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Signatures and higher signatures of S^1-quotients

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Abstract. We define and study the signature, $\widehat{A}$-genus and higher signatures of the quotient space of an S^1-action on a closed oriented manifold. We give applications to questions of positive scalar curvature and to an Equivariant Novikov Conjecture.

1. Introduction

The signature $\sigma(M)$ is a classical invariant of a closed oriented manifold M. If M is smooth, the Hirzebruch signature theorem expresses $\sigma(M)$ in terms of the L-class of M [20]. It is of interest to extend Hirzebruch's formula to various types of singular spaces. For example, Thom defined the L-class of a PL-manifold in a way so that the Hirzebruch formula still holds [42]. A large generalization of Thom's result was given by Cheeger and Goresky-MacPherson [14,18], who defined the signatures and homology L-classes of so-called Witt spaces.

In this paper we define and study the signatures of certain singular spaces which arise in transformation group theory, namely quotients of closed oriented smooth manifolds M by S^1-actions. This class of spaces includes oriented manifolds-with-boundary, but also contains spaces with much more drastic singularities. If the group action is semifree, meaning that each isotropy subgroup is $\{e\}$ or S^1, then any point in the quotient space $S^1\backslash M$ which is in the singular stratum has a neighborhood which is homeomorphic to $D^k \times \mathrm{cone}(\mathbb{C}P^N)$, for some k and N. If N is even then the quotient space is not a Witt space.

Our motivation to study such spaces comes from the Equivariant Novikov Conjecture. The usual Novikov Conjecture hypothesizes that the higher signatures of a closed oriented manifold are oriented-homotopy invariants. When one studies compact group actions, one wants to know what the possible equivariant homotopy invariants are. In particular, in view of the importance of the Novikov conjecture in surgery theory, one wants to know if there are equivariant higher signatures and an Equivariant Novikov Conjecture. There are two candidate

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Equivariant Novikov Conjectures, one based on classifying spaces for compact group actions [38] and one based on classifying spaces for proper group actions [5, Section 8]. We describe these in detail in Subsection 3.1. In the special case of free S^1-actions on simply-connected manifolds, the first conjecture is false (as was pointed out in [38]) and the second conjecture is true but vacuous. Since the usual signature of the quotient space of a free S^1-action is an oriented S^1-homotopy invariant, there is clearly something missing in these conjectures. Hence it is a serious conceptual problem to even give a good notion of equivariant higher signatures.

We start out by considering the case when there are no fundamental group complications. In Section 2 we define the equivariant signature $\sigma_{S^1}(M) \in \mathbb{Z}$ of an S^1-action. In the special case when $S^1\backslash M$ is a manifold (possibly with boundary), $\sigma_{S^1}(M)$ equals the usual signature of $S^1\backslash M$.

Note that the fixed-point-set M^{S^1} embeds in $S^1\backslash M$. Let $\int_{S^1\backslash M}$ denote integration over $\left(S^1\backslash M\right) - M^{S^1}$.

Theorem 1. *$\sigma_{S^1}(M)$ is an oriented S^1-homotopy invariant. Suppose that the S^1-action is semifree. If M is equipped with an S^1-invariant Riemannian metric, give $\left(S^1\backslash M\right) - M^{S^1}$ the quotient metric. Then*

$$\sigma_{S^1}(M) = \int_{S^1\backslash M} L\left(T(S^1\backslash M)\right) + \eta\left(M^{S^1}\right), \tag{1.1}$$

where $\eta\left(M^{S^1}\right)$ is the Atiyah-Patodi-Singer eta-invariant of the tangential signature operator on M^{S^1} [3].

We also give the extension of (1.1) to general S^1-actions. If $S^1\backslash M$ is a Witt space, we show that $\sigma_{S^1}(M)$ equals the intersection-homology signature of $S^1\backslash M$.

In Subsection 2.4 we define the $\widehat{A}$-genus $\widehat{A}_{S^1}(M) \in \mathbb{Z}$ of the quotient space of an even semifree S^1-action on a spin manifold M. This has applications to questions of positive scalar curvature. Let us recall the result of Bérard-Bergery that if S^1 acts freely on a compact manifold M then M has an S^1-invariant metric of positive scalar curvature if and only if $S^1\backslash M$ has a metric of positive scalar curvature [6, Theorem C]. As a consequence, if S^1 acts freely and evenly on a spin manifold M then the usual $\widehat{A}$-genus of $S^1\backslash M$ is an obstruction to having an S^1-invariant metric on M of positive scalar curvature. We extend this to a statement about semifree S^1-actions.

Theorem 2. *Suppose that S^1 acts semifreely and evenly on a spin manifold M. If M admits an S^1-invariant metric of positive scalar curvature and M^{S^1} has no connected components of codimension 2 in M then $\widehat{A}_{S^1}(M) = 0$.*

The codimension assumption in Theorem 2 is probably not necessary; see the remark after the proof of Theorem 2.

In the case of an odd semifree S^1-action on a spin manifold M, we define the $\widehat{A}$-genus $\widehat{\mathcal{A}}_{S^1}(M) \in \mathbb{Z}$ corresponding to a spin^c-structure on the quotient space. We show that $\widehat{A}_{S^1}(M)$ and $\widehat{\mathcal{A}}_{S^1}(M)$ are metric-independent provided that there is no spectral flow for the Dirac operator on M^{S^1}.

In Section 3 we construct equivariant higher signatures for S^1-actions, using [29–31]. Let Γ' be a finitely-generated discrete group and let $\rho : \pi_1(M, m_0) \to \Gamma'$ be a surjective homomorphism. Let $o \in \pi_1(M, m_0)$ be the homotopy class of the S^1-orbit of a basepoint m_0 and let $\widehat{\Gamma}$ be the quotient of Γ' by the central cyclic subgroup generated by $\rho(o)$. The equivariant higher signatures will involve the group cohomology of $\widehat{\Gamma}$.

In order to construct the equivariant higher signatures we make a certain (S^1-homotopy invariant) assumption about M^{S^1}. Namely, if F is a connected component of M^{S^1}, let Γ_F be the image of $\pi_1(F)$ in $\widehat{\Gamma}$. Let $\overline{\mathcal{D}}$ be the canonical flat $C_r^* \Gamma_F$-bundle on F. We assume that $\mathrm{H}^*(F; \overline{\mathcal{D}})$ vanishes in the middle degree if F is even-dimensional, or in the middle two degrees if F is odd-dimensional. We also assume that Γ_F is virtually nilpotent or Gromov-hyperbolic.

There is a space $\widehat{M}$ on which $\widehat{\Gamma}$ acts properly and cocompactly, with $\widehat{\Gamma} \backslash \widehat{M} = S^1 \backslash M$. We needd two pieces of additional data : an S^1-invariant Riemannian metric g on M and a compactly-supported function H on $\widehat{M}$ satisfying $\sum_{\gamma \in \widehat{\Gamma}} \gamma \cdot H = 1$. Given $[\tau] \in \mathrm{H}^k \left(\widehat{\Gamma}; \mathbb{R}\right)$, represent it by a cocycle $\tau \in Z^k \left(\widehat{\Gamma}; \mathbb{R}\right)$. There is a corresponding cyclic cocycle $Z_\tau \in ZC^k(\mathbb{R}\widehat{\Gamma})$. Using g, H and Z_τ, we define a closed orbifold form $\omega_\tau \in \Omega^k \left(\left(S^1 \backslash M\right) - M^{S^1} \right)$. In Subsection 3.3, we use ω_τ to give a differential form proof of a result of Browder and Hsiang [12, Theorem 1.1], in the case of S^1-actions. Now suppose that the S^1-action is semifree. Then using the higher eta-invariant $\widetilde{\eta}$ of [30], the equivariant higher signature is defined to be

$$\langle \sigma_{S^1}(M), [\tau] \rangle = \int_{S^1 \backslash M} L(T(S^1 \backslash M)) \wedge \omega_\tau + c(k) \langle \widetilde{\eta}(M^{S^1}), Z_\tau \rangle \in \mathbb{R}. \quad (1.2)$$

Here $c(k)$ is a certain nonzero constant.

Theorem 3. $\langle \sigma_{S^1}(M), [\tau] \rangle$ *is independent of* g *and* H.

Thus $\langle \sigma_{S^1}(M), [\tau] \rangle$ is a (smooth) topological invariant of the S^1-action. If S^1 acts freely on M then we recover the Novikov higher signatures of $S^1 \backslash M$ in full generality. (Note that the assumptions just involve M^{S^1}). We also give the extensions of (1.2) and Theorem 3 to general S^1-actions. We conjecture that $\langle \sigma_{S^1}(M), [\tau] \rangle$ is an oriented S^1-homotopy invariant of M. In Appendix A we outline a proof of this when $S^1 \backslash M$ is a manifold-with-boundary whose fundamental group is virtually nilpotent or Gromov-hyperbolic.

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2. Signatures of S^1-quotients

2.1. S^1-homotopy invariance

Let G be a compact Lie group and let $G-\mathrm{Man}$ be the category whose objects are closed oriented smooth manifolds on which G acts on the left by orientation-preserving diffeomorphisms, and whose morphisms are smooth G-maps. If H is a closed subgroup of G, let M^H denote the points of M which are fixed by H.

The most basic G-homotopy invariant information of a G-manifold M is the collection of finite sets $\{\pi_0(M^H)\}$. To organize this information coherently, let Or_G be the orbit category of G, with objects given by G-homogeneous spaces G/H, H closed, and morphisms given by G-maps. Let Fin be the category whose objects are isomorphism classes of finite sets and whose morphisms are set maps. Then there is a functor $F : G-\mathrm{Man} \to \mathrm{Func}(\mathrm{Or}_G^{op}, \mathrm{Fin})$ where $F(M) \in \mathrm{Func}(\mathrm{Or}_G^{op}, \mathrm{Fin})$ sends $G/H \in \mathrm{Or}_G^{op}$ to $\{\pi_0(M^H)\}$. Given $\mu \in \mathrm{Func}(\mathrm{Or}_G^{op}, \mathrm{Fin})$, the set of G-manifolds M such that $F(M) = \mu$ is closed under G-homotopy equivalence. For example, the notion of an action being free or semifree is G-homotopy invariant.

We now restrict to the case $G = S^1$. Suppose that M has dimension $4k+1$. Let X be the vector field on M which generates the S^1-action. Let $i_X : \Omega^*(M) \to \Omega^{*-1}(M)$ be interior multiplication by X and let $\mathcal{L}_X : \Omega^*(M) \to \Omega^*(M)$ be Lie differentiation by X. Let $e : M^{S^1} \to M$ be the inclusion of the fixed-point-set.

Definition 1. *Define the basic forms on M and $\left(M, M^{S^1}\right)$ by*

$$\begin{aligned} \Omega^{*,basic}(M) &= \{\omega \in \Omega^*(M) : i_X\omega = \mathcal{L}_X\omega = 0\}, \\ \Omega^{*,basic}\left(M, M^{S^1}\right) &= \{\omega \in \Omega^{*,basic}(M) : e^*\omega = 0\}. \end{aligned} \tag{2.1}$$

Let $\Omega_c^{*,basic}\left(M - M^{S^1}\right)$ be the complex of compactly-supported basic forms on $M - M^{S^1}$. Let $\mathrm{H}^{*,basic}(M)$, $\mathrm{H}^{*,basic}\left(M, M^{S^1}\right)$ and $\mathrm{H}_c^{*,basic}\left(M - M^{S^1}\right)$ be the corresponding cohomology groups.

Proposition 1. $\mathrm{H}^{*,basic}(M) \cong \mathrm{H}^*(S^1\backslash M; \mathbb{R})$ *and*

$$\mathrm{H}^{*,basic}\left(M, M^{S^1}\right) \cong \mathrm{H}_c^{*,basic}\left(M - M^{S^1}\right) \cong \mathrm{H}^*\left(S^1\backslash M, M^{S^1}; \mathbb{R}\right). \tag{2.2}$$

Proof. The fact that $\mathrm{H}^{*,basic}(M) \cong \mathrm{H}^*(S^1\backslash M; \mathbb{R})$ was proven in [25]. Let us briefly recall the proof. By a Mayer-Vietoris argument, we can reduce to the case when H is a subgroup of S^1, V is a representation space of H, DV is the unit ball in V and $M = S^1 \times_H DV$. By a product formula, we can also reduce to the case when V has no trivial subrepresentations. Let SV be the unit sphere in

V. Then $S^1\backslash M = H\backslash DV$ is a cone over the orbifold $H\backslash SV$. A Poincaré lemma gives

$$\mathrm{H}^{*,basic}(M) \cong \mathrm{H}^*(\mathrm{pt};\mathbb{R}) \cong \mathrm{H}^*(S^1\backslash M;\mathbb{R}), \tag{2.3}$$

which proves the claim.

We now do a similar argument for $\mathrm{H}^{*,basic}\left(M, M^{S^1}\right)$. We can reduce to the case $M = S^1 \times_H DV$ as above. The Poincaré lemma gives

$$\begin{aligned}\mathrm{H}^{*,basic}\left(M, M^{S^1}\right) &= \mathrm{H}^{*,basic}\left(S^1 \times_H DV, \mathrm{pt.}\right) = 0 \\ &= \mathrm{H}^*(H\backslash DV, \mathrm{pt};\mathbb{R}) = \mathrm{H}^*\left(S^1\backslash M, M^{S^1};\mathbb{R}\right).\end{aligned} \tag{2.4}$$

Finally, there is an obvious cochain inclusion $\Omega_c^{*,basic}\left(M - M^{S^1}\right) \to \Omega^{*,basic}\left(M, M^{S^1}\right)$. Using the Poincaré lemma, one can construct a homotopy inverse $\Omega^{*,basic}\left(M, M^{S^1}\right) \to \Omega_c^{*,basic}\left(M - M^{S^1}\right)$. The proposition follows. $\square$

Example: Let X^{4k} be a compact manifold-with-boundary. Let M be the manifold obtained by spinning X. That is, $M = \partial(D^2 \times X) = \left(D^2 \times \partial X\right) \cup_{S^1\times\partial X} \left(S^1 \times X\right)$, with the induced S^1-action. Then $\mathrm{H}^{*,basic}\left(M, M^{S^1}\right) \cong \mathrm{H}^*(X, \partial X;\mathbb{R}) \cong \mathrm{H}_c^*(\mathrm{int}(X);\mathbb{R})$.

Proposition 2. $\mathrm{H}^{*,basic}\left(M, M^{S^1}\right)$ *is an S^1-homotopy invariant.*

Proof. Let $f : M \to N$ be an S^1-homotopy equivalence, with S^1-homotopy inverse $g : N \to M$. Then $f\left(M^{S^1}\right) \subset N^{S^1}$. Hence the pullback $f^* : \Omega^{*,basic}\left(N, N^{S^1}\right) \to \Omega^{*,basic}\left(M, M^{S^1}\right)$ is well-defined. Let $F : [0,1] \times M \to M$ be an S^1-homotopy from the identity to $g \circ f$. Then there is a pullback $F^* : \Omega^{*,basic}\left(M, M^{S^1}\right) \to \Omega^*([0,1]) \otimes \Omega^{*,basic}\left(M, M^{S^1}\right)$. One can construct the cochain-homotopy equivalence between $\Omega^{*,basic}\left(M, M^{S^1}\right)$ and $\Omega^{*,basic}\left(N, N^{S^1}\right)$ by the standard argument. $\square$

Remark: It is not surprising that $\mathrm{H}^{*,basic}\left(M, M^{S^1}\right)$ is unchanged by an S^1-isovariant homotopy equivalence, as this would correspond to a stratum-preserving homotopy equivalence between M/S^1 and N/S^1. However, it is perhaps less obvious that it is unchanged by an S^1-equivariant homotopy equivalence.

In the rest of the paper, we will deal with $\mathrm{H}_c^{*,basic}\left(M - M^{S^1}\right)$ instead of the equivalent $\mathrm{H}^{*,basic}\left(M, M^{S^1}\right)$. Give M an S^1-invariant Riemannian metric. Let $X^* \in \Omega^1(M)$ be the dual 1-form to X, using the Riemannian metric. Define $\eta \in \Omega^1\left(M - M^{S^1}\right)$ by $\eta = X^*/|X|^2$.

Proposition 3. *$d\eta$ is a basic 2-form on $M - M^{S^1}$.*

Proof. By construction, $\mathcal{L}_X\eta = 0$ and hence $\mathcal{L}_X d\eta = 0$. Also by construction, $i_X\eta = 1$. Hence $i_x d\eta = \mathcal{L}_X\eta - di_X\eta = 0$. □

Proposition 4. *If $\sigma \in \Omega_c^{4k-1,basic}\left(M - M^{S^1}\right)$ then $\int_M \eta \wedge d\sigma = 0$.*

Proof. We have

$$\int_M \eta \wedge d\sigma = \int_M d\eta \wedge \sigma. \tag{2.5}$$

As $d\eta$ and σ are basic, $d\eta \wedge \sigma$ is basic and so the $(4k+1)$-form $d\eta \wedge \sigma$ vanishes in $\Omega_c^{4k+1}\left(M - M^{S^1}\right)$. □

Definition 2. *The S^1-fundamental class of M is the map $\tau : \mathrm{H}_c^{4k,basic}\left(M - M^{S^1}\right) \to \mathbb{R}$ given by $\tau(\omega) = \int_M \eta \wedge \omega$.*

By Proposition 4, the S^1-fundamental class is well-defined.

Proposition 5. *The S^1-fundamental class of M is independent of the choice of Riemannian metric.*

Proof. Let η_1 and η_2 be the 1-forms coming from two Riemannian metrics. Then $\eta_1 - \eta_2$ is basic. Hence $\int_M (\eta_1 - \eta_2) \wedge \omega = 0$ for any $\omega \in \Omega_c^{4k,basic}\left(M - M^{S^1}\right)$. □

Definition 3. *The intersection form on $\Omega_c^{2k,basic}\left(M - M^{S^1}\right)$ is*

$$(\omega_1, \omega_2) = \int_M \eta \wedge \omega_1 \wedge \omega_2. \tag{2.6}$$

Clearly $(\cdot, \cdot)$ is symmetric. By Proposition 4, it extends to a bilinear form on $\mathrm{H}_c^{2k,basic}\left(M - M^{S^1}\right)$.

Definition 4. *$\sigma_{S^1}(M)$ is the signature of $(\cdot, \cdot)$. That is, if the symmetric form $(\cdot, \cdot)$ is diagonalized then $\sigma_{S^1}(M)$ is (the number of positive eigenvalues) minus (the number of negative eigenvalues).*

Remark: The symmetric form $(\cdot, \cdot)$ on $\mathrm{H}_c^{2k,basic}\left(M - M^{S^1}\right)$ may be degenerate. For example, if M comes from spinning an oriented compact manifold-with-boundary X then the intersection form $(\cdot, \cdot)$ on $\mathrm{H}_c^{2k,basic}\left(M - M^{S^1}\right)$ is the same as that on $\mathrm{H}^{2k}(X, \partial X)$, which may be degenerate. In this case, $\sigma_{S^1}(M) = \sigma(X)$.

Proposition 6. *If $f : M \to N$ is an orientation-preserving S^1-homotopy equivalence then $\sigma_{S^1}(M) = \sigma_{S^1}(N)$.*

Proof. It suffices to show that the S^1-fundamental class on M pushes forward to the S^1-fundamental class on N. Let η_N be the 1-form constructed from a Riemannian metric on N. For $\omega \in \Omega_c^{4k,basic}\left(N - N^{S^1}\right)$, we have

$$\int_N \eta_N \wedge \omega = \int_M f^*\eta_N \wedge f^*\omega. \tag{2.7}$$

However, $f^*\eta_N - \eta_M$ is a basic 1-form on M and so

$$\int_M \left(f^*\eta_N - \eta_M\right) \wedge f^*\omega = 0. \tag{2.8}$$

The proposition follows. □

2.2. *Fixed-point-free actions*

Let S^1 act effectively on M without fixed points. Then $S^1 \backslash M$ is an oriented orbifold. If M has an S^1-invariant Riemannian metric then $S^1 \backslash M$ is a Riemannian orbifold. To write the formula for $\sigma_{S^1}(M)$, we first describe a certain set of suborbifolds $\mathcal{O}$ of $S^1 \backslash M$. We construct these suborbifolds by describing their intersections with orbifold coordinate charts in $S^1 \backslash M$; the suborbifolds can then be defined by patching together these intersections. Given $x \in S^1 \backslash M$, let Γ be a finite group and let $U \subset \mathbb{R}^n$ be a domain with a Γ-action such that (Γ, U) is an orbifold coordinate chart for $S^1 \backslash M$ around x. In particular, $\Gamma \backslash U$ can be identified with a neighborhood of x. Put

$$\widehat{U} = \{(g, u) \in \Gamma \times U : gu = g\}. \tag{2.9}$$

Define a Γ-action on $\widehat{U}$ by $\gamma \cdot (g, u) = (\gamma g \gamma^{-1}, gu)$. Let $\pi : \widehat{U} \to \Gamma \backslash \widehat{U}$ be the quotient map. Let $\langle \Gamma \rangle$ denote the set of conjugacy classes of Γ. There are projection maps $p_1 : \Gamma \backslash \widehat{U} \to \langle \Gamma \rangle$ and $p_2 : \Gamma \backslash \widehat{U} \to \Gamma \backslash U$. Then the intersections of the $\mathcal{O}$'s with $\Gamma \backslash U$ are $\{p_2 p_1^{-1}(\langle g \rangle)\}_{\langle g \rangle \in \langle \Gamma \rangle}$.

Define the multiplicity $m_x \in \mathbb{Z}^+$ of x by $\frac{1}{m_x} = \frac{|(p_2 \circ \pi)^{-1}(x)|}{|\Gamma|}$. This is independent of the choice of orbifold coordinate chart.

The Atiyah-Singer equivariant L-class $L(g) \in \Omega^{even}(U^g, o(TU^g))$ [4] is the pullback of a differential form $L(\langle g \rangle)$ on the image of U^g in $\Gamma \backslash U$. Given a suborbifold $\mathcal{O}$, define $L(\mathcal{O}) \in \Omega^{even}(\mathcal{O}, o(T\mathcal{O}))$ by

$$L(\mathcal{O})\big|_{\mathcal{O} \cap (\Gamma \backslash U)} = \sum_{\langle g \rangle : p_2 p_1^{-1}(\langle g \rangle) = \mathcal{O} \cap (\Gamma \backslash U)} L(\langle g \rangle). \tag{2.10}$$

If $\mathcal{O}$ is one of the suborbifolds then m_x is constant on the regular part of $\mathcal{O}$ and so we can define the multiplicity $m_{\mathcal{O}} \in \mathbb{Z}^+$ of $\mathcal{O}$. From [24], it follows that

$$\sigma_{S^1}(M) = \sum_{\mathcal{O}} \frac{1}{m_{\mathcal{O}}} \int_{\mathcal{O}} L(\mathcal{O}). \tag{2.11}$$

By definition, $\sigma_{S^1}(M) \in \mathbb{Z}$. In fact, it equals the signature of $S^1 \backslash M$ as a rational homology manifold. In the orbifold world it may be more natural to consider the $\mathbb{Q}$-valued orbifold signature $\int_{S^1 \backslash M} L(T(S^1 \backslash M))$. However, this is definitely a different object and is a single term in (2.11).

Remark: In the case of fixed-point-free actions, $\sigma_{S^1}(M)$ comes from the index of a signature operator which is transversally elliptic in the sense of [2]. This transversally elliptic signature operator only exists in the fixed-point-free case.

2.3. Semifree actions

Suppose that S^1 acts effectively and semifreely on M. Let $(S^1 \backslash M) - M^{S^1}$ have the quotient Riemannian metric. We write $L\left(T(S^1 \backslash M)\right)$ for the L-form and $\int_{S^1 \backslash M} L\left(T(S^1 \backslash M)\right)$ for its integral over $(S^1 \backslash M) - M^{S^1}$.

Theorem 4.

$$\sigma_{S^1}(M) = \int_{S^1 \backslash M} L\left(T(S^1 \backslash M)\right) + \eta\left(M^{S^1}\right). \tag{2.12}$$

Proof. Let F be a connected component of the fixed-point-set M^{S^1}. It is an oriented odd-dimensional manifold, say of dimension $4k - 2N - 1$. Let NF be the normal bundle of F in M. It has an S^1-action by orthogonal automorphisms, which is free on $NF - F$. Furthermore, the disk bundle DNF is S^1-diffeomorphic to a neighborhood of F in M. Let SNF be the sphere bundle of NF. Then $S^1 \backslash SNF$ is the total space of a Riemannian fiber bundle $\mathcal{F}$ over F whose fibers Z are copies of $\mathbb{C}P^N$. The quotient space $S^1 \backslash DNF$ is homeomorphic to the mapping cylinder of the projection $\pi : \mathcal{F} \to F$.

Let us first pretend that for each F, a neighborhood of F in M is S^1-isometric to DNF. For simplicity, we suppose that there is only one connected component F of M^{S^1}; the general case is similar. For $r > 0$, let $N_r(F)$ be the r-neighborhood of F in $S^1\backslash M$. Then for small r, $\sigma_{S^1}(M) = \sigma\left((S^1\backslash M) - N_r(F)\right)$. By the Atiyah-Patodi-Singer theorem,

$$\sigma\left((S^1\backslash M) - N_r(F)\right) = \int_{(S^1\backslash M)-N_r(F)} L\left(T(S^1\backslash M)\right) + \int_{\partial N_r(F)} \widetilde{L}(\partial N_r(F)) + \eta\left(\partial N_r(F)\right), \tag{2.13}$$

where $\widetilde{L}(\partial N_r(F))$ is a local expression on $\partial N_r(F)$ which involves the second fundamental form and the curvature tensor [17, Section 3.10] and we give $\partial N_r(F)$ the orientation induced from that of $N_r(F)$. We will compute the limit of the right-hand-side of (2.13) as $r \to 0$.

We use the notation of [10, Section III(c)] to describe the geometry of the fiber bundle $\mathcal{F}$. In particular, the second fundamental form of the fibers and the curvature of the fiber bundle are parts of the connection 1-form component $\omega^i_\alpha = \omega^i_{\alpha j}\tau^j + \omega^i_{\alpha\beta}\tau^\beta$. Let R^{TZ} be the curvature of the Bismut connection on TZ. Define $\widehat{\Omega} \in \Omega^2([0,1] \times \mathcal{F}) \otimes \mathrm{End}(TZ \oplus \mathbb{R})$, a skew-symmetric matrix of 2-forms, by

$$\begin{aligned} \widehat{\Omega}^i_j &= \left(R^{TZ}\right)^i_j - t^2\tau^i \wedge \tau^j, \\ \widehat{\Omega}^i_r &= dt \wedge \tau^i - t\omega^i_\alpha \wedge \tau^\alpha. \end{aligned} \tag{2.14}$$

Definition 5. *The transgressed L-class, $\widehat{L}(F) \in \Omega^{odd}(F)$, is given by*

$$\widehat{L}(F) = \int_Z \int_0^1 L\left(\widehat{\Omega}\right). \tag{2.15}$$

We first compute the curvature of $S^1\backslash DNF$ in terms of the geometric invariants of the fiber bundle $\mathcal{F}$. Let $\{\tau^i\}_{i=1}^{dim(Z)}$, $\{\tau^\alpha\}_{\alpha=1}^{dim(F)}$ be a local orthonormal basis of 1-forms on $\mathcal{F}$ as in [10, Section III(c)]. Then a local orthonormal basis of 1-forms on $S^1\backslash DNF$ is given by

$$\begin{aligned} \widehat{\tau}^r &= dr, \\ \widehat{\tau}^i &= r\tau^i, \\ \widehat{\tau}^\alpha &= \tau^\alpha. \end{aligned} \tag{2.16}$$

Let ω^I_J be the connection matrix of $\mathcal{F}$. The structure equations $0 = d\widehat{\tau}^I + \widehat{\omega}^I_J \wedge \widehat{\tau}^J$ give the connection matrix $\widehat{\omega}$ of $S^1\backslash DNF$ to be

$$\begin{aligned} \widehat{\omega}^i_j &= \omega^i_j, \\ \widehat{\omega}^i_r &= \tau^i, \\ \widehat{\omega}^i_\alpha &= r\omega^i_\alpha, \\ \widehat{\omega}^\alpha_\beta &= r^2\omega^\alpha_{\beta i}\tau^i + \omega^\alpha_{\beta\gamma}\tau^\gamma, \\ \widehat{\omega}^\alpha_r &= 0. \end{aligned} \tag{2.17}$$

In the limit when $r \to 0$, the curvature matrix of $S^1 \backslash DNF$ becomes

$$\begin{aligned}\widehat{\Omega}^i_j &= (R^{TZ})^i_j - \tau^i \wedge \tau^j,\\ \widehat{\Omega}^i_r &= -\omega^i_\alpha \wedge \tau^i,\\ \widehat{\Omega}^i_\alpha &= dr \wedge \omega^i_\alpha,\\ \widehat{\Omega}^\alpha_\beta &= (R^{TF})^\alpha_\beta,\\ \widehat{\Omega}^\alpha_r &= 0.\end{aligned} \tag{2.18}$$

(By way of illustration, let us check the $\widehat{\Omega}^i_r$-term explicitly. We have

$$\begin{aligned}\widehat{\Omega}^i_r &= d\widehat{\omega}^i_r + \widehat{\omega}^i_j \wedge \widehat{\omega}^j_r\\ &= d\tau^i + \omega^i_j \wedge \tau^j\\ &= -\omega^i_j \wedge \tau^j - \omega^i_\alpha \wedge \tau^\alpha + \omega^i_j \wedge \tau^j\\ &= -\omega^i_\alpha \wedge \tau^\alpha.)\end{aligned} \tag{2.19}$$

It is now clear that

$$\int_{S^1\backslash M} L\left(T\left(S^1\backslash M\right)\right) = \lim_{r\to 0} \int_{(S^1\backslash M)-N_r(F)} L\left(T(S^1\backslash M)\right) \tag{2.20}$$

exists.

Restricted to $\partial N_r(F)$, as $r \to 0$ the curvature matrix has nonzero entries

$$\begin{aligned}\widehat{\Omega}^i_j &= (R^{TZ})^i_j - \tau^i \wedge \tau^j,\\ \widehat{\Omega}^i_r &= -\omega^i_\alpha \wedge \tau^i,\\ \widehat{\Omega}^\alpha_\beta &= (R^{TF})^\alpha_\beta.\end{aligned} \tag{2.21}$$

The second fundamental form of $\partial N_r(F)$ enters in the connection matrix element

$$\widehat{\omega}^i_r = \tau^i = \frac{1}{r}\widehat{\tau}^i. \tag{2.22}$$

That is, with respect to the orthogonal decomposition $T(\partial N_r(F)) = TZ \oplus \pi^* TF$, the shape operator of $\partial N_r(F)$ is $\frac{1}{r}\mathrm{Id}_{TZ} \oplus 0$.

To compute $\widetilde{L}(\partial N_r(F))$ for small r, we construct a 1-parameter family of connections which interpolate between the Riemannian connection of $S^1\backslash DNF$ (pulled back to $\partial N_r(F)$) and the Riemannian connection of a product metric, at least as $r \to 0$. For $t \in [0, 1]$, put

$$\begin{aligned}\widehat{\omega}^i_j(t) &= \omega^i_j,\\ \widehat{\omega}^i_r(t) &= t\tau^i,\\ \widehat{\omega}^\alpha_\beta(t) &= \omega^\alpha_{\beta\gamma}\tau^\gamma.\end{aligned} \tag{2.23}$$

Then $\widehat{\omega}(0)$ is the product connection 1-form and by (2.17), $\widehat{\omega}(1)$ is the limit of the pullback connection 1-form on $\partial N_r(F)$ as $r \to 0$. The curvature of (2.23) on $[0, 1] \times \mathcal{F}$ has nonzero components

$$\begin{aligned} \widehat{\Omega}^i_j(t) &= (R^{TZ})^i_j - t^2 \tau^i \wedge \tau^j, \\ \widehat{\Omega}^i_r(t) &= dt \wedge \tau^i - t\omega^i_\alpha \wedge \tau^\alpha, \\ \widehat{\Omega}^\alpha_\beta(t) &= (R^{TF})^\alpha_\beta. \end{aligned} \tag{2.24}$$

Then

$$\lim_{r \to 0} \int_{\partial N_r(F)} \widetilde{L}(\partial N_r(F)) = \int_{[0,1]\times\mathcal{F}} L(\widehat{\Omega}(t)) = \int_F \widehat{L}(F) \wedge L(TF). \tag{2.25}$$

Now the fiber bundle $S^1 \backslash SNF$ is associated to a principal bundle P over F with compact structure group. Hence $\widehat{L}(F)$ can be computed by equivariant methods [7, Section 7.6]. Such a calculation will necessarily give it as a polynomial in the curvature form of P, and in particular as an even form on F. However, by parity reasons, $\widehat{L}(F)$ is an odd form on F. Thus

$$\lim_{r \to 0} \int_{\partial N_r(F)} \widetilde{L}(\partial N_r(F)) = 0. \tag{2.26}$$

From [16],

$$\lim_{r \to 0} \eta(\partial N_r(F)) = \int_F \widetilde{\eta} \wedge L(TF) + \eta(F; \mathrm{Ind}(D_Z)) + \tau_F, \tag{2.27}$$

where τ_F is a signature correction term [16, p. 268]. Again, we can compute $\widetilde{\eta}$ by equivariant methods to obtain an even form on F, while by parity reasons $\widetilde{\eta}$ is an odd form. Thus

$$\int_F \widetilde{\eta} \wedge L(TF) = 0. \tag{2.28}$$

Next, the index bundle $\mathrm{Ind}(D_Z)$ on F is the difference of the vector bundles $\mathrm{H}^N_+(Z)$ and $\mathrm{H}^N_-(Z)$ of self-dual and anti-self-dual cohomology groups. As $Z = \mathbb{C}P^N$, $\mathrm{H}^N_\pm(Z; \mathbb{R})$ vanishes unless N is even, in which case $\mathrm{H}^N_+(Z; \mathbb{R}) = \mathbb{R}$ and $\mathrm{H}^N_-(Z; \mathbb{R}) = 0$. Then $\mathrm{H}^N_+(Z)$ is a trivial real line bundle on F with a flat Euclidean metric. Thus $\eta(F; \mathrm{Ind}(D_Z)) = \eta(F)$.

Finally, the Leray-Hirsch theorem implies that the Leray-Serre spectral sequence for $\mathrm{H}^*(\mathcal{F}; \mathbb{R})$ degenerates at the E_2-term [11, p. 170, 270]. Hence $\tau_F = 0$.

This proves the proposition if a neighborhood of F in M is S^1-isometric to DNF. If a neighborhood of F in M is not S^1-isometric to DNF, nevertheless as one approaches F the Riemannian metric on M is better and better approximated by that of DNF. The above calculations will still be valid in this limit. □

Example: If M is obtained by spinning a compact oriented manifold-with-boundary X then $M^{S^1} = -\partial X$, when one takes orientations into account. The boundary ∂X in $X = S^1 \backslash M$ is totally geodesic. In this case, Theorem 4 reduces to the Atiyah-Patodi-Singer formula for $\sigma(X)$.

Proposition 7. *Let W be a semifree S^1-cobordism between M_1 and M_2. Then*

$$\sigma_{S^1}\left(M^1\right) - \sigma_{S^1}\left(M^2\right) = \sigma\left(W^{S^1}\right). \tag{2.29}$$

Proof. We take $\partial W = M_1 \cup (-M_2)$. Let $N_r\left(W^{S^1}\right)$ be the r-neighborhood of W^{S^1} in $S^1 \backslash W$. Then for r small, $\left(\left(S^1 \backslash W\right) - N_r\left(W^{S^1}\right), \partial N_r\left(W^{S^1}\right)\right)$ is a cobordism of manifold pairs from $\left(\left(S^1 \backslash M_1\right) - N_r\left(M_1^{S^1}\right), \partial N_r\left(M_1^{S^1}\right)\right)$ to $\left(\left(S^1 \backslash M_2\right) - N_r\left(M_2^{S^1}\right), \partial N_r\left(M_2^{S^1}\right)\right)$. Hence $\partial N_r\left(W^{S^1}\right) \cup \left(\left(S^1 \backslash M_1\right) - N_r\left(M_1^{S^1}\right)\right) \cup - \left(\left(S^1 \backslash M_2\right) - N_r\left(M_2^{S^1}\right)\right)$ is an oriented boundary, where $\partial N_r\left(W^{S^1}\right)$ has the boundary orientation coming from $N_r\left(W^{S^1}\right)$. Giving it the other orientation, we obtain

$$\sigma_{S^1}\left(M^1\right) - \sigma_{S^1}\left(M^2\right) = \sigma\left(\partial N_r\left(W^{S^1}\right)\right). \tag{2.30}$$

Now $\partial N_r\left(W^{S^1}\right)$ is the total space of a fiber bundle with fiber Z, a complex projective space, and base W^{S^1}. By the same calculation as at the end of the proof of Theorem 4, the boundary fibration over $\partial W^{S^1} = M_1^{S^1} \cup (-M_2^{S^1})$ has vanishing signature correction τ. Then by [16, Theorem 0.4b, p. 315],

$$\sigma\left(\partial N_r\left(W^{S^1}\right)\right) = \sigma(Z) \cdot \sigma\left(W^{S^1}\right) = \sigma\left(W^{S^1}\right). \tag{2.31}$$

The proposition follows. $\square$

By way of comparison, the S^1-semifree cobordism-invariant information of M essentially consists of the cobordism classes of the components of M^{S^1}, listed by dimension, along with their normal data [43].

If the codimension of M^{S^1} in M is divisible by four then $S^1 \backslash M$ is a Witt space in the sense of [40]. Hence it has an L-class in $\mathrm{H}_*(S^1 \backslash M; \mathbb{Q})$. Also, as $\dim\left(M^{S^1}\right) \equiv 1 \bmod 4$, the eta-invariant of M^{S^1} vanishes. We use the differential-form description of the homology of Witt spaces given in [9, Section 4].

Proposition 8. *In this case, the homology L-class of $S^1 \backslash M$ is represented by $L(T(S^1 \backslash M))$.*

Proof. We can deform the metric of $S^1\backslash M$ to make it strictly conical in a neighborhood of M^{S^1}. By [9, Theorem 5.7], the homology L-class is represented by the pair of forms $\left(L(T(S^1\backslash M)),\, L(TM^{S^1}) \wedge \widetilde{\eta}\right)$, where $\widetilde{\eta}$ is the eta-form of the $\mathbb{C}P^N$-bundle over M^{S^1}. By the method of proof of Theorem 4, $\widetilde{\eta} = 0$. $\square$

Corollary 1. *In this case,* $\sigma_{S^1}(M) = \int_{S^1\backslash M} L(T(S^1\backslash M))$ *equals the intersection homology signature of* $S^1\backslash M$.

One can give a more direct proof of the corollary. For small $r > 0$, let $N_r\left(M^{S^1}\right)$ be the r-tubular neighborhood of M^{S^1} in $S^1\backslash M$. As in [40, Proposition 3.1], there is a Witt cobordism which pinches $\partial N_r\left(M^{S^1}\right)$ to a point. Letting X_1 be the coning of $(S^1\backslash M) - N_r\left(M^{S^1}\right)$ and X_2 be the coning of $\overline{N_r\left(M^{S^1}\right)}$, it follows that

$$\sigma(S^1\backslash M) = \sigma(X_1) + \sigma(X_2), \tag{2.32}$$

where σ denote the intersection homology signature. Now $\sigma(X_1) = \sigma_{S^1}(M)$. Let X_3 be the mapping cylinder of the projection $\overline{N_r\left(M^{S^1}\right)} \to M^{S^1}$. A further Witt cobordism shows that $\sigma(X_2) = \sigma(X_3)$. It is well-known that the signature of the total space of an oriented fiber bundle vanishes if the fiber and base have odd dimension. One can extend this fact to the fibration $X_3 \to M^{S^1}$, whose fiber is a Witt space, as in [13, p. 545-546]. (Strictly speaking, [13] deals with the more interesting case of even-dimensional fiber and base.) The corollary follows.

We expect that for a general semifree effective S^1-action, $\sigma_{S^1}(M)$ will be the signature of the intersection pairing on the image of the (lower middle perversity) middle-dimensional intersection homology in the (upper middle perversity) middle-dimensional intersection homology.

2.4. $\widehat{A}$-genus

We wish to construct an analog of the $\widehat{A}$-genus for $S^1\backslash M$. If there were a Dirac operator on $S^1\backslash M$ then this $\widehat{A}$-genus should be its index. Although we will not actually construct a Dirac operator on $S^1\backslash M$, it is nevertheless worth considering the topological conditions to have such an operator. Suppose that M is spin, with a free S^1-action. It does not follow that $S^1\backslash M$ is spin. For example, if $M = S^{4k+1}$ has the Hopf action then $S^1\backslash M = \mathbb{C}P^{2k}$, which is not spin. The problem in this case is that the S^1-action on the oriented orthonormal frame bundle of M does not lift to an S^1-action on the principal spin bundle. Recall that an S^1-action is said to be even if it lifts to the principal spin bundle and odd if it does not [26, p. 295]. We will consider the two cases separately.

Lemma 1. *Let M be a spin manifold with a fixed spin structure and a semifree S^1-action. If F is a connected component of M^{S^1}, let* $\operatorname{codim}(F)$ *be its codimension in M.*
1. If the S^1-action is even then $\operatorname{codim}(F) = 2$ *or* $\operatorname{codim}(F) \equiv 0 \bmod 4$.
2. If the S^1-action is odd then $\operatorname{codim}(F) \equiv 2 \bmod 4$.

Proof. Let NF be the normal bundle to F and let SNF be its sphere bundle, with fiber S^{2N+1}. Then $\operatorname{codim}(F) = 2N + 2$. As S^1 acts trivially on F, if the S^1-action on M is even (odd) then the Hopf action on S^{2N+1} is even (odd). (Note that S^{2N+1} has a unique spin structure if $N > 0$.) If the Hopf action on S^{2N+1} is even then either $N = 0$ and the spin structure on S^1 is the one which does not extend to D^2, or N is odd. Thus F satisfies conclusion 1. of the lemma. If the Hopf action on S^{2N+1} is odd then N is even, so F satisfies conclusion 2. of the lemma. □

2.4.1. Even semifree S^1-actions Suppose that the spin manifold M has an even effective S^1-action. Let SM be the spinor bundle of M. If $\dim(M) = 4k+1$ then $\dim_{\mathbb{C}} SM = 2^{2k}$. If the S^1-action is free then $S^1\backslash M$ acquires a spin structure, with spinor bundle $S(S^1\backslash M) = S^1\backslash SM$. If the S^1-action is semifree, let $M^{S^1}_{(2)}$ denote the submanifold of M^{S^1} which has codimension 2 in M. As $M^{S^1}_{(2)}$ appears as a boundary component in a compactification of $(S^1\backslash M) - M^{S^1}$, it acquires a spin structure. Let $D_{M^{S^1}_{(2)}}$ denote the Dirac operator on $M^{S^1}_{(2)}$.

Definition 6.

$$\widehat{A}_{S^1}(M) = \int_{S^1\backslash M} \widehat{A}\left(T\left(S^1\backslash M\right)\right) + \frac{1}{2}\left[\eta\left(D_{M^{S^1}_{(2)}}\right) + \dim\left(\operatorname{Ker}\left(D_{M^{S^1}_{(2)}}\right)\right)\right]. \tag{2.33}$$

Proposition 9. *The number $\widehat{A}_{S^1}(M)$ is an integer. If $\{g(\epsilon)\}_{\epsilon\in[0,1]}$ is a smooth 1-parameter family of S^1-invariant metrics on M and* $\dim\left(\operatorname{Ker}\left(D_{M^{S^1}_{(2)}}\right)\right)$ *is constant in ϵ then $\widehat{A}_{S^1}(M)$ is constant in ϵ.*

Proof. For small $r > 0$, let $N_r\left(M^{S^1}\right)$ be the r-neighborhood of M^{S^1} in $S^1\backslash M$. The manifold-with-boundary $(S^1\backslash M) - N_r\left(M^{S^1}\right)$ is spin and one can talk about the index $\operatorname{Ind}_r \in \mathbb{Z}$ of its Dirac operator. By the method of proof of Theorem 4, one finds that in $\mathbb{R}/\mathbb{Z}$,

$$\int_{S^1\backslash M} \widehat{A}\left(T\left(S^1\backslash M\right)\right) + \lim_{r\to 0}\frac{1}{2}\left[\eta\left(\partial N_r(M^{S^1})\right) + \dim\left(\operatorname{Ker}\left(\partial N_r(M^{S^1})\right)\right)\right] \equiv 0. \tag{2.34}$$

(The spectral invariants in the above equation are with respect to Dirac operators.) Let F be a connected component of M^{S^1} whose codimension in M is divisible by four. Then $\partial N_r(F)$ is a fiber bundle whose fiber is $\mathbb{C}P^N$ for some odd N. As $\mathbb{C}P^N$ is a spin manifold with positive scalar curvature, it follows from [8] that

$$\lim_{r\to 0} \eta\left(\partial N_r(M^{S^1})\right) = \int_F \widetilde{\eta} \wedge \widehat{A}(TF) \tag{2.35}$$

and

$$\lim_{r\to 0} \dim\left(\mathrm{Ker}\left(\partial N_r(M^{S^1})\right)\right) = 0. \tag{2.36}$$

As in the proof of Theorem 4, $\widetilde{\eta} = 0$.

If F is a connected component of M^{S^1} whose codimension in M is two then $\partial N_r(F)$ is a Riemannian manifold which is topologically the same as F and which approaches F metrically as $r \to 0$. Thus in $\mathbb{R}/\mathbb{Z}$,

$$\lim_{r\to 0} \frac{1}{2}\left[\eta\left(\partial N_r(F)\right) + \dim\left(\mathrm{Ker}\left(\partial N_r(F)\right)\right)\right] \equiv \frac{1}{2}\left[\eta\left(D_F\right) + \dim\left(\mathrm{Ker}\left(D_F\right)\right)\right]. \tag{2.37}$$

It follows that $\widehat{A}_{S^1}(M)$ is an integer.

Let $\{g(\epsilon)\}_{\epsilon\in[0,1]}$ be a family of metrics as in the statement of the proposition. Let $I_{S^1}(M)$ denote the first term in the right-hand-side of (2.33). We first compute $I_{S^1}(M)\big|_{\epsilon=1} - I_{S^1}(M)\big|_{\epsilon=0}$. As in the proof of Theorem 4, we compactify $\left(S^1\backslash M\right) - M^{S^1}$ by $\cup_F \left(S^1\backslash SNF\right)$, where F ranges over the connected components of M^{S^1}. Let $\widehat{\omega}(\epsilon)$ be the connection on $S^1\backslash SNF$, as in (2.17). We can compute $I_{S^1}(M)\big|_{\epsilon=1} - I_{S^1}(M)\big|_{\epsilon=0}$ as the integral over $\cup_F \left(S^1\backslash SNF\right)$ of a transgressed characteristic class. Namely,

$$I_{S^1}(M)\big|_{\epsilon=1} - I_{S^1}(M)\big|_{\epsilon=0} = -\sum_F \int_0^1 \int_{S^1\backslash SNF} \widehat{A}\left(\widehat{\Omega}(\epsilon) + d\epsilon \wedge \partial_\epsilon \widehat{\omega}\right). \tag{2.38}$$

(The minus sign on the right-hand-side of (2.38) comes from the different orientations of $S^1\backslash SNF$.) Let $\widehat{\omega}_V$ be the i and r-components of (2.17). Then from the structure of (2.17),

$$I_{S^1}(M)\big|_{\epsilon=1} - I_{S^1}(M)\big|_{\epsilon=0} = -\sum_F \int_0^1 \int_F \widehat{A}\left(R^{TF}(\epsilon) + d\epsilon \wedge \partial_\epsilon \omega^{TF}\right) \wedge \int_Z \widehat{A}\left(\widehat{\Omega}_V(\epsilon) + d\epsilon \wedge \partial_\epsilon \widehat{\omega}_V\right). \tag{2.39}$$

Let us write

$$\begin{aligned} \widehat{A}\left(R^{TF}(\epsilon) + d\epsilon \wedge \partial_\epsilon \omega^{TF}\right) &= a_1 + d\epsilon \wedge a_2, \\ \widehat{A}\left(\widehat{\Omega}_V(\epsilon) + d\epsilon \wedge \partial_\epsilon \widehat{\omega}_V\right) &= b_1 + d\epsilon \wedge b_2, \end{aligned} \tag{2.40}$$

where a_1, a_2, b_1 and b_2 depend on ϵ. Then

$$\begin{aligned} & I_{S^1}(M)\big|_{\epsilon=1} - I_{S^1}(M)\big|_{\epsilon=0} \\ & \quad = -\sum_F \int_0^1 d\epsilon \wedge \left(\int_F a_1 \wedge \int_Z b_2 + \int_F a_2 \wedge \int_Z b_1 \right). \end{aligned} \tag{2.41}$$

Now b_1 and b_2 can be computed by equivariant means, and the result will be a polynomial in the curvature of the principal bundle underlying $S^1 \backslash SNF$. In particular, they will be even forms. However, by parity considerations, b_2 is an odd form. Thus $b_2 = 0$ and

$$I_{S^1}(M)\big|_{\epsilon=1} - I_{S^1}(M)\big|_{\epsilon=0} = -\sum_F \int_0^1 d\epsilon \wedge \int_F a_2 \wedge \int_Z b_1. \tag{2.42}$$

From (2.40),

$$b_1 = \widehat{A}\left(\widehat{\Omega}_V(\epsilon)\right). \tag{2.43}$$

The Atiyah-Singer families index theorem gives an equality in $\mathrm{H}^{even}(F; \mathbb{R})$:

$$\mathrm{ch}(\mathrm{Ind}(D_Z)) = \int_Z b_1, \tag{2.44}$$

where D_Z is the family of vertical Dirac operators on the fiber bundle $S^1 \backslash SNF \to F$. If $\dim(Z) > 0$ then Z is a spin manifold with positive scalar curvature and so $\mathrm{Ind}(D_Z) = 0$. If $\dim(Z) = 0$ then Z is a point and $\int_Z b_1 = 1$. Thus

$$I_{S^1}(M)\big|_{\epsilon=1} - I_{S^1}(M)\big|_{\epsilon=0} = -\int_0^1 d\epsilon \wedge \int_{M^{S^1}_{(2)}} a_2. \tag{2.45}$$

On the other hand, from [3], as there is no spectral flow,

$$\begin{aligned} \frac{1}{2} \quad & \left[\eta\left(D_{M^{S^1}_{(2)}}\right) + \dim\left(\mathrm{Ker}\left(D_{M^{S^1}_{(2)}}\right)\right)\right]\Big|_{\epsilon=1} \\ & -\frac{1}{2}\left[\eta\left(D_{M^{S^1}_{(2)}}\right) + \dim\left(\mathrm{Ker}\left(D_{M^{S^1}_{(2)}}\right)\right)\right]\Big|_{\epsilon=0} \\ = & \int_0^1 d\epsilon \wedge \int_{M^{S^1}_{(2)}} a_2. \end{aligned} \tag{2.46}$$

The proposition follows. □

Theorem 5. *If M admits an S^1-invariant metric of positive scalar curvature and $M^{S^1}_{(2)} = \emptyset$ then $\widehat{A}_{S^1}(M) = 0$.*

Proof. We may assume that M has dimension $4k+1$. Suppose that it has an S^1-invariant Riemannian metric of positive scalar curvature. Let g^* be the quotient Riemannian metric on $(S^1\backslash M) - M^{S^1}$. Let $l \in C^\infty\left((S^1\backslash M) - M^{S^1}\right)$ be the function which assigns to a point $x \in (S^1\backslash M) - M^{S^1}$ the length of the S^1-orbit over x. Put $\widetilde{g} = l^{\frac{2}{4k-1}} g^*$. Then $\widetilde{g}$ has positive scalar curvature [6, p. 22].

Let us first suppose that for each connected component F of M^{S^1}, a neighborhood of F in M is S^1-isometric to DNF. Then as in (2.16), a local orthonormal basis of 1-forms on $S^1\backslash DNF$ for $\widetilde{g}$ is given by

$$\begin{aligned} \widehat{\tau}^r &= r^{\frac{1}{4k-1}} dr, \\ \widehat{\tau}^i &= r^{\frac{4k}{4k-1}} \tau^i, \\ \widehat{\tau}^\alpha &= r^{\frac{1}{4k-1}} \tau^\alpha. \end{aligned} \tag{2.47}$$

Changing variable to $u = r^{\frac{4k}{4k-1}}$, we obtain the local orthonormal basis

$$\begin{aligned} \widehat{\tau}^u &= \frac{4k-1}{4k} du, \\ \widehat{\tau}^i &= u\tau^i, \\ \widehat{\tau}^\alpha &= u^{\frac{1}{4k}} \tau^\alpha. \end{aligned} \tag{2.48}$$

Let us consider a more general class of bases given by

$$\begin{aligned} \widehat{\tau}^u &= \frac{1}{f(u)} du, \\ \widehat{\tau}^i &= u\tau^i, \\ \widehat{\tau}^\alpha &= u^{\frac{1}{4k}} \tau^\alpha \end{aligned} \tag{2.49}$$

for some positive function f. Let $\phi \in C^\infty(0,\infty)$ be a nondecreasing function such that $\phi(x) = x$ if $x \in (0, \frac{1}{2})$ and $\phi(x) = 1$ if $x \geq 1$. Given a small $\epsilon > 0$, define

$$f(u) = \frac{4k}{4k-1} \phi\left(\frac{u-\epsilon}{\epsilon^{1/2}}\right) \tag{2.50}$$

for $u > \epsilon$. Then one can check that the metric for which (2.49) is an orthonormal basis is complete with positive scalar curvature. In effect, a change of variable to $s = -\ln(u-\epsilon)$ shows that the metric is asymptotically cylindrical, with cross-section $S^1\backslash SNF$ having $\mathbb{C}P^N$-fibers of diameter proportionate to ϵ and base F of diameter proportionate to $\epsilon^{\frac{1}{4k}}$. As $N > 0$, the positive scalar curvature of the $\mathbb{C}P^N$ fibers ensures that the metric will have positive scalar curvature for small ϵ. Truncate the cylinder at a large distance and smooth the metric to a product near the boundary, while keeping positive scalar curvature. Let N_ϵ denote the corresponding manifold-with-boundary. Applying the Atiyah-Patodi-Singer theorem and the Lichnerowicz vanishing theorem to N_ϵ, we obtain

$$0 = \int_{N_\epsilon} \widehat{A}\,(TN_\epsilon) + \frac{1}{2}\eta\left(D_{\partial N_\epsilon}\right). \tag{2.51}$$

Now

$$\lim_{\epsilon \to 0} \int_{N_\epsilon} \widehat{A}\,(TN_\epsilon) = \int_{S^1\backslash M} \widehat{A}\left(T(S^1\backslash M)\right). \tag{2.52}$$

As in the proof of Proposition 9, since $M^{S^1}_{(2)} = \emptyset$,

$$\lim_{\epsilon \to 0} \eta\left(D_{\partial N_\epsilon}\right) = 0. \tag{2.53}$$

The proposition follows in this case.

In general, a neighborhood of F in M may not be S^1-isometric to DNF. Nevertheless, we can use the distance function from F to write $\widetilde{g}$ as $\left(\frac{4k-1}{4k}\right)^2 du^2 + h(u)$ where for $u > 0$, $h(u)$ is a metric on $S^1\backslash SNF$. For small u, $\widetilde{g}$ will be well-approximated by the metric of the form (2.48). Then we can deform $\widetilde{g}$ for small u to obtain a metric of positive scalar curvature and precisely of the form (2.48) for small u, to which we can apply the previous argument. $\square$

Remark : Suppose that M^{S^1} has codimension two in M. Then the orthonormal frame (2.48) becomes

$$\begin{aligned} \widehat{\tau}^u &= \frac{4k}{4k-1}du, \\ \widehat{\tau}^\alpha &= u^{\frac{1}{4k}}\tau^\alpha. \end{aligned} \tag{2.54}$$

We no longer have the benefit of the positive scalar curvature coming from $\mathbb{C}P^N$. Metrically with respect to $\widetilde{g}$, $S^1\backslash M$ has a "puffy" cone over $M^{S^1}_{(2)}$. If one could prove an index theorem for Dirac operators on such spaces, along with a vanishing theorem in the case of positive scalar curvature, one could remove the codimension restriction in Theorem 5.

2.4.2. Even or odd semifree S^1-actions Suppose that the spin manifold M has an S^1-action which is even or odd. If the S^1-action is free then $S^1\backslash M$ may not have a spin structure, but it always has a canonical spin^c structure. Namely, if the S^1-action is even, put $S(S^1\backslash M) = \mathbb{C} \times_{S^1} SM$, where $\mathbb{C}$ has the standard S^1-action. If the S^1-action is odd, let $\widehat{S^1}$ be the double cover of S^1. It acts on M through the quotient map $\widehat{S^1} \to S^1$. Consider the standard action of $\widehat{S^1}$ on $\mathbb{C}$. The infinitesimal action of $u(1)$ on $\mathbb{C} \times SM$ integrates to an $\widehat{S^1}$-action, so we can put $S(S^1\backslash M) = \mathbb{C} \times_{\widehat{S^1}} SM$. In either case, $S(S^1\backslash M)$ is the spinor bundle of a spin^c structure on $S^1\backslash M$.

Now suppose that the S^1-action is effective and semifree.

Lemma 2. *M^{S^1} is* spin^c.

Proof. Let F be a connected component of M^{S^1}, with normal bundle NF. We know that F is oriented. As the total space NF is diffeomorphic to a neighborhood of F in M, TNF inherits a spin structure. Let $p : NF \to F$ be projection to the base. Then $TNF = p^*NF \oplus p^*TF$. As NF has a complex structure, it has a canonical spin^c structure. Then p^*TF acquires a spin^c structure, and so does TF. □

Let $\xi^{S^1\backslash M}$ be the complex line bundle on $\left(S^1\backslash M\right) - M^{S^1}$ associated to the spin^c structure. It has an induced connection $\nabla^{\xi^{S^1\backslash M}}$. Let $c_1\left(\xi^{S^1\backslash M}\right) \in \Omega^2\left((S^1\backslash M) - M^{S^1}\right)$ be the corresponding characteristic form. Let $D_{M^{S^1}}$ be the spin^c Dirac operator on M^{S^1}.

Definition 7.

$$\widehat{\mathcal{A}}_{S^1}(M) = \int_{S^1\backslash M} \widehat{A}\left(T\left(S^1\backslash M\right)\right) \wedge e^{\frac{c_1(\xi^{S^1\backslash M})}{2}} \tag{2.55}$$
$$+\frac{1}{2}\left[\eta\left(D_{M^{S^1}}\right) + \dim\left(\operatorname{Ker}\left(D_{M^{S^1}}\right)\right)\right].$$

Proposition 10. *The number $\widehat{\mathcal{A}}_{S^1}(M)$ is an integer. If $\{g(\epsilon)\}_{\epsilon\in[0,1]}$ is a smooth* 1-*parameter family of S^1-invariant metrics on M and* $\dim\left(\operatorname{Ker}\left(D_{M^{S^1}}\right)\right)$ *is constant in ϵ then $\widehat{\mathcal{A}}_{S^1}(M)$ is constant in ϵ.*

Proof. For notational convenience, put

$$\widehat{\mathcal{A}}(TX) = \widehat{A}(TX) \wedge e^{\frac{c_1(\xi^X)}{2}}. \tag{2.56}$$

For small $r > 0$, let $N_r\left(M^{S^1}\right)$ be the r-neighborhood of M^{S^1} in $S^1\backslash M$. The manifold-with-boundary $(S^1\backslash M) - N_r\left(M^{S^1}\right)$ is spin^c and one can talk about the index $\mathrm{Ind}_r \in \mathbb{Z}$ of its Dirac operator. By the method of proof of Theorem 4, one finds that in $\mathbb{R}/\mathbb{Z}$,

$$\int_{S^1\backslash M} \widehat{\mathcal{A}}\left(T\left(S^1\backslash M\right)\right) \tag{2.57}$$
$$+\lim_{r\to 0}\frac{1}{2}\left[\eta\left(\partial N_r(M^{S^1})\right) + \dim\left(\operatorname{Ker}\left(\partial N_r(M^{S^1})\right)\right)\right] \equiv 0.$$

(The spectral invariants in the above equation are with respect to spin^c Dirac operators.) Let F be a connected component of M^{S^1}. Then $\partial N_r(F)$ is a fiber bundle over F. In terms of the complex structure on a fiber Z, we can write

$$D_Z = \overline{\partial} + \overline{\partial}^* : \Omega^{0,even}(Z) \to \Omega^{0,odd}(Z). \tag{2.58}$$

As the fiber Z is a complex projective space, $\mathrm{Ker}(D_Z) = \Omega^{0,0}(Z) = \mathbb{C}$ consists of the constant functions on the fibers and $\mathrm{Ker}(D_Z^*) = 0$. Hence $\mathrm{Ind}(D_Z)$ is a trivial complex line bundle on F. It follows from [16] that in $\mathbb{R}/\mathbb{Z}$,

$$\lim_{r\to 0} \frac{1}{2} \left[\eta\left(\partial N_r(F)\right) + \dim\left(\mathrm{Ker}\left(\partial N_r(F)\right)\right)\right] \tag{2.59}$$
$$\equiv \frac{1}{2}\left[\int_F \widetilde{\eta} \wedge \widehat{\mathcal{A}}(TF) + \eta(D_F) + \dim\left(\mathrm{Ker}\left(D_F\right)\right)\right].$$

As in the proof of Theorem 4, $\widetilde{\eta} = 0$. Thus $\widehat{\mathcal{A}}_{S^1}(M)$ is an integer.

The rest of the proof of Proposition 10 is similar to that of Proposition 9. We omit the details. □

2.5. *General S^1-actions*

Let S^1 act effectively on M. There are suborbifolds $\mathcal{O}$ of $(S^1\backslash M) - M^{S^1}$ defined as in Subsection 2.2.

Proposition 11.

$$\sigma_{S^1}(M) = \sum_{\mathcal{O}} \frac{1}{m_{\mathcal{O}}} \int_{\mathcal{O}} L(\mathcal{O}) + \eta\left(M^{S^1}\right). \tag{2.60}$$

Proof. The proof is a combination of those of (2.11) and Theorem 4. Let F be a connected component of M^{S^1}. Let NF be the normal bundle of F in M. It has an S^1-action by orthogonal automorphisms, which is fixed-point-free on $NF - F$. Let SNF be the sphere bundle of NF. Then $S^1\backslash SNF$ is an orbifold. For $r > 0$, let $N_r(F)$ be the r-neighborhood of F in $S^1\backslash M$. Then for small r, $\partial N_r(F)$ is an orbifold. We define the η-invariant of $\partial N_r(F)$ using the tangential signature operator on orbifold-differential forms on $\partial N_r(F)$, i.e. S^1-basic differential forms on the preimage of $\partial N_r(F)$ in M. Then the method of proof of Theorem 4 goes through with minor changes. □

3. Equivariant higher indices

3.1. *Equivariant Novikov conjectures*

Let M^n be a closed oriented connected manifold. Let Γ' be a countable discrete group and let $\rho : \pi_1(M) \to \Gamma'$ be a surjective homomorphism. There is an induced continuous map $\nu : M \to B\Gamma'$, defined up to homotopy. Let $L \in \mathrm{H}_{n-4*}(M;\mathbb{Q})$ be the homology L-class of M, i.e. the Poincaré dual of the cohomology L-class. The Novikov Conjecture hypothesizes that $\nu_*(L) \in \mathrm{H}_{n-4*}(B\Gamma';\mathbb{Q})$ is an oriented homotopy invariant of M. Another way to state this is to let $D \in \mathrm{KO}_n(M)$ be the KO-homology class of the signature operator. The Novikov Conjecture says that $\nu_*(D) \otimes_{\mathbb{Z}} 1 \in \mathrm{KO}_n(B\Gamma') \otimes_{\mathbb{Z}} \mathbb{Q}$ should be an oriented homotopy invariant of M. It is usually assumed that $\Gamma' = \pi_1(M)$, although this is not necessary.

Now suppose that a compact Lie group G acts on M in an orientation-preserving way. One would like to extend the Novikov Conjecture to the G-equivariant setting. One approach is to extend the classifying space construction. The idea is that $B\pi_1(M)$ has exactly the information about $\pi_0(M)$ and $\pi_1(M)$. In the equivariant case one wants a space with a G-action, constructed from the data $\{\pi_0(M^H)\}$ and $\{\pi_1(M^H)\}$ as H runs over the closed subgroups of G. Such a space $B\pi(M)$ is constructed in [34]. It has the property that each connected component of $B\pi(M)^H$ is aspherical, and there is a G-map $\nu : M \to B\pi(M)$, unique up to G-homotopy, which induces an isomorphism from $\pi_0(M^H)$ to $\pi_0(B\pi(M)^H)$ and an isomorphism on π_1 of each connected component of M^H. Choosing a G-invariant Riemannian metric on M, there is a G-invariant signature operator $D \in \mathrm{KO}_n^G(M)$. Then one Equivariant Novikov Conjecture would be that $\nu_*(D) \otimes_{\mathbb{Z}} 1 \in \mathrm{KO}_n^G(B\pi(M)) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an oriented G-homotopy invariant of M [38].

As was pointed out in [38, p. 31], this conjecture is false in the case of free S^1-actions. In that case $B\pi(M) = S^\infty$, $\mathrm{KO}_n^G(B\pi(M)) = \mathrm{KO}_{n-1}(\mathbb{C}P^\infty)$ and $\mathrm{KO}_n^G(B\pi(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{H}_{n-1-4*}(\mathbb{C}P^\infty; \mathbb{Q})$. The principal S^1-bundle M is classified by a map $f : (S^1 \backslash M) \to \mathbb{C}P^\infty$, and $\nu_*(D) \otimes_{\mathbb{Z}} 1 = f_*(L(S^1\backslash M)) \in \mathrm{H}_{n-1-4*}(\mathbb{C}P^\infty; \mathbb{Q})$. If X is a homotopy-$\mathbb{C}P^N$, let M be the total space of the S^1-bundle associated to the standard generator of $\mathrm{H}^2(X; \mathbb{Z}) = \mathrm{H}^2\left(\mathbb{C}P^N; \mathbb{Z}\right)$. Then $\nu_*(D) \otimes_{\mathbb{Z}} 1$ can be identified with the rational homology L-class of X. If $N > 2$ then it follows from surgery theory that there is an infinite number of nonhomeomorphic homotopy-$\mathbb{C}P^N$'s $\{X_i\}_{i=1}^\infty$ with distinct rational homology L-classes. The S^1-actions on the corresponding homotopy-spheres $\{M_i\}_{i=1}^\infty$ will be mutually homotopy equivalent, showing the falsity of the conjecture. The rest of [38] is devoted to looking at the conjecture under some finiteness assumptions on $B\pi(M)$.

Another Equivariant Novikov Conjecture uses the classifying space $\underline{E}G'$ for proper G'-actions, where G' is a Lie group with a countable number of connected components [5]. Let Γ' and ρ be as above. There is an induced connected normal Γ'-covering M' of M. Let $\pi : M' \to M$ be the projection map. Define a group G' by

$$G' = \{(\phi, g) \in \mathrm{Diff}\left(M'\right) \times G : \pi \circ \phi = g \cdot \pi\}. \tag{3.1}$$

There is a G'-invariant signature operator $D \in \mathrm{KO}_n^{G'}(M')$. The conjecture states that $\nu_*(D) \in \mathrm{KO}_n^{G'}(\underline{E}G')$ is an oriented G-homotopy invariant of M [5], [38, Proposition 2.10].

This conjecture is very reasonable. However, it seems to be more useful when G is finite. Suppose, for example, that $G = S^1$ and $\Gamma' = \{e\}$. Then $G' = S^1$,

$\underline{E}S^1$ is a point and if n is divisible by four then $\mathrm{KO}_n^{S^1}(\mathrm{pt.})$ is a countable sum of $\mathbb{Z}$'s, while it vanishes rationally otherwise. If n is divisible by four then the only information in $\nu_n(D) \in \mathrm{KO}_n^{S^1}(\mathrm{pt.})$ is the ordinary signature of M. If S^1 acts freely on M then $M = \partial \left(D^2 \times_{S^1} M\right)$ and so its signature vanishes. Thus in the case of free S^1-actions, the second Equivariant Novikov Conjecture is true but vacuous.

In order to construct higher signatures of $S^1 \backslash M$, we will use the higher eta-invariant of [30]. We now recall the construction of [30], with some modifications. We will let groups act on the left, as in [31], instead of on the right, as in [30]. The differential form conventions will be as in [31].

3.2. Higher eta-invariant

Let Γ be a finitely generated discrete group and let $C_r^* \Gamma$ be the reduced group C^*-algebra.

Assumption 1 *There is a Fréchet locally m-convex *-algebra $\mathfrak{B}$ such that*

1. $\mathbb{C}\Gamma \subset \mathfrak{B} \subset C_r^* \Gamma$.
2. *$\mathfrak{B}$ is stable under the holomorphic functional calculus in $C_r^* \Gamma$.*
3. *For each $\tau \in \mathrm{H}^q(\Gamma; \mathbb{C})$, there is a representative cocycle $\tau \in Z^q(\Gamma; \mathbb{C})$ such that the ensuing cyclic cocycle $Z_\tau \in ZC^q(\mathbb{C}\Gamma)$ extends to a continuous cyclic cocycle on $\mathfrak{B}$.*

It is know that such "smooth subalgebras" $\mathfrak{B}$ exist if Γ is virtually nilpotent or Gromov-hyperbolic [15, Section III.5], [21].

Let F be a closed oriented Riemannian manifold of dimension n. Let $\rho : \pi_1(F) \to \Gamma$ be a surjective homomorphism. There is an induced connected normal Γ-covering F' of F, on which $g \in \Gamma$ acts on the left by $L_g \in \mathrm{Diff}\left(F'\right)$. Let $\pi : F' \to F$ be the projection map.

Put $\mathcal{D} = \mathfrak{B} \times_\Gamma F'$, a $\mathfrak{B}$-vector bundle on F, and put $\overline{\mathcal{D}} = (C_r^* \Gamma) \times_\Gamma F'$, a $C_r^* \Gamma$-vector bundle on F. Both $\mathcal{D}$ and $\overline{\mathcal{D}}$ are local systems.

Assumption 2 *If n is even then* $\mathrm{H}^{\frac{n}{2}}(F; \overline{\mathcal{D}}) = 0$. *If n is odd then* $\mathrm{H}^{\frac{n \pm 1}{2}}(F; \overline{\mathcal{D}}) = 0$.

The cohomology involved in Assumption 2 is ordinary unreduced cohomology; that is, we quotient by $\mathrm{Im}(d)$, not its closure. Equivalent formulations are:

1. If n is even then the spectrum of the L^2-Laplacian on F' is strictly positive in degree $\frac{n}{2}$. If n is odd then the spectrum of the L^2-Laplacian on F' is strictly positive in degrees $\frac{n \pm 1}{2}$.
2. If n is even then the Laplacian on $\Omega^{\frac{n}{2}}(F; \overline{\mathcal{D}})$ is invertible. If n is odd then the Laplacians on $\Omega^{\frac{n \pm 1}{2}}(F; \overline{\mathcal{D}})$ are invertible.
3. If n is even then the Laplacian on $\Omega^{\frac{n}{2}}(F; \mathcal{D})$ is invertible. If n is odd then the Laplacians on $\Omega^{\frac{n \pm 1}{2}}(F; \mathcal{D})$ are invertible.

4. If n is even then $\mathrm{H}^{\frac{n}{2}}(F;\mathcal{D}) = 0$. If n is odd then $\mathrm{H}^{\frac{n\pm1}{2}}(F;\mathcal{D}) = 0$.

We use the notions of Hermitian complex and regular Hermitian complex from [22] and [33]. Using [31, Section 4.1 and Proposition 10], one can generalize the results of [22] from $C_r^*\Gamma$-complexes to $\mathfrak{B}$-complexes.

Proposition 12. *There is a cochain complex W^* of finitely-generated projective $\mathfrak{B}$-modules such that*

1. *W^* is a regular Hermitian complex.*
2. *$W^{\frac{n}{2}} = 0$ if n is even and $W^{\frac{n\pm1}{2}} = 0$ if n is odd.*
3. *The complex $\Omega^*(F;\mathcal{D})$ of smooth $\mathcal{D}$-valued differential forms on F is homotopy equivalent to W^*.*

Proof. We will implicitly use results from [31, Proposition 10 and Section 6.1] concerning spectral analysis involving $\mathfrak{B}$. First, let K be a triangulation of F. Then $\Omega^*(F;\mathcal{D})$ is homotopy equivalent to the simplicial cochain complex $C^*(K;\mathcal{D})$. The latter is a Hermitian complex of finitely-generated free $\mathfrak{B}$-modules. By [22, Proposition 2.4], it is homotopy equivalent to a regular Hermitian complex V^* of finitely-generated projective $\mathfrak{B}$-modules. Suppose that n is even. We have $\mathrm{H}^{\frac{n}{2}}(V) = 0$. Put

$$W^i = \begin{cases} V^i & \text{if } i < \frac{n}{2} - 1, \\ \mathrm{Ker}\left(d : V^{\frac{n}{2}-1} \longrightarrow V^{\frac{n}{2}}\right) & \text{if } i = \frac{n}{2} - 1, \\ 0 & \text{if } i = \frac{n}{2}, \\ \mathrm{Im}\left(d : V^{\frac{n}{2}} \longrightarrow V^{\frac{n}{2}+1}\right)^{\perp} & \text{if } i = \frac{n}{2} + 1, \\ V^i & \text{if } i > \frac{n}{2} + 1. \end{cases} \tag{3.2}$$

Then W^* is a regular Hermitian complex. There are homotopy equivalences $V^* \longrightarrow W^*$ and $W^* \longrightarrow V^*$ given by

$$\begin{array}{ccccccccccc} \ldots \longrightarrow & V^{\frac{n}{2}-2} & \longrightarrow & V^{\frac{n}{2}-1} & \longrightarrow & V^{\frac{n}{2}} & \longrightarrow & V^{\frac{n}{2}+1} & \longrightarrow & V^{\frac{n}{2}+2} & \longrightarrow \ldots \\ & \mathrm{Id.} \downarrow & & p \downarrow & & 0 \downarrow & & p \downarrow & & \mathrm{Id.} \downarrow & \\ \ldots \longrightarrow & W^{\frac{n}{2}-2} & \longrightarrow & W^{\frac{n}{2}-1} & \longrightarrow & 0 & \longrightarrow & W^{\frac{n}{2}+1} & \longrightarrow & W^{\frac{n}{2}+2} & \longrightarrow \ldots \end{array} \tag{3.3}$$

and

$$\begin{array}{ccccccccccc} \ldots \longrightarrow & W^{\frac{n}{2}-2} & \longrightarrow & W^{\frac{n}{2}-1} & \longrightarrow & 0 & \longrightarrow & W^{\frac{n}{2}+1} & \longrightarrow & W^{\frac{n}{2}+2} & \longrightarrow \ldots \\ & \mathrm{Id.} \downarrow & & i \downarrow & & 0 \downarrow & & i \downarrow & & \mathrm{Id.} \downarrow & \\ \ldots \longrightarrow & V^{\frac{n}{2}-2} & \longrightarrow & V^{\frac{n}{2}-1} & \longrightarrow & V^{\frac{n}{2}} & \longrightarrow & V^{\frac{n}{2}+1} & \longrightarrow & V^{\frac{n}{2}+2} & \longrightarrow \ldots, \end{array} \tag{3.4}$$

where p denotes orthogonal projection and i is inclusion. The cochain homotopy operators are

$$\dots \longleftarrow V^{\frac{n}{2}-2} \xleftarrow{0} V^{\frac{n}{2}-1} \xleftarrow{d^*\triangle^{-1}} V^{\frac{n}{2}} \xleftarrow{\triangle^{-1}d^*} V^{\frac{n}{2}+1} \xleftarrow{0} V^{\frac{n}{2}+2} \longleftarrow \dots \tag{3.5}$$

and

$$\dots \longleftarrow W^{\frac{n}{2}-2} \xleftarrow{0} W^{\frac{n}{2}-1} \xleftarrow{0} 0 \xleftarrow{0} W^{\frac{n}{2}+1} \xleftarrow{0} W^{\frac{n}{2}+2} \longleftarrow \dots \tag{3.6}$$

If n is odd, we have $\mathrm{H}^{\frac{n\pm1}{2}}(V) = 0$. Put

$$W^i = \begin{cases} V^i & \text{if } i < \frac{n-3}{2}, \\ \mathrm{Ker}\left(d : V^{\frac{n-3}{2}} \longrightarrow V^{\frac{n-1}{2}}\right) & \text{if } i = \frac{n-3}{2}, \\ 0 & \text{if } i = \frac{n\pm1}{2}, \\ \mathrm{Im}\left(d : V^{\frac{n+1}{2}} \longrightarrow V^{\frac{n+3}{2}}\right)^{\perp} & \text{if } i = \frac{n+3}{2}, \\ V^i & \text{if } i > \frac{n+3}{2}. \end{cases} \tag{3.7}$$

Then W^* is a regular Hermitian complex. There are homotopy equivalences $V^* \longrightarrow W^*$ and $W^* \longrightarrow V^*$ given by

$$\begin{array}{ccccccccccc} \dots \longrightarrow & V^{\frac{n-3}{2}} & \longrightarrow & V^{\frac{n-1}{2}} & \longrightarrow & V^{\frac{n+1}{2}} & \longrightarrow & V^{\frac{n+3}{2}} & \longrightarrow \dots \\ & p \downarrow & & 0 \downarrow & & 0 \downarrow & & p \downarrow & \\ \dots \longrightarrow & W^{\frac{n-3}{2}} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & W^{\frac{n+3}{2}} & \longrightarrow \dots \end{array} \tag{3.8}$$

and

$$\begin{array}{ccccccccccc} \dots \longrightarrow & W^{\frac{n-3}{2}} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & W^{\frac{n+3}{2}} & \longrightarrow \dots \\ & i \downarrow & & 0 \downarrow & & 0 \downarrow & & i \downarrow & \\ \dots \longrightarrow & V^{\frac{n-3}{2}} & \longrightarrow & V^{\frac{n-1}{2}} & \longrightarrow & V^{\frac{n+1}{2}} & \longrightarrow & V^{\frac{n+3}{2}} & \longrightarrow \dots \end{array} \tag{3.9}$$

The cochain homotopy operators are

$$\dots \xleftarrow{0} V^{\frac{n-3}{2}} \xleftarrow{d^*\triangle^{-1}} V^{\frac{n-1}{2}} \xleftarrow{d^*\triangle^{-1}} V^{\frac{n+1}{2}} \xleftarrow{\triangle^{-1}d^*} V^{\frac{n+3}{2}} \xleftarrow{0} \dots \tag{3.10}$$

and

$$\dots \xleftarrow{0} W^{\frac{n-3}{2}} \xleftarrow{0} 0 \xleftarrow{0} 0 \xleftarrow{0} W^{\frac{n+3}{2}} \xleftarrow{0} \dots \tag{3.11}$$

The proposition follows. $\square$

We briefly review some notation from [30] and [31]. Let $\Omega^*(\mathfrak{B})$ be the universal graded differential algebra of $\mathfrak{B}$ and let $\overline{\Omega}^*(\mathfrak{B})$ be the quotient by (the Fréchet closure of) the graded commutator. Let $\overline{\mathrm{H}}^*(\mathfrak{B})$ denote the cohomology of the complex $\overline{\Omega}^*(\mathfrak{B})$. If E is a complex vector bundle on F, put $\mathcal{E} = \mathcal{D} \otimes E$. There is a bigraded complex $\Omega^{*,*}(F, \mathfrak{B})$ which, roughly speaking, consists of differential forms on F along with noncommutative differential forms on $\mathfrak{B}$.

Let $h \in C_0^\infty(F')$ be a real-valued function satisfying $\sum_{g\in\Gamma} L_g^* h = 1$. One obtains a connection

$$\nabla^{\mathcal{D}} : C^\infty(F; \mathcal{D}) \to \Omega^{1,0}(F, \mathfrak{B}; \mathcal{D}) \oplus \Omega^{0,1}(F, \mathfrak{B}; \mathcal{D}) \tag{3.12}$$

on $\mathcal{D}$. The $(1, 0)$-part of the connection comes from the flat structure of $\mathcal{D}$ as a vector bundle on F. The $(0, 1)$-part of the connection is constructed using h. Locally on F, using the flat structure on $\mathcal{D}$, one can write

$$\nabla^{\mathcal{D}} = \sum_{\mu=1}^{dim(F)} dx^\mu \partial_\mu + \sum_{g\in\Gamma} dg \nabla_g. \tag{3.13}$$

Suppose that $\dim(F)$ is even. Take $E = \Lambda^*(T^*F)$, a vector bundle on F with a $\mathbb{Z}_2$-grading coming from Hodge duality. The signature operator $d + d^* : C^\infty(F; E) \to C^\infty(F; E)$ couples to $\mathcal{D}$ to give a Dirac-type operator

$$Q : C^\infty(F; \mathcal{E}) \to C^\infty(F; \mathcal{E}) \tag{3.14}$$

which commutes with the left-action of $\mathcal{B}$. We can "quantize" the dx^μ-variables in (3.13) to obtain a superconnection

$$D : C^\infty(F; \mathcal{E}) \to C^\infty(F; \mathcal{E}) \oplus \Omega^{0,1}(F, \mathfrak{B}; \mathcal{E}) \tag{3.15}$$

given by

$$D = Q + \nabla^{\mathcal{E},0,1}. \tag{3.16}$$

Given $s > 0$, we rescale the Clifford variables in (3.16) to obtain

$$D_s = sQ + \nabla^{\mathcal{E},0,1}. \tag{3.17}$$

It extends by Leibniz' rule to an odd map

$$D_s : \Omega^{0,*}(F, \mathfrak{B}; \mathcal{E}) \to \Omega^{0,*}(F, \mathfrak{B}; \mathcal{E}). \tag{3.18}$$

(This is like a superconnection on a fiber bundle whose base is the noncommutative space specified by $\mathfrak{B}$.) Using the supertrace TR_s on integral operators on F, one can define

$$\mathrm{TR}_s\left(e^{-D_s^2}\right) \in \overline{\Omega}^{even}(\mathfrak{B}). \tag{3.19}$$

A form of the local Atiyah-Singer index theorem says

$$\lim_{s\to 0} \mathrm{TR}_s \left(e^{-D_s^2}\right) = \int_F L(TF) \wedge \mathrm{ch}\left(\nabla^{\mathcal{D}}\right), \tag{3.20}$$

where $\mathrm{ch}\left(\nabla^{\mathcal{D}}\right) \in \Omega^*\left(F; \overline{\Omega}^*(\mathfrak{B})\right)$ is the Chern character.

Consider

$$\mathrm{TR}_s \left(Q e^{-D_s^2}\right) \in \overline{\Omega}^{odd}(\mathfrak{B}). \tag{3.21}$$

We would like to define the noncommutative eta-form by

$$\int_0^\infty \mathrm{TR}_s \left(Q e^{-D_s^2}\right) ds. \tag{3.22}$$

As shown in [30, Proposition 26], there is no problem with the small-s integration. In [30, Section 4.7] we argued that the large-s integration is also well-defined, because of Hodge duality. However, Eric Leichtnam and Paolo Piazza pointed out to me that there are technical problems with the argument in [30, Section 4.7]. Consequently, we do not know whether or not the integral in (3.22) is convergent for large-s. We now present a way to get around this problem.

Take n of either parity. For $-1 \le i \le n+1$, put

$$\widehat{W}^i = \begin{cases} W^{i+1} & \text{if } -1 \le i < \frac{n}{2}, \\ 0 & \text{if } n \text{ is even and } i = \frac{n}{2}, \\ W^{i-1} & \text{if } \frac{n}{2} < i \le n+1. \end{cases} \tag{3.23}$$

and

$$C^i = \Omega^i(F; \mathcal{D}) \oplus \widehat{W}^i. \tag{3.24}$$

Let $f : \Omega^*(F; \mathcal{D}) \to W^*$ be a homotopy equivalence of Hermitian complexes. Let $g : W^* \to \Omega^*(F; \mathcal{D})$ be the adjoint of f with respect to the nondegenerate Hermitian form $H_W : W^i \otimes W^{n-i} \to \mathfrak{B}$. Given $\epsilon \in \mathbb{R}$, define a differential d^C on C^* by

$$d_i^C = \begin{cases} \begin{pmatrix} d & \epsilon g \\ 0 & -d \end{pmatrix} & \text{if } i < \frac{n}{2}, \\ d & \text{if } n \text{ is even and } i = \frac{n}{2}, \\ \begin{pmatrix} d & 0 \\ \epsilon f & -d \end{pmatrix} & \text{if } i > \frac{n}{2}. \end{cases} \tag{3.25}$$

There is a nondegenerate form H on C^* given by

$$H((\omega, w), (\omega', w')) = \int_F \omega \wedge \overline{\omega'} + (-1)^{i+1} H_W(w, w') \tag{3.26}$$

for $(\omega, w) \in C^i$, $(\omega', w') \in C^{n-i}$, if $i < \frac{n}{2}$, and

$$H(\omega, \omega') = \int_F \omega \wedge \overline{\omega'} \tag{3.27}$$

if $\omega, \omega' \in \Omega^{\frac{n}{2}}(F; \mathcal{D})$. Then one can check that C^* is a regular Hermitian complex.

If $\epsilon \neq 0$ then C^* has vanishing cohomology, as the complex is the mapping cone of g in degrees less than $\frac{n}{2}$ and the adjoint in degrees greater than $\frac{n}{2}$. It follows that if $\epsilon \neq 0$ then the Laplacian $d^C \left(d^C\right)^* + \left(d^C\right)^* d^C$ of C^* has a bounded inverse.

Let $\nabla^W : W^* \to \Omega^1(\mathfrak{B}) \otimes_{\mathfrak{B}} W^*$ be a self-dual connection on W^*. There is a direct sum connection

$$\nabla^C = \nabla^{\mathcal{D},0,1} \oplus \nabla^W \tag{3.28}$$

on C^*.

Suppose that n is even. Put $Q^C = d^C + \left(d^C\right)^*$. We define a superconnection $D_s(\epsilon)$ on C^* by

$$D_s(\epsilon) = s Q^C + \nabla^C. \tag{3.29}$$

Let $\epsilon(s)$ be a smooth function of $s \in \mathbb{R}^+$ which is identically zero for $s \in (0, 1]$ and identically one for $s \geq 2$. Put

$$\widetilde{\eta}(s) = \mathrm{TR}_s \left(\frac{d D_s(\epsilon(s))}{ds} e^{-D_s^2(\epsilon(s))} \right). \tag{3.30}$$

Proposition 13. *For $s \in (0, 1]$,*

$$\widetilde{\eta}(s) = \mathrm{TR}_s \left(Q e^{-D_s^2} \right), \tag{3.31}$$

as in (3.22).

Proof. As $\epsilon(s) = 0$, the factors $\Omega^*(F; \mathcal{D})$ and $\widehat{W}^*$ in C^* completely decouple and it is enough to show that the analog of $\widetilde{\eta}(s)$ for $\widehat{W}^*$,

$$\mathrm{TR}_s \left(Q^{\widehat{W}} e^{-(D_s^{\widehat{W}})^2} \right), \tag{3.32}$$

vanishes. This follows from Hodge duality as in [30, p. 227]. Namely, define $T \in \mathrm{End}(\widehat{W}^*)$ to be multiplication by sign $\left(i - \frac{n}{2}\right)$ on $\widehat{W}^i$. It is odd with respect to the Hodge duality on $\widehat{W}^*$. Then

$$\begin{aligned} \mathrm{TR}_s \left(Q^{\widehat{W}} e^{-(D_s^{\widehat{W}})^2} \right) &= \mathrm{TR}_s \left(T^{-1} T Q^{\widehat{W}} e^{-(D_s^{\widehat{W}})^2} \right) \\ &= -\mathrm{TR}_s \left(T Q^{\widehat{W}} e^{-(D_s^{\widehat{W}})^2} T^{-1} \right) \\ &= -\mathrm{TR}_s \left(T T^{-1} Q^{\widehat{W}} e^{-(D_s^{\widehat{W}})^2} \right) \\ &= -\mathrm{TR}_s \left(Q^{\widehat{W}} e^{-(D_s^{\widehat{W}})^2} \right) = 0. \end{aligned} \tag{3.33}$$

□

Definition 8. *Define* $\widetilde{\eta} \in \overline{\Omega}^{odd}(\mathfrak{B})/\mathrm{Im}(d)$ *by*

$$\widetilde{\eta} = \int_0^\infty \widetilde{\eta}(s) ds. \tag{3.34}$$

By [30, Proposition 26], the integrand of (3.34) is integrable for small-s. Using the techniques of [31, Section 6.1] and [32, Section 4], one can show that it is also integrable for large-s. This uses the invertibility of the Laplacian of C^* for $s \geq 2$, i.e. $\epsilon = 1$. Note that $\widetilde{\eta}$ is defined modulo $\mathrm{Im}(d)$. It is not hard to show that $\widetilde{\eta}$ is independent of the choice of $\epsilon(s)$.

Proposition 14. $\widetilde{\eta}$ *is independent of the choice of* W.

Proof. Let W' be another regular Hermitian complex which is homotopy equivalent to $\Omega^*(F;\mathcal{D})$, with $W'^{,\frac{n}{2}} = 0$. Let $h : W' \to W$ be a homotopy equivalence.
For $-1 \leq i \leq n+1$, put

$$D^i = \begin{cases} \Omega^i(F;\mathcal{D}) \oplus W^i \oplus W^{i+1} \oplus W'^{,i+1} & \text{if } -1 \leq i < \frac{n}{2}, \\ \Omega^{\frac{n}{2}}(F;\mathcal{D}) & \text{if } n \text{ is even and } i = \frac{n}{2}, \\ \Omega^i(F;\mathcal{D}) \oplus W^i \oplus W^{i-1} \oplus W'^{,i-1} & \text{if } \frac{n}{2} < i \leq n+1. \end{cases} \tag{3.35}$$

Given $\begin{pmatrix} \epsilon_1 & \epsilon_2 \\ \epsilon_3 & \epsilon_4 \end{pmatrix} \in M_2(\mathbb{R})$, define a differential d^D on D^* by

$$d_i^D = \begin{cases} \begin{pmatrix} d & 0 & \epsilon_1 g & \epsilon_2 g \circ h \\ 0 & d & \epsilon_3 & \epsilon_4 h \\ 0 & 0 & -d & 0 \\ 0 & 0 & 0 & -d \end{pmatrix} & \text{if } i < \frac{n}{2}, \\ d & \text{if } n \text{ is even and } i = \frac{n}{2}, \\ \begin{pmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ \epsilon_1 g^* & \epsilon_3 & -d & 0 \\ \epsilon_2 h^* \circ g^* & \epsilon_4 h^* & 0 & -d \end{pmatrix} & \text{if } i > \frac{n}{2}. \end{cases} \tag{3.36}$$

As

$$\begin{pmatrix} \epsilon_1 g & \epsilon_2 g \circ h \\ \epsilon_3 & \epsilon_4 h \end{pmatrix} = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \epsilon_1 & \epsilon_2 \\ \epsilon_3 & \epsilon_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}, \tag{3.37}$$

the complex D^* has vanishing cohomology if $\begin{pmatrix} \epsilon_1 & \epsilon_2 \\ \epsilon_3 & \epsilon_4 \end{pmatrix}$ is invertible. Given $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R})$, put

$$\begin{pmatrix} \epsilon_1 & \epsilon_2 \\ \epsilon_3 & \epsilon_4 \end{pmatrix} = \epsilon(s) A, \tag{3.38}$$

where $\epsilon(s)$ is as before. Define the noncommutative eta-form of D^* as in (3.34).

There is a smooth path in $\mathrm{GL}(2, \mathbb{R})$ from $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. It follows from [30, (50)] that the corresponding eta-forms of D^* differ by something in $\mathrm{Im}(d)$. Consider the eta-form coming from $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. In this case, D^* splits into the sum of two complexes, one involving Ω^* and W^*, the other involving W^* and $W^{\prime,*}$. By the Hodge duality argument of Proposition 13, the eta-form of the second subcomplex vanishes. Hence when $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we recover the eta-form of the complex C^* constructed from Ω^* and W^*. Similarly, when $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we recover the eta-form constructed from Ω^* and $W^{\prime,*}$. The proposition follows. $\square$

If n is odd then one can define the higher eta-form using an extra Clifford variable as in [30, Definitions 2,11].

3.3. "Moral" fundamental group of $S^1\backslash M$

Let M be a closed oriented smooth manifold with an effective S^1-action. Let Γ' be a finitely generated discrete group and let $\rho : \pi_1(M) \to \Gamma'$ be a surjective homomorphism. There is an induced connected normal Γ'-covering M' of M, on which $\gamma' \in \Gamma'$ acts on the left by $L_{\gamma'} \in \mathrm{Diff}(M')$. Let $\pi : M' \to M$ be the projection map.

Define a Lie group G' as in (3.1), with $G = S^1$. As the generator of the S^1-action on M can be lifted to a vector field on M', there is a short exact sequence

$$1 \longrightarrow \Gamma' \longrightarrow G' \longrightarrow S^1 \longrightarrow 1. \tag{3.39}$$

The homotopy exact sequence of this fibration gives

$$1 \longrightarrow \pi_1\left(G'\right) \longrightarrow \mathbb{Z} \longrightarrow \Gamma' \longrightarrow \pi_0\left(G'\right) \longrightarrow 1. \tag{3.40}$$

Put $\widehat{\Gamma} = \pi_0\left(G'\right)$. We will think of $\widehat{\Gamma}$ as the "moral" fundamental group of $S^1\backslash M$, although it may not be the same as $\pi_1(S^1\backslash M)$; $\widehat{\Gamma}$ also appears in the work of Browder-Hsiang [12]. Fixing a basepoint $m_0 \in M$, let o be the homotopy class of the orbit of m_0 in $\pi_1(M, m_0)$. From (3.40), $\widehat{\Gamma} = \Gamma'/\rho(\langle o\rangle)$, where $\langle o\rangle$ is the central subgroup of $\pi_1(M, m_0)$ generated by o. If $M^{S^1} \neq \emptyset$ then it is natural to take $m_0 \in M^{S^1}$, showing that $\langle o\rangle = \{e\}$.

Let G'_0 be the connected component of the identity of G'. It is a copy of either S^1 or $\mathbb{R}$. Put $\widehat{M} = G'_0\backslash M'$. Then $\widehat{\Gamma}$ acts properly and cocompactly on $\widehat{M}$, with $\widehat{\Gamma}\backslash\widehat{M} = S^1\backslash M$. Let $p : \widehat{M} \to S^1\backslash M$ be the quotient map. Putting

$\widehat{M^{S^1}} = p^{-1}\left(M^{S^1}\right)$, we can describe $\widehat{M^{S^1}}$ as the cover of M^{S^1} induced from the composite map

$$\pi_1\left(M^{S^1}\right) \longrightarrow \pi_1(M) \xrightarrow{\rho} \Gamma' \longrightarrow \widehat{\Gamma}. \tag{3.41}$$

The complement $\widehat{M} - \widehat{M^{S^1}}$ has a natural orbifold structure.

We construct certain differential forms on the strata of $S^1\backslash M$. Let $h \in C_0^\infty\left(\widehat{M^{S^1}}\right)$ satisfy

$$\sum_{\widehat{\gamma}\in\widehat{\Gamma}} L^*_{\widehat{\gamma}} h = 1; \tag{3.42}$$

it is easy to construct such functions. Let N be a small neighborhood of M^{S^1} in $S^1\backslash M$ which is diffeomorphic to the mapping cylinder of a fiber bundle, whose fibers are weighted complex projective spaces and whose base is M^{S^1}. Let $\widehat{N}$ be the preimage of N in $\widehat{M}$, with projection $\widehat{q} : \widehat{N} \to \widehat{M^{S^1}}$. Consider $\widehat{q}^* h$ on $\widehat{N}$. It can be extended to a compactly-supported function H on $\widehat{M}$ which is smooth, in the orbifold sense, on $\widehat{M} - \widehat{M^{S^1}}$ and which satisfies

$$\sum_{\widehat{\gamma}\in\widehat{\Gamma}} L^*_{\widehat{\gamma}} H = 1. \tag{3.43}$$

Consider the group cochains

$$\begin{aligned} C^k(\widehat{\Gamma}) = \Big\{\tau : \widehat{\Gamma}^{k+1} \to \mathbb{R} : \tau \text{ is skew and for all } (\widehat{\gamma}_0, \ldots, \widehat{\gamma}_k) \in \widehat{\Gamma}^{k+1} \\ \text{and } z \in \widehat{\Gamma}, \tau\left(\widehat{\gamma}_0 z, \widehat{\gamma}_1 z, \ldots, \widehat{\gamma}_k z\right) = \tau\left(\widehat{\gamma}_0, \widehat{\gamma}_1, \ldots, \widehat{\gamma}_k\right)\Big\}. \end{aligned} \tag{3.44}$$

Suppose that τ is a cocycle, i.e.

$$\sum_{j=0}^{k+1} (-1)^j \tau\left(\widehat{\gamma}_0, \ldots, \widehat{\widehat{\gamma}_j}, \ldots, \widehat{\gamma}_{k+1}\right) = 0. \tag{3.45}$$

Definition 9. *Define an orbifold form $\widehat{\omega} \in \Omega^k\left(\widehat{M} - \widehat{M^{S^1}}\right)$ by*

$$\widehat{\omega} = \sum_{\widehat{\gamma}_1,\ldots,\widehat{\gamma}_k} \tau\left(\widehat{\gamma}_1, \ldots, \widehat{\gamma}_k, e\right) L^*_{\widehat{\gamma}_1} dH \wedge \ldots \wedge L^*_{\widehat{\gamma}_k} dH. \tag{3.46}$$

Define $\widehat{\mu} \in \Omega^k\left(\widehat{M^{S^1}}\right)$ by

$$\widehat{\mu} = \sum_{\widehat{\gamma}_1,\ldots,\widehat{\gamma}_k} \tau\left(\widehat{\gamma}_1, \ldots, \widehat{\gamma}_k, e\right) L^*_{\widehat{\gamma}_1} dh \wedge \ldots \wedge L^*_{\widehat{\gamma}_k} dh. \tag{3.47}$$

Proposition 15. *There are closed forms* $\omega \in \Omega^k\left((S^1\backslash M) - M^{S^1}\right)$ *and* $\mu \in \Omega^k\left(M^{S^1}\right)$ *such that* $\widehat{\omega} = p^*\omega$ *and* $\widehat{\mu} = p^*\mu$.

Proof. The proof is as in [29, Lemma 4]. We omit the details. □

Let N be a small neighborhood of M^{S^1} in $S^1\backslash M$ as above, with projection $q : N \to M^{S^1}$. By construction, $\omega\big|_N = q^*\mu$. By [41, §7], the pair (ω, μ) represents a class in $\mathrm{H}^k(S^1\backslash M; \mathbb{R})$. We obtain a map $\phi : \mathrm{H}^*\left(\widehat{\Gamma}; \mathbb{R}\right) \longrightarrow \mathrm{H}^*\left(S^1\backslash M; \mathbb{R}\right)$ given by $\phi([\tau]) = [(\omega, \mu)]$.

Proposition 16. *(Browder-Hsiang [12, Theorem 1.1]) There is a commutative diagram*

$$\begin{array}{ccc} \mathrm{H}^*\left(\widehat{\Gamma}; \mathbb{R}\right) & \xrightarrow{\phi} & \mathrm{H}^*\left(S^1\backslash M; \mathbb{R}\right) \\ \alpha \downarrow & & \beta \downarrow \\ \mathrm{H}^*\left(\Gamma'; \mathbb{R}\right) & \xrightarrow{\gamma} & \mathrm{H}^*\left(M; \mathbb{R}\right) \end{array} \tag{3.48}$$

where the bottom row of (3.48) comes from the map $M \to B\Gamma'$ *induced by* ρ, *the left column of (3.48) comes from the homomorphism* $\Gamma' \to \widehat{\Gamma}$ *and the right column of (3.48) is pullback.*

Proof. Given $[\tau] \in \mathrm{H}^k\left(\widehat{\Gamma}; \mathbb{R}\right)$, represent it by a cocycle $\tau \in Z^k\left(\widehat{\Gamma}; \mathbb{R}\right)$. Let $\alpha(\tau) \in Z^k\left(\Gamma'; \mathbb{R}\right)$ be its pullback to Γ'. Let $r : M' \to \widehat{M}$ be the quotient map. Then $(\beta \circ \phi)[\tau]$ is characterized by the Γ'-invariant closed form

$$r^*\widehat{\omega} = \sum_{\widehat{\gamma}_1,\ldots,\widehat{\gamma}_k \in \widehat{\Gamma}} \tau\left(\widehat{\gamma}_1, \ldots, \widehat{\gamma}_k, e\right) L^*_{\widehat{\gamma}_1} dr^*H \wedge \ldots L^*_{\widehat{\gamma}_k} dr^*H \tag{3.49}$$

on M'. Let $K \in C_0^\infty\left(M'\right)$ satisfy

$$\sum_{g \in \rho(\langle o \rangle)} L_g^* K = r^*H. \tag{3.50}$$

Let $\widetilde{\Gamma} \subset \Gamma'$ be a set of representatives for the cosets $\rho(\langle o \rangle)\backslash\Gamma'$. Then

$$\begin{aligned} r^*\widehat{\omega} &= \sum_{\widetilde{\gamma}_1,\ldots,\widetilde{\gamma}_k \in \widetilde{\Gamma}} \alpha(\tau)\left(\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_k, e\right) L^*_{\widetilde{\gamma}_1} dr^*H \wedge \ldots L^*_{\widetilde{\gamma}_k} dr^*H \\ &= \sum_{\gamma'_1,\ldots,\gamma'_k \in \Gamma'} \alpha(\tau)\left(\gamma'_1, \ldots, \gamma'_k, e\right) L^*_{\gamma'_1} dK \wedge \ldots L^*_{\gamma'_k} dK. \end{aligned} \tag{3.51}$$

By [29, Proposition 14], the last term in (3.51) is the lift of a closed k-form on M which represents $\gamma(\alpha([\tau]))$. □

Remark: Theorem 1.1 of [12] is phrased in terms of rational cohomology and is valid for any compact connected Lie group, not just S^1. We expect that our proof could be extended to these cases.

If $\widehat{\Gamma}$ satisfies Assumption 1, let $\mathcal{D}$ be the canonical $\mathfrak{B}$-vector bundle on M^{S^1}. As in (3.12), the function h gives a partially-flat connection on $\mathcal{D}$. Let $[\tau] \in \mathrm{H}^k(\Gamma; \mathbb{C})$ be represented by a cocycle $\tau \in Z^k(\Gamma; \mathbb{C})$ as in Assumption 1. Then the Chern form $\mathrm{ch}\left(\nabla^{\mathcal{D}}\right) \in \Omega^*\left(M^{S^1}; \overline{\Omega}^*(\mathfrak{B})\right)$ satisfies

$$\mu = c(k)\langle \mathrm{ch}\left(\nabla^{\mathcal{D}}\right), Z_\tau\rangle \tag{3.52}$$

for some nonzero $c(k) \in \mathbb{R}$ [31, Proposition 3].

3.4. *Fixed-point-free actions II*

Suppose that the S^1-action has no fixed-points. There is a $C_r^*\widehat{\Gamma}$-Hilbert module of orbifold differential forms on $\widehat{M}$ and a (tangential) signature operator, which has an index $\sigma_{S^1}(M) \in K_*\left(C_r^*\widehat{\Gamma}\right)$. Suppose that $\widehat{\Gamma}$ satisfies Assumption 1. Then for any $[\tau] \in \mathrm{H}^*\left(\widehat{\Gamma}; \mathbb{R}\right)$, we can consider the pairing $\langle \sigma_{S^1}(M), Z_\tau\rangle \in \mathbb{R}$.

Proposition 17. *Construct $\phi([\tau]) \in \mathrm{H}^*(S^1\backslash M; \mathbb{R})$ as in the previous subsection. Given a suborbifold $\mathcal{O}$ of $S^1\backslash M$ as in Subsection 2.2, let $\phi([\tau])\big|_{\mathcal{O}}$ denote the pullback of $\phi[\tau]$ to $\mathcal{O}$. Then*

$$\langle \sigma_{S^1}(M), Z_\tau\rangle = \sum_{\mathcal{O}} \frac{1}{m_{\mathcal{O}}} \int_{\mathcal{O}} L(\mathcal{O}) \cup \phi([\tau])\big|_{\mathcal{O}}. \tag{3.53}$$

Proof. The method of proof is the same as in [29], which dealt with the case when $\widehat{\Gamma}$ acts freely on a smooth $\widehat{M}$. The only difference is that the local analysis must now be done on orbifolds, as in [24]. We omit the details. □

Remark: It seems likely that Proposition 17 follows from a general localization result and is true whenever $\widehat{\Gamma}$ satisfies the Strong Novikov Conjecture; compare [39, Theorem 2.6].

3.5. *Semifree actions II*

Suppose that S^1 acts effectively and semifreely on M. If F is a connected component of M^{S^1}, put

$$\Gamma_F = \mathrm{Im}\left(\pi_1(F) \longrightarrow \pi_1(M) \longrightarrow \Gamma' \longrightarrow \widehat{\Gamma}\right). \tag{3.54}$$

Suppose that Γ_F satisfies Assumption 1, with smooth subalgebra $\mathfrak{B}_F$ of $C_r^*\Gamma_F$, and that F satisfies Assumption 2 with respect to $C_r^*\Gamma_F$. Construct $\widetilde{\eta} \in \overline{\Omega}^*(\mathfrak{B}_F)/\mathrm{Im}(d)$ for the manifold F as in Subsection 3.2.

Definition 10. *Given $[\tau] \in \mathrm{H}^k\left(\widehat{\Gamma};\mathbb{R}\right)$, represent it by a cocycle $\tau \in Z^k\left(\widehat{\Gamma};\mathbb{R}\right)$. Construct $\omega_\tau \in \Omega^k\left((S^1\backslash M) - M^{S^1}\right)$ as in Proposition 15. Given a connected component F of M^{S^1}, let $\tau_F \in Z^k(\Gamma_F;\mathbb{R})$ be the restriction of τ. Suppose that the cyclic cocycle Z_{τ_F} extends to a cyclic cocycle on $\mathfrak{B}_F$. Put*

$$\langle\sigma_{S^1}(M),[\tau]\rangle = \int_{S^1\backslash M} L(T(S^1\backslash M))\wedge\omega_\tau + c(k)\sum_F\langle\widetilde{\eta}, Z_{\tau_F}\rangle \in \mathbb{R}. \tag{3.55}$$

We assume that $k \equiv \dim(M) - 1 \bmod 4$ so that the integral in (3.55) can be nonzero.

Theorem 6. *$\langle\sigma_{S^1}(M),[\tau]\rangle$ is independent of the choices of S^1-invariant Riemannian metric on M and function $H \in C_0^\infty\left(\widehat{M}\right)$ on $\widehat{M}$.*

Proof. The method of proof is that of Proposition 10. Define $\mu_\tau \in \Omega^*\left(M^{S^1}\right)$ as in Proposition 15. Let $J_{S^1}(M)$ denote the first term in the right-hand-side of (3.55). We first show the metric independence. Let $\{g(\epsilon)\}_{\epsilon\in[0,1]}$ be a smooth 1-parameter family of S^1-invariant metrics on M. For simplicity, assume that M^{S^1} has one connected component F. As in (2.40), let us write

$$\begin{aligned} L\left(R^{TF}(\epsilon) + d\epsilon\wedge\partial_\epsilon\omega^{TF}\right) &= a_1 + d\epsilon\wedge a_2, \\ L\left(\widehat{\Omega}_V(\epsilon) + d\epsilon\wedge\partial_\epsilon\widehat{\omega}_V\right) &= b_1 + d\epsilon\wedge b_2. \end{aligned} \tag{3.56}$$

Then as in (2.42),

$$J_{S^1}(M)\big|_{\epsilon=1} - J_{S^1}(M)\big|_{\epsilon=0} = -\int_0^1 d\epsilon\wedge\int_F a_2\wedge\mu_\tau\wedge\int_Z b_1. \tag{3.57}$$

The Atiyah-Singer families index theorem gives an equality in $\mathrm{H}^{even}(F;\mathbb{R})$:

$$\mathrm{ch}(\mathrm{Ind}(d+d^*)) = \int_Z b_1, \tag{3.58}$$

where $d+d^*$ denotes the family of vertical signature operators on the fiber bundle $S^1\backslash SNF \to F$.

Case I. $\dim(M) - \dim(F) \equiv 2 \bmod 4$.

As $Z = \mathbb{C}P^{2N}$ for some N, $\mathrm{Ind}(d+d^*)$ is a trivial complex line bundle on F. Then

$$J_{S^1}(M)\big|_{\epsilon=1} - J_{S^1}(M)\big|_{\epsilon=0} = -\int_0^1 d\epsilon\wedge\int_F a_2\wedge\mu_\tau. \tag{3.59}$$

On the other hand, from [30, Proposition 27],

$$c(k)\langle\widetilde{\eta}, Z_{\tau_F}\rangle\big|_{\epsilon=1} - c(k)\langle\widetilde{\eta}, Z_{\tau_F}\rangle\big|_{\epsilon=0} = \int_0^1 d\epsilon\wedge\int_F a_2\wedge\mu_\tau. \tag{3.60}$$

The proposition follows in this case.

Case II. $\dim(M) - \dim(F) \equiv 0 \bmod 4$.

As $Z = \mathbb{C}P^{2N+1}$ for some N, $\mathrm{Ind}(d + d^*) = 0$. Then

$$J_{S^1}(M)\big|_{\epsilon=1} - J_{S^1}(M)\big|_{\epsilon=0} = 0. \tag{3.61}$$

Equation (3.60) is again valid. As a_2 is concentrated in degree congruent to $-1 \bmod 4$ and $k \equiv \dim(F) - 1 \bmod 4$, we have $\int_F a_2 \wedge \mu_\tau = 0$. The proposition follows in this case.

Now fix the metric and suppose that $\{H(\epsilon)\}_{\epsilon\in[0,1]}$ is a smooth 1-parameter family of functions H constructed as in (3.43). Construct the corresponding form $\mu_\tau \in \Omega^*([0,1] \times F)$. Write

$$\mu_\tau = a_1 + d\epsilon \wedge a_2, \tag{3.62}$$

where $a_1, a_2 \in \Omega^*(F)$ depend on ϵ. Then

$$J_{S^1}(M)\big|_{\epsilon=1} - J_{S^1}(M)\big|_{\epsilon=0} = -\int_0^1 d\epsilon \wedge \int_F L\left(R^{TF}\right) \wedge a_2. \tag{3.63}$$

From [30, Proposition 27],

$$c(k)\langle\widetilde{\eta}, Z_{\tau_F}\rangle\big|_{\epsilon=1} - c(k)\langle\widetilde{\eta}, Z_{\tau_F}\rangle\big|_{\epsilon=0} = \int_0^1 d\epsilon \wedge \int_F L\left(R^{TF}\right) \wedge a_2. \tag{3.64}$$

The proposition follows. $\square$

3.6. General S^1-actions II

Let S^1 act effectively on M. For each connected component F of M^{S^1}, define Γ_F as in (3.54). Suppose that Γ_F satisfies Assumption 1, with smooth subalgebra $\mathfrak{B}_F \subset C_r^*\Gamma_F$, and that F satisfies Assumption 2 with respect to $C_r^*\Gamma_F$.

Definition 11. *Given $[\tau] \in \mathrm{H}^k\left(\widehat{\Gamma}; \mathbb{R}\right)$, represent it by a cocycle $\tau \in Z^k\left(\widehat{\Gamma}; \mathbb{R}\right)$. Construct $\omega_\tau \in \Omega^k\left((S^1\backslash M) - M^{S^1}\right)$ as in Proposition 15. Given a connected component F of M^{S^1}, let $\tau_F \in Z^k(\Gamma_F; \mathbb{R})$ be the restriction of τ. Suppose that the cyclic cocycle Z_{τ_F} extends to a cyclic cocycle on $\mathfrak{B}_F$. Put*

$$\langle\sigma_{S^1}(M), [\tau]\rangle = \sum_{\mathcal{O}} \frac{1}{m_{\mathcal{O}}} \int_{\mathcal{O}} L(\mathcal{O}) \wedge \omega_\tau\big|_{\mathcal{O}} + \sum_F \langle\widetilde{\eta}, Z_{\tau_F}\rangle \in \mathbb{R}. \tag{3.65}$$

As in Theorem 6, $\langle\sigma_{S^1}(M), [\tau]\rangle$ is independent of the choices of S^1-invariant metric and H.

Conjecture 1. $\langle \sigma_{S^1}(M), [\tau] \rangle$ is an S^1-homotopy invariant of M.

One may want to assume that $\widehat{\Gamma}$ satisfies Assumption 1. In this case, if the S^1-action has no fixed-points then the conjecture follows from Proposition 17, along with the homotopy invariance of the index $\sigma_{S^1}(M) \in K_*\left(C_r^* \widehat{\Gamma}\right)$. If the S^1-action is semifree and the codimension of M^{S^1} in M is at most two then an outline of a proof of the conjecture is given in Appendix A.

4. Remarks

1. One may wonder whether Assumption 2 is really necessary. To see that some assumption is necessary to define equivariant higher signatures, consider the special case when the quotient space is a manifold-with-boundary. So consider compact oriented manifolds-with-boundary equipped with a map to a classifying space $B\pi$. As Shmuel Weinberger pointed out to me, if one had a reasonable higher signature for such manifolds then one would expect to have Novikov additivity for the higher signatures of closed oriented manifolds. That is, if M is a closed oriented manifold with a map to $B\pi$ and N is a hypersurface in M which cuts it into two pieces M_1 and M_2 then the higher signatures of M would be the sum of those of M_1 and M_2, for the same reasons that the Atiyah-Patodi-Singer theorem implies the Novikov additivity of the usual signature. In particular, the higher signatures of closed oriented manifolds would give invariants of the cut-and-paste group $SK_*(B\pi)$ [23]. However, it is known for some groups π that the only cut-and-paste invariants of $B\pi$ are the Euler characteristic and the usual signature. For example, it easy to show that this is the case when $\pi = \mathbb{Z}$ and it then follows from [35, Lemma 8] that it is also the case when $\pi = \mathbb{Z}^k$. Thus in general one needs some assumption in order to define the higher signatures.

As a side remark, in some cases it is possible to define higher signatures of manifolds-with-boundary without any extra assumptions. For example, let M be a compact oriented manifold-with-boundary such that $4 | \dim(M)$. Let $\nu : M \to B\pi$ be a continuous map. Suppose that we are given a homomorphism $\rho : \pi \to SO(p,q)$ for some $p, q > 0$. Let $BSO(p,q)_\delta$ be the classifying space for $SO(p,q)$ with the discrete topology. There is a canonical flat real vector bundle V on $BSO(p,q)_\delta$ of rank $p+q$. The pullback $(B\rho \circ \nu)^* V$ is a flat real vector bundle on M with a flat symmetric form. Hence one can consider the twisted signature $\sigma(M, (B\rho \circ \nu)^* V) \in \mathbb{Z}$. This is an oriented-homotopy invariant of M by construction. On the other hand, if M is closed then it is also a higher signature of M involving the pullback of a Borel class from $BSO(\infty,\infty)_\delta$ [33]. It follows from the usual Novikov additivity argument that this higher signature is a cut-and-paste invariant. For example, if M is closed, $4 | \dim(M)$ and $B\pi$ is a closed oriented hyperbolic manifold of dimension $\dim(M)$ then one finds that the degree of the map ν gives a nontrivial invariant of $SK_{dim(B\pi)} B\pi$. If

$\dim(M) \equiv 2 \bmod 4$ then one can do a similar construction in which $SO(p,q)$ is replaced by $Sp(2n)$. In general, it seems to be an interesting question as to which higher signatures of closed manifolds are cut-and-paste invariants.

2. Although we have defined the signature of an S^1-quotient, we have not defined a signature operator of which the signature is the index. If M^{S^1} has codimension in M divisible by four then there is a signature operator on $S^1 \backslash M$ by the work of Cheeger [14]. If the S^1-action is semifree and M^{S^1} has codimension two in M then $S^1 \backslash M$ is a manifold-with-boundary and one has the Atiyah-Patodi-Singer signature operator on $S^1 \backslash M$. For a general semifree S^1-action, the quotient space will contain families of cones over complex projective spaces. We note that there is a topological obstruction to having a self-adjoint signature operator on a singular space with a single cone over $\mathbb{C}P^N$, N even [28]. However, in our case such cones occur in odd-dimensional families and this fact may allow one to construct the signature operator.

3. Suppose that a compact Lie group G acts effectively on an oriented closed manifold M. Let M^{sing} be the set of points in M whose isotropy subgroup has positive dimension. Then we can define $\Omega^{*,basic}\left(M, M^{sing}\right)$ and $\mathrm{H}^{*,basic}\left(M, M^{sing}\right)$ as in Definition 1. There is again an intersection form on $\mathrm{H}^{*,basic}\left(M, M^{sing}\right)$ which comes from integrating on the orbifold $(G\backslash M) - (G\backslash M^{sing})$, and its signature $\sigma_G(M)$ is a G-homotopy invariant of M. One can ask for an explicit formula for $\sigma_G(M)$, as was done in this paper when $G = S^1$. If the G-action is semifree then the analog of Theorem 4 holds and the proof is virtually the same as that of Theorem 4. However, if the action is not semifree then the situation is more involved. Suppose, for simplicity, that all isotropy groups are connected. In principle, one can follow the proof of Theorem 4 by applying the Atiyah-Patodi-Singer formula to a sequence of compact manifolds-with-boundary that exhaust $(G\backslash M) - (G\backslash M^{sing})$. However, the limiting formula must be more complicated than in Theorem 4. For example take $G = SU(2)$. If $m \in M^{sing} - M^{SU(2)}$ then a neighborhood of $\overline{m} \in SU(2)\backslash M$ is like an S^1-quotient of the type studied in Section 2. In analogy to Theorem 4, we expect that there will be a contribution to $\sigma_G(M)$ of the form $\eta\left((G\backslash M^{sing}) - (G\backslash M^{SU(2)})\right)$. However, $(G\backslash M^{sing}) - (G\backslash M^{SU(2)})$ is a space with conical singularities like those in Section 2 and it is not immediately clear how to define its eta-invariant; this is related to the preceding remark.

Appendix A. Homotopy invariance of higher signatures of manifolds-with-boundary

Suppose that we have a compact oriented manifold-with-boundary A, a finitely generated discrete group $\widehat{\Gamma}$ and a surjective homomorphism $\pi_1(A) \to \widehat{\Gamma}$. For simplicity, suppose that ∂A just has one connected component. Put

$$\Gamma_F = \mathrm{Im}\left(\pi_1(\partial A) \longrightarrow \pi_1(A) \longrightarrow \widehat{\Gamma}\right). \tag{A.1}$$

Put $n = \dim(\partial A)$. Suppose that $\widehat{\Gamma}$ satisfies Assumption 1, with smooth subalgebra $\mathfrak{B}$ of $C_r^* \widehat{\Gamma}$. Put

$$\mathfrak{B}_F = \{T \in \mathfrak{B} : T\left(l^2(\Gamma_F)\right) \subset l^2(\Gamma_F)\}. \tag{A.2}$$

Then $\mathbb{C}\Gamma_F \subset \mathfrak{B}_F \subset C_r^*\Gamma_F$, with $\mathfrak{B}_F$ closed under the holomorphic functional calculus in $C_r^*\Gamma_F$. Let $i : \overline{\Omega}^*(\mathfrak{B}_F)/\mathrm{Im}(d) \to \overline{\Omega}^*(\mathfrak{B})/\mathrm{Im}(d)$ be the obvious map. Suppose that ∂A satisfies Assumption 2 with respect to $C_r^*\Gamma_F$. Construct $\widetilde{\eta} \in \overline{\Omega}^*(\mathfrak{B}_F)/\mathrm{Im}(d)$ for ∂A as in Subsection 3.2. Let $\mathcal{D}$ be the canonical flat $\mathfrak{B}$-vector bundle on A. We have the higher signature

$$\sigma(A) = \int_A L(TA) \wedge \mathrm{ch}\left(\nabla^{\mathcal{D}}\right) + i(\widetilde{\eta}) \in \overline{\mathrm{H}}^*(\mathfrak{B}). \tag{A.3}$$

We want to realize $\sigma(A)$ as the Chern character of an index. We first describe the "unperturbed" setting. Without loss of generality, suppose that A is metrically a product near ∂A. Put $B = A \cup_{\partial A} ([0, \infty) \times \partial A)$. We extend $\mathcal{D}$ over B as a product over the cylindrical end. Consider the $\mathfrak{B}$-module $\Omega^*(B; \mathcal{D})$ of smooth compactly-supported $\mathcal{D}$-valued forms on B. This is one component of the unperturbed situation.

We would like to interpret $\sigma(A)$ as the index of the signature operator on the $C_r^*\widehat{\Gamma}$-completion of $\Omega^*(B; \mathcal{D})$. However, there is the problem that this signature operator need not be Fredholm in the $C_r^*\widehat{\Gamma}$-sense, because the signature operator on $\Omega^*(\partial A; \mathcal{D})$ may not be invertible. This is why we proceed as follows.

The other component of the unperturbed situation is an algebraic analog of a half-infinite cylinder which is coned off. More precisely, let W^* be a cochain complex of finitely generated projective $\mathfrak{B}$-modules which is homotopy equivalent to $\Omega^*(\partial A; \mathcal{D})$ as in Subsection 3.2. Let $\widehat{W}^*$ be as in (3.23). Let $\phi \in C^\infty([0, \infty))$ be a nondecreasing function such that

$$\phi(r) = \begin{cases} r & \text{if } r \le \frac{1}{2}, \\ 1 & \text{if } r \ge 2. \end{cases} \tag{A.4}$$

Define a $\mathfrak{B}$-inner product on the $\mathfrak{B}$-cochain complex $\Omega^*((0, \infty)) \otimes \widehat{W}^*$ such that if $w^i \in C_0^\infty((0, \infty)) \otimes \widehat{W}^i$ and $w^j \in C_0^\infty((0, \infty)) \otimes \widehat{W}^j$ then

$$\begin{aligned} \langle w^i, w^j \rangle &= \int_0^\infty \phi(r)^{dim(\partial A) - i - j} \langle w^i(r), w^j(r) \rangle_{\widehat{W}} dr, \qquad \text{(A.5)} \\ \langle w^i, dr \wedge w^j \rangle &= 0, \\ \langle dr \wedge w^i, dr \wedge w^j \rangle &= \int_0^\infty \phi(r)^{dim(\partial A) - i - j} \langle w^i(r), w^j(r) \rangle_{\widehat{W}} dr. \end{aligned}$$

This is the second component of the unperturbed situation. Formally, one would expect from Hodge duality that the index of a signature operator on $\Omega^*((0,\infty))\otimes\widehat{W}^*$ should vanish. Hence the index of a signature operator on $\Omega^*(B;\mathcal{D}) \oplus \left(\Omega^*((0,\infty))\otimes\widehat{W}^*\right)$ is formally the same as that of $\Omega^*(B;\mathcal{D})$.

We now perturb $\Omega^*(B;\mathcal{D}) \oplus \left(\Omega^*((0,\infty))\otimes\widehat{W}^*\right)$ to obtain a Fredholm operator. Let d denote the total differential on $\Omega^*(B;\mathcal{D}) \oplus \left(\Omega^*((0,\infty))\otimes\widehat{W}^*\right)$, where we switch the sign on the $\widehat{W}^*$-differential as in (3.25). Let $\epsilon \in C^\infty([0,\infty))$ be a nondecreasing function such that

$$\epsilon(r) = \begin{cases} 0 & \text{if } r \le 1, \\ 1 & \text{if } r \ge 2. \end{cases} \tag{A.6}$$

Given $\alpha > 1$, define an operator D on $\Omega^*(B;\mathcal{D}) \oplus \left(\Omega^*((0,\infty))\otimes\widehat{W}^*\right)$ by saying that on the degree-i subspace,

$$D = d + \begin{cases} \begin{pmatrix} 0 & \epsilon(r/\alpha)g \\ 0 & 0 \end{pmatrix} & \text{if } i < \frac{n}{2}, \\ \begin{pmatrix} 0 & 0 \\ \epsilon(r/\alpha)f & 0 \end{pmatrix} & \text{if } i > \frac{n}{2}. \end{cases} \tag{A.7}$$

Note that $D^2 \neq 0$ because ϵ is a nonconstant function of r. If $n+1$ is even, put $T = D + D^*$. If $n+1$ is odd, put $T = \pm(*D - D*)$. Then we expect that it will be possible to show the following :

1. The operator T extends to a Fredholm operator in the $C_r^*\widehat{\Gamma}$-sense. Its index $\mathrm{Ind}(T)$ is independent of α.
2. In analogy to [27], $\mathrm{ch}(\mathrm{Ind}(T)) = \sigma(A)$.
3. In analogy to [19], $\mathrm{Ind}(T)$ is a smooth homotopy invariant of the pair $(A, \partial A)$. (That is, the homotopy equivalence is not required to be a diffeomorphism on ∂A.)

To relate this to S^1-actions, let M have a semifree S^1-action such that M^{S^1} is nonempty and has codimension two. Then $S^1\backslash M$ is a manifold-with-boundary A, with $\partial A = M^{S^1}$. By [37, Proposition 1.2], $\pi_1(M) = \pi_1(A)$. If $\rho : \pi_1(M) \to \Gamma'$ is a surjective homomorphism as in Section 3.3 then $\widehat{\Gamma} = \Gamma'$.

Extending point 3. above, we mean that $\mathrm{Ind}(T)$ should be an S^1-homotopy invariant of M. Given an S^1-homotopy equivalence $h : M \to N$, put $A = S^1\backslash M$ and $B = S^1\backslash N$. We obtain a homotopy equivalence $\overline{h} : A \to B$ on the quotient spaces. It may not be a proper map, in that $\partial A = M^{S^1}$ may be properly contained in the preimage of $\partial B = N^{S^1}$. Nevertheless, we can extend $\overline{h}$ to a smooth map $h' : A \cup_{\partial A} ([0,\infty) \times \partial A) \to B \cup_{\partial B} ([0,\infty) \times \partial B)$ which is a product map on $[0,\infty) \times \partial A$. It should be possible to use h', as in [19], to compare the signature operators of A and B. The analog of the almost-flat connection of [19] is the fact that although $D^2 \neq 0$, by taking α large we can make the norm of D^2 as small as we want. Regarding point 3. above, it may be more convenient to work with a conical end than a cylindrical end. This would correspond to multiplying the metric on $[0,\infty)\times\partial A$ by a conformal factor which is asymptotically e^{-2cr} for large r, and similarly changing the inner product on $\left(\Omega^*((0,\infty))\otimes\widehat{W}^*\right)$. Here c is some positive constant.

Remark: In the topological setting, with similar assumptions one has a symmetric signature $s(A)\in L^*\left(\mathbb{Z}\widehat{\Gamma}\right)$. To describe this, assume for simplicity that $\Gamma_F=\widehat{\Gamma}$. Following [44], assume that ∂A is antisimple, meaning that the chain complex $C_*\left(\partial A;\mathbb{Z}\widehat{\Gamma}\right)$ is homotopy equivalent to a chain complex P_* of finitely generated projective $\mathbb{Z}\widehat{\Gamma}$-modules, with $P_{\frac{n}{2}}=0$ if n is even and $P_{\frac{n\pm 1}{2}}=0$ if n is odd. Let $P_<$ denote the truncation of P_* at $\left[\frac{n}{2}\right]$. Then the map $C_*\left(\partial A;\mathbb{Z}\widehat{\Gamma}\right)\to P_<$ defines an algebraic Poincaré pair in the sense of [36, p. 134]. The (closed) algebraic Poincaré complex $C_*\left(A;\mathbb{Z}\widehat{\Gamma}\right)\cup_{C_*\left(\partial A;\mathbb{Z}\widehat{\Gamma}\right)}P_<$ has a symmetric signature $s(A)\in L^*\left(\mathbb{Z}\widehat{\Gamma}\right)$ which will be a homotopy invariant of the pair $(A,\partial A)$. If $\widehat{\Gamma}$ satisfies Assumption 1 then we can construct $\mathrm{ch}(s(A))\in\overline{\mathrm{H}}_*(\mathfrak{B})$; compare with (A.3).

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