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BLD-mappings in $W^{2,2}$ are locally invertible

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Abstract. We prove that mappings of bounded length distortion are local homeomorphisms if they have L^2-integrable weak second derivatives.

1. Introduction

In this note, we establish the following theorem:

Theorem 1.1. *Every BLD-mapping that belongs to the Sobolev space $W^{2,2}$ is a local homeomorphism.*

Mappings of bounded length distortion, abbreviated BLD-mappings, were introduced and studied by Martio and Väisälä in [MV]: a mapping f from an open subset Ω of $\mathbf{R}^n$, $n \geq 2$, into $\mathbf{R}^n$ is a *BLD-mapping* if

$$f \in L^{1,\infty}(\Omega) \tag{1.2}$$

and

$$\det df(x) \geq c > 0 \tag{1.3}$$

for some constant c and for almost every x in Ω. Here $L^{1,\infty}(\Omega)$ is the Sobolev space of (continuous) functions with essentially bounded first distributional derivatives; thus (1.2) is equivalent to the requirement that f is locally uniformly Lipschitz: there is a constant $L \geq 1$ such that

$$|f(x) - f(y)| \leq L|x - y| \tag{1.4}$$

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whenever x and y lie in a ball contained in Ω. Lipschitz functions are almost everywhere differentiable, and in (1.3) $df(x)$ denotes the total derivative of f.

BLD-mappings form an interesting subclass of general *quasiregular mappings* [Re], [Ri], and they can be characterized by a quasipreserving property of lengths of paths [MV]. (See also [HKM, Chapter 14] and [HR].)

Theorem 1.1 was informally conjectured by Dennis Sullivan in a discussion with the first author. The conjecture was based on the philosophy of [S], and corroborated by the fact that the "winding map" $(r, \theta, w) \mapsto (r, 2\theta, w)$ in cylindrical coordinates belongs (locally) to $W^{2,p}$ for each $p < 2$ and fails to be a local homeomorphism on the $(n-2)$-dimensional (linear) subspace $\{r = 0\}$ of $\mathbf{R}^n$, $n \geq 2$.

We recall another conjecture (due to Olli Martio) which predicts the extremality of the winding map: the so-called inner dilatation K_I of the winding map is 2 and, conjecturally, in dimensions $n \geq 3$, every nonconstant quasiregular map with inner dilatation strictly less than 2 is a local homeomorphism. (See [Ri, I.3.1].) In Sect. 3 below we shall show that the conjecture is true for quasiregular maps whose dilatation tensor belongs to the Sobolev space $W^{1,2}$.

2. Proof of Theorem 1.1

Let $f : \Omega \to \mathbf{R}^n$, $n \geq 2$, be a mapping that satisfies (1.2) and (1.3), and assume that f has second distributional derivatives in $L^2(\Omega)$. By a theorem of Reshetnyak [Re, Thm. II.6.3], f is an open mapping with discrete fibers (where the latter means that for each $y \in \mathbf{R}^n$ the preimage $f^{-1}(y)$ consists of isolated points). The branch set B_f is the closed set in Ω, where f does not define a local homeomorphism.

We assume that $B_f \neq \emptyset$, and then show that this leads to a contradiction. By the general theory of discrete and open mappings, the Hausdorff $(n-2)$-measure of $f(B_f)$ is positive [Ri, III.5.3]; therefore, because f is locally Lipschitz, we have that

$$\mathcal{H}_{n-2}(B_f) > 0 \tag{2.1}$$

as well, where $\mathcal{H}_{n-2}$ denotes the Hausdorff $(n-2)$-measure in $\mathbf{R}^n$. (Note that (2.1) is unknown for general quasiregular mappings with nonempty branch set B_f in dimensions $n \geq 4$. See [Ri, III. 5.4.2] and Sect. 3 below.)

The key point in our argument is the following fact: there is an exceptional set E of Hausdorff $(n-2)$-measure zero in Ω such that

$$\lim_{r\to 0} \fint_{B(x,r)} |df(y) - (df)_{x,r}|\, dy = 0 \tag{2.2}$$

for $x \in \Omega \setminus E$, where the barred integral sign denotes the integral average, $B(x, r)$ is an open n-ball with center x and radius $r > 0$, and

$$(df)_{x,r} = \fint_{B(x,r)} df(y)\, dy\,.$$

This fact follows from the Poincaré inequality

$$\left(\fint_{B(x,r)} |df(y) - (df)_{x,r}|\, dy\right)^2 \leq c(n)r^2 \fint_{B(x,r)} |d^2 f(y)|^2\, dy\,,$$

and the following well-known application of the Lebesgue differentiation theorem and basic covering arguments: if $u \in L^1(\Omega)$, then

$$\mathcal{H}_{n-2}\left(\left\{x \in \Omega\,:\, limsup_{r\to 0}\frac{1}{r^{n-2}}\int_{B(x,r)} |u(y)|\, dy > 0\right\}\right) = 0\,.$$

See [EG, p. 141 and p. 77].

By the above discussion, and by (2.1), there is a point $x \in B_f$ such that (2.2) holds. We shall show that this is impossible. The method is a standard blow-up and normal families argument, used before in the study of branch sets, cf. [Re, II.§10], [MRV], [GMRV].

Without loss of generality, we assume that $x = 0 = f(x)$. Consider mappings

$$\begin{aligned} &f_r : \mathbf{B}^n \to \mathbf{R}^n,\\ &f_r(x) = r^{-1} f(rx), \quad r > 0, \end{aligned}$$

where $\mathbf{B}^n$ is the open unit ball in $\mathbf{R}^n$. Because f is locally uniformly Lipschitz, there is a constant $L > 0$ such that

$$f_r(\mathbf{B}^n) \subset B(0, L)$$

for all $r > 0$ small enough. On the other hand, one also has that

$$B(0, 1/L) \subset f_r(\mathbf{B}^n)$$

by standard properties of BLD-mappings ([MV, Lemma 4.6]). The mappings f_r are uniformly BLD for all small $r > 0$, and it follows that for a subsequence $f_k = f_{r_k}$, the limit

$$\lim_{k\to\infty} f_k = F$$

defines a BLD-mapping, $F : \mathbf{B}^n \to B(0, L)$, such that $B(0, 1/L) \subset F(\mathbf{B}^n)$ ([MV, Theorem 4.7]). Because the convergence of the sequence f_k is locally uniform, it follows from the basic degree theory that $0 \in B_F$; that is, F is not a local homeomorphism at 0.

Next,

$$\fint_{B(0,r_k)} |df(y) - (df)_{0,r_k}|\, dy = \fint_{\mathbf{B}^n} |df_k(y) - (df_k)_{0,1}|\, dy \to 0$$

by assumption. By passing to another subsequence, we may assume that

$$(df_k)_{0,1} \to M$$

as $k \to \infty$, where M is an invertible $n \times n$-matrix and that

$$df_k(y) \to M \text{ at a.e. } y \in \mathbf{B}^n,$$

as $k \to \infty$. Note that all the matrices in question lie in a compact set of $n \times n$-matrices with definite positive distance from the zero locus of the determinant function.

Because $f_k \to F$ uniformly and $df_k \to M$ a.e., we must have that

$$dF(y) = M \text{ for a.e. } y \in \mathbf{B}^n\,.$$

But this means that, up to a linear change of coordinates, F is a conformal mapping if $n \geq 3$, or an analytic function if $n = 2$. (This follows from the generalized Liouville theorem for $n \geq 3$ [Re, II §5.9], and from Weyl's lemma if $n = 2$.) In either case F must be a local homeomorphism, contradicting the result that $0 \in B_F$. The proof of the theorem is therefore complete.

3. Remarks on the quasiregular case

The argument in Sect. 2 works for general quasiregular mappings with some limitations. We use the terminology of [Ri].

For a (nonconstant) quasiregular map f in $\mathbf{R}^n$ it is natural to consider the *dilatation tensor*

$$G_f(x) = \det df(x)^{-2/n} df(x)^* df(x),$$

defined almost everywhere in the domain of f. Thus, G_f is a bounded matrix valued measurable function with $\det G_f(x) = 1$ almost everywhere. As in Sect. 2, we find that if $G_f \in W^{1,p}$ for some $1 \leq p \leq n$, then

$$\lim_{r\to 0} \fint_{B(x,r)} |G_f(y) - (G_f)_{x,r}|\, dy = 0 \tag{3.1}$$

for x outside an exceptional set of Hausdorff $(n - p)$-measure zero. By a recent result of Martio et. al. [MRV, Lemma 3.2], f is locally invertible in a neighborhood of each point x such that (3.1) holds, provided $n \geq 3$. The proof in [MRV] is a blow-up and normal families argument similar to that in the previous section. (Note that $G_f = Id$ for each holomorphic function f so that the dimensional restriction $n \geq 3$ is necessary.) It follows in particular that every nonconstant quasiregular map of a domain in $\mathbf{R}^n$, $n \geq 3$, is a local homeomorphism if $G_f \in W^{1,n}$. This can be improved somewhat:

Proposition 3.2. *Let f be a nonconstant quasiregular map of a domain in $\mathbf{R}^n$, $n \geq 3$, with a nonempty branch set B_f. If f is Hölder continuous of order α on B_f and if G_f belongs to the Sobolev space $W^{1,p}$, then $p < n - \alpha(n-2)$.*

Proof. Because $\mathcal{H}_{n-2}(f(B_f)) > 0$ [Ri, III.5.3], the α-Hölder continuity of f on B_f implies that $\mathcal{H}_{\alpha(n-2)}(B_f) > 0$, so that we must have $n - p > \alpha(n-2)$ by the above discussion. □

Note that every quasiregular mapping f is locally Hölder continuous of order $\alpha = K_I^{1/(1-n)}$, where $K_I = K_I(f)$ is the *inner dilatation* of f ([Ri, I.2.1, III.1.11]). Thus Proposition 3.2 is never vacuous.

If $n = 3$, then $B_f = \emptyset$ or $\mathcal{H}_1(B_f) > 0$ for all discrete and open maps, in particular for nonconstant quasiregular maps, by [MR, 2.20]. Moreover, if $K_I(f) < 2$, then f is locally Lipschitz continuous on B_f by a theorem of Martio [Ri, III.4.7]. We thus have the following result:

Theorem 3.3. *Let f be a nonconstant quasiregular map of a domain in $\mathbf{R}^n$ with dilatation tensor G_f in the Sobolev space $W^{1,2}$. If either $n = 3$, or $n \geq 4$ and the inner dilatation K_I of f is less than 2, then f is locally invertible.*

For $n = 3$ Theorem 3.3 improves earlier results of Iwaniec [I], Manfredi [M], Gutlyanskii et. al. [GMRV], and Martio et. al. [MRV]. Because the dilatation tensor of the winding map belongs (locally) to $W^{1,p}$ for all $p < 2$, it is tempting to believe that each nonconstant quasiregular map f with $G_f \in W^{1,2}$ is locally invertible in all dimensions $n \geq 3$.

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