligned}$$

and since χ has compact support, $dd^c(\chi\tau)$ has measure coefficients. Hence $\|\frac{f_*^n(\chi\tau)}{d^n}\| \to < dd^c(\chi\tau), G > = < \chi\tau, dd^c G >$. Here G_n, G are local potentials. The last statement follows easily. $\square$

Proposition 5.4 *Let $f \in \mathcal{H}_d$, $d \geq 2$, and let R be a positive closed $(1,1)$ current in an open set $U \subset \mathbb{P}^2$. Let $\psi \in \mathcal{C}_0^\infty(U)$, $\psi \geq 0$. Define $R_n := \frac{(f^n)_*(\psi R)}{d^n}$. The sequence $R_n \to 0$ if and only if $c := \int \psi R \wedge T = 0$. If $c \neq 0$, all subsequences have limit points and all the limit points are nonzero positive closed currents of mass c. If $U \cap S_\mu = \emptyset$, then the limits are supported on K^-. Moreover ∂R_n and $\partial\overline{\partial} R_n$ converge to zero in the strong sense of measures.*

Proof. We show first that $\|R_n\|$ converge to $< \psi R, T >=: c$. By definition, $\|R_n\| =< R_n, \omega >=< \psi R, \frac{(f^n)^*\omega}{d^n} >$. We know from Proposition 5.3 that $\|R_n\|$ will be bounded. We can assume that U is contained in a canonical coordinate chart. So $\frac{(f^n)^*\omega}{d^n} = dd^c u_n$ where u_n are smooth plurisubharmonic functions converging uniformly on compact sets to a function u, satisfying $dd^c u = T$ ([FS3]). We have

$$\begin{aligned}
< \psi R, \frac{(f^n)^*\omega}{d^n} > &= < \psi R, dd^c u_n > \\
&= < dd^c \psi \wedge R, u_n > \\
&\to < dd^c \psi \wedge R, u > \\
&= < \psi R, dd^c u > \\
&= < \psi R, T > .
\end{aligned}$$

Hence any weak limit of $\{R_n\}$ has mass c. In particular, $R_n \to 0$ if and only if $c = 0$.

Assume $c \neq 0$.

Let σ be a cluster point of $\{R_n\}$, $\sigma = \lim_{k\to\infty} R_{n_k}$. We prove next that σ is closed. It suffices to show that $\partial\sigma = 0$. Let ϕ be a $(0,1)$ test form and let $\chi \in \mathcal{C}_0^\infty(U)$ be a nonnegative function, $\chi \equiv 1$ on $\mathrm{supp}(\psi)$.

Then

$$\begin{aligned}
< \partial\sigma, \phi > &:= < \sigma, \partial\phi > \\
&= \lim_{k\to\infty} < R_{n_k}, \partial\phi > \\
&= \lim_{k\to\infty} < \partial R_{n_k}, \phi >
\end{aligned}$$

Using that ∂ commutes with f^* and Schwarz' inequality, as for Hénon maps ([BS2]), we get

$$\begin{aligned}
\left|\int \partial R_n \wedge \phi\right| &= \left|\int R_n \wedge \partial\phi\right| \\
&= \left|\int \frac{f_*^n(\psi R)}{d^n} \wedge \partial\phi\right| \\
&= \left|\int \psi R \wedge \frac{(f^n)^*(\partial\phi)}{d^n}\right|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{d^n} \left| \int \psi R \wedge \partial (f^n)^* \phi \right| \\
&= \frac{1}{d^n} \left| \int \partial(\psi R) \wedge (f^n)^* \phi \right| \\
&= \frac{1}{d^n} \left| \int \partial \psi \wedge R \wedge (f^n)^* \phi \right| \\
&\leq \frac{1}{d^n} \left| \int R \wedge \partial \psi \wedge i \overline{\partial} \psi \right|^{1/2} \\
&* \left| \int_{\mathrm{supp}(\psi)} \chi R \wedge (f^n)^* \phi \wedge \overline{(f^n)^* \phi} \right|^{1/2} \\
&= \frac{1}{d^{n/2}} \left| \int R \wedge \partial \psi \wedge i \overline{\partial} \psi \right|^{1/2} \left| \int \frac{(f^n)_* (\chi R)}{d^n} \wedge \phi \wedge \overline{\phi} \right|^{1/2}
\end{aligned}
$$

We have seen that the mass of $\frac{(f^n)_*(\chi R)}{d^n}$ is uniformly bounded, (Proposition 5.3). Hence we have shown that $\partial \sigma = 0$. Notice that the above inequality holds uniformly in ϕ. Hence we also have shown that $\|\partial R_n\| = O(\frac{1}{d^{n/2}})$, hence goes to zero when $n \to \infty$.

If θ is a test function, $< \partial \overline{\partial} R_n, \theta >= \frac{1}{d^n} < \partial \overline{\partial} \psi \wedge R, \theta(f^n) >$.
Hence $\|\partial \overline{\partial} R_n\| = O(\frac{1}{d^n}) \to 0$ as $n \to \infty$.

Assume $U \cap S_\mu = \emptyset$. If $x_0 \notin K^-$ and θ is supported near x_0, support$(f^n)^* \theta$ is arbitrarily close to S_μ, so $< (f^n)_*(\psi R), \theta >= 0$, for large n and if σ is any limit point of R_n, we have $< \sigma, \theta >= 0$. □

Remark 5.5 If $\psi R \wedge T$ gives no mass to S_μ, then cluster points of $R_n := \frac{(f^n)_*(\psi R)}{d^n}$ are supported on K^-.

Proof. Let $\epsilon > 0$. Then there exists $\psi_1 \in \mathscr{C}^\infty(\mathbb{P}^2)$, $\psi_1 \geq 0$, $\psi_1 \equiv 1$ on a neighborhood of S_μ so that $< \psi_1 \psi R, T >< \epsilon$. Using Proposition 5.4 it follows that any weak limit of $\{\frac{f^n_*(\psi_1 \psi R)}{d^n}\}$ has mass $< \epsilon$.

On the other hand, by Proposition 5.4, any weak limit of $\{\frac{f^n_*((1-\psi_1)\psi R)}{d^n}\}$ is supported on K^-.

Hence any weak limit of $\frac{f^n_*(\psi R)}{d^n}$ has no mass on $\mathbb{P}^2 \setminus K^-$. □

Corollary 5.6 *Let R be a positive closed $(1,1)$ current in an open set U of $\mathbb{P}^2$. Let $\psi \in \mathscr{C}_0^\infty(U)$, $\psi \geq 0$. Assume $\{\sigma_i = \frac{(f^{n_i})_*(\psi R)}{d^{n_i}}\}$ converges to S. Then if $\theta \in \mathscr{C}_0^\infty(V)$ is a test function and u is a continuous plurisubharmonic function on V, some open set, $< \theta dd^c u, \sigma_i > \to < \theta dd^c u, S >$.*

Proof.

$$
\begin{aligned}
< \theta dd^c u, \sigma_i > &= < u, dd^c(\theta \sigma_i) > \\
&= < u, dd^c \theta \wedge \sigma_i > + < u, \theta dd^c \sigma_i > +2 < u, d^c \theta \wedge d \sigma_i >
\end{aligned}
$$

so

$$\begin{aligned} \lim < \theta dd^c u, \sigma_i > &= < u, dd^c\theta \wedge \sigma > \\ &= < \theta dd^c u, \sigma > \qquad \square \end{aligned}$$

Corollary 5.7 *Assume $f \in \mathscr{H}_d$ is $s-$ hyperbolic. Let S_1^1 be a minimal set. There is a positive closed $(1,1)$ current σ supported on $W^u(S_1^1) \cup S_0$, $\|\sigma\| = 1$, $f_*\sigma = d \cdot \sigma$. In fact σ can be taken to be a multiple of a cluster point of the sequence $\sigma_N := \frac{1}{N}\sum_0^{N-1} \frac{f_*^n[\psi D]}{d^n}$ where D is any local unstable disc centered at x for some $\tilde{x} \in \tilde{S}_1^1$ and $\psi \geq 0$ is a test function vanishing in a neighborhood of ∂D, $\psi(x) > 0$,*

Proof. We can apply Proposition 5.5 to the current $R = [D]$. Let G be a continuous local potential for T, i.e. $dd^c G = T$. Then $< T, \psi[D] >= \int_D \psi dd^c G$. This integral is not zero because if G were harmonic on D, then by ([FS2]), $f^n_{|D}$ will be normal, but D is an unstable disc and this is impossible. Since $W^u(S_1^1) \cup S_0$ is closed, see Theorem 4.4, we get a current σ with the required properties.

5.2 Laminar structure on J for $s-$hyperbolic maps

We want to show that locally on J the current T is an integral of currents of integration on stable varieties (a laminar current). For that purpose we first show that T admits a transverse measure on the stable set of a basic set S_1^j, i.e. on $W^s(S_1^j)$.

We introduce some notation and recall some notions.

Let $f \in \mathscr{H}_d$, $d \geq 2$, be an $s-$ hyperbolic map and S_1^j a basic set of unstable dimension 1. Let $B \subset S_1^j$ be a flow box centered at a point p of a minimal basic set S_1^j. It can be described as follows:

We can assume that B is relatively open in $W^s(S_1^j)$ and is homeomorphic to $E * \Delta_2$ where E is contained in a holomorphic disc Δ_1 and Δ_2 is a holomorphic disc. There is a homeomorphism $\pi = (\pi_1, \pi_2)$, $\pi : B \to E * \Delta_2$, such that for $x \in E \subset \Delta_1$, $\pi_1^{-1}(x)$ is a local stable manifold of a point in S_1^j which is a holomorphic graph over Δ_2. We can assume that π_2 is the restriction to B of a holomorphic projection and hence is defined on a neighborhood of B, but π_1 is just continuous, we will identify B and $E * \Delta_2$.

Given two points $\zeta, \zeta' \in \Delta_2$, there is a natural map $\chi_{\zeta',\zeta} : \pi_2^{-1}(\zeta) \to \pi_2^{-1}(\zeta')$. The point $\chi_{\zeta',\zeta}(y)$ is determined by the relation $\pi_1(\chi_{\zeta',\zeta}(y)) = \pi_1(y)$.

Assume that for each $\zeta \in \Delta_2$, μ_ζ is a Radon measure on $\pi_2^{-1}(\zeta)$. Such a family is said to be a transverse measure if and only if $\mu_{\zeta'} = (\chi_{\zeta',\zeta})_*\mu_\zeta$ i.e. $\mu_{\zeta'}$ is equal to the direct image of μ_ζ under the map $\chi_{\zeta',\zeta}$.

Given a positive closed $(1,1)$ current τ on a neighborhood of a flow box, we define for almost every $\zeta \in \Delta_2$ the slice of τ for π_2 on $\pi_2^{-1}(\zeta)$, i.e. $< \tau, \pi_2, \zeta >$ as follows. If u is a potential for τ, then $< \tau, \pi_2, \zeta >= dd^c(u_{|\pi_2^{-1}(\zeta)})$. It is a positive measure. The slice is defined for those ζ for which $u_{|\pi_2^{-1}(\zeta)}$ is not identically $-\infty$. If u is continuous, which is the case for the current T, then the slice is always defined. The slicing formula ([Fe]) in this context asserts that for a test function θ

$$< \tau \wedge id\zeta \wedge d\overline{\zeta}, \theta > = \int < \theta, < \tau, \pi_2, \zeta >> d\lambda(\zeta)$$

where λ denotes the Lebesgue measure on Δ_2.

Theorem 5.8 *Let $f \in \mathscr{H}_d,\ d \geq 2$, be an $s-$hyperbolic map on $\mathbb{P}^2$. If B is a flow box around a point in a minimal S_1^j as described above, then $\mu_\zeta := < T, \pi_2, \zeta >$ is a transverse measure on B and*

$$(1) \quad T_{|B} = \int [V_y] d\mu_{\zeta_0}(y)$$

where $[V_y]$ denotes the current of integration on the local stable manifold $\pi_1^{-1}(y) = [V_y]$ and ζ_0 is any point in Δ_2.

Proof. We prove that $\mu_\zeta := < T, \pi_2, \zeta >$ is a transverse measure.

Let D, D' be two relatively compact open sets in transverse discs, $\Delta,\ \Delta'$, and corresponding to each other under the map $\chi_{\zeta',\ \zeta}$ i.e. $D \subset \pi_2^{-1}(\zeta),\ D' \subset \pi_2^{-1}(\zeta')$ and $D' \cap B = \chi_{\zeta',\zeta}(D) \cap B$. The map $\chi_{\zeta',\zeta}$ is not defined on the whole disc, only on the intersection with $W^s(S_1^j)$. We want to show that

$$\mu_\zeta(D) = \mu_{\zeta'}(D').$$

The idea is to use the fact that if one restricts to $W^s(S_1^j)$, $f^n(D) =: D^n$ and $f^n(D') =: D'_n$ are very close though they have been expanded in the unstable direction. We can write $D^n = \cup D_i^n$ where $f_i^{-n} : D_i^n \to D_i \subset D$ is well defined and the D_i^n are disjoint.

We know that if G is the Green function for $\pi^* T$ then $G(F^n) = d^n G$. We identify D with a section over D in $\mathbb{C}^3$. We lift D^n by applying F^n to the lift of D and identifying with f^n. So if f_i^{-n} is defined we get $G(f_i^{-n}) = G/d^n$. It follows, using the change of variables formula, that

$$\frac{1}{d^n} \int_{D_i^n} dd^c G = \int_{D_i^n} (f_i^{-n})^* (dd^c G) = \int_{D_i} dd^c G.$$

Summing over i, we get

$$\frac{1}{d^n} \int_{f^n(D)} (dd^c G) = \int_D dd^c G.$$

Let $\mu_\zeta := < T, \pi_2, \zeta >$.

Let $\Sigma_n := f^n(\Delta), \Sigma'_n := f^n(\Delta')$. Let χ be a test function on Σ_n with value one on D_n and supported on a small neighborhood of $\overline{D_n} \cap W^s(S_1^j)$ in Σ_n. Since (D_n, Σ_n) and (D'_n, Σ'_n) are close and we have uniform expansion, we can assume that there is a function χ' close to χ in $\mathscr{C}^2$ norm with the same properties with respect to (D'_n, Σ'_n). Because

$$
\begin{aligned}
\int_{D_n} dd^c G &\leq \int \chi dd^c G \\
&= \int dd^c \chi G \\
&= \int dd^c \chi' G + \epsilon \mathrm{vol}(D_n') \\
&= \int \chi' dd^c G + \epsilon \mathrm{vol}(D_n') \\
&\leq \int_{D_n'} dd^c G + 2\epsilon \mathrm{vol}(D_n')
\end{aligned}
$$

if χ has support close enough to D_n. Hence $d^n \mu_\zeta(D_n') = d^n \mu_\zeta(D_n') + 2\epsilon \mathrm{vol}(D_n')$. So since $\mathrm{vol}(D_n') = O(d^n)$, (see Proposition 5.4) we get $\mu_\zeta(D) = \mu_\zeta(D') + o(1)$. But n can be chosen arbitrarily large, hence $\mu_\zeta(D) = \mu_\zeta(D')$.
i.e. $\{\mu_\zeta\}$ is a transverse measure.

We prove now that T is laminar on $W^s(S_1^j)$ near S_1^j, S_1^j minimal, i.e. that the representation formula given by the theorem is valid.

The slicing formula for T in B gives that for a test function θ

$$
< T \wedge id\zeta \wedge d\overline{\zeta}, \theta >= \int < \theta, < \tau, \pi_2, \zeta >> d\lambda(\zeta) = \int (\int \theta d\mu_\zeta(y)) d\lambda(\zeta).
$$

Since the measure (μ_ζ) is transverse, we get that $\mu_0 := (\pi_1)_* < T, \pi_2, \zeta >$ is independent of ζ. If we identify $T \wedge id\zeta \wedge d\overline{\zeta}$ with a measure, Fubini's theorem gives that

$$
(1) < \pi_*(T \wedge id\zeta \wedge d\overline{\zeta}), \phi >= \int (\int \phi d\lambda(\zeta)) d\mu_0(y).
$$

Hence $< T, \theta id\zeta \wedge d\overline{\zeta} >= \int_E (\int_{\pi_1^{-1}(y)} \theta d\lambda(\zeta)) d\mu_0(y)$. So $< T, \theta id\zeta \wedge d\overline{\zeta} >= \int < \theta id\zeta \wedge d\overline{\zeta}, [V_y] > d\mu_0(y)$. Perturbing the projection π_2 we show that this identity holds as well for all $(1,1)$ test forms. Hence $T = \int [V_y] d\mu_0(y)$. □

Remark 5.9 We use the same hypothesis as in Theorem 5.9. Let U be a relatively open set in $W^s(S_1^j)$ (and hence in J) on which some f^n is a homeomorphism (where S_1^j is minimal), so that $f^n(U) = B$, a flow box, centered at a point of S_1^j. Furthermore we assume that U and Δ are disjoint from $\cup_{j=0}^n f^{-j}(C)$, where C is the critical set. Let Δ be a disc cutting U such that $f^n(\Delta \cap U)$ is a level set of π_2 in B, cutting all across B and f^n is $1-1$ on Δ. a. We have

$$
d\mu_\Delta := dd^c(G_{|\Delta}) = \frac{1}{d^n} dd^c((G \circ f^n)_{|\Delta}) = \frac{(f^n)^*}{d^n} [dd^c(G_{|f^n(\Delta)})]_{|\Delta}.
$$

So using the functional equation for T

$$\begin{aligned} T_{|U} &= ((f^n)^* T/d^n)_{|U} \\ &= ((f^n)^* T_{|B}/d^n)_{|U} \\ &= (\frac{(f^n)^*}{d^n}\{\int [V_y] d\mu_0(y)\})_{|U} \\ &= \int (f^n)^*[V_y] dd^c G_{|\Delta} \\ &= \int [W_y] d\mu_\Delta(y). \end{aligned}$$

Here W_y denotes the preimage of V_y under f^n.

5.3 Convergence results for currents on J^-

Theorem 5.10 *Let $f \in \mathscr{H}_d$, $d \geq 2$, be an s−hyperbolic map on $\mathbb{P}^2$. Assume that S_1^j is minimal. There is a positive closed $(1,1)$ current σ supported on $\overline{W^u(S_1^j)}$ satisfying $f_*\sigma = d \cdot \sigma$, $\|\sigma\| = 1$, and a neighborhood U of S_1^j in $\mathbb{P}^2$ with the following property: If $D \subset\subset R$ is a relatively open subset of a Riemann surface R in U, R is transverse to the stable direction at every point in $R \cap J$, and $T_{|R}$ has no mass on ∂D, then $(\frac{f_*^n([D])}{d^n})$ converges to $(\int D \wedge T)\sigma$.*

Proof. We let σ be given as in Corollary 5.7. In particular the proof will show that σ is unique. Because of Proposition 5.4, it suffices to consider the case when R is a local graph over an unstable manifold, transverse to the stable direction on $R \cap J$. Observe that if R does not intersect J, then $\frac{f_*^n([D])}{d^n} \to 0$. Hence we can assume that R is a local graph over the unstable direction.

We want to prove that if $D \subset\subset R$ and $D' \subset\subset R'$ are two such regions with $\int D \wedge T = \int D' \wedge T$, then the limits of

$$\frac{f_*^n(D)}{d^n} \text{ and } \frac{f_*^n(D')}{d^n}$$

exist and are the same.

Assume first that D and D' are two such regions cutting across the flow box B and $D \cap J$, $D' \cap J \subset B$. Denote $\sigma_n := \frac{f_*^n(D)}{d^n}$, $\sigma_n' := \frac{f_*^n(D')}{d^n}$.

We want to show that $\sigma_n - \sigma_n' \to 0$ weakly. By the contraction in the stable direction, the sets $f^n(D) \cap W^s(S_j^1)$ and $f^n(D') \cap W^s(S_1^j)$ get closer and closer. The areas of $f^n(D)$ and $f^n(D')$ are bounded above by Cd^n and bounded below by $c_1 d^n$, $(c_1 > 0)$ by Proposition 5.3. Fix $0 < \epsilon << 1$. Let $K \subset\subset D$, $K' \subset\subset D'$ be smaller regions so that $\int_{D\setminus K} T < \epsilon$, $\int_{D'\setminus K'} T < \epsilon$ with $T_{|R}$ having no mass on ∂K, and similarly for $\partial K'$. There is a fixed $r > 0$ such that for n sufficiently large and for every $x \in f^n(K) \cap J$ there is a disc of radius r, $\Delta(x,r) \subset f^n(D)$ which is a graph over a similar disc $\Delta(x',r) \subset f^n(D')$.

Let ϕ be a test form in a small neighborhood of S_1^1. We can assume that the support of ϕ has diameter less than $\frac{r}{10}$. Then $f^n(D) \supset \cup\Delta_j \supset f^n(K \cap J)$ and

$f^n(D') \supset \cup \Delta'_j \supset f^n(K' \cap J)$ with Δ_j and Δ'_j such that Δ_j is a graph over Δ'_j close to Δ'_j.

We then get

$$< \frac{f^n_*(D)}{d^n}, \phi >= \frac{1}{d^n} \sum < \Delta_j, \phi > + < A_n, \phi >$$

where $| < A_n, \phi > | \leq \frac{\|f^n_*(D \setminus K)\|}{d^n} \leq 2\epsilon$ for large n.

We have a similar formula for $\frac{(f^n)_*(D')}{d^n}$ and $\Delta_j,\ \Delta'_j$ are uniformly close. It follows that $\sigma_n - \sigma'_n \to 0$ in a neighborhood of S^j_1.

Since this is true for all subsequences, using that the limit set of $\{\sigma_n\}$ respectively $\{\sigma'_n\}$ is forward invariant and that positive closed currents have no mass at isolated points, we get that $\sigma_n - \sigma'_n \to 0$ on $\mathbb{P}^2$.

Hence

$$\begin{aligned}
\int D \wedge T &= \lim_n \int D \wedge \frac{(f^n)^* \omega}{d^n} \\
&= \lim_n \int \frac{f^n_* D}{d^n} \wedge \omega \\
&= \lim_n \int \sigma_n \wedge \omega \\
&= \lim_n \int \sigma'_n \wedge \omega \\
&= \int D' \wedge T.
\end{aligned}$$

By varying the flow boxes and letting D' be a slice of π_2, we see that $T_{|R}$ agrees with the push forward of the transverse measure to R.

Next we consider the case where $D \cap J$ and $D' \cap J$ are contained respectively in two flow boxes B and B', but not necessarily cutting all across. We can cover a neighborhood V of S^j_1 by a finite union of not necessarily disjoint small flow boxes.

We choose the flow boxes B by using small discs δ in local unstable manifolds for which $\partial\delta$ has 0 transverse measure, and then the base of B is $\delta \cap J$.

Fix a $0 < r << 1$ but r much larger than the diameter of the flow boxes. We will define a refined collection $\mathscr{C}$ of flow boxes: If B' is a flow box closer to B than r, there is a natural projection of B' to the local unstable manifold containing δ. We can let $B_1,\ B_2$ be the two flow boxes in B obtained by taking as base the projection of B', $\pi(B') \cap \delta$ and $\delta \setminus \overline{\pi(B')}$ respectively.

We allow all flow boxes $\subset B$ which are intersections of such subsets. Then $\mathscr{C}$ contains finitely many flow boxes.

By having the original flow boxes B extended in the stable direction, we may assume that if $x \in f^n(R) \cap B$ for some flow box B, then $f^n(R)$ contains a neighborhood of x cutting all across <u>some</u> flow box (x might be too near the "top" of B for $f^n(R)$ to cut across B).

Fix $0 < \epsilon << 1$. Let A_ϵ be a neighborhood of ∂D in D of width ϵ. Since the mass of $T_{|R}$ on $\overline{A_\epsilon}$ can be chosen arbitrarily small, it follows that the mass of $\frac{f_*^n(A_\epsilon)}{d^n}$ is also arbitrarily small (see Proposition 5.4). Fix n_0 such that if $f^{n_0+\ell}(D \setminus A_\epsilon)$ hits a flow box B_i in x, then $f^{n_0+\ell}(D)$ contains a disc through x cutting all across some B_j.

By transitivity of f on S_1^j, we can choose N such that $f^{n+N}(D \setminus A_\epsilon)$ hits all the flow boxes for any $n \geq 0$. In the discs D, D' we have a pairing between regions which end up as a union of the same number of graphs in the same flow box, we know by the first step that these pieces give the same limit, we cover in this way in D and D' a region $R = \cup D_i$, $R' = \cup D_i'$ such that $\int R \wedge T = \int R' \wedge T \geq \frac{\alpha}{d^{n_0+N}}$ where α is a fixed constant. Indeed $(\int R \wedge T) d^{n_0+N} = \int R \wedge (f^{n_0+N})^* T = \int f_*^{n_0+N} R \wedge T \geq \alpha > 0$ because $f_*^{n_0+N} R$ cuts all the flow boxes.

We continue the process by considering the image of D, D' under the map $f^{2(n_0+N)}$, we have new regions in $D \setminus R$, $D' \setminus R'$ which end up as graphs in the same flow box. We pair them extending the regions that are already paired.

In a finite number of steps we exhaust arbitrarily large fractions of $\int T \wedge (D \setminus A_\epsilon)$ or of $\int T \wedge (D' \setminus A'_\epsilon)$. Then the remaining regions B_ϵ, B'_ϵ satisfy $\int T \wedge B_\epsilon = \int T \wedge B'_\epsilon = o(1)$ (going to zero when $\epsilon \to 0$). So we get that for any test form ϕ

$$| < \sigma_n, \phi > - < \sigma'_n, \phi > | = o(1).$$

Since ϵ is arbitrary $\sigma_n - \sigma'_n \to 0$.

Instead of two discs we consider the family of currents $\frac{f_*^m[D]}{d^m}$. We want to show that the sequence is convergent. We consider an annulus A_ϵ in D as before and observe that if $f^m(D \setminus A_\epsilon)$ intersects a flow box B and $m \geq n_0$ then $f^m(D) \cap B$ continues across some B'. We can apply the above procedure to the sets $d^{m_1-m_2} f_*^{m_2}(D)$, (i.e. $d^{m_1-m_2}$ copies of $f_*^{m_2}(D)$) and $f_*^{m_1}(D)$, $m_1 > m_2 > \ell N + n_0$ where ℓ is the number of pairings needed to exhaust most of $\int D \wedge T$. After dividing the sequences by $\frac{1}{d^{m_1}}$ we get that $\frac{f^{m_1}(D)}{d^{m_1}}$ and $\frac{f^{m_2}(D)}{d^{m_2}}$ are close in the sense of currents.

It follows that $\frac{f_*^n(D)}{d^n}$ has as limit σ_c which must necessarily be $(\int D \wedge T)\sigma$. □

Corollary 5.11 *If θ is a test function and D is as in Theorem 5.11, the sequence*

$$\frac{f_*^n(\theta[D])}{d^n} \to c\sigma, \ c = \int \theta[D] \wedge T.$$

Proof. We can replace $\theta[D]$ by a finite sum of the type $\sum c_j[D_j]$, c_j constants approximating θ on D_j. Theorem 5.10 applies because we have an a priori estimate of the errors under push forward, Proposition 5.4. □

5.4 Convergence results for currents on J

Theorem 5.12 *Let S_1^j be a minimal basic set for an $s-$ hyperbolic map $f \in \mathscr{H}_d$, $d \geq 2$. Let $D \subset\subset R \subset \mathbb{P}^2$ be a region in a Riemann surface R, D*

intersecting S_1^j at a point a. Then $[D]$ is a closed current in some open set U. We assume U is contained in a coordinate chart. Let $\rho \geq 0$ be a test function supported in U. Assume $D \cap supp(\rho)$ is not contained in $\overline{W^u(S_1^j)}$. We suppose that the tangent space of D is close to the stable direction at every point of $D \cap J$. Let u be a local potential for a current σ (as in Theorem 5.11) associated to S_1^j in U. Then the sequence of currents $\tau_n := \frac{(f^n)^*(\rho[D])}{d^n}$ converges to cT on the open set $\Omega := \mathbb{P}^2 \setminus (S_2 \cup_{j \neq 1} W^s(S_1^j))$ where $c = \int_D \rho dd^c u$.

Proof. We first prove the result for a $(1,1)$ current τ which is positive closed in an open set V and such that $\tau = \int [D_\alpha] d\nu(\alpha)$ where D_α are discs and ν a probability measure, and the discs D_α are graphs over a disc D_0, $(w = h_\alpha(z))$. We will also assume that the potential v, $v(z,w) = \int \log|w - h_\alpha(z)| d\nu(\alpha)$, such that $dd^c v = \tau$, is continuous in V. We assume that τ extends in this way to a neighborhood of $\overline{V}$.

Let Δ be a disc in Ω. Let θ be a nonnegative test function supported in Ω, $\theta = 0$ on a neighborhood of $\partial\Delta$. Define $\sigma_n := \frac{f_*^n(\theta[\Delta])}{d^n}$. We assume that for some large n, $f^n(\Delta)$ satisfies the condition of Theorem 5.11. The condition on θ implies that (σ_n) converges to $c'\sigma$ where $c' = \int T \wedge \theta[\Delta]$ (see Corollary 5.12).

We then have $< \frac{(f^n)^*(\rho\tau)}{d^n}, \theta[\Delta] >=< \rho\tau, \sigma_n >$, which converges to $c' < \rho\tau, \sigma >$, because the potential of τ is continuous, see Corollary 5.7. We get, if $\tau_n := \frac{(f^n)^*(\rho\tau)}{d^n}$, that $< \tau_n, \theta[\Delta] > \to < T, \theta[\Delta] >< \rho\tau, \sigma >$.

Observe that the convergence is uniform with respect to Δ as soon as $f^{n_0}(\Delta)$ satisfies the conditions of Theorem 5.11.

We next show that $\tau_n \to < \rho\tau, \sigma > T$. First we show that $(\|\tau_n\|)$ is bounded. We observe that $\|\frac{f_*^n\omega}{d^n}\| = 1$. Indeed

$$\begin{aligned} \|\frac{f_*^n\omega}{d^n}\| &= < \frac{f_*^n\omega}{d^n}, \omega > \\ &= < \omega, \frac{(f^n)^*\omega}{d^n} >= 1 \end{aligned}$$

If S is the limit of some subsequence $(\frac{f_*^{n_i}\omega}{d^{n_i}})$, we have

$$\begin{aligned} \|\tau_{n_i}\| &= < \frac{(f^{n_i})^*\rho\tau}{d^{n_i}}, \omega > \\ &= < \rho\tau, \frac{(f^{n_i})_*\omega}{d^{n_i}} > \\ &= < \rho\tau, S > . \end{aligned}$$

The last equality is a consequence of Corollary 5.6, since the potential of τ is continuous. Hence $(\|\tau_n\|)$ is bounded.

We can fatten $[\Delta]$ to a current $[\tilde{\Delta}]$ such that $\theta[\tilde{\Delta}]$ has continuous coefficients. If $[\Delta] = 1_{|z_1|<1}\delta_0 dz_2 \wedge d\overline{z_2}$ in local coordinates; we can take $[\tilde{\Delta}] = 1_{|z_1|<1}\chi(z_2) dz_2 \wedge d\overline{z_2}$ where χ is a positive test function.

Let $\tilde{\tau}$ be a cluster point of the sequence τ_n in $\mathbb{P}^2$. We get from the previous discussion that $\tilde{\tau}$ and $< \rho\tau, \sigma > T$ coincide on forms of type $\theta[\tilde{\Delta}]$. We do as observed above. The vector space generated by such forms is dense in the space of $(1,1)$ forms with continuous coefficients near S_1^1. Hence $\tilde{\tau}$ coincide with $< \rho\tau, \sigma > T$ near S_1^1. The cluster values of the sequence $\{\tau_n\}$ are invariant under f^*, since all of them coincide near S_1^1, we have that $\tilde{\tau} =< \rho\tau, \sigma > T$ in Ω.

When $\tau = [D]$ we cannot say that $< \rho\tau, \sigma_n > \to c' < \rho\tau, \sigma >$ because we only have $\sigma_n \to \sigma$ weakly, and the potential of τ is not continuous. We fatten D into $\tilde{\tau} = \int [D_\alpha] d\nu(\alpha)$ in order that $\tilde{\tau}$ has a local continuous potential. Under $(f^n)^*$ each disc D_α is expanded along the stable manifold and they get closer and closer near S_1^1. Let $\int_{D_\alpha} \rho dd^c u =: c(\alpha)$. We can assume the $c(\alpha)'s$ are arbitrarily close to c. So if $\rho\tilde{\tau}_n \to \tilde{c}T$ then because the (D_α) have the same behavior $\frac{(f^n)^*(\rho D_\alpha)}{d^n} \to c(\alpha)T$. The (D_α) get closer, the limits are the same up to multiplicative constants and in particular $\frac{(f^n)^*(\rho[D])}{d^n} \to cT$. □

Corollary 5.13 *For $a \in S_1^1$, $W^s(a)$ is dense in $W^s(S_1^1)$.*

Proof. Let D be a disc centered at a and contained in $W^s(a)$. Let u be a local potential of σ, u is pluriharmonic out of $W^u(S_1^1) \cup S_0$. The disc D is not completely contained in $W^u(S_1^1)$ hence u is not identically $-\infty$ on D. Let $\tau_n = \frac{(f^n)^*(D\rho)}{d^n}$. We know that $\tau_n \to cT$, $c \neq 0$ in $\Omega = \mathbb{P}^2 \setminus (S_2 \cup_{j\neq 1} W^s(S_1^j))$, since $W^s(S_1^j)$ $j \neq 1$ is disjoint from $W^s(S_1^1)$ we get the result. □

5.5 Structure of the invariant current σ on $W^u(S_1^1)$.

The structure of the current σ which is supported on $W^u(S_1^1) \cup S_0$, is not necessarily laminar. Indeed the unstable manifolds might intersect and hence do not give a foliation of $W^u(S_1^1)$. We show here that σ is locally an integral of analytic graphs. We first describe ”flow boxes” on $W^u(S_1^1)$.

Proposition 5.14 *Let $x^0 \in S_1^1$, a minimal hyperbolic set for an $s-$ hyperbolic map f. There are neighborhoods $\Delta = \Delta_1 * \Delta_2 \subset \Delta_1 * \Delta_2' = \Delta'$ in local coordinates of x^0 such that the local unstable manifolds intersecting Δ are graphs over the Δ_1 axis with values in Δ_2'.*

Proof. We first observe that the unstable direction and the stable direction never coincide at a point $x \in S_1^1$ since they span the tangent space at x. Since the splitting of the tangent bundle is continuous, we get a positive angle between the stable and unstable direction at x^0 and uniformly in x^0.

We can assume $\Delta = \Delta_1 * \Delta_2$ where Δ_1 is along the stable direction at x^0. Using the graph transform, and the construction of unstable manifolds we see that the local unstable manifolds are graphs over Δ_1. □

Let $A(\Delta_1, \Delta_2)$ denote the space of holomorphic maps from Δ_1 with values in Δ_2. We give this space the topology of uniform convergence. For $g \in A(\Delta_1, \Delta_2)$ we will denote by V_g the graph of g and $[V_g]$ the current of integration on V_g.

Theorem 5.15 *Let f be an s-hyperbolic map on $\mathbb{P}^2$. Let $x^0 \in S_1^1$, a minimal basic set. In the notation of Proposition 5.15 there is a "flow box" Δ' and a positive measure λ on $A(\Delta_1, \Delta_2')$ such that on Δ, $\sigma = \int [V_g] d\lambda(g)_{|\Delta}$, where V_g are the local unstable manifolds intersecting Δ.*

Proof. Let $\Delta = \Delta_1 * \Delta_2$ be the neighborhood containing x^0, introduced in Proposition 5.15. Let $\Delta'' = \Delta_1'' * \Delta_2'$ where $\Delta_1'' \subset\subset \Delta_1$. We define $C(\Delta_1, \Delta_2)$ as the space of currents R in $\Delta_1 * \Delta_2$ which can be represented as $R = \int [V_g] d\rho(g)$ where ρ is a finite positive measure on $A(\Delta_1, \Delta_2)$.

Let $g \in A(\Delta_1, \Delta_2')$. There is a uniform bound on the derivative of g in Δ_1''. It follows that there is a constant C such that if $R \in C(\Delta_1 * \Delta_2')$,

$$\frac{1}{C}\mathrm{Mass}(\rho) \leq \|R\chi_{\Delta''}\| \leq C\,\mathrm{Mass}(\rho). \quad (1)$$

We next observe that if R_n is a sequence of currents in $C(\Delta_1 * \Delta_2')$ which converges in the sense of currents to a current R in $\Delta_1 * \Delta_2''$ for some $\Delta_2" \supset \overline{\Delta_2'}$, then the restriction of R to Δ'' is in $C(\Delta_1'' * \Delta_2')$: From (1) we deduce that the corresponding measures ρ_n have bounded mass. We can assume $\rho_n \to \rho$, a positive measure. Since the problem is local, we can assume we are in $\mathbb{C}^2$ and that $\Delta_1 * \Delta_2$ is the unit polydisc. Let $u_n(z, w) = \int \log|w - g_\alpha(z)| d\rho_n(\alpha)$ be the potential of R_n in $\Delta_1 * \mathbb{C}$. For $z_0 \in \Delta_1$ let $\theta_{z_0} : A(\Delta_1, \Delta_2) \to \Delta_2$ be the evaluation at z_0, i.e. $\theta_{z_0}(g) = g(z_0)$. We want to show that u_n converges in L^1_{loc} to $u(z, w) = \int \log|w - g(z)| d\rho(g)$. The function $u_n(z_0, \cdot)$ is the potential for the measure $(\theta_{z_0})_*(\rho_n)$. If $\rho_n \to \rho$, $(\theta_{z_0})_*\rho_n \to (\theta_{z_0})_*\rho$, the support of $(\theta_{z_0})_*\rho_n$ are contained in a fixed compact. Hence $u_n(z_0, \cdot)$ converges to the potential of the measure $(\theta_{z_0})_*\rho$, which proves the claim.

As a consequence $R = \int [V_g] d\rho(g)$.

The current σ is obtained as a limit of $\frac{[f_*^n(D)]}{d^n}$ where D is a Riemann surface in the unstable direction. As in Theorem 5.11 we can consider an annulus A_ϵ near ∂D, the mass of $\frac{f_*^n(A_\epsilon)}{d^n}$ is small (Proposition 5.4)for large n and if $f^n(D \setminus A_\epsilon)$ hits $\Delta_1 * \Delta_2$ at p for large n, then there is a component of $f^n(D) \cap (\Delta_1 * \Delta_2')$ which is a graph over Δ_1 and contains p. It follows that $\frac{f_*^n(D)}{d^n}$ restricted to $\Delta_1 * \Delta_2'$ is approximable in the mass norm by currents in $C(\Delta_1, \Delta_2')$. Hence, by the above arguments, $\sigma = \int [V_g] d\lambda(g)$. □

6 Invariant measures on minimal invariant sets

Let $f \in \mathscr{H}_d$ be an $s-$ hyperbolic map. For any $S \in \mathscr{S}$ (see the definition before Theorem 5.3) we can define the measure $\nu_s := T \wedge S$. The wedge product is well defined since T has locally a continuous potential. It follows from ([FS4]) that ν_s is a probability measure.

Let σ be the current supported on $W^u(S_1^1) \cup S_0$ in Theorem 5.10.

Proposition 6.1 *Define $\nu = T \wedge \sigma$. Then ν is a forward invariant probability measure supported on S_1^1.*

Proof. We prove the invariance. Let ϕ be a test function. Then

$$
\begin{aligned}
< f_*\nu, \phi > &= < T \wedge \sigma, f^*\phi > \\
&= < \frac{f^*T}{d} \wedge \sigma, f^*\phi > \\
&= < \frac{\sigma}{d}, f^*(T\phi) > \\
&= \lim_n < \frac{\sigma}{d}, f^*((\frac{(f^n)^*\omega}{d^n})\phi) > \\
&= \lim < \frac{f_*\sigma}{d}, \frac{(f^n)^*\omega}{d^n}\phi >
\end{aligned}
$$

because the convergence of the local potentials of $\frac{(f^n)^*\omega}{d^n}$ is uniform in the above limit. So

$$
\begin{aligned}
< f_*\nu, \phi > &= < \sigma, T\phi > \\
&= < T \wedge \sigma, \phi > \\
&= < \nu, \phi > .
\end{aligned}
$$

We know that $\overline{W^u(S_1^1)} \cap J \subset S_1^1$. So the support of ν is contained in S_1^1. $\square$

Theorem 6.2 *The measure ν is mixing.*

Proof. Let ϕ and ψ be two positive test functions with small support in a neighborhood U of S_1^1. We want to show that

$$I_n := (\int \psi(f^n)\phi d\nu) \to (\int \psi d\nu)(\int \phi d\nu).$$

$$
\begin{aligned}
I_n &= \int \psi(f^n)\phi T \wedge \sigma \\
&= \int (f^n)^*\psi \frac{(f^n)^*T}{d^n} \wedge \phi\sigma \\
&= < \frac{(f^n)^*(\psi T)}{d^n}, \phi\sigma > \\
&= < \psi T, \frac{f_*^n(\phi\sigma)}{d^n} > .
\end{aligned}
$$

The current σ was obtained by pushing a disc and averaging. We have seen that the limit does not depend on the disk we start with and that the convergence is uniform for $(\frac{f_*^m(D)}{d^m})$ in U. It follows that $\sigma_n := \frac{f_*^n(\phi\sigma)}{d^n} \to c\sigma$ with $c = \int T \wedge \phi\sigma$. Let G be a potential for T in a chart containing supp ψ.

$I_n =< \psi T, \sigma_n >=< dd^c G, \psi\sigma_n >=< G, dd^c\psi\sigma_n >$

Using Proposition 5.5, since $dd^c\sigma_n \to 0,\ d\sigma_n \to 0$, in mass norm we get

$$\begin{aligned} \lim I_n &= < G, dd^c\psi\sigma > c \\ &= < T\psi, \sigma > c \\ &= < \nu, \psi >< \nu, \phi > \qquad \square \end{aligned}$$

Theorem 6.3 *Let f be an $s-$ hyperbolic map in $\mathscr{H}_d$ and let ν be the measure of Proposition 6.1. Then support $\nu = S_1^1$.*

Proof. We will cover S_1^1 by flow boxes $B_i = B_{j,\ell}$. We can assume that $B_{j,\ell} \cap J$ is a union of stable manifolds which are graphs over the Δ'_ℓ axis.

Let $D \subset R$ be a region cutting all across a flow box such that $D \wedge T \neq 0$, as in Theorem 5.11. Let A_ϵ be a neighborhood of ∂D of width ϵ. Given $0 < \epsilon << 1$ we choose n_0 large enough so that for $n \geq n_0$ if $f^n(D \setminus A_\epsilon)$ intersects B_i then the components of the intersection of $f^n(D)$ with B_i extend as graphs over the Δ axis (unstable direction) contained in $\tilde{B}_i$ (flow boxes slightly enlarged in the Δ'_ℓ direction).

Let M be the number of boxes B_i. There is an index $i(n)$ such that $\int \frac{f_*^n(D\setminus A_\epsilon)\chi_{B_i}}{d^n} \wedge T \geq \frac{c}{M}$ for all large n where c is a fixed strictly positive constant. Hence $f^n(D)$ cuts completely across $\tilde{B}_{i(n)}$ at least $c'd^n$ times. Since f is transitive and expanding, for some fixed m, $f^{n+m}(D)$ cuts across any B_j at least $c'd^n$ times. Hence $\int_{\tilde{B}_j} \frac{f_*^{n+m}(D)\wedge T}{d^{n+m}} \geq \frac{c''}{d^m}$.. Since m is fixed, we get $(\sigma \wedge T)(\tilde{B}_j) \geq \frac{c''}{d^m}$. We can take the boxes $(\tilde{B}_j)$ of arbitrarily small diameter. So supp(ν)= S_1^1. $\square$

We consider now the decomposition of the invariant measure ν.

Theorem 6.4 *Let B be a small neighborhood of a point $x \in S_1^1$. Then on B, $T = \int [V_y] d\mu_0(y)$, $\sigma = \int [V_g] d\lambda(g)$ and $\nu = T \wedge \sigma = \int ([V_y] \wedge [V_g]) d\mu_0(y) d\lambda(g)$, where the measures μ_0 and λ are as in Theorems 5.9 and 5.15 respectively.*

Proof. We have already proved the representation formulas for T (Theorem 5.9) and σ (Theorem 5.15). Since T has continuous potential, if $\sigma_i \to \sigma$ then $T \wedge \sigma_i \to T \wedge \sigma$, so using an approximation to the integral we get that

$$T \wedge \sigma = \int T \wedge [V_g] d\lambda(g).$$

If u is a potential for T in a neighborhood of B, then $T \wedge [V_g] = dd^c u_{|V_g}$. So

$$T \wedge \sigma = \int dd^c u_{|V_g} d\lambda(g).$$

We consider that $u = \lim v_n$ where v_n are the local potentials for $\sum c_n [V_{g_n}]$, c_n corresponding to an approximation of μ_0 by point masses. Then $dd^c u_{|V_g} = \int [V_y] \wedge [V_g] d\mu_0(y)$. This proves the formula for ν. $\square$

Corollary 6.5 *Assume S_1^1 is minimal. Let $W^u_{\tilde{x},R}$ be the local unstable manifold for the prehistory $\tilde{x}$ with $x_0 \in S_1^1$. Then $W^u_{\tilde{x}} = \cup_{n\geq 0} f^n(W^u_{\tilde{x},R})$ is dense in $W^u(S_1^1)$.*

Proof. Let $D = W^u_{\tilde{x},R}$. Since D is not contained in S^1_1 and $W^u(S^1_1) \cap J = S^1_1$ we have that $\int T \wedge D = c \neq 0$.

We know $\frac{f^n_*(D)}{d^n} \to c\sigma$ and support σ contains $W^u(S^1_1)$, Theorem 5.10. Hence $W^u_{\tilde{x}}$ is dense in $W^u(S^1_1)$. □

Theorem 6.6 *Assume that $f \in \mathscr{H}_d$, $d \geq 2$, is an $s-$ hyperbolic map. Let S^1_1 be a minimal basic set. The support of the current σ in Theorem 5.11 is equal to $W^u(S^1_1) \cup S'_0$ with $S'_0 \subset S_0$.*

Proof. We know from Theorem 6.4 that the support of σ contains S^1_1.

The support of σ is not contained in S^1_1 because there is an algebraic curve A which does not intersect S^1_1 (see Definition 3.2). But the complement of the support of a nonzero, positive closed $(1,1)$ current is Stein ([FS4]) and hence cannot contain A. Hence the support of σ must intersect A.

Since the support of $f_*\sigma = f(\text{supp}) = \text{supp}(\sigma)$, then the support is forward invariant. The transitivity of f on S^1_1 implies that for any neighborhood V of a point in S^1_1, $V \setminus S^1_1$ has mass for σ.

We have to consider the possibility that near a point $x_0 \in S^1_1$ the mass of σ is concentrated near an unstable manifold $W^u_{\tilde{x},R}$ corresponding to a prehistory $\tilde{x} = (x_0, x_{-1}, \ldots)$ but there is no mass along the unstable manifold corresponding to another prehistory $\tilde{x}' = (x_0, x'_{-1}, \ldots)$. By the invariance property all the global unstable manifolds $W^u_{\tilde{x}}$ is in the support of σ. Since f^n collapses the direction of unstable manifolds because of contraction in the stable direction, it follows that $f^n(W^u_{\tilde{x}})$ and $f^n(W^u_{\tilde{x}'})$ are along the same directions so there is also mass along $W^u_{\tilde{x}'}$. Hence supp $\sigma \supset W^u(S^1_1)$. The support being closed, we add $S'_0 \subset S_0$ to get support σ. □

7 Examples

Example 7.1 Let $f \in \mathscr{H}_d$ be defined by

$$f[z : w : t] = [P(z : t) : Q(w : t) : t^d]$$

where P, Q are polynomials of degree d in one variable. Assume that the critical points of $P(z)$ and $Q(w)$ are in the basin of attraction of attracting cycles, i.e. P, Q are hyperbolic on their Julia sets, J_P, J_Q. If we use the inhomogeneous coordinate $t = 1$, we get that $S_2 = J_P(z) * J_Q(w)$. The basic sets for S_1 are in $t = 1$ $\{$periodic sinks for $P\} * J_Q$ or $J_P * \{$periodic sinks for $Q\}$. We also have the basic set in $t = 0$ corresponding to the Julia set for $f_0 = [P(z:0) : Q(w:0)]$. Under these assumptions f is $s-$hyperbolic. In these examples the unstable manifold for all prehistories in S^1_1 of a given point coincide and are contained in complex lines. So J^- is a union of analytic varieties.

Example 7.2 Let Φ be the Segre map from $\mathbb{P}^1 * \mathbb{P}^1 \to \mathbb{P}^2$. In homogeneous coordinates $\Phi([z_0 : z_1], [w_0 : w_1]) = [z_0 w_0 : z_1 w_1 : z_0 w_1 + z_1 w_0]$. Let $f_0 : \mathbb{P}^1 \to \mathbb{P}^1$ be a holomorphic map of degree d. There is $f \in \mathscr{H}_d$ such that $\Phi(f_0, f_0) =$

$f \circ \Phi$, see Ueda ([U]). If f_0 is hyperbolic, then f is hyperbolic. In this case $S_2 = \Phi(J_0, J_0)$ where J_0 denotes the Julia set of f_0. The basic sets for S_1 are of the form Φ(periodic sink) $* J_0$). In this case also J^- is an algebraic variety. In these examples the no-cycle condition is satisfied.

Example 7.3 Perturbations of these examples are quite different. Consider the family of mappings, $0 < \epsilon << 1$.

$$f_\epsilon[z : w : t] = [z^2 : \frac{1}{10}wt + \frac{zt}{2} + \epsilon w^2 : t^2].$$

There are three basic sets for S_1, a circle in $(t = 0)$, a quasicircle in $(z = 0)$ and in $(t = 1)$ we get a solenoid in the region $1 - \delta < |z| < 1 + \delta,\ |w| < \delta$, where δ is small if ϵ is small. The map f_ϵ is injective in a neighborhood of the solenoid so again the unstable manifolds do not have self intersection and indeed there is just one prehistory for a given point.

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