Hölder estimates on convex domains of finite type

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1 Introduction

This article contains a natural and important application of the holomorphic support functions for convex domains of finite type in $\mathbb{C}^n$ constructed in [DiFo]. Namely, we use these functions to get $\overline{\partial}$-solving Cauchy-Fantappié kernels for $\overline{\partial}$-closed $(0, q)$-forms, such that the solutions given by them on bounded forms satisfy the best possible uniform Hölder estimates. More precisely we show:

Theorem 1.1 *Let $D \subset\subset \mathbb{C}^n$ be a linearly convex domain with $\mathcal{C}^\infty$-smooth boundary of finite type m. We denote by $L^\infty_{(0,q)}(D)$ the Banach space of $(0, q)$-forms with bounded coefficients on D and by $\Lambda^{1/m}_{(0,q)}(D)$ the Banach space of $(0, q)$-forms whose coefficients are uniformly Hölder continuous of order $1/m$ on D. Then there are bounded linear operators*

$$T_q : L^\infty_{(0,q+1)}(D) \to \Lambda^{1/m}_{(0,q)}(D)$$

such that $\overline{\partial} T_q f = f$ for all $f \in L^\infty_{(0,q+1)}(D)$ with $\overline{\partial} f = 0$.

A different proof for this result has already been announced 1997 by A. Cumenge in [Cu]. In fact, A. Cumenge uses certain approximate holomorphic peak functions obtained from the Bergman kernel of D (see (1)) in her construction of a $\overline{\partial}$-solving kernel of Andersson-Berndtsson type. In order to get the desired estimates for this kernel, she applies the very precise estimates for the Bergman kernel and its derivatives proved by J.D. McNeal in [Mc2] by using the detailed information on the complex geometry of convex domains of finite type from [Mc1] and the complete machinery of the

$\overline{\partial}$-Neumann problem. Our approach seems to be in some sense more direct. Of course, the complex geometry from [Mc1] is again an essential tool (as it was already in [DiFo] for the construction of the support functions). However, no other deep analysis is needed. The construction of our $\overline{\partial}$-solving Cauchy-Fantappié kernels from the support functions follows well-known routines. For estimating them, the estimates for the support functions S from Theorem 2.3 of [DiFo] and the fact, that also the imaginary part of S can be easily controlled in the usual way, play the essential role.

Concerning this question of the imaginary part of our S, we would like to mention here the following: In [DiHe] a bounded pseudoconvex domain $D_1 \subset\subset \mathbb{C}^3$ with smooth polynomial boundary of finite type and a linearly convex domain $D_2 \subset\subset \mathbb{C}^4$ with smooth $\mathcal{C}^1$-boundary of finite type have been constructed with the following property: if, in analogy to [Cu], we define the approximate peak functions

$$P_j(z, \zeta) := \frac{K_{D_j}(z, \zeta)}{K_{D_j}(\zeta, \zeta)} \tag{1}$$

for $(z, \zeta) \in D_j \times D_j$ (K_{D_j} denotes, of course, the Bergman kernel of the domain D_j, $j = 1, 2$), then there are boundary points $z^j \in \partial D_j$, such that $P_j(z, \zeta)$ has zeros for points (z, ζ) arbitrarily close to (z^j, z^j). It follows from this, in particular, that the imaginary parts of the P_j do not satisfy the properties which are needed in the usual estimates of the Cauchy-Fantappié kernels. Notice, that the D_j are not of the type of the domains considered in the above Theorem, but rather close to them. However, it seems to be difficult to imagine, how one could prove, that, nevertheless, the approximate peak functions P defined as in (1) behave nicely on bounded linearly convex domains D of finite type with $\mathcal{C}^\infty$-smooth boundaries.

For more details about other relevant work and the history concerning the problem considered in Theorem 1.1, we refer the reader to [DiFo]. Further results, concerning the construction of solution operators which satisfy estimates with respect to other norms will be given in another paper.

This article is organized in the following way: in Sect. 2 we recall the support functions S constructed in [DiFo], write down a Leray decomposition Q for them and give the construction of our $\overline{\partial}$-solving Cauchy-Fantappié kernels. We also start with the Hölder estimates of these kernels which will be continued in all the remaining sections. In Sect. 3 we collect some basic geometric tools for convex domains of finite type. In Sect. 4 we use these tools to prove the needed estimates for the support functions S. The estimates for the Leray decomposition Q and some first order derivatives of them are given in Sect. 5. In Sect. 6 we finally write down the integral estimates for the Cauchy-Fantappié kernels as defined in Sect. 2.

2 Solution operators

Let $l_\zeta(z)$ be a smooth family of coordinate changes as defined in [DiFo]. We write $l_\zeta(z) = \Phi(\zeta)(z - \zeta)$, where $\Phi(\zeta)$ is a unitary matrix depending smoothly on $\zeta \in \partial D$ such that the unit outer normal vector to ∂D will be turned into $(1, 0, \ldots, 0)$. The inverse transformation then is $l_\zeta^{-1}(w) = \zeta + \Phi^{-1}(\zeta)w = \zeta + \bar{\Phi}^T(\zeta)w$. The following definitions are as in [DiFo]:

$$r_\zeta(w) := \varrho(l_\zeta^{-1}(w)),$$

$$S_\zeta(w) := 3w_1 + Kw_1^2 - c \sum_{j=2}^m M^{2^j} \sigma_j \sum_{\substack{|\alpha|=j \\ \alpha_1=0}} \frac{1}{\alpha!} \frac{\partial^j r_\zeta}{\partial w^\alpha}(0)w^\alpha \qquad (2)$$

for $M > 0$ suitably large, $c > 0$ suitably small (both independent of ζ), and put

$$S(z, \zeta) := S_\zeta(l_\zeta(z)). \qquad (3)$$

Next we want to define n functions $Q_j(z, \zeta)$ such that

$$S(z, \zeta) = \langle Q(z, \zeta), z - \zeta \rangle = \sum_{j=1}^n Q_j(z, \zeta)(z_j - \zeta_j)$$

for $Q := (Q_1, \ldots, Q_n)$. We will do this by first defining $Q_\zeta^k(w)$ with $S_\zeta(w) = \langle Q_\zeta(w), w \rangle$. Then we have the computation

$$S_\zeta(w) = \langle Q_\zeta(w), w \rangle$$
$$S_\zeta(l_\zeta(z)) = \langle Q_\zeta(l_\zeta(z)), l_\zeta(z) \rangle$$
$$S(z, \zeta) = \langle Q_\zeta(l_\zeta(z)), \Phi(\zeta)(z - \zeta) \rangle$$

Thus

$$Q(z, \zeta) := \Phi^T(\zeta)Q_\zeta(l_\zeta(z)) \qquad (4)$$

will have the required property, once we will have found the $Q_\zeta^k(w)$ as above. For this we just define

$$Q_\zeta^1(w) := 3 + Kw_1 \qquad (5)$$

and for $k > 1$

$$Q_\zeta^k(w) := -c \sum_{j=2}^m M^{2^j} \sigma_j \sum_{\substack{|\alpha|=j \\ \alpha_1=0, \, \alpha_k>0}} \frac{\alpha_k}{j\alpha!} \frac{\partial^j r_\zeta}{\partial w^\alpha}(0) \frac{w^\alpha}{w_k}. \qquad (6)$$

The equation $S_\zeta(w) = \langle Q_\zeta(w), w \rangle$ then follows. It is also important to mention that the definition of $Q(z, \zeta)$ in fact does not depend on the choice of the transformation Φ. To be more precise we have the following lemma.

Lemma 2.1 *Let $A(\zeta)$ be a unitary matrix of the form*

$$A(\zeta) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A'(\zeta) & \\ 0 & & & \end{pmatrix}$$

and let $\Psi(\zeta) = A(\zeta)\Phi(\zeta)$. If we define $\tilde{Q}(z,\zeta)$ in the same way as $Q(z,\zeta)$ but with Ψ instead of Φ then we get

$$\tilde{Q}(z,\zeta) = Q(z,\zeta) \quad \text{for all} \quad z, \zeta.$$

Proof. To see this we just have to observe that the term

$$\sum_{\substack{|\alpha|=j \\ \alpha_1=0}} \frac{1}{\alpha!} \frac{\partial^j r_\zeta}{\partial w^\alpha}(0) w^\alpha$$

is rotation invariant. The term in the definition of Q_ζ^k is not. But then there is some additional A^T if we transform Q into $\tilde{Q}$ and this makes it rotation invariant again. □

Now we define Cauchy-Fantappié integral operators R_q based on the support function S and its Leray decomposition $Q(z,\zeta)$. We define the Cauchy-Fantappié form

$$W(z,\zeta) := \sum_i \frac{Q_i(z,\zeta)}{S(z,\zeta)} d\zeta_i.$$

Let $B = \dfrac{b}{|\zeta - z|^2} = \sum_i \dfrac{\bar{\zeta}_i - \bar{z}_i}{|\zeta - z|^2} d\zeta_i$ be the usual Martinelli Bochner form and let K_q be the well known Martinelli Bochner operator. Further define

$$R_q f := \sum_{k=0}^{n-q-2} c_k^q \int_{\zeta \in \partial D} f \wedge W \wedge B \wedge (\bar{\partial}_\zeta W)^k \wedge (\bar{\partial}_\zeta B)^{n-q-k-2} \wedge (\bar{\partial}_z B)^q$$

$$= \sum_{k=0}^{n-q-2} c_k^q \int_{\zeta \in \partial D} f \wedge \frac{Q \wedge b \wedge (\bar{\partial}_\zeta^T Q)^k \wedge (\bar{\partial}_\zeta^T b)^{n-q-k-2} \wedge (\bar{\partial}_z b)^q}{S^{k+1} |\zeta - z|^{2(n-k-1)}}.$$

In the last line we used the convention of denoting the $(1,0)$-form $\sum_i Q_i(z, \zeta) d\zeta_i$ again by Q. That we only have to apply the complex tangential components $\bar{\partial}_\zeta^T$ of the operator $\bar{\partial}_\zeta$, follows from the fact that the integral is already saturated with respect to $d\zeta$. It is also well known (see for instance [Ra] or [DiFoWi]) that the operators $T_q = R_q + K_q$ are solution operators, that means $\bar{\partial} T_q f = f$ for all $\bar{\partial}$-closed $(0, q+1)$-forms on D.

The usual way to prove Hölder estimates, is to use the Hardy-Littlewood lemma which states that a function $f \in C^1(D)$ also belongs to $\Lambda^\alpha(D)$ for $0 < \alpha \leq 1$ if it satisfies the estimate

$$|df(z)| \leq C \operatorname{dist}(z, \partial D)^{\alpha-1} \quad \text{for all} \quad z \in D.$$

Due to the fact that $\operatorname{dist}(z, \partial D) \approx |\varrho(z)|$ and because it is well known that K_q maps $L^\infty_{(0,q+1)}(D)$ to $\Lambda^\alpha_{(0,q)}(D)$ for all $\alpha < 1$ it remains to show that

$$|d_z R_q f(z)| \lesssim \|f\|_\infty \, |\varrho(z)|^{\frac{1}{m}-1}.$$

To compute $d_z R_q f$ we just have to put the derivative on each of the factors of the kernel. If we keep in mind that $|b| \leq |\zeta - z|$, that $d_z \bar\partial_\zeta b = d_z \bar\partial_z b = 0$ and that d_z of all the other terms are bounded we get that

$$
\begin{aligned}
|d_z R_q f(z)| \lesssim \sum_{k=0}^{n-q-2} \|f\|_\infty \Big(& \int_{\partial D} \frac{|d_z Q \wedge (\bar\partial_\zeta^T Q)^k|}{|S^{k+1}| \, |\zeta - z|^{2(n-k-1)-1}} d\sigma_{2n-1} \\
+ & \int_{\partial D} \frac{|Q \wedge (\bar\partial_\zeta^T Q)^k|}{|S^{k+1}| \, |\zeta - z|^{2(n-k-1)}} d\sigma_{2n-1} \\
+ & \int_{\partial D} \frac{k|Q \wedge (\bar\partial_\zeta Q)^{k-1}|}{|S^{k+1}| \, |\zeta - z|^{2(n-k-1)-1}} d\sigma_{2n-1} \\
+ & \int_{\partial D} \frac{|-k-1| \, |Q \wedge (\bar\partial_\zeta Q)^k|}{|S^{k+2}| \, |\zeta - z|^{2(n-k-1)-1}} d\sigma_{2n-1} \\
+ & \int_{\partial D} \frac{|2(n-k-1)| \, |Q \wedge (\bar\partial_\zeta Q)^k|}{|S^{k+1}| \, |\zeta - z|^{2(n-k-1)+1-1}} d\sigma_{2n-1} \Big)
\end{aligned}
$$

and the third term only appears for $k > 0$. Since

$$
\begin{aligned}
\int_{\partial D} \frac{|Q \wedge (\bar\partial_\zeta Q)^{k-1}| d\sigma_{2n-1}}{S^{k+1} |\zeta - z|^{2n-2k-3}} &= \int_{\partial D} \frac{|Q \wedge (\bar\partial_\zeta Q)^l| d\sigma_{2n-1}}{S^{l+2} |\zeta - z|^{2n-2l-5}} \\
&\lesssim \int_{\partial D} \frac{|Q \wedge (\bar\partial_\zeta Q)^l| d\sigma_{2n-1}}{S^{l+2} |\zeta - z|^{2n-2l-3}}
\end{aligned}
$$

it remains to show that for each $k = 0, \ldots, n - q - 2$ the following three integrals

$$
\begin{aligned}
&\int_{\zeta \in \partial D} \frac{|Q \wedge (\bar\partial_\zeta^T Q)^k|}{S^{k+2} |\zeta - z|^{2n-2k-3}} d\sigma_{2n-1} \\
&\int_{\zeta \in \partial D} \frac{|Q \wedge (\bar\partial_\zeta^T Q)^k|}{S^{k+1} |\zeta - z|^{2n-2k-2}} d\sigma_{2n-1} \\
&\int_{\zeta \in \partial D} \frac{|d_z Q \wedge (\bar\partial_\zeta^T Q)^k|}{S^{k+1} |\zeta - z|^{2n-2k-2}} d\sigma_{2n-1}.
\end{aligned}
\tag{7}
$$

can be estimated by $|\varrho(z)|^{\frac{1}{m}-1}$. For this purpose we need estimates for S, Q, $d_z Q$ and $\bar{\partial}_\zeta^T Q$. They, in turn, are based on some known facts about convex domains of finite type, which we recall first.

3 Basic geometric tools for convex domains

Let $D = \{\varrho < 0\} \subset \mathbb{C}^n$ be a bounded convex domain with $\mathcal{C}^\infty$-boundary of finite type m. We may assume, that the defining function ϱ has been chosen to be convex on $\mathbb{C}^n$ and smooth on $\mathbb{C}^n \backslash \{0\}$. We define some sort of complex directional boundary distances by

$$\tau(\zeta, v, \varepsilon) := \max\{c : |\varrho(\zeta + \lambda v) - \varrho(\zeta)| < \varepsilon \quad \text{for all} \quad \lambda \in \mathbb{C}, |\lambda| < c\}.$$

For a fixed point ζ and a fixed radius ε we define the ε-extremal basis $(v_1, \ldots, v_n)$ centered at ζ as in [Mc2]. If it is important to mention the dependence on ζ and ε of the coordinates with respect to this basis, we denote their components by $z_{k,\zeta,\varepsilon}$. Let v_k be a unit vector in the $z_{k,\zeta,\varepsilon}$-direction and write $\tau_k(\zeta, \varepsilon) := \tau(\zeta, v_k, \varepsilon)$. We can now define the polydiscs

$$AP_\varepsilon(\zeta) := \{z \in \mathbb{C}^n : |z_{k,\zeta,\varepsilon}| \leq A\tau_k(\zeta, \varepsilon) \forall k\}.$$

(Note that the factor A in front means blowing up the polydisc around its center and not just multiplying each point by A.)

Using these polydiscs we define the pseudodistance

$$d(z, \zeta) := \inf\{\varepsilon : z \in P_\varepsilon(\zeta)\}.$$

The following statements can be found in the literature (see for instance [Mc1], [Mc2], [BrNaWa], [BrChDu]):

Proposition 3.1   *(i) For each constant K there are constants $c(K)$ and $C(K)$ only depending on K such that*

$$P_{c(K)\varepsilon}(\zeta) \subset KP_\varepsilon(\zeta) \subset P_{C(K)\varepsilon}(\zeta),$$
$$c(K)P_\varepsilon(\zeta) \subset P_{K\varepsilon}(\zeta) \subset C(K)P_\varepsilon(\zeta).$$

for ζ near ∂D and all $\varepsilon > 0$ small enough.
(ii) There are constants $C_1 > 1$, $c_2 < 1$ and c_3 (independent of ζ and ε) such that

$$C_1 P_{\varepsilon/2}(\zeta) \supset \tfrac{1}{2}P_\varepsilon(\zeta) \quad \textit{for all} \quad \zeta, \varepsilon, \qquad (8)$$
$$C_1 P_t(\zeta) \subset P_\varepsilon(\zeta) \quad \textit{for all} \quad t < c_2\varepsilon, \zeta, \varepsilon, \qquad (9)$$
$$c_3 P_{|\varrho(\zeta)|}(\zeta) \subset D \quad \textit{for all} \quad \zeta \in D. \qquad (10)$$

(iii) *If $v = \sum_{j=1}^{n} a_j v_j$, where $(v_1, \ldots, v_n)$ is the ε-extremal basis at ζ, then we have*

$$\frac{1}{\tau(\zeta, v, \varepsilon)} \approx \sum_{j=1}^{n} \frac{|a_j|}{\tau_j(\zeta, \varepsilon)}.$$

In particular for every unit vector v we have $\tau(\zeta, v, \varepsilon) \lesssim \tau_k(\zeta, \varepsilon)/|a_k|$ for all k.

(iv) *For every $z \in P_\varepsilon(\zeta)$ we have $\tau(\zeta, v, \varepsilon) \approx \tau(z, v, \varepsilon)$.*

(v) *We have $\tau_1(\zeta, \varepsilon) \approx \varepsilon$ and $\tau(\zeta, v, \varepsilon) \lesssim \varepsilon^{\frac{1}{m}}$ for every unit vector v. If v is a unit vector in complex tangential direction then we also have $\varepsilon^{\frac{1}{2}} \lesssim \tau(\zeta, v, \varepsilon)$.*

(vi) *Let v be a unit vector and let*

$$a_{ij}(z, v) := \frac{\partial^{i+j}}{\partial \lambda^i \partial \bar{\lambda}^j} \varrho(z + \lambda v)_{|\lambda=0}.$$

Then we have

$$\sum_{1 \leq i+j \leq m} |a_{ij}(z, v)| \tau(z, v, \varepsilon)^{i+j} \approx \varepsilon$$

uniformly for all z, v and ε.

(vii) *Let w be any orthonormal coordinate system centered at z and let v_j be the unit vector in the w_j-direction. Then we have*

$$\left| \frac{\partial^{|\alpha+\beta|} \varrho(z)}{\partial w^\alpha \partial \bar{w}^\beta} \right| \lesssim \frac{\varepsilon}{\prod_j \tau(z, v_j, \varepsilon)^{\alpha_j + \beta_j}}$$

for all multiindices α and β with $|\alpha + \beta| \geq 1$.

(viii) *The pseudodistance $d(z, \zeta)$ satisfies the properties*

$$d(z, \zeta) \approx d(\zeta, z),$$
$$d(z, \zeta) \lesssim d(z, w) + d(w, \zeta).$$

(ix) *If $\pi(z)$ is the projection of a point z to the boundary ∂D then $d(z, \pi(z)) \approx |\varrho(z)|$; $z \in P_\varepsilon(\zeta)$ implies $d(z, \zeta) \leq \varepsilon$; $z \notin P_\varepsilon(\zeta)$ implies $d(z, \zeta) \gtrsim \varepsilon$ (not $\geq \varepsilon$); $d(z, \zeta) \leq \varepsilon$ implies $z \in P_t(\zeta)$ for all $t \gtrsim \varepsilon$ and $d(z, \zeta) \geq \varepsilon$ implies $z \notin P_t(\zeta)$ for all $t \lesssim \varepsilon$.*

For later use we define a family of polyannuli based on the polydiscs from above. Using the constant C_1 from Proposition 3.1 (ii) we put

$$P_\varepsilon^i(\zeta) := C_1 P_{2^{-i}\varepsilon}(\zeta) \backslash \tfrac{1}{2} P_{2^{-i}\varepsilon}(\zeta).$$

It follows from (8) that these polyannuli cover the full punctured polydisc

$$\bigcup_{i=0}^{\infty} P_\varepsilon^i(\zeta) \supset P_\varepsilon(\zeta) \backslash \{0\} \tag{11}$$

Moreover if $i_0(\varepsilon)$ is the smallest integer larger than $-\log_2(c_2\varepsilon)$ then $2^{-i_0(\varepsilon)} < c_2\varepsilon$ and it follows from (9) that $P_1^{i_0}(\zeta) \subset C_1 P_{2^{-i_0}}(\zeta) \subset P_\varepsilon(\zeta)$ and consequently we have a finite covering

$$\bigcup_{i=0}^{i_0} P_1^i(\zeta) \supset P_1(\zeta) \backslash P_\varepsilon(\zeta). \tag{12}$$

Note also that

$$i_0(\varepsilon) < 2 - \log_2(c_2\varepsilon) = -\log_2(c_2\varepsilon/4). \tag{13}$$

4 Estimates for S

The following Proposition is proved in [DiFo].

Proposition 4.1 *Let n_ζ be the normal unit vector to ∂D at the boundary point ζ and let v be a complex tangential unit vector. Define*

$$a_{\alpha\beta}(\zeta, v) := \frac{\partial^{\alpha+\beta}}{\partial\lambda^\alpha\partial\bar\lambda^\beta}\varrho(\zeta + \lambda v)_{|\lambda=0}.$$

For points z of the form $z = \zeta + \mu n_\zeta + \lambda v$ with $\mu, \lambda \in \mathbb{C}$, we have

$$Re\, S(z, \zeta) \le \frac{Re\,\mu}{2} - \frac{K}{2}(Im\,\mu)^2 - c\hat{c}\sum_{j=2}^{m}\sum_{\alpha+\beta=j}|a_{\alpha\beta}(\zeta, v)||\lambda|^j,$$

where $\hat{c}$ is a constant not depending on ζ or v.

We also need the following

Lemma 4.2 *Let $z \in D$ be close enough to the boundary and assume that ε is small enough. Then one has*

$$|S(z, \zeta)| \gtrsim \varepsilon \quad \text{for all} \quad \zeta \in \partial D \cap P_\varepsilon^0(\pi(z)) \tag{14}$$

$$|S(z, \zeta)| \gtrsim |\varrho(z)| \quad \text{for all} \quad \zeta \in \partial D \cap P_{|\varrho(z)|}(\pi(z)) \tag{15}$$

Proof. Let $\zeta \in \partial D$ and ε be fixed, $0 < k < K$ some fixed constants and k_0 a small constant to be chosen later. Write $z = \mu n_\zeta + \lambda v$, where n_ζ is the normal vector at ζ and v is a unit vector complex tangential to ∂D at ζ. First we define

$$\tilde{P}_\varepsilon^0(\zeta) := \{z : |Re\,\mu| < k_0,\ (z - Re\,\mu n_\zeta) \in KP_\varepsilon(\zeta)$$
$$\text{and} \quad (z - Re\,\mu n_\zeta) \notin kP_\varepsilon(\zeta)\}.$$

We will show that

$$|S(z, \zeta)| \gtrsim \varepsilon \quad \text{for all} \quad z \in \tilde{P}_\varepsilon^0(\zeta). \tag{16}$$

uniform in the choice of ζ and v.

Using Proposition 3.1 (iii) it is clear that there is a constant k_1' such that $\operatorname{Im}\mu \leq k_1'\tau(\zeta, n_\zeta, \varepsilon)$ and $|\lambda| \leq k_1'\tau(\zeta, v, \varepsilon)$ implies $(z - \operatorname{Re}\mu n_\zeta) \in kP_\varepsilon(\zeta)$. Thus we have either $|\lambda| \geq k_1'\tau(\zeta, v, \varepsilon)$ or $\operatorname{Im}\mu \geq k_1'\tau(\zeta, n_\zeta, \varepsilon)$ or both. Let $k_1 < k_1'$ be a constant to be chosen later. If $|\lambda| \geq k_1\tau(\zeta, v, \varepsilon)$ then we can use the estimate from Proposition 4.1 and Proposition 3.1 (vi) and get

$$
\begin{aligned}
|S(z, \zeta)| &\geq -\operatorname{Re}S(z, \zeta) \\
&\geq c\hat{c}\sum_{j=2}^m \sum_{\alpha+\beta=j} |a_{\alpha\beta}(\zeta, v)||\lambda|^j \\
&\gtrsim \sum_{j=2}^m \sum_{\alpha+\beta=j} |a_{\alpha\beta}(\zeta, v)|\tau(\zeta, v, \varepsilon)^j \\
&\gtrsim \varepsilon.
\end{aligned}
$$

If $\lambda \leq k_1\tau(\zeta, v, \varepsilon)$ then we must have $\operatorname{Im}\mu \geq k_1'\tau(\zeta, n_\zeta, \varepsilon) \geq k_2\varepsilon$. Now we have to consider the imaginary part of S and get

$$
\begin{aligned}
|S(z, \zeta)| &\geq |\operatorname{Im}S(z, \zeta)| \\
&\geq |3\operatorname{Im}\mu| - |2K\operatorname{Re}\mu\operatorname{Im}\mu| - c\hat{c}\sum_{j=2}^m \sum_{\alpha+\beta=j} |a_{\alpha\beta}(\zeta, v)||\lambda|^j.
\end{aligned}
$$

Using the estimate for λ and again Proposition 3.1 (vi) the last term can be estimated from above by $k_1^2 c\varepsilon$. Now we can choose k_1 so small that $k_1^2 c\varepsilon < k_2\varepsilon$. By the definition of $\tilde{P}_\varepsilon^0(\zeta)$ we also have that $\operatorname{Im}\mu < C_1\tau(\zeta, n_\zeta, \varepsilon) \leq k_3\varepsilon$. So the second term can be estimated by $2Kk_0k_3\varepsilon$ and k_0 can be chosen so small that $2Kk_0k_3\varepsilon < k_2\varepsilon$. Altogether we have

$$
|S(z, \zeta)| \geq 3k_2\varepsilon - k_2\varepsilon - k_2\varepsilon \gtrsim \varepsilon
$$

and the proof of (16) is complete.

To prove (14) we just have to observe that $\zeta \in P_\varepsilon^0(\pi(z))$ means $\zeta \in C_1P_\varepsilon(\pi(z))$ and $\zeta \notin \frac{1}{2}P_\varepsilon(\pi(z))$. Using Proposition 3.1 (i) and (ix) this implies the inequalities $c\varepsilon \leq d(\zeta, \pi(z)) \leq C\varepsilon$ for certain constants c and C. By Proposition 3.1 (viii) we also get $c\varepsilon \leq d(\pi(z), \zeta) \leq C\varepsilon$ for some other constants c and C. Using Proposition 3.1 (ix) and (i) again we get that $\pi(z) \in CP_\varepsilon(\zeta)$ and $\pi(z) \notin cP_\varepsilon(\zeta)$. If z is close enough to the boundary and ε is small enough this implies $z \in \tilde{P}_\varepsilon^0(\zeta)$ for still some other constants k and K. The first statement of the Lemma now follows from (16).

The estimate (15) also follows from (16) because we have $z \in \tilde{P}_{|\varrho(z)|}^0(\zeta)$ for all $\zeta \in \partial D \cap P_{|\varrho(z)|}(\pi(z))$. To see this, first observe that by (10) $c_3P_{|\varrho(z)|}(z) \cap \partial D = \emptyset$ and consequently $d(\zeta, z) > c|\varrho(z)|$ for all $\zeta \in \partial D$. Using Proposition 3.1 (viii), (ix) and (i) this implies $z \notin kP_{|\varrho(z)|}(\zeta)$ for all

$\zeta \in \partial D$. On the other hand we have $d(z, \zeta) \lesssim d(z, \pi(z)) + d(\pi(z), \zeta)$. Using again Proposition 3.1 (ix) and (i) this implies $d(z, \zeta) < C|\varrho(z)|$ and $z \in KP_{|\varrho(z)|}(\zeta)$ for all $\zeta \in P_{|\varrho(z)|}(\pi(z))$. So z belongs to $\tilde{P}^0_{|\varrho(z)|}(\zeta)$ for all $\zeta \in \partial D \cap P_{|\varrho(z)|}(\pi(z))$ and the proof of the Lemma is complete.   □

5 Estimates for Q

We now come to the decisive estimates for the components of Q, $d_z Q$ and $\bar{\partial}^T_\zeta Q$. First we fix a point $z_0 \in D$ close enough to the boundary, set $\zeta_0 := \pi(z_0)$ and $\varrho = |\varrho(z_0)|$ and choose a small number ε. Now we want to write all forms with respect to the ε-extremal coordinates at ζ_0, which we denote by w^*. We choose a unitary transformation Φ^* such that $w^* = \Phi^*(\zeta - \zeta_0)$. If we define

$$Q^*(w^*) := \bar{\Phi}^* Q(z_0, \zeta_0 + (\bar{\Phi}^*)^T w^*)$$

then we have $\sum_i Q_i(z_0, \zeta) d\zeta_i = \sum_k Q_k^*(w^*) dw_k^*$ and

$$\bar{\partial}_\zeta Q = \sum_{lk} \frac{\partial}{\partial \bar{w}_l^*} Q_k^*(w^*) d\bar{w}_l^* \wedge dw_k^*.$$

Lemma 5.1 *For all w^* with $|w_j^*| < \tau_j(\zeta_0, \varepsilon)$ we have*

$$|Q_k^*(w^*)| \lesssim \frac{\varepsilon}{\tau_k(\zeta_0, \varepsilon)}$$

$$\left| \frac{\partial}{\partial z_j} Q_k^*(w^*) \right| \lesssim \frac{\varepsilon}{\tau_k(\zeta_0, \varepsilon)}$$

$$\left| \frac{\partial}{\partial \bar{w}_j^*} Q_k^*(w^*) \right| \lesssim \frac{\varepsilon}{\tau_j(\zeta_0, \varepsilon) \tau_k(\zeta_0, \varepsilon)}$$

and the involved constants are independent of ζ_0 and ε. Note that according to Proposition 3.1 (v) *we have $\varepsilon/\tau_1(\zeta_0, \varepsilon) \lesssim 1$.*

Before we prove this lemma we want to make use of Lemma 2.1 and choose a convenient transformation Φ for the definition of Q. We may assume that $|\nabla \varrho(\zeta)| = 1$ for all $\zeta \in \partial D$ and that ε is so small that $|\partial \varrho(\zeta_0 + (\bar{\Phi}^*)^T w^*)/\partial w_1^*|^2 \geq c$ for all w^* with $|w_j^*| < \tau_j(\zeta_0, \varepsilon)$. We define

$$\nu_j := \frac{\partial}{\partial w_j^*} \varrho(\zeta_0 + (\bar{\Phi}^*)^T w^*),$$

and

$$A_j := 1 - \sum_{l=2}^{j} |\nu_l|^2.$$

It is clear that $A_j \geq |\nu_1|^2 \geq c$ for all j. Now we set

$$\psi_{1l} := \nu_l$$

$$\psi_{jl} := \frac{1}{\sqrt{A_{j-1}A_j}} \begin{cases} -\nu_j\nu_l & \text{for} \quad l = 1 \\ 0 & \text{for} \quad 1 < l < j \\ A_j & \text{for} \quad l = j \\ -\nu_j\nu_l & \text{for} \quad l > j \end{cases} \quad \text{for} \quad j > 1.$$

Obviously we have $\Psi(0) = \mathrm{Id}$ and it is easy to check that $\Phi(\zeta) := \Psi(\Phi^*(\zeta - \zeta_0))\Phi^*$ has the desired properties. Moreover we have $\Phi(\zeta_0 + (\bar{\Phi}^*)^T w^*) = \Psi(w^*)\Phi^*$.

Using the definitions of Q^*, Q and Φ we get

$$Q^*(w^*) = \Psi^T(w^*)Q_{\zeta_0+(\bar{\Phi}^*)^T w^*}(\Phi(\zeta_0 + (\bar{\Phi}^*)^T w^*)(z_0 - \zeta_0 - (\bar{\Phi}^*)^T w^*))$$

$$= \Psi^T(w^*)Q_{\zeta_0+(\bar{\Phi}^*)^T w^*}(\Psi(w^*)(\Phi^*(z_0 - \zeta_0) - w^*)).$$

Therefore we have

$$\frac{\partial}{\partial z_j}Q_k^*(w^*) = \sum_{\nu=1}^n \psi_{\nu k}(w^*) \sum_{\lambda=1}^n \frac{\partial}{\partial \omega_\lambda}Q_\zeta^\nu(\omega)\frac{\partial \omega_\lambda}{\partial z_j} \tag{17}$$

and

$$\frac{\partial}{\partial \bar{w}_j^*}Q_k^*(w^*)$$

$$= \sum_{\nu=1}^n \left(\frac{\partial}{\partial \bar{w}_j^*}\psi_{\nu k}(w^*)\right) Q_{\zeta_0+(\bar{\Phi}^*)^T w^*}^\nu(\Psi(w^*)(\Phi^*(z_0 - \zeta_0) - w^*))$$

$$+ \sum_{\nu=1}^n \psi_{\nu k}(w^*)\left(\frac{\partial}{\partial \bar{w}_j^*}Q_{\zeta_0+(\bar{\Phi}^*)^T w^*}^\nu(\omega)\right)$$

$$+ \sum_{\nu=1}^n \psi_{\nu k}(w^*)\left(\sum_{\lambda=1}^n \frac{\partial}{\partial \omega_\lambda}Q_\zeta^\nu(\omega)\frac{\partial}{\partial \bar{w}_j^*}\omega_\lambda\right) \tag{18}$$

with $\omega = \Psi(w^*)(\Phi^*(z_0-\zeta_0)-w^*)$. In order to prove Lemma 5.1 we need estimates for $Q_\zeta^\nu(\omega)$, $\frac{\partial}{\partial \bar{w}_j^*}Q_{\zeta_0+(\bar{\Phi}^*)^T w^*}^\nu(\omega)$, $\frac{\partial}{\partial \omega_\lambda}Q_\zeta^\nu(\omega)$, $\psi_{\nu k}(w^*)$, $\frac{\partial}{\partial \bar{w}_j^*}\psi_{\nu k}(w^*)$ and $\frac{\partial}{\partial \bar{w}_j^*}\omega_\lambda$. These estimates are given in the following lemmas.

Lemma 5.2 *For all w^* with $|w_j^*| < \tau_j(\zeta_0, \varepsilon)$ we get*

$$c \leq |\psi_{kk}(w^*)| \leq 1 \quad and \quad |\psi_{\nu k}(w^*)| \lesssim \frac{\varepsilon^2}{\tau_\nu(\zeta_0, \varepsilon)\tau_k(\zeta_0, \varepsilon)} \quad for \quad \nu \neq k \tag{19}$$

$$\left| \frac{\partial}{\partial \bar{w}_j^*} \psi_{kk}(w^*) \right| \lesssim \frac{\varepsilon}{\tau_j(\zeta_0, \varepsilon) \tau_k(\zeta_0, \varepsilon)}$$

$$\left| \frac{\partial}{\partial \bar{w}_j^*} \psi_{\nu k}(w^*) \right| \lesssim \frac{\varepsilon^2}{\tau_j(\zeta_0, \varepsilon) \tau_\nu(\zeta_0, \varepsilon) \tau_k(\zeta_0, \varepsilon)} \quad for \quad \nu \neq k$$

$$\left| \frac{\partial}{\partial \bar{w}_j^*} \omega_1(w^*) \right| \lesssim \frac{\varepsilon + \varrho}{\tau_j(\zeta_0, \varepsilon)},$$

$$\left| \frac{\partial}{\partial \bar{w}_j^*} \omega_l(w^*) \right| \lesssim \frac{\varepsilon}{\tau_j(\zeta_0, \varepsilon) \tau_l(\zeta_0, \varepsilon)} \quad for \quad l > 1$$

Proof. Since Ψ is a unitary matrix the estimate $|\psi_{kk}| \leq 1$ is obvious. The estimate for $|\psi_{\nu k}|$ follows from the facts that $|A_k| \geq c$ and $\sqrt{A_{k-1} A_k} \leq 1$. It follows immediately from Proposition 3.1 (vii) and (iv) that $|\nu_k| \lesssim \frac{\varepsilon}{\tau_k(\zeta_0, \varepsilon)}$. Together with Proposition 3.1 (v) we get

$$|\psi_{1k}| = |\nu_k| \lesssim \frac{\varepsilon^2}{\tau_1(\zeta_0, \varepsilon) \tau_k(\zeta_0, \varepsilon)}.$$

Using the fact that $\sqrt{A_{l-1} A_l} > c$ we also get

$$|\psi_{lk}| \lesssim |\nu_l \nu_k| \lesssim \frac{\varepsilon^2}{\tau_l(\zeta_0, \varepsilon) \tau_k(\zeta_0, \varepsilon)}$$

for $l > 1$ and $l \neq k$.

The estimate $\left| \frac{\partial}{\partial \bar{w}_j^*} \nu_k \right| \lesssim \frac{\varepsilon}{\tau_k(\zeta_0, \varepsilon) \tau_j(\zeta_0, \varepsilon)}$ is also an immediate consequence of Proposition 3.1 (vii) and (iv). Since $\varepsilon / \tau_1(\zeta_0, \varepsilon) \lesssim 1$ this gives the desired estimate for $|\frac{\partial}{\partial \bar{w}_j^*} \psi_{1k}|$.

We also compute

$$\left| \frac{\partial}{\partial \bar{w}_j^*} A_k \right| = \left| -\sum_{\kappa=2}^{k} \left(\frac{\partial}{\partial \bar{w}_j^*} \nu_\kappa \right) \bar{\nu}_\kappa + \left(\frac{\partial}{\partial \bar{w}_j^*} \bar{\nu}_\kappa \right) \nu_\kappa \right|$$

$$\lesssim \sum_{\kappa=2}^{k} \frac{\varepsilon^2}{\tau_j(\zeta_0, \varepsilon) \tau_\kappa(\zeta_0, \varepsilon) \tau_\kappa(\zeta_0, \varepsilon)} \lesssim \frac{1}{\tau_j(\zeta_0, \varepsilon)}.$$

Now for $l = 1$ this gives us

$$\left| \frac{\partial}{\partial \bar{w}_j^*} \omega_1 \right| = \left| \sum_k \frac{\partial}{\partial \bar{w}_j^*} \nu_k \left[\Phi^*(z_0 - \zeta_0) - w^* \right]_k \right|$$

$$\lesssim \sum_k \frac{\varepsilon}{\tau_j(\zeta_0, \varepsilon) \tau_k(\zeta_0, \varepsilon)} \left| \left[\Phi^*(z_0 - \zeta_0) - w^* \right]_k \right|.$$

Since for all w^* with $|w_j^*| < \tau_j(\zeta_0, \varepsilon)$ we have $|[\Phi^*(z_0 - \zeta_0) - w^*]_k| \lesssim \varrho + \tau_k(\zeta_0, \varepsilon)$ and therefore

$$\frac{|[\Phi^*(z_0 - \zeta_0) - w^*]_k|}{\tau_k(\zeta_0, \varepsilon)} \lesssim \frac{\varepsilon + \varrho}{\varepsilon}.$$

this gives the desired result for $l = 1$.

For $l > 1$ we compute

$$\left| \frac{\partial}{\partial \bar{w}_j^*} \psi_{ll} \right| = \left| \frac{\partial}{\partial \bar{w}_j^*} \left(1 - \frac{\nu_l}{\sqrt{A_{l-1}}} \right) \right| \lesssim \frac{\varepsilon}{\tau_j(\zeta_0, \varepsilon)\tau_l(\zeta_0, \varepsilon)}$$

and with $k \neq l$

$$|\frac{\partial}{\partial \bar{w}_j^*} \psi_{lk}| = \left| \frac{\partial}{\partial \bar{w}_j^*} \left(\nu_l \nu_k A_l^{-\frac{1}{2}} A_{l-1}^{-\frac{1}{2}} \right) \right|$$

$$\lesssim \frac{\varepsilon^2}{\tau_j(\zeta_0, \varepsilon)\tau_l(\zeta_0, \varepsilon)\tau_k(\zeta_0, \varepsilon)} \lesssim \frac{\varepsilon}{\tau_j(\zeta_0, \varepsilon)\tau_l(\zeta_0, \varepsilon)}.$$

Since

$$\left| \frac{\partial}{\partial \bar{w}_j^*} \omega_l \right| = \left| \sum_k \frac{\partial}{\partial \bar{w}_j^*} \psi_{lk} \left[\Phi^*(z_0 - \zeta_0) - w^* \right]_k \right|$$

and $[\Phi^*(z_0 - \zeta_0) - w^*]_k$ is bounded this completes the proof. $\qquad \square$

We also need the following lemma.

Lemma 5.3 *Let $\zeta = \zeta_0 + (\bar{\Phi}^*)^T w^*$, let ω be as above and let $v_j(w^*)$ be the unit vector in $\omega_j(w^*)$ direction. Then for every w^* with $|w_j^*| < \tau_j(\zeta_0, \varepsilon)$ we have*

$$\tau(\zeta, v_j(w^*), \varepsilon) \approx \tau_j(\zeta_0, \varepsilon).$$

Proof. Using Proposition 3.1 (iv) and (iii) we get for all w^* with $|w_j^*| < \tau_j(\zeta_0, \varepsilon)$

$$\tau(\zeta, v_j(w^*), \varepsilon) \approx \tau(\zeta_0, v_j(w^*), \varepsilon) \approx \left(\sum_{l=1}^n \frac{|\psi_{jl}(w^*)|}{\tau_l(\zeta_0, \varepsilon)} \right)^{-1}$$

$$\approx \min_{l=1,\dots,n} \frac{\tau_l(\zeta_0, \varepsilon)}{|\psi_{jl}(w^*)|}. \tag{20}$$

For $j = 1$ we derive from (19)

$$\frac{\tau_1(\zeta_0, \varepsilon)}{|\psi_{11}(w^*)|} \lesssim \varepsilon \quad \text{and} \quad \frac{\tau_l(\zeta_0, \varepsilon)}{|\psi_{1l}(w^*)|} \gtrsim \frac{\tau_l(\zeta_0, \varepsilon)\varepsilon}{\tau_l(\zeta_0, \varepsilon)} \gtrsim \varepsilon.$$

Therefore the minimum is comparable to $\tau_1(\zeta_0, \varepsilon)$.

If $j > 1$ it follows from (19) that for $l \neq j$ we get

$$|\psi_{jl}(w^*)| \lesssim \frac{\varepsilon}{\tau_j(\zeta_0, \varepsilon)} \frac{\varepsilon}{\tau_l(\zeta_0, \varepsilon)} \lesssim \left(\frac{\varepsilon}{\tau_l(\zeta_0, \varepsilon)}\right)^2 \frac{\tau_l(\zeta_0, \varepsilon)}{\tau_j(\zeta_0, \varepsilon)}.$$

Since $\varepsilon/\tau_l(\zeta_0, \varepsilon)$ is bounded and $|\psi_{jj}(w^*)| \geq c$ this implies

$$\frac{\tau_j(\zeta_0, \varepsilon)}{|\psi_{jj}(w^*)|} \lesssim \frac{\tau_l(\zeta_0, \varepsilon)}{|\psi_{jl}(w^*)|} \quad \text{for all} \quad l \neq j$$

and together with (20) this proves the lemma.   □

Lemma 5.4 *Using again the abbreviation* $\omega = \Psi(w^*)(\Phi^*(z_0 - \zeta_0) - w^*)$ *we get for all* w^* *with* $|w_j^*| < \tau_j(\zeta_0, \varepsilon)$

$$|Q^1_{\zeta_0 + (\bar{\Phi}^*)^T w^*}(\omega)| \lesssim 1$$

$$\left|\frac{\partial}{\partial \omega_l} Q^1_{\zeta_0 + (\bar{\Phi}^*)^T w^*}(\omega)\right| \lesssim 1$$

$$\frac{\partial}{\partial \bar{w}_j^*} Q^1_{\zeta_0 + (\bar{\Phi}^*)^T w^*}(\omega) \equiv 0$$

and for $k > 1$ *we have*

$$|Q^k_{\zeta_0 + (\bar{\Phi}^*)^T w^*}(\omega)| \lesssim \frac{\varepsilon}{\tau_k(\zeta_0, \varepsilon)}$$

$$\left|\frac{\partial}{\partial \omega_l} Q^k_{\zeta_0 + (\bar{\Phi}^*)^T w^*}(\omega)\right| \lesssim \frac{\varepsilon}{\tau_l(\zeta_0, \varepsilon)\tau_k(\zeta_0, \varepsilon)}$$

$$\left|\frac{\partial}{\partial \bar{w}_j^*} Q^k_{\zeta_0 + (\bar{\Phi}^*)^T w^*}(\omega)\right| \lesssim \frac{\varepsilon}{\tau_j(\zeta_0, \varepsilon)\tau_k(\zeta_0, \varepsilon)}$$

Proof. By definition we have $Q^1_{\zeta_0 + (\bar{\Phi}^*)^T w^*}(\omega) = 3 + K\omega_1$ and the first three statements are therefore obvious.

In $Q^k_\zeta(w^*)$, $k > 1$ there are only the coordinates ω_l with $l > 1$ involved. We observe that $|\omega_l(w^*)| = |\sum_\mu \psi_{l\mu}(w^*)[\Phi^*(z_0 - \zeta_0) - w^*]_\mu| \lesssim |\psi_{l1}(w^*)|(\varrho + |w_1^*|) + \sum_{\mu>1} |\psi_{l\mu}(w^*)| |w_\mu^*|$. Using (19) we see that $\sum_{\mu>1} |\psi_{l\mu}(w^*)| |w_\mu^*| \lesssim \tau_l(\zeta_0, \varepsilon)$ for all w^* with $|w_j^*| < \tau_j(\zeta_0, \varepsilon)$. Since by (19) and Proposition 3.1 (v) $|\psi_{l1}(w^*)| \lesssim \varepsilon/\tau_l(\zeta_0, \varepsilon) \lesssim \varepsilon^{\frac{1}{2}}$ we also have $|\psi_{l1}(w^*)|(\varrho + |w_1^*|) \lesssim \varepsilon^{\frac{1}{2}}(\varrho + \varepsilon) \lesssim \varepsilon^{\frac{1}{2}} \lesssim \tau_l(\zeta_0, \varepsilon)$. Therefore we get for all $l > 1$

$$|\omega_l(w^*)| \lesssim \tau_l(\zeta_0, \varepsilon) \quad \text{for all} \quad w^* : |w_j^*| < \tau_j(\zeta_0, \varepsilon).$$

Now it follows from Proposition 3.1 (vii) and Lemma 5.3 that for $k > 1$

$$
\begin{aligned}
|Q_\zeta^k(\omega)| &\leq \left| -c \sum_{j=2}^m M^{2^j} \sigma_j \sum_{\substack{|\alpha|=j \\ \alpha_1=0,\, \alpha_k>0}} \frac{\alpha_k}{j\alpha!} \frac{\partial^j r_\zeta(0)}{\partial\omega^\alpha} \frac{\omega^\alpha}{\omega_k} \right| \\
&\leq c \sum_{j=2}^m M^{2^j} \sum_{\substack{|\alpha|=j \\ \alpha_1=0,\, \alpha_k>0}} \left| \frac{\alpha_k}{j\alpha!} \frac{\partial^j r_\zeta(0)}{\partial\omega^\alpha} \right| \frac{|\omega^\alpha|}{|\omega_k|} \\
&\lesssim c \sum_{j=2}^m M^{2^j} \sum_{\substack{|\alpha|=j \\ \alpha_1=0,\, \alpha_k>0}} \frac{\varepsilon}{\prod_\nu \tau(\zeta, v_\nu(w^*), \varepsilon)^{\alpha_\nu}} \frac{|\omega^\alpha|}{|\omega_k|} \\
&\lesssim c \sum_{j=2}^m M^{2^j} \sum_{\substack{|\alpha|=j \\ \alpha_1=0,\, \alpha_k>0}} \frac{\varepsilon}{\tau^\alpha(\zeta_0, \varepsilon)} \frac{|\omega^\alpha|}{|\omega_k|} \\
&\lesssim \frac{\varepsilon}{\tau_k(\zeta_0, \varepsilon)}
\end{aligned}
$$

which completes the proof of the first statement for $k > 1$.

The second statement can be proved exactly in the same way. Except for the fact that the additional derivative gives an additional factor $\tau(\zeta, v_l(w^*), \varepsilon) \approx \tau_l(\zeta_0, \varepsilon)$ in the denominator.

To prove the third statement for $k > 1$ we first have to rewrite the $\bar{w}_j$ derivative. Observe that

$$
\frac{\partial}{\partial \bar{w}_j^*} r_{\zeta_0 + (\bar{\Phi}^*)^T w^*}(\omega)_{|\omega=0} = \frac{\partial}{\partial \bar{w}_j^*} \varrho(\zeta_0 + (\bar{\Phi}^*)^T w^* + (\bar{\Phi}^*)^T \bar{\Psi}^T(w^*)\omega)_{|\omega=0}.
$$

From this it is easy to see that

$$
\frac{\partial}{\partial \bar{w}_j^*} r_\zeta(\omega)_{|\omega=0} = \sum_{l=1}^n \bar{\psi}_{lj}(w) \frac{\partial}{\partial \bar{\omega}_l} r_\zeta(\omega)_{|\omega=0}.
$$

So we can write the $\bar{w}_j$ derivative as a sum of $\bar{\omega}_l$ derivatives. Then we proceed as in the proof of the second statement and get

$$
\left| \frac{\partial}{\partial \bar{w}_j^*} Q_{\zeta_0 + (\bar{\Phi}^*)^T w^*}^k(\omega) \right| \lesssim \sum_{l=1}^n |\bar{\psi}_{lj}(w)| \frac{\varepsilon}{\tau_l(\zeta_0, \varepsilon)\tau_k(\zeta_0, \varepsilon)}.
$$

Together with (19) this completes the proof of the lemma. □

Proof of Lemma 5.1. In view of (17) and (18) the statements of the lemma are now straight forward consequences of Lemma 5.2 and Lemma 5.4 and Proposition 3.1 (v). □

Before we come to the estimates of the integrals we want to state the estimates of this section in there final form as they will be needed below.

Lemma 5.5 *Let $[Q]$ denote either Q or $d_z Q$. Then for all w^* with $|w_j^*| < \tau_j(\zeta_0, \varepsilon)$ the term*

$$\left| [Q] \wedge (\bar\partial_\zeta^T Q)^k \right|$$

can be estimated by a sum of products of the form

$$\frac{\varepsilon^k}{\prod_{i=1}^k \tau_{\mu_i}(\zeta_0, \varepsilon)\tau_{\nu_i}(\zeta_0, \varepsilon)}$$

where μ_i and ν_i are greater than 1 and each index appears at most once.

Proof. As done in the beginning of this section we can write $[Q] \wedge (\bar\partial_\zeta Q)^k$ with respect to the w^* coordinates and get

$$\sum_{\mu_1,\ldots,\mu_k,\nu_0,\ldots,\nu_n} (Q_{\nu_0}^*(w^*)$$

$$\text{or} \quad \frac{\partial}{\partial z_j}Q_{\nu_0}^*(w^*))dw_{\nu_0}^* \wedge \bigwedge_{i=1}^k \frac{\partial}{\partial \bar w_{\mu_i}^*}Q_{\nu_i}^*(w^*)d\bar w_{\mu_i}^* \wedge dw_{\nu_i}^*$$

Now it is clear that all the μ_j and ν_j must be different from each other. However there might be one of the ν_j being equal to 1. If it is ν_0 the first term can be estimated by a constant, if it is some other ν_j the corresponding term still gives an estimate of the form $1/\tau_{\mu_j}(\zeta_0, \varepsilon)$. Finally there are k indices $\nu_j > 1$ left and Lemma 5.1 now almost gives the desired estimate. The only remaining problem is that one of the μ_j might be equal to 1. In this case we would get an estimate $\varepsilon/(\tau_1(\zeta_0, \varepsilon)\tau_{\nu_j}(\zeta_0, \varepsilon))$. However we have to observe that dw_1^* is the normal direction at ζ_0 and only has a small tangential component in $P_\varepsilon(\zeta_0)$. To compute the precise amount we may assume that $\bar\partial\varrho, d\bar w_2^*, \ldots, d\bar w_n^*$ is a basis for the $(0,1)$-forms near ζ_0. With respect to this basis we have

$$\frac{\partial\varrho}{\partial\bar w_1^*}dw_1^* = \bar\partial\varrho - \sum_{j=2}^n \frac{\partial\varrho}{\partial\bar w_j^*}dw_j^*$$

Since $\left|\frac{\partial\varrho}{\partial\bar w_1^*}\right|$ is bounded from below and $\left|\frac{\partial\varrho}{\partial\bar w_j^*}\right| \lesssim \frac{\varepsilon}{\tau_j(\zeta_0, \varepsilon)}$ for all $\zeta \in P_\varepsilon(\zeta_0)$ we see that the tangential component of the form under consideration can be estimated by a sum terms of the form

$$\frac{\varepsilon}{\tau_1(\zeta_0, \varepsilon)\tau_{\nu_j}(\zeta_0, \varepsilon)}\frac{\varepsilon}{\tau_j(\zeta_0, \varepsilon)} \lesssim \frac{\varepsilon}{\tau_j(\zeta_0, \varepsilon)\tau_{\nu_j}(\zeta_0, \varepsilon)}$$

where j is some index which is not 1 and different from all the other μ_j. $\square$

6 Integral estimates

Now we come to the final step in the proof of Theorem 1.1. Let us introduce the notation (See Lemma 5.5 for the definition of $[Q]$.)

$$I_1(X) := \int_{\partial D \cap X} \frac{|[Q] \wedge (\bar\partial_\zeta^T Q)^k|}{|S^{k+2}| \, |\zeta - z|^{2n-2k-3}} d\sigma_{2n-1}$$

$$I_2(X) := \int_{\partial D \cap X} \frac{|[Q] \wedge (\bar\partial_\zeta^T Q)^k|}{|S^{k+1}| \, |\zeta - z|^{2n-2k-2}} d\sigma_{2n-1}.$$

According to Sect. 2 it remains to show that for each $k = 0, \ldots, n - q - 2$ the two integrals $I_{1/2}(\partial D)$ can be estimated by $|\varrho(z)|^{\frac{1}{m}-1}$.

Since the only singularity of the integrals occurs for $\zeta = z$ it is clear that $I_{1/2}(\partial D) \le C$ if $\operatorname{dist}(z, \partial D) \ge c$ or if the integration is only over the boundary outside some small neighborhood U of $\pi(z)$. This neighborhood always should be chosen small enough, so that we have on it nice local coordinates and several of the other properties discussed above. For simplicity let us assume that $P_1(\pi(z))$ is such a neighborhood.

Now let z_0 be a fixed point close enough to the boundary, let $\zeta_0 = \pi(z_0)$ be the projection to the boundary and set $\varrho = |\varrho(z_0)|$. In order to estimate $I_{1/2}(P_1(\zeta_0))$ we first give an estimate for some auxiliary integral over $\partial D \cap P_\varepsilon(\zeta_0)$ for arbitrary $\varepsilon < 1$. After that we consider the two parts $I_{1/2}(P_\varrho(\zeta_0))$ and $I_{1/2}(P_1(\zeta_0) \backslash P_\varrho(\zeta_0))$ separately.

First we want to estimate integrals of the form

$$\int_{\partial D \cap P_\varepsilon(\zeta_0)} \frac{|[Q] \wedge (\bar\partial_\zeta^T Q)^k|}{|\zeta - z_0|^{2n-2k-3}} d\sigma_{2n-1}.$$

Since z and ε are fixed, we can change to the ε-extremal coordinates at ζ_0, write the integrand with respect to these coordinates and use the estimates from Lemma 5.5. We also want to mention that all the involved constants can be chosen to be independent of z and ε. What we finally have to deal with are integrals of the form

$$\int_{|v_1| < \tau_1(\zeta_0, \varepsilon)} \int_{|w_2| < \tau_2(\zeta_0, \varepsilon)} \cdots \int_{|w_n| < \tau_n(\zeta_0, \varepsilon)} \frac{\varepsilon^k \, dv_1 \, du_2 \, dv_2 \cdots du_n \, dv_n}{\prod_{j=1}^k \tau_{\mu_j}(\zeta_0, \varepsilon) \tau_{\nu_j}(\zeta_0, \varepsilon) (\sum |w_l|)^{2n-2k-3}},$$

where $\mu_j > 1$ and $\nu_j > 1$ and each index appears at most once.

First we integrate with respect to v_1 and get a constant factor $\tau_1(\zeta_0, \varepsilon) \lesssim \varepsilon$ which together with the other ε already gives us ε^{k+1}. Now we still have to

integrate over $n-1$ complex discs but there are only $2n-3$ factors in the denominator. Therefore the following integrals may occur

$$I_a := \int_{|w_l|<\tau_l(\zeta_0,\varepsilon)} \frac{du_l\,dv_l}{\tau_l(\zeta_0,\varepsilon)^2} \lesssim 1$$

$$I_b := \int_{|w_l|<\tau_l(\zeta_0,\varepsilon)} \frac{du_l\,dv_l}{\tau_l(\zeta_0,\varepsilon)|w_l|} \lesssim 1$$

$$I_c := \int_{|w_l|<\tau_l(\zeta_0,\varepsilon)} \frac{du_l\,dv_l}{\tau_l(\zeta_0,\varepsilon)} \lesssim \tau_l(\zeta_0,\varepsilon) \lesssim \varepsilon^{\frac{1}{m}}$$

$$I_d := \int_{|w_{l_1}|<\tau_{l_1}(\zeta_0,\varepsilon)} \cdots \int_{|w_{l_i}|<\tau_{l_i}(\zeta_0,\varepsilon)} \frac{du_{l_1}\,dv_{l_1}\ldots du_{l_i}\,dv_{l_i}}{(\sum|w_{l_j}|)^{2i-1}}$$

$$\lesssim \int_0^{\varepsilon^{\frac{1}{m}}} \frac{r^{2i-1}dr}{r^{2i-1}} \lesssim \varepsilon^{\frac{1}{m}}$$

However I_c and I_d may occur at most once and only one of them will be present. So finally we get the following result.

$$\int_{\partial D\cap P_\varepsilon(\zeta_0)} \frac{|[Q]\wedge(\bar\partial_\zeta^T Q)^k|}{|\zeta-z_0|^{2n-2k-3}}d\sigma_{2n-1} \lesssim \varepsilon^{\frac{1}{m}+k+1}. \tag{21}$$

Now we want to estimate the integrals $I_{1/2}(P_\varrho(\zeta_0))$. It follows from Lemma 4.2 that $|S^{k+2}| \gtrsim \varrho\varepsilon^{k+1}$ and $|S^{k+1}||\zeta-z| \gtrsim \varrho\varepsilon^{k+1}$ for every ζ in $P_\varepsilon^0(\zeta_0)$. Using the covering (11) and estimate (21) from above we now can write

$$I_{1/2}(P_\varrho(\zeta_0)) \leq \sum_{j=0}^\infty I_{1/2}(P_\varrho^j(\zeta_0))$$

$$\leq \sum_{j=0}^\infty \frac{1}{\varrho(2^{-j}\varrho)^{k+1}} \int_{\partial D\cap P_{2^{-j}\varrho}(\zeta_0)} \frac{|[Q]\wedge(\bar\partial_\zeta^T Q)^k|}{|\zeta-z_0|^{2n-2k-3}}d\sigma_{2n-1}$$

$$\lesssim \sum_{j=0}^\infty \frac{(2^{-j}\varrho)^{\frac{1}{m}+k+1}}{\varrho(2^{-j}\varrho)^{k+1}} \lesssim \varrho^{\frac{1}{m}-1}$$

which is the desired result.

It remains to consider $I_{1/2}(P_1(\zeta_0)\backslash P_\varrho(\zeta_0))$. Now we use the estimates $|S^{k+2}| \gtrsim \varepsilon^{k+2}$ and $|S^{k+1}||\zeta-z| \gtrsim \varepsilon^{k+2}$ in $P_\varepsilon^0(\zeta_0)$ which also follow from Lemma 4.2. Using the second covering (12) and again the estimate (21) we get

$$I_{1/2}(P_1(\zeta_0)\backslash P_\varrho(\zeta_0))$$

$$\leq \sum_{j=0}^{i_0(\varrho)} I_{1/2}(P_1^j(\zeta_0))$$

$$\leq \sum_{j=0}^{i_0(\varrho)} \frac{1}{(2^{-j})^{k+2}} \int_{\partial D \cap P_{2^{-j}}(\zeta_0)} \frac{|[Q] \wedge (\bar\partial_\zeta^T Q)^k|}{|\zeta - z_0|^{2n-2k-3}} d\sigma_{2n-1}$$

$$\lesssim \sum_{j=0}^{i_0(\varrho)} \frac{(2^{-j})^{\frac{1}{m}+k+1}}{(2^{-j})^{k+2}} \lesssim \sum_{j=0}^{i_0(\varrho)} (2^{-j})^{\frac{1}{m}-1}$$

$$= \frac{1 - (2^{1-\frac{1}{m}})^{i_0(\varrho)+1}}{1 - 2^{1-\frac{1}{m}}} \lesssim 2^{(1-\frac{1}{m})i_0(\varrho)}.$$

Using the fact that $i_0(\varrho) < -\log_2(c_2\varrho/4)$ we also get

$$I_{1/2}(P_1(\zeta_0)\backslash P_\varrho(\zeta_0)) \leq 2^{(\frac{1}{m}-1)\log_2(c_2\varrho/4)} \lesssim \varrho^{\frac{1}{m}-1}$$

which is again what we wanted.

References

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Noted added in Proofs. In December 1998 the preprint "Sharp estimates for $\bar\partial$ on convex domains of finite type" by Anne Comenge appeared containing details of her proofs for the results announced in [Cu] and some additional results.