Algebraic varieties on which the classical Phragmén-Lindelöf estimates hold for plurisubharmonic functions

Rüdiger W. Braun[1]**, Reinhold Meise**[1]**, B.A. Taylor**[2]

[1] Mathematisches Institut, Heinrich-Heine-Universität, Universitätsstraße 1, 40225 Düsseldorf, Germany (e-mail: braun@cs.uni-duesseldorf.de / meise@cs.uni-duesseldorf.de)
[2] Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA (e-mail: taylor@umich.edu)

Abstract. Algebraic varieties V are investigated on which the natural analogue of the classical Phragmén-Lindelöf principle for plurisubharmonic functions holds. For a homogeneous polynomial P in three variables it is shown that its graph has this property if and only if P has real coefficients, no elliptic factors, is locally hyperbolic in all real characteristics, and the localizations in these characteristics are square-free. The last condition is shown to be necessary in any dimension.

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1. Introduction

An algebraic variety V in $\mathbb{C}^n$, $n \geq 2$, has the property (SPL) if there exists a constant $A \geq 1$ such that for each plurisubharmonic function u on V the estimates

$$(1.1) \qquad u(z) \leq |z| + o(|z|), z \in V, \quad \text{and} \quad u(z) \leq 0, \ z \in V \cap \mathbb{R}^n,$$

imply

$$(1.2) \qquad\qquad u(z) \leq A|\operatorname{Im} z|, \ z \in V.$$

By the classical Phragmén-Lindelöf Theorem, $V = \mathbb{C}^n$ satisfies (SPL) with $A = 1$. Thus the varieties with the property (SPL) are the ones for which a natural extension of the Phragmén-Lindelöf Theorem holds. This, however, is not the only reason why they are of interest. By results of Meise, Taylor, and Vogt [10], [11], [15], the zero variety

$$V(P_m) = \{z \in \mathbb{C}^n : P_m(z) = 0\}$$

of a homogeneous polynomial P_m of degree $m \geq 2$ in n variables satisfies (SPL) if and only if the differential operator $P(D) : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ admits a continuous linear right inverse. Moreover, results of Hörmander [8] imply that for each $Q \in \mathbb{C}[z_1, \ldots, z_n]$ with $\deg Q < m$ the operator $(P_m + Q)(D)$ acts surjectively on the space $\mathcal{A}(\mathbb{R}^n)$ of all real-analytic functions if $V(P_m)$ has (SPL). It is reasonable to ask why $A > 1$ is allowed in (1.2). Proposition 2.8 explains why the case $A = 1$, while interesting, is too restrictive. For example, $V(P_m)$ satisfies (SPL) with $A = 1$ if and only if P_m is a product of real linear forms.

To consider non-homogeneous polynomials, let $P_m \in \mathbb{C}[z_1, \ldots, z_n]$ be homogeneous of degree $m \geq 2$ and define $P \in \mathbb{C}[z_1, \ldots, z_{n+1}]$ by

$$P(z', z_{n+1}) := P_m(z') - z_{n+1}.$$

In this situation, Meise and Taylor [9] have shown recently that if $V(P)$ satisfies (SPL) then P_m is square-free and has real coefficients. They also proved that $V(P)$ satisfies (SPL) if P_m is of real principal type and none of its irreducible factors is elliptic. For $n = 2$ this condition is a characterization and is equivalent to P_m being strictly hyperbolic or P_m being a product of m distinct real linear forms. Moreover, Meise and Taylor [9] showed that for real homogeneous polynomials P_m, $V(P)$ has (SPL) if and only if both varieties

$$V_\pm(P_m) := \{z \in \mathbb{C}^n : P_m(z) = \pm 1\}$$

have (SPL) and that $V(P)$ has (SPL) if and only if the operator $P(D) : C^\infty(\mathbb{R}^{n+1}) \to C^\infty(\mathbb{R}^{n+1})$ admits a continuous linear right inverse.

In the present paper we improve the necessary condition of P_m being square-free and use the improvement to characterize in dimension $n = 3$ when $V(P)$ has (SPL). To formulate the result, denote by $(P_m)_\theta$ the lowest order homogeneous polynomial in the expansion of $z \mapsto P_m(\theta + z)$. Using the concept of quasihomogeneity and a result of [4], we show that if $V(P)$ has (SPL) then $(P_m)_\theta$ is square-free for each $\theta \in V(P_m) \cap \mathbb{R}^n$, $|\xi| = 1$. Of course, this condition is a lot more restrictive than the requirement that P_m is square-free. In fact, for $n = 3$ it leads to the following characterization:

Theorem 1.1. *Let $P_m \in \mathbb{C}[z_1, z_2, z_3]$ be homogeneous of degree $m \geq 2$ and define $P \in \mathbb{C}[z_1, \ldots, z_4]$ by $P(z) := P_m(z_1, z_2, z_3) - z_4$. The following are equivalent*:

(1) $V(P)$ *has* (SPL),
(2) $P(D) : C^\infty(\mathbb{R}^4) \to C^\infty(\mathbb{R}^4)$ *admits a continuous linear right inverse*,
(3) $V_+(P_m)$ *and* $V_-(P_m)$ *satisfy* (SPL),
(4) $V(P_m)$ *has* (SPL), P_m *has real coefficients, and* $(P_m)_\theta$ *is square-free for each* $\theta \in V(P_m) \cap \mathbb{R}^3$ *with* $|\theta| = 1$.

Examples like Newton's knot and the Cartesian leaf in homogeneous coordinates

$$P_3(x, y, z) = y^2 z - x(x - z)^2 \quad \text{and} \quad P_3(x, y, z) = x^3 + y^3 - 3xyz$$

satisfy these conditions (see Example 4.11). Hence for $n \geq 3$ there are homogeneous polynomials $P_m \in \mathbb{R}[z_1, \ldots, z_n]$ for which $V(P)$ has (SPL), while $V(P_m) \cap \mathbb{R}^n$ contains non-zero singular points, contrary to the situation for $n = 2$.

Because of the results of Meise and Taylor [9], the essential point in proving Theorem 1.1 is to show that (4) implies (3). To do this we apply a recent result of [5], by which there exists a constant $A_0 \geq 1$ such that each $u \in \text{PSH}(V_\pm(P_m))$ which satisfies (1.1) already satisfies

(1.3) $$u(z) \leq A_0|z|, \quad z \in V_\pm(P_m).$$

From this it follows that it suffices to prove the estimate (1.2) for u only at points $z \in V_\pm(P_m)$ for which $|\text{Im } z|$ is small compared to $|z|$. Since both $V_+(P_m)$ and $V_-(P_m)$ are manifolds, points in a fixed ball of arbitrary size are easy to handle. Since points $z \in V_\pm(P_m)$ with large $|z|$ are close to points in $V(P_m)$, it suffices to consider $V_\pm(P_m)$ in cones around real lines spanned by $\xi \in V(P_m) \cap \mathbb{R}^3, |\xi| = 1$. To handle $V_\pm(P_m)$ in such cones we use a result of Braun [1] by which there exist suitable coordinates so that in some neighborhood of ξ the variety $V(P_m)$ is a finite union of graphs of holomorphic functions which are real-valued for real arguments. Assuming $\xi = (1, 0, 0)$ it follows from this that for large $t > 0$ the varieties $V_\pm(P_m)$ near $(t\xi, 0, 0)$ can be approximated sufficiently well by $V_\pm(Q)$, where $Q(z_1, z_2, z_3) = z_1^{m-\nu} Q_0(z_2, z_3)$ for some $\nu \in \mathbb{N}$ and some $Q_0 \in \mathbb{R}[z_2, z_3]$ which is strictly hyperbolic. Knowing this, we use compactness and scaling arguments together with bounds for particular harmonic functions to derive the estimate (1.2) for u in a cone around $\mathbb{R}\xi$ from the fact that we already know (1.3).

2. Preliminaries

In this section we introduce the basic definitions that will be used subsequently.

Definition 2.1. Let V be an algebraic variety in $\mathbb{C}^n$ and Ω be an open subset of V. A function $u : \Omega \to [-\infty, \infty[$ will be called plurisubharmonic if it is locally bounded above, plurisubharmonic in the usual sense on Ω_{reg}, the set of all regular points of V in Ω, and satisfies

$$u(z) = \limsup_{\xi \in \Omega_{\mathrm{reg}}, \xi \to z} u(\xi)$$

at the singular points of V in Ω. By $\mathrm{PSH}(\Omega)$ we denote the set of all plurisubharmonic functions on Ω.

Definition 2.2. Let V be an algebraic variety in $\mathbb{C}^n$, $n \geq 2$, and $A \geq 1$, $B \geq 0$. We say:

(a) V satisfies the condition $\mathrm{SPL}(A, B)$ if for each $u \in \mathrm{PSH}(V)$ the conditions (α) and (β) imply (γ), where
   - (α) $u(z) \leq |z| + o(|z|)$, $z \in V$,
   - (β) $u(z) \leq 0$, $z \in V \cap \mathbb{R}^n$,
   - (γ) $u(z) \leq A|\mathrm{Im}\, z| + B$, $z \in V$.
(b) V satisfies the condition (SPL) (the strong Phragmén-Lindelöf condition) if V satisfies $\mathrm{SPL}(A, 0)$ for some $A \geq 1$.

Remark. By the classical Phragmén-Lindelöf Theorem, $\mathbb{C}^n$ satisfies $\mathrm{SPL}(1, 0)$. Hence the algebraic varieties satisfying (SPL) are those on which the obvious extension of the classical Phragmén-Lindelöf condition holds. To indicate why these varieties are of interest also in a different context, we recall the following notions.

Definition 2.3. Let $\omega : [0, \infty[\to]0, \infty[$ be continuous and increasing and assume that it has the following properties:

$(\alpha)\quad \omega(2t) = O(\omega(t))$ $\qquad\qquad$ $(\beta)\quad \displaystyle\int_1^\infty \frac{\omega(t)}{t^2}\, dt < \infty$

$(\gamma)\quad \log t = O(\omega(t))$ as t tends to infinity $\quad(\delta)\quad x \mapsto \omega(e^x)$ is convex.

Then $\omega : z \mapsto \omega(|z|)$, $z \in \mathbb{C}^n$, $n \in \mathbb{N}$, will be called a weight function. Also we will assume without restriction that $\omega(0) \geq 1$. For examples we refer to [3].

Definition 2.4. Let V be an algebraic variety in $\mathbb{C}^n$ and ω a weight function. V satisfies the condition $\mathrm{PL}(\mathbb{R}^n, \omega)$ of Phragmén-Lindelöf type, if the following holds:

There exists $A \geq 1$ so that for each $\rho > 0$ there exists $B_\rho > 0$ so that each $u \in \mathrm{PSH}(V)$ which satisfies (α) and (β) also satisfies (γ), where

(α) $u(z) \leq |\mathrm{Im}\, z| + O(\omega(z))$, $z \in V$,
(β) $u(z) \leq \rho|\mathrm{Im}\, z|$, $z \in V$,
(γ) $u(z) \leq A|\mathrm{Im}\, z| + B_\rho \omega(z)$, $z \in V$.

If $\omega(t) = \log(2 + t)$ then we will write $\mathrm{PL}(\mathbb{R}^n, \log)$.

One relation between the conditions $\mathrm{SPL}(A, B)$ and $\mathrm{PL}(\mathbb{R}^n, \omega)$ is explained in the following lemma.

Lemma 2.5. *For each algebraic variety V in $\mathbb{C}^n$ the following assertions hold*:

(a) *If V satisfies $\mathrm{SPL}(A, B)$ for some $A \geq 1, B \geq 0$, then V satisfies $\mathrm{PL}(\mathbb{R}^n, \omega)$ for each weight function ω.*
(b) *If V is homogeneous and satisfies $\mathrm{PL}(\mathbb{R}^n, \omega)$ for some weight function ω then V satisfies (SPL).*

In particular, (SPL) and $\mathrm{PL}(\mathbb{R}^n, \omega)$ are equivalent if V is homogeneous.

Proof. (a) Since each weight function ω satisfies $\omega(t) = o(t)$ as t tends to infinity, this is easily checked.
(b) This follows from Meise, Taylor, and Vogt [15], Theorem 3.1, in connection with Meise and Taylor [9], Lemma 3.2. $\qquad\square$

2.6. Differential operators and Phragmén-Lindelöff conditions. For $P \in \mathbb{C}[z_1, \ldots, z_n]$, $P(z) = \sum_{|\alpha| \leq m} a_\alpha z^\alpha$, we define the differential operator $P(D)$ with symbol P by

$$P(D) := \sum_{|\alpha| \leq m} a_\alpha \frac{1}{i^{|\alpha|}} \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

When $\sum_{|\alpha|=m} |a_\alpha| \neq 0$, P has degree m and its principal part P_m is defined as $P_m(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$. Sometimes, P_m will just denote an arbitrary homogeneous polynomial of degree m. We will use the notation

$$V(P) := \{z \in \mathbb{C}^n : P(z) = 0\}$$

and $V_\pm(P_m) := V(P_m \mp 1)$.

By Meise, Taylor, and Vogt [11], [14], the variety $V(P)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \log)$ (resp. $\mathrm{PL}(\mathbb{R}^n, \omega)$) if and only if the differential operator $P(D)$ admits a continuous linear right inverse on $C^\infty(\mathbb{R}^n)$ and/or $\mathcal{D}'(\mathbb{R}^n)$ (resp. on

$\mathcal{E}_{(\omega)}(\mathbb{R}^n)$ and/or $\mathcal{D}'_{(\omega)}(\mathbb{R}^n)$). Palamodov [16] has shown that the splitting of a differential complex of C^∞-functions or distributions over $\mathbb{R}^n$ is equivalent to $\mathrm{PL}(\mathbb{R}^n, \log)$ holding for the corresponding varieties.

By Meise, Taylor, and Vogt [15], Theorem 4.1, $V(P_m)$ has $\mathrm{PL}(\mathbb{R}^n, \log)$ and hence (SPL), whenever $V(P)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega)$ for some weight function ω. Therefore it is reasonable to treat general differential operators $P(D)$ as perturbations of their principal part $P_m(D)$. In this connection, Meise and Taylor [9] showed recently that the condition (SPL) plays an important role when P_m is perturbed by an independent variable. From [9], Theorem 3.4, we recall the following theorem.

Theorem 2.7. *Let* $P_m \in \mathbb{R}[z_1, \ldots, z_n]$ *be homogeneous of degree* $m \geq 2$ *and let* $P(z) := P_m(z') - z_{n+1}$ *for* $z = (z', z_{n+1}) \in \mathbb{C}^{n+1}$. *Then the following assertions are equivalent*:

(1) $V(P)$ *satisfies* $\mathrm{PL}(\mathbb{R}^{n+1}, \log)$,
(2) $V(P)$ *satisfies* $\mathrm{PL}(\mathbb{R}^{n+1}, \omega)$ *for some weight function* ω *with* $\omega(t) = o(t^{1/m})$,
(3) $V_\pm(P_m)$ *both satisfy* (SPL),
(4) $V(P)$ *satisfies* (SPL),
(5) $V(P)$ *satisfies* $\mathrm{PL}(\mathbb{R}^{n+1}, \omega)$ *for each weight function* ω.

It is natural to ask why we consider $\mathrm{SPL}(A, 0)$ for constants $A > 1$. The case $A = 1$ is also interesting but it is very restrictive. For example, there are no polynomials of degree $m > 1$ with an irreducible principal part that can satisfy SPL with $A = 1$. To explain this in a bit more detail, we include the following proposition that is essentially well-known.

Proposition 2.8. *Let* $P \in \mathbb{C}[z_1, \ldots, z_n]$ *be of degree* $m \geq 1$ *and denote by* P_m *its principal part. If* $V(P)$ *satisfies* $\mathrm{SPL}(1, 0)$ *then* P *and* P_m *are hyperbolic with respect to each vector* $N \in \mathbb{R}^n \setminus \{0\}$ *satisfying* $P_m(N) \neq 0$. *In particular, up to a complex factor,* P_m *is the product of* m *real linear forms.*

Proof. For $\rho > 0$ let $K_\rho := \{x \in \mathbb{R}^n : |x| \leq \rho\}$. If $V(P)$ satisfies $\mathrm{SPL}(1, 0)$ then it also satisfies the condition $\widetilde{\mathrm{PL}}(K_1, K_2)$ defined in Franken and Meise [7], 2.6(c). Hence it follows from [7], 2.9, 2.11, and 2.13 that $P(D) : C^\infty(\overset{\circ}{K}_1) \to C^\infty(\overset{\circ}{K}_1)$ admits a continuous linear right inverse. By Meise, Taylor, and Vogt [11], 3.8, this implies the conclusion. $\square$

3. Local conditions

In this section we introduce the localization $(P)_\theta$ of a given polynomial P at a point $\theta \in V(P)$, which is used in the next section to formulate one

of the main results. To derive properties of $(P)_\theta$ from those of P, we also consider local Phragmén-Lindelöf conditions and investigate some of their properties.

Definition 3.1. Let V be an algebraic variety in $\mathbb{C}^n$ and $\xi \in V \cap \mathbb{R}^n$.

(a) V satisfies the condition $\mathrm{PL}_{\mathrm{loc}}(\xi)$ if there exist constants $r_1 > r_2 > 0$ and $A > 0$ such that each plurisubharmonic function u on $V \cap \{z \in \mathbb{C}^n : |z - \xi| < r_1\}$ that satisfies (α) and (β) also satisfies (γ), where
  (α) $u(z) \leq 1$, $z \in V \cap \{z \in \mathbb{C}^n : |z - \xi| < r_1\}$,
  (β) $u(z) \leq 0$, $z \in V \cap \mathbb{R}^n \cap \{z \in \mathbb{C}^n : |z - \xi| < r_1\}$,
  (γ) $u(z) \leq A|\mathrm{Im}\, z|$, $z \in V \cap \{z \in \mathbb{C}^n : |z - \xi| < r_2\}$.

(b) Let $\eta : \mathbb{C}^n \to [0, \infty[$ be a function which satisfies $\eta(z) = o(|z|)$ as $|z|$ tends to zero. Then we say that V satisfies the condition $\mathrm{PL}_{\mathrm{loc}}(\xi, \eta)$ if there exist constants $r_1 > r_2 > 0$ and $A > 0$ such that for every $\rho > 0$ there is a constant $B_\rho > 0$ such that each plurisubharmonic function u on $V \cap \{z \in \mathbb{C}^n : |z - \xi| < r_1\}$ which satisfies (α) and (β) also satisfies (γ), where
  (α) $u(z) \leq 1$, $z \in V \cap \{z \in \mathbb{C}^n : |z - \xi| < r_1\}$,
  (β) $u(z) \leq \rho|\mathrm{Im}\, z|$, $z \in V \cap \{z \in \mathbb{C}^n : |z - \xi| < r_1\}$,
  (γ) $u(z) \leq A|\mathrm{Im}\, z| + B_\rho \eta(z - \xi)$, $z \in V \cap \{z \in \mathbb{C}^n : |z - \xi| < r_2\}$.

Clearly, $\mathrm{PL}_{\mathrm{loc}}(\xi)$ implies $\mathrm{PL}_{\mathrm{loc}}(\xi, \eta)$ for each function η as in 3.1(b).

Remark 3.2. Classical estimates for the harmonic measure of the real line in the unit disk show that $\mathrm{PL}_{\mathrm{loc}}(\xi)$ holds at $\xi \in V \cap \mathbb{R}^n$ whenever V is a manifold in ξ and $\dim_{\mathbb{R}} V \cap \mathbb{R}^n = \dim_{\mathbb{C}} V$ in ξ (see, e.g., Meise, Taylor, and Vogt [11], Corollary 4.8).

From Meise and Taylor [9], Proposition 4.4, we recall the following result:

Proposition 3.3. *If an algebraic variety V in $\mathbb{C}^n$ satisfies* (SPL)*, then it also satisfies* $\mathrm{PL}_{\mathrm{loc}}(\xi)$ *for each $\xi \in V \cap \mathbb{R}^n$.*

The converse of Proposition 3.3 is false, since $V := \{(z_1, z_2) \in \mathbb{C}^2 : z_1^2 = z_2\}$ satisfies $\mathrm{PL}_{\mathrm{loc}}(\xi)$ at each $\xi \in V \cap \mathbb{R}^2$ but it does not satisfy (SPL) by [9], Proposition 2.3 and Theorem 3.4.

Definition 3.4. Let $\theta \in \mathbb{C}^n$ and a function P holomorphic in some neighborhood of θ be given. Then the lowest order homogeneous polynomial $(P)_\theta$ in the Taylor series expansion of $z \mapsto P(\theta + z)$ will be called the localization of P in the point θ.

In Chapter 7 of Whitney [17] several possibilities to introduce tangent varieties for analytic varieties are discussed. Here we only recall [17], 7.1G, the one which is defined in terms of limiting directions.

Definition 3.5. Let V be an analytic variety in $\mathbb{C}^n$ and $\zeta \in V$. The tangent cone $V_{\tan}(\zeta)$ in ζ is defined as the set of all $v \in \mathbb{C}^n$ such that for each $\varepsilon > 0$ there exist $z \in V$ and $t \in \mathbb{C}$ satisfying $|z - \zeta| < \varepsilon$ and $|t(z - \zeta) - v| < \varepsilon$.

Remark 3.6. In Whitney [17], Theorem 7.4D, it is shown that $V_{\tan}(\zeta)$ can also be described as the common zero set of all localizations $(f)_\zeta$ of functions f which are holomorphic in some neighborhood U_f of ζ and vanish on $U_f \cap V$. In particular, $V_{\tan}(\zeta)$ is a homogeneous algebraic variety.

In Sect. 5 below we need to know a result (Corollary 3.8) which can be easily derived from the following theorem.

Theorem 3.7. *If an algebraic variety V in $\mathbb{C}^n$ satisfies the condition $\mathrm{PL}_{\mathrm{loc}}$ (ξ, η) at $\xi \in V \cap \mathbb{R}^n$ for some function η on $\mathbb{C}^n$ satisfying $\eta(z) = o(|z|)$, then the variety $V_{\tan}(\xi)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$ and* (SPL).

We do not include a proof of Theorem 3.7, because it is similar to published proofs of other results which show that certain Phragmén-Lindelöf conditions holding on an algebraic variety carry over to limit varieties. The first result of this type is due to Hörmander [8], who showed that the global Phragmén-Lindelöf condition that characterizes surjectivity of constant coefficient partial differential operators $P(D)$ on real analytic functions not only carries over to the cone of limiting directions of $V(P)$ at infinity, but is equivalent to the condition holding on this cone. In Meise, Taylor, and Vogt [15], Theorem 4.1, it was shown that the global Phragmén-Lindelöf condition $\mathrm{PL}(\mathbb{R}^n, \omega)$ carries over to the cone of limiting directions at infinity, while Braun [2], Theorem 4.1, proves the corresponding result for a local Phragmén-Lindelöf condition which is quite similar to $\mathrm{PL}_{\mathrm{loc}}$. Note that by Meise, Taylor, and Vogt [15], Theorem 3.3, $V_{\tan}(\xi)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$ if and only if $V_{\tan}(\xi)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \log)$, which is equivalent to (SPL), as we have noted in Lemma 2.5.

Corollary 3.8. *For $P \in \mathbb{C}[z_1, \ldots, z_n]$ let $V(P)$ satisfy $\mathrm{PL}_{\mathrm{loc}}(\theta)$ at some $\theta \in V(P) \cap \mathbb{R}^n$. Then each irreducible factor of $(P)_\theta$ is a constant multiple of a polynomial with real coefficients.*

Proof. If $V(P)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(\theta)$ then

$$V_{\tan}(\theta) = \{z \in \mathbb{C}^n : (P)_\theta(z) = 0\}$$

satisfies (SPL) by Theorem 3.7 and hence $\mathrm{PL}(\mathbb{R}^n, \log)$. Consequently, it follows from Meise, Taylor, and Vogt [13], Lemma 2, that every irreducible factor of $(P)_\theta$ has real coefficients, up to a complex constant. $\square$

Next we provide two lemmas which will be applied in the following section and in which localizations are used.

Lemma 3.9. *For $n \geq 2$ let $P_m \in \mathbb{C}[z_1, \ldots, z_n]$ be homogeneous of degree $m \geq 2$, let $\theta := (0, \ldots, 0, 1)$, and assume $P_m(\theta) = 0$. Then there exist $\nu \in \mathbb{N}$, $\nu \leq m$, and polynomials $Q_j \in \mathbb{C}[z_1, \ldots, z_{n-1}]$, homogeneous of degree j, $\nu \leq j \leq m$, such that $P_m(z', z_n) = \sum_{j=\nu}^m z_n^{m-j} Q_j(z')$. Moreover, $(P_m)_\theta(z', z_n) = Q_\nu(z')$.*

Proof. Applying Taylor's formula at θ, $P_m(\theta) = 0$ implies

$$(3.1) \qquad P_m(z', 1) = P_m(\theta + (z', 0)) = \sum_{j=\nu}^m Q_j(z')$$

where $Q_j \in \mathbb{C}[z_1, \ldots, z_{n-1}]$ is zero or homogeneous of degree j and where $Q_\nu \neq 0$. From $P_m(\theta) = 0$ and $P_m \neq 0$ it follows that $1 \leq \nu \leq m$. Since P_m is homogeneous, (3.1) implies for $z_n \neq 0$:

$$(3.2) \qquad P_m(z', z_n) = z_n^m P_m\left(\frac{z'}{z_n}, 1\right) = \sum_{j=\nu}^m z_n^{m-j} Q_j(z').$$

By continuity, this holds also when $z_n = 0$. From (3.2) we get for $z = (z', z_n)$

$$P_m(\theta + z) = P_m(z', 1 + z_n) = \sum_{j=\nu}^m (1 + z_n)^{m-j} Q_j(z')$$

$$= \sum_{j=\nu}^m \sum_{k=0}^{m-j} \binom{m-j}{k} z_n^k Q_j(z')$$

and hence $(P_m)_\theta(z', z_n) = Q_\nu(z')$. $\qquad \square$

Lemma 3.10. *Let $P_m \in \mathbb{C}[z_1, \ldots, z_n]$ be homogeneous of degree $m \geq 2$. If $\xi := (1, 0, \ldots, 0) \in V(P_m)$ and if the localization $(P_m)_\xi$ of P_m at ξ has degree ν, then*

$$s : (\tau, z) \mapsto P_m(z_1 \tau^{-\nu}, z_2 \tau^{m-\nu}, \ldots, z_n \tau^{m-\nu}),$$
$$(\tau, z) \in (\mathbb{C} \setminus \{0\}) \times \mathbb{C}^n,$$

extends to a polynomial $\widetilde{s} \in \mathbb{C}[\tau, z_1, \ldots, z_n]$ with $\widetilde{s}(0, z) = z_1^{m-\nu}(P_m)_\xi (z_2, \ldots, z_n)$.

Proof. It is easy to check that for $z = (z_1, z')$

$$P_m(z_1, z') = \sum_{j=\nu}^m z_1^{m-j} q_j(z'),$$

where the polynomials q_j are either zero or homogeneous of degree j. By definition, $q_\nu = (P_m)_\xi$. From this representation we get

$$s(\tau, z) = \sum_{j=\nu}^{m} z_1^{m-j} \tau^{-\nu(m-j)} \tau^{j(m-\nu)} q_j(z') = \sum_{j=\nu}^{m} \tau^{m(j-\nu)} z_1^{m-j} q_j(z').$$

This shows that s is the restriction of a polynomial $\widetilde{s}$ to $(\mathbb{C} \setminus \{0\}) \times \mathbb{C}^n$ and that

$$s(0, z) = z_1^{m-\nu} q_\nu(z') = z_1^{m-\nu} (P_m)_\xi(z').$$

$\square$

4. A necessary condition for (SPL)

Let $P \in \mathbb{C}[z_1, \ldots, z_n]$ have real principal part P_m of degree $m \geq 2$ and define the polynomial S by $S(z_1, \ldots, z_{n+1}) := P(z_1, \ldots, z_n) - z_{n+1}$. In this section we use the concept of quasihomogeneity and a result of [4] to derive a necessary condition in terms of P_m for $V(S)$ to have $\mathrm{PL}(\mathbb{R}^{n+1}, \log)$. This extends a result of Meise and Taylor [9] and explains some of their counterexamples. The precise result is stated in the following theorem.

Theorem 4.1. *Let $P \in \mathbb{C}[z_1, \ldots, z_n]$ have degree $m \geq 2$ and principal part $P_m \in \mathbb{R}[z_1, \ldots, z_n]$. Define $S \in \mathbb{C}[z_1, \ldots, z_{n+1}]$ by $S(z', z_{n+1}) := P(z') - z_{n+1}$. If $V(S)$ satisfies $\mathrm{PL}(\mathbb{R}^{n+1}, \log)$, then for each $\theta \in V(P_m) \cap \mathbb{R}^n, |\theta| = 1$, the localization $(P_m)_\theta$ of P_m in θ is square-free.*

The proof of this theorem is prepared by several lemmas for which we need the following definition.

Definition 4.2. For $d = (d_1, \ldots, d_n) \neq (0, \ldots, 0)$ with d_j non-negative and rational for $1 \leq j \leq n$, a given polynomial $P \in \mathbb{C}[z_1, \ldots, z_n]$ is said to be d-quasihomogeneous of degree $m \geq 0$ if

$$P(z) = \sum_{\langle d, \alpha \rangle = m} a_\alpha z^\alpha, \quad z \in \mathbb{C}^n,$$

where $\langle d, \alpha \rangle = \sum_{j=1}^{n} d_j \alpha_j$ and where not all a_α vanish. The zero polynomial is considered to be d-quasihomogeneous of degree $-\infty$.

For an arbitrary polynomial $Q \in \mathbb{C}[z_1, \ldots, z_n]$ the d-quasihomogeneous principal part of Q is defined as the highest degree d-quasihomogeneous part of Q.

The concept of quasihomogeneity is relevant for our considerations because of the following lemma which we recall from [4], Lemma 3.2.

Lemma 4.3. *Let* $P \in \mathbb{C}[z_1, \ldots, z_n]$ *be* d-*quasihomogeneous of degree* $m > 0$ *and let* $Q \in \mathbb{C}[z_1, \ldots, z_n]$ *be a sum of* d-*quasihomogeneous polynomials of degrees less than* m. *Assume further that the following conditions are fulfilled*:

(1) $d_1 < d_j$ *for* $2 \leq j \leq n$,
(2) *there exists* $\zeta = (\zeta_1, \zeta'') \in V(P)$ *with* $\zeta_1 \notin \mathbb{R}$, $\zeta'' \in \mathbb{R}^{n-1}$, *and* $\zeta'' \neq 0$,
(3) *the polynomial* $\lambda \mapsto P(\lambda, \zeta'')$ *does not vanish identically*.

If $V(P + Q)$ *satisfies* $\mathrm{PL}(\mathbb{R}^n, \omega)$ *for some weight function* ω *and* $D := \max\{d_j : \zeta_j \neq 0\}$, *then* ω *satisfies* $t^{d_1/D} = O(\omega(t))$ *as* t *tends to infinity*.

Lemma 4.4. *For* $n \geq 3$ *let* $P \in \mathbb{C}[z_1, \ldots, z_n]$ *be a polynomial of degree* $m \geq 2$ *with principal part* P_m *which satisfies* $P_m(\theta) = 0$ *for* $\theta = (0, \ldots, 0, 1) \in \mathbb{R}^n$. *Assume that there exist* $k \geq 2$, $g \geq 1$, *and* $Q, R \in \mathbb{C}[z_1, \ldots, z_{n-1}]$ *such that the following conditions are satisfied*:

(a) $(P_m)_\theta (z', z_n) = Q(z')^k R(z')$ *for* $z = (z', z_n) \in \mathbb{C}^n$,
(b) $Q(z') = \sum_{|\alpha|=g} a_\alpha(z')^\alpha$ *where* $a_{1,g-1,0,\ldots,0} \neq 0$ *and* $a_\alpha = 0$ *for all* $\alpha \in \mathbb{N}_0^{n-1}$ *satisfying* $\alpha_1 = 0$ *and* $\alpha_2 \geq g - 1$.

Then the zero variety of the polynomial $S \in \mathbb{C}[z_1, \ldots, z_{n+1}]$ *defined by*

$$S(z_1, \ldots, z_{n+1}) = P(z_1, \ldots, z_n) - z_{n+1}$$

does not satisfy $\mathrm{PL}(\mathbb{R}^{n+1}, \log)$.

Proof. For P_m and θ choose $\nu \in \mathbb{N}$, $1 \leq \nu \leq m$, and polynomials $Q_j \in \mathbb{C}[z_1, \ldots, z_{n-1}]$, $\nu \leq j \leq m$, according to Lemma 3.9. Because of (a) we get from Lemma 3.9 that S is of the following form:

$$\begin{aligned} S(z', z_n, z_{n+1}) = {} & z_n^{m-\nu} Q(z')^k R(z') \\ & + \sum_{j=\nu+1}^{m} z_n^{m-j} Q_j(z') + \widetilde{P}(z', z_n) - z_{n+1}, \end{aligned}$$

where $\deg \widetilde{P} < m$. In order to apply Lemma 4.3, we have to find an appropriate quasihomogeneity d such that the d-quasihomogeneous principal part of S admits roots whose imaginary parts have a special form. The definition of the first n weights is

$$d_1 := 1, \quad d_2 := 1 + \frac{1}{m^2}, \quad d_3 := \cdots := d_{n-1} := 1 + \frac{1}{3m^2},$$
$$d_n := 1 + \frac{1}{m} + \frac{1}{m^2}, \quad d' := (d_1, \ldots, d_{n-1}).$$

Denote the d'-quasihomogeneous principal part of R by B and let $A = d'$-deg B. Note that A is not smaller than the classical degree of B, i.e.,

(4.1) $$A \geq \nu - kg.$$

Now define

(4.2)
$$d_{n+1} := (m - \nu)d_n + k\left(g + \frac{g-1}{m^2}\right) + A, \quad d := (d', d_n, d_{n+1}).$$

Note that (4.1) implies $d_{n+1} \geq m$. We claim:

($*$) The d'-quasihomogeneous principal part of Q is $a_{1,g-1,0,\ldots,0}z_1 z_2^{g-1}$, whose d'-degree is $g + (g-1)m^{-2}$.

To prove this claim, let $\alpha \neq (1, g-1, 0, \ldots, 0)$ with $a_\alpha \neq 0$ be given. There are three cases:

Case $\alpha_1 = 0$ Then $\alpha_2 \leq g-2$ because of (b). Since $d_2 > d_j$ for $2 < j < n$, we have the following estimate for the d'-degree of z^α:

$$\sum_{j=2}^{n-1} \alpha_j d_j \leq (g-2)d_2 + 2d_3 = g + \frac{g-2}{m^2} + \frac{2}{3m^2}$$

$$= g + \frac{g - 4/3}{m^2} < g + \frac{g-1}{m^2}.$$

Case $\alpha_1 = 1$ The case $\alpha_2 = g - 1$ being excluded, we have again $\alpha_2 \leq g - 2$. Since $d_2 > d_j > d_1$ for $2 < j < n$, the d'-degree of z^α is clearly smaller than that of $z_1 z_2^{g-1}$.

Case $\alpha_1 \geq 2$ The d'-degree of z^α is

$$\alpha_1 d_1 + \sum_{j=2}^{n-1} \alpha_j d_j \leq 2d_1 + (g-2)d_2 = g + \frac{g-2}{m^2} < g + \frac{g-1}{m^2}.$$

This proves ($*$). The next claim is:

($**$) The d-quasihomogeneous principal part of S is

$$S_q(z_1, \ldots, z_{n+1}) := z_n^{m-\nu} a_{1,g-1,0,\ldots,0} z_1^k z_2^{k(g-1)} B(z') - z_{n+1}.$$

To prove ($**$), note first that because of (4.2) the polynomial S_q is d-quasihomogeneous of d-degree d_{n+1}. Furthermore, since $d_j \leq d_n$ for $1 \leq j \leq n$,

$$d\text{-deg}\,\widetilde{P} \leq (m-1)d_n = m - 1 + (m-1)\frac{m+1}{m^2} = m - \frac{1}{m^2} < d_{n+1}.$$

To show that $z_n^{m-j}Q_j(z')$, $\nu < j \leq m$, does not belong to the d-quasihomogeneous principal part of S, consider:

$$d_{n+1} - d\text{-deg }z_n^{m-j}Q_j(z') \geq (m-\nu)d_n + k\left(g + \frac{g-1}{m^2}\right)$$
$$+ A - (m-j)d_n - jd_2$$
$$\geq (j-\nu)d_n + \nu - jd_2$$
$$= \frac{j-\nu}{m} - \frac{\nu}{m^2} \geq \frac{1}{m} - \frac{m-1}{m^2} = \frac{1}{m^2}\ .$$

So $(**)$ is shown. Choose $\zeta' \in \mathbb{R}^{n-1} \setminus \{0\}$ such that $p\colon \lambda \mapsto S_q(\lambda, \zeta', 0)$, $\lambda \in \mathbb{C}$, is a polynomial of degree at least $k \geq 2$. Fix now a value $b \in \mathbb{R}$ that $p|_{\mathbb{R}}$ obtains at most once, let ζ_1 be a non-real solution of $p(\zeta_1) = b$, and set $\zeta'' = (\zeta', b)$. Then the claim follows from Lemma 4.3. $\qquad\square$

Lemma 4.5. *For $n \geq 2$ let $Q \in \mathbb{R}[z_1, \ldots, z_n]$ be square-free and homogeneous of degree $g \geq 1$. Then there exists a real linear change of coordinates such that in the new coordinates $Q(z) = \sum_{|\alpha|=g} a_\alpha z^\alpha$, where $a_\alpha \neq 0$ for $\alpha = (g-1, 1, 0, \ldots, 0)$ and $a_\alpha = 0$ whenever $\alpha_1 \geq g - 1$ and $\alpha_2 = 0$.*

Proof. Since Q is square-free, we can find $b_1 \in \mathbb{R}^n$ satisfying $Q(b_1) = 0$ and $\operatorname{grad} Q(b_1) \neq 0$. Next choose $b_2 \in \mathbb{R}^n$ such that $\langle b_2, \operatorname{grad} Q(b_1)\rangle \neq 0$. Then Euler's rule, $\langle x, \operatorname{grad} Q(x)\rangle = gQ(x)$, implies that b_1 and b_2 are linearly independent. By our choice of b_1 and b_2, the function $g : \mathbb{R} \to \mathbb{R}$, $g(\zeta) := Q(b_1 + \zeta b_2)$ satisfies

$$(4.3) \qquad g(0) = Q(b_1) = 0 \quad \text{and} \quad g'(0) = \langle \operatorname{grad} Q(b_1), b_2\rangle \neq 0.$$

Next choose $b_3, \ldots, b_n$ such that $\{b_1, \ldots, b_n\}$ is a basis of $\mathbb{R}^n$ and consider the following expansion

$$(4.4) \qquad Q\left(\sum_{k=1}^n \zeta_k b_k\right) = \sum_{j=0}^g q_j(\zeta_2, \ldots, \zeta_n)\zeta_1^{g-j}$$

where the polynomials q_j are homogeneous of degree j. In particular, (4.4) implies

$$g(\zeta) = Q(b_1 + \zeta b_2) = \sum_{j=0}^g q_j(\zeta, 0, \ldots, 0) = \sum_{j=0}^g q_j(1, 0, \ldots, 0)\zeta^j.$$

By (4.3) and the homogeneity properties of q_0, we conclude from this $q_0 = 0$ and $q_1(1, 0, \ldots, 0) \neq 0$. Now choose $a \in \mathbb{R}^{n-1}$ such that $q_1(\xi) = \langle \xi, a\rangle$

and note that $a \neq 0$. Hence there is $M \in \mathrm{GL}(n-1, \mathbb{R})$ such that $M(a) = (1, 0, \ldots, 0)$. Define

$$p\Big(\sum_{j=1}^{n} \zeta_j b_j \Big) := Q\Big(\zeta_1 b_1 + \sum_{i=2}^{n} \Big(\sum_{j=2}^{n} M_{j-1,i-1} \zeta_j \Big) b_i \Big).$$

Then the linear term of this polynomial is

$$q_1 \Big(\Big(\sum_{j=2}^{n} M_{j-1,i-1} \zeta_j \Big)_{i=2}^{n} \Big) = \langle M^t(\zeta_2, \ldots, \zeta_n), a \rangle$$

$$= \langle (\zeta_2, \ldots, \zeta_n), Ma \rangle = \zeta_2.$$

Hence p has the desired form. $\square$

Proof of Theorem 4.1. Note first that the principal part S_m of S is the polynomial P_m, regarded as an element of $\mathbb{R}[z_1, \ldots, z_{n+1}]$. Since $V(S)$ satisfies $\mathrm{PL}(\mathbb{R}^{n+1}, \log)$, it follows from Meise, Taylor, and Vogt [15], Theorem 4.1, that $V(S_m) = V(P_m) \times \mathbb{C}$ has the same property. Hence $V(P_m)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \log)$ and consequently (SPL), by Proposition 2.5(b). Assume there exists $\theta \in V(P_m) \cap \mathbb{R}^n$, $|\theta| = 1$, such that $(P_m)_\theta$ is not square-free. After a suitable real linear change of variables we may assume $\theta = (0, \ldots, 0, 1)$. By Lemma 3.9 and our assumption there exist $Q, R \in \mathbb{C}[z_1, \ldots, z_{n-1}]$, Q square-free, and $k \geq 2$ such that $\deg Q \geq 1$, $R \neq 0$, and

$$(P_m)_\theta(z', z_n) = Q(z')^k R(z').$$

Since P_m has real coefficients, so does $(P_m)_\theta$. As $V(P_m)$ satisfies (SPL), it follows from Corollary 3.8 that it is no restriction to assume that Q and R have real coefficients. If $n \geq 3$, then Lemma 4.5 shows that there exists a change of variables in the z' variables such that Q has the special form given in condition (b) of Lemma 4.4. Hence Lemma 4.4 implies that $V(S)$ does not satisfy $\mathrm{PL}(\mathbb{R}^{n+1}, \log)$ in contradiction to the hypothesis. This proves Theorem 4.1 for $n \geq 3$.

If $n = 2$ then it follows from Lemma 3.9 that

$$P_m(z_1, z_2) = z_2^{m-\nu}(P_m)_\theta(z_1) + \sum_{j=\nu+1}^{m} z_2^{m-j} Q_j(z_1)$$

for some $\nu \in \mathbb{N}$ and suitable polynomials Q_j which are homogeneous of degree j. The assumption on $(P_m)_\theta$ implies $\nu \geq 2$. Hence

$$S(z_1, z_2, z_3) = \sum_{j=\nu}^{m} a_j z_2^{m-j} z_1^j + \sum_{k=0}^{m-1} \sum_{|\alpha|=k} a_\alpha z_1^{\alpha_1} z_2^{\alpha_2} - z_3,$$

where $a_\nu \in \mathbb{R} \setminus \{0\}$. To apply Lemma 4.3 again, let

$$d_1 := 1, d_2 := \frac{m+1}{m}, \text{ and } d_3 := d_2(m-\nu) + \nu.$$

Then it is easy to check that the d-quasihomogeneous principal part of S is the polynomial $Q(z_1, z_2, z_3) := a_\nu z_2^{m-\nu} z_1^\nu - z_3$. Now let $\zeta_2 := 1$, $\zeta_3 = -\operatorname{sign} a_\nu$, and note that $\nu \geq 2$. Hence there exists $\zeta_1 \in \mathbb{C} \setminus \mathbb{R}$ such that $Q(\zeta_1, \zeta_2, \zeta_3) = 0$. Consequently, Lemma 4.3 implies that $V(S)$ does not satisfy $\mathrm{PL}(\mathbb{R}^3, \log)$, in contradiction to the hypothesis. This proves Theorem 4.1 also if $n = 2$. □

Theorem 4.1 extends Meise and Taylor [9], Lemma 2.2, as the following corollary shows.

Corollary 4.6. *Let $P \in \mathbb{C}[z_1, \ldots, z_n]$ have degree $m \geq 2$ and principal part $P_m \in \mathbb{R}[z_1, \ldots, z_n]$. Define $S \in \mathbb{C}[z_1, \ldots, z_{n+1}]$ by $S(z', z_{n+1}) := P(z') - z_{n+1}$. If $V(S)$ satisfies $\mathrm{PL}(\mathbb{R}^{n+1}, \log)$, then P_m is square-free.*

Proof. As in the proof of Theorem 4.1 it follows that $V(P_m)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \log)$. If we assume that P_m is not square-free then there exist $Q, R \in \mathbb{C}[z_1, \ldots, z_n]$ and $k \geq 2$ such that $P_m = Q^k R$, where $\deg Q \geq 1$. Since $V(P_m)$ has $\mathrm{PL}(\mathbb{R}^n, \log)$, also $V(Q)$ has this property by Meise, Taylor, and Vogt [15], Proposition 2.6, and hence the real zero set of Q has dimension $n - 1$ by [15], Theorem 3.13. In particular there exists $\theta \in V(Q) \cap \mathbb{R}^n$, $|\theta| = 1$. Now it is easy to check that

$$(P_m)_\theta = (Q_\theta)^k R_\theta.$$

Hence Theorem 4.1 implies that $V(S)$ does not satisfy $\mathrm{PL}(\mathbb{R}^{n+1}, \log)$ in contradiction to the hypothesis. □

5. A characterization in dimension 3

In this section we show that the necessary condition which we derived in Theorem 4.1 leads to a characterization of those homogeneous polynomials P_m in three variables for which $V_+(P_m)$ and $V_-(P_m)$ have (SPL). The precise result is the following theorem which implies Theorem 1.1.

Theorem 5.1. *Let $P_m \in \mathbb{C}[z_1, z_2, z_3]$ be homogeneous of degree $m \geq 2$ and define $P \in \mathbb{C}[z_1, \ldots, z_4]$ by $P(z_1, \ldots, z_4) := P_m(z_1, z_2, z_3) - z_4$. Then the following conditions are equivalent:*

(1) $V_+(P_m)$ *and* $V_-(P_m)$ *satisfy* (SPL),
(2) $V(P)$ *satisfies* (SPL),
(3) $V(P)$ *satisfies* $\mathrm{PL}(\mathbb{R}^4, \log)$,

(4) $V(P_m)$ *satisfies* (SPL)*, P_m has real coefficients, and for each $\theta \in$
   $V(P_m) \cap \mathbb{R}^3, |\theta| = 1$, the localization of P_m at θ is square-free.*

The theorem is proved by showing that $V_+(P_m)$ and $V_-(P_m)$ inherit
(SPL) from two simpler varieties. One of them is of course $V(P_m)$, the
cone of limiting directions. The hypotheses imply that $V(P_m)$ is locally
hyperbolic near $V(P_m) \cap \mathbb{R}^3 \setminus \{0\}$. The a priori estimate of Theorem 5.2 al-
lows us to restrict our attention to cones spanned by neighborhoods of points
$\xi \in V(P_m)$. Assume $\xi = (1, 0, 0)$ and denote the vanishing order of P_m in
ξ by ν. In Lemma 5.6 we show that inside the above mentioned cone, but
outside a tubular neighborhood U of $\mathbb{R} \cdot \xi$, the two varieties $V(S)$ and $V(P_m)$
are so close that the standard method for proving Phragmén-Lindelöf condi-
tions for hyperbolic polynomials, namely taking maxima over the fibers of
a suitable projection, works also for $V(S)$. Inside U, however, $V(S)$ is still
a manifold, while $V(P_m)$ may be singular in ξ. Here, we compare $V(S)$
to another manifold $V(Q_0 - 1)$, where $Q_0(z_2, z_3) := (P_m)_\xi(z)$. We show
in Lemma 5.10 that coordinate patches for $V(Q_0 - 1)$ induce coordinate
patches for $V(S) \cap U$ and use standard estimates on these patches. At first
we recall a result from [5].

Theorem 5.2. *Let $P_m \in \mathbb{R}[z_1, \ldots, z_n]$ be homogeneous of degree m and
let $Q \in \mathbb{R}[z_1, \ldots, z_n]$ be of degree less than m. If P_m is square-free and
if $V(P_m)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \log)$ then there exist $A_0 \geq 1$, $B_0 \geq 0$ such that
each $u \in \mathrm{PSH}(V(P_m + Q))$ which satisfies the conditions (α) and (β) of
2.2(a) also satisfies $u(z) \leq A_0|z| + B_0$, $z \in V(P_m + Q)$.*

Proof. Since $V(P_m)$ has $\mathrm{PL}(\mathbb{R}^n, \log)$, it follows from Meise, Taylor, and
Vogt [15], Theorem 3.13, that each irreducible factor of P_m has an $(n-1)$-
dimensional set of real zeros. Therefore, the present theorem follows from
[5], Theorem 1.1.   □

If P_m satisfies the hypotheses in 5.1(4) then Theorem 5.2 and Lemma
2.5 imply that on $V_\pm(P_m)$ the estimate needed for (SPL) holds whenever
$z \in V_\pm(P_m)$ satisfies $|z| \leq \lambda|\mathrm{Im}\, z|$ for some constant $\lambda > 0$. Hence we
only have to consider those points in $V_\pm(P_m)$ for which $|\mathrm{Im}\, z|/|z|$ is small.
This will be done in cones around real lines in $V(P_m)$. To handle these we
need several lemmas. Furthermore, we will use the following notation.

Notation. For $w \in \mathbb{C}^n$ and $r = (r_1, \ldots, r_n), r_j > 0, 1 \leq j \leq n$, we denote
the polydisk with center w and polyradius r by $B(w; r)$ or $B(w; r_1, \ldots, r_n)$.
For $\rho > 0$ we will use the abbreviation $B(w; \rho) = B(w; \rho, \ldots, \rho)$.

Lemma 5.3. *Let $P_m \in \mathbb{R}[z_1, z_2, z_3]$ be homogeneous of degree $m \geq 2$
and assume that $V(P_m)$ has $\mathrm{PL}(\mathbb{R}^3, \log)$. Then for each $\xi \in V(P_m) \cap \mathbb{R}^3$,
$|\xi| = 1$, there exist a real linear change of variables, $\nu \in \mathbb{N}$, $\eta > 0$, $\sigma > 0$,*

and holomorphic functions $\beta_j : B((1,0); \eta, \eta) \to \mathbb{C}$, $1 \leq j \leq \nu$, such that in the new coordinates the following holds:

(1) *$\xi = (1, 0, 0)$, and $(0, 0, 1)$ is non-characteristic for P_m,*
(2) *$V(P_m) \cap B(\xi; \eta, \eta, \sigma) = \bigcup_{j=1}^{\nu}\{(z_1, z_2, \beta_j(z_1, z_2)) : (z_1, z_2) \in B((1,0); \eta, \eta)\}$,*
(3) *$\beta_j(z_1, z_2)$ is real for real z_1, z_2, $1 \leq j \leq \nu$.*

Proof. It is easy to check that (1) can be achieved by a real linear change of variables. Since P_m is real and $(0, 0, 1)$ is non-characteristic for P_m, up to a real constant we have

$$P_m(z_1, z_2, z_3) = \prod_{j=1}^{m}(z_3 - \beta_j(z_1, z_2)), \quad (z_1, z_2, z_3) \in \mathbb{C}^3,$$

where $\beta_1(z_1, z_2), \dots, \beta_m(z_1, z_2)$ denote the (not necessarily distinct) m roots of $z_3 \mapsto P_m(z_1, z_2, z_3)$ for fixed $(z_1, z_2) \in \mathbb{C}^2$. By hypothesis $P_m(1, 0, 0) = 0$, therefore there exists ν with $1 \leq \nu \leq m$ such that (without restriction)

$$\beta_j(1, 0) = 0 \text{ for } 1 \leq j \leq \nu \text{ and } \beta_j(1, 0) = a_j \neq 0 \text{ for } \nu + 1 \leq j \leq m.$$

Since $V(P_m)$ satisfies $\mathrm{PL}(\mathbb{R}^3, \log)$ it follows from Braun [1], Corollary 12, that there are $0 < \eta, \sigma \leq 1$, and holomorphic functions $\beta_j : B((1, 0); \eta, \eta) \to \mathbb{C}, 1 \leq j \leq \nu$, such that

$$\begin{aligned}
&V(P_m) \cap B(\xi; \eta, \eta, \sigma) \\
&= \bigcup_{j=1}^{\nu}\{(z_1, z_2, \beta_j(z_1, z_2)) : (z_1, z_2) \in B((1, 0); \eta, \eta)\}.
\end{aligned}$$

Moreover, $\beta_j(z_1, z_2)$ is real when (z_1, z_2) is real for $1 \leq j \leq \nu$. Hence (2) and (3) hold. $\qquad\square$

The easy proof of the following lemma is left to the reader.

Lemma 5.4. *Let $\nu \in \mathbb{N}$ and distinct numbers $b_1, \dots, b_\nu \in \mathbb{R} \setminus \{0\}$ be given and let $Q_0(z_2, z_3) := \prod_{j=1}^{\nu}(z_3 - b_j z_2)$. Then there exist $\rho_0 \geq 1$ and $D_0 > 0$ such that for each $\mu \in \mathbb{C}$, $|\mu| \leq 2$ and each $(w_2, w_3) \in \mathbb{C}^2$ satisfying $Q_0(w_2, w_3) = \mu$ and $|w_2| \geq \rho_0$ there exists a unique l, $1 \leq l \leq \nu$, such that*

$$|w_3 - b_l w_2| \leq \frac{D_0}{|w_2|^{\nu-1}}.$$

Lemma 5.5. *For $p \in \mathbb{N}$ there exists $C \geq 1$ such that for any choice of A and $a \in \mathbb{C}^p$ satisfying $|a_j| < |A_j|/2$ for $1 \leq j \leq p$, the following estimate holds*:

$$\Big| \prod_{j=1}^{p}(A_j + a_j) - \prod_{j=1}^{p} A_j \Big| \leq C \max_{1 \leq k \leq p} \Big(|a_k| \prod_{j \neq k} A_j \Big).$$

Proof. Writing the difference as a telescoping sum, the estimate is obtained easily. □

Lemma 5.6. *Let $P_m \in \mathbb{R}[z_1, z_2, z_3]$ be homogeneous of degree $m \geq 2$ and satisfy condition 5.1(4). Assume that $\xi = (1, 0, 0) \in V(P_m)$ and that the conclusions of Lemma 5.3 hold. Then there exist $\rho_0 \geq 1$, r, $s > 0$, $D > 0$, and $R_1 > 1$ such that for each $t \geq R_1$ and $(z_1, z_2) \in \mathbb{C}^2$ satisfying $|z_1| < rt$ and $t^{1-m/\nu}\rho_0 < |z_2| < rt$ the equation $P_m(z_1, z_2, z_3) = \pm 1$ has exactly ν distinct roots $z_3^1(z_1, z_2), \ldots, z_3^\nu(z_1, z_2)$ satisfying $|z_3^j(z_1, z_2)| \leq st$ and $|z_3^j(z_1, z_2)| \leq D|z_2|$. Moreover, each $z_3^j(z_1, z_2)$ is real for real z_1, z_2.*

Proof. We will give the proof only for the case $P_m(z) = +1$, since the one for $P_m = -1$ is the same, up to obvious modifications. In the proof we will use the notation that was introduced in Lemma 5.3. Since $(P_m)_\xi$ is square-free, the real numbers

$$b_j := \frac{\partial \beta_j}{\partial z_2}(1, 0), \quad 1 \leq j \leq \nu,$$

are distinct. Since the functions β_j, $1 \leq j \leq \nu$, are holomorphic, we have the following power series expansions

$$\beta_j(1, \lambda) = b_j \lambda + \sum_{k=2}^{\infty} b_{j,k} \lambda^k, \quad |\lambda| < \eta, \quad 1 \leq j \leq \nu.$$

Next define the holomorphic functions

$$F, U : B(\xi; \eta, \eta, \sigma) \to \mathbb{C}, \quad F(z) = \prod_{j=1}^{\nu}(z_3 - \beta_j(z_1, z_2)),$$

$$U(z) := \frac{P_m(z)}{F(z)}.$$

Since $(0, 0, 1)$ is non-characteristic for P_m, the equation $P_m(1, 0, z_3) = 0$ has m roots, ν of which are zero. The other roots, $a_{\nu+1}, \ldots, a_m$, are all non-zero and satisfy $|a_j| > \sigma$ by 5.3(2). It is no restriction to assume $P_m(1, 0, z_3) = z_3^\nu \prod_{j=\nu+1}^{m}(-a_j)$ and $\prod_{j=\nu+1}^{m}(-a_j) = 1$, which implies

$$U(1, 0, 0) = \prod_{j=\nu+1}^{m}(-a_j) = 1.$$

Shrinking η and σ if necessary, we may assume $|a_j| > 3\sigma$ for $\nu+1 \le j \le m$. Next note that for $z \in B(\xi; \eta, \eta, \sigma)$ we have

$$F(z_1, z_2, z_3) = \prod_{j=1}^{\nu} \left(z_3 - z_1 \beta_j(1, \frac{z_2}{z_1}) \right)$$

(5.1)
$$= \prod_{j=1}^{\nu} (z_3 - z_2 b_j - g_j(z_1, z_2)),$$

where

$$g_j(z_1, z_2) = z_1 \sum_{k=2}^{\infty} b_{j,k} \left(\frac{z_2}{z_1} \right)^k.$$

From this it follows that, shrinking $\eta > 0$ again, there exists $M > 0$ such that

(5.2) $|g_j(z_1, z_2)| \le M|z_2|^2, \quad (z_1, z_2) \in B((1,0); \eta, \eta), \quad 1 \le j \le \nu.$

Now define $Q_0 \in \mathbb{R}[z_2, z_3]$ and $Q \in \mathbb{R}[z_1, z_2, z_3]$ by

(5.3) $Q_0(z_2, z_3) := \prod_{j=1}^{\nu} (z_3 - b_j z_2), \quad Q(z_1, z_2, z_3) = z_1^{m-\nu} Q_0(z_2, z_3).$

To prove the lemma we will apply a scaling argument and Rouché's theorem. The basic observation for the scaling is that $P_m(z) = 1$ is equivalent to $P_m(z/t) = t^{-m}$. If $z/t \in B(\xi; \eta, \eta, \sigma)$, this equation is equivalent to $F(z/t) = t^{-m}/U(z/t)$. To get the desired information on the zeros of $P_m = 1$ we approximate F by Q_0 and U by $z_1^{m-\nu}$. To do so, we need a number of estimates. To derive them, denote by ρ_0 and D_0 the numbers which exist for the polynomial Q_0 by Lemma 5.4. Furthermore, define

$$\kappa := \min\{|b_j - b_k| : 1 \le j, \ k \le \nu, \ j \ne k\} > 0,$$
(5.4) $\qquad M_0 := \max_{1 \le j \le \nu} |b_j|.$

Enlarging ρ_0 if necessary, we may assume $D_0/\rho_0^{\nu} \le \min(\kappa/2, M_0, (\kappa/2)^{\nu}$ $(4(\nu-1)(3M_0)^{\nu-1})^{-1})$. Then fix $t > (2\rho_0/\eta)^{\nu/m}$ and $z \in B(\xi; \eta, \eta, \sigma)$ satisfying $|z_2| > \rho_0 t^{-m/\nu}$. If $Q(z) = t^{-m}$ then the definition of Q implies

$$Q_0(t^{m/\nu} z_2, t^{m/\nu} z_3) = \frac{1}{z_1^{m-\nu}}.$$

Provided that η is small enough, we have

$$\frac{1}{|z_1|^{m-\nu}} \le \left(\frac{1}{1-\eta} \right)^{m-\nu} \le 2.$$

Since $|t^{m/\nu}z_2| > \rho_0$, Lemma 5.4 implies the existence of a unique $l, 1 \leq l \leq \nu$, satisfying $|t^{m/\nu}z_3 - b_l t^{m/\nu}z_2| \leq D_0/|t^{m/\nu}z_2|^{\nu-1}$ and hence, using $|z_2| > \rho_0 t^{-m/\nu}$:

$$(5.5) \qquad |z_3 - b_l z_2| \leq \frac{D_0}{t^m |z_2|^{\nu-1}} \leq \frac{D_0}{\rho_0^\nu}|z_2|.$$

By the definition of κ and M_0 in (4.1), it follows from (5.5) and $|z_2| > \rho_0 t^{-m/\nu}$ that for $j \neq l$ and $1 \leq j \leq \nu$ we have the following estimates:

$$(5.6) \qquad |z_3 - b_j z_2| \leq \left(\frac{D_0}{\rho_0^\nu} + 2M_0\right)|z_2| \leq 3M_0|z_2|,$$

$$(5.7) \qquad |z_3 - b_j z_2| \geq \left(\kappa - \frac{D_0}{\rho_0^\nu}\right)|z_2| \geq \frac{\kappa}{2}|z_2|.$$

Together with (5.5) they imply

$$(5.8) \qquad \begin{aligned}
\left|\frac{\partial Q_0}{\partial z_3}(z_2, z_3)\right| &= \left|\sum_{k=1}^{\nu}\prod_{j\neq k}(z_3 - b_j z_2)\right| \\
&\geq \left(\frac{\kappa}{2}|z_2|\right)^{\nu-1} - (\nu-1)(3M_0)^{\nu-2}\frac{D_0}{\rho_0^\nu}|z_2|^{\nu-1} \\
&\geq \frac{1}{2}\left(\frac{\kappa}{2}|z_2|\right)^{\nu-1}.
\end{aligned}$$

By homogeneity there exists $K_\nu \in \mathbb{N}$ such that

$$(5.9) \qquad \left|\left(\frac{\partial}{\partial z_3}\right)^p Q_0(z_2, z_3)\right| \leq K_\nu|z_2|^{\nu-p}, \; p \geq 2.$$

Next choose $\delta > 0$ so small that $\delta \leq \kappa/4$ and $K_\nu \sum_{p=2}^{\nu}\delta^p < \frac{1}{4}\left(\frac{\kappa}{2}\right)^\nu \delta$. Since $Q(z) = t^{-m}$ it follows from (5.8) and (5.9) that for $\zeta \in \mathbb{C}, |\zeta| = \delta|z_2|$, we have the following estimate

$$|Q(z_1, z_2, b_l z_2 + \zeta) - t^{-m}| = |Q(z_1, z_2, b_l z_2 + \zeta) - Q(z)|$$

$$= \left| \sum_{p=1}^{\nu} \left(\frac{\partial}{\partial z_3} \right)^p Q_0(z_2, z_3) z_1^{m-\nu} \zeta^p \right|$$

$$\geq \left(\frac{1}{2} \left(\frac{\kappa}{2} |z_2| \right)^{\nu-1} \delta |z_2| \right.$$

(5.10)

$$\left. - K_\nu \sum_{p=2}^{\nu} |z_2|^{\nu-p} \delta^p |z_2|^p \right) |z_1|^{m-\nu}$$

$$\geq \left(\frac{1}{4} \left(\frac{\kappa}{2} \right)^{\nu-1} \delta |z_2|^\nu \right) |z_1|^{m-\nu}$$

$$> \frac{1}{8} \left(\frac{\kappa}{2} \right)^{\nu-1} \delta |z_2|^\nu.$$

To derive an upper bound for $P_m(z_1, z_2, \cdot) - Q(z_1, z_2, \cdot)$ on the same circle, note that $Q = Q_0 \cdot Q_1$, where $Q_1(z) = z_1^{m-\nu}$ and that

(5.11) $\qquad P_m - Q = FU - Q_0 Q_1 = F(U - Q_1) + (F - Q_0) Q_1.$

To estimate the terms on the right hand side, we use (5.3), (5.4), and Lemma 5.5 to get for $\lambda = b_l z_2 + \zeta$:

$$|((F - Q_0) Q_1)(z_1, z_2, \lambda)|$$

$$\leq \left| \prod_{j=1}^{\nu} (\lambda - b_j z_2 - g_j(z_1, z_2)) - \prod_{j=1}^{\nu} (\lambda - b_j z_2) \right| |z_1|^{m-\nu}$$

(5.12)

$$\leq C \max_{1 \leq k \leq \nu} \left(|g_j(z_1, z_2)| \left| \prod_{j \neq k} (\lambda - b_j z_2) \right| \right) |z_1|^{m-\nu}$$

$$\leq CM |z_2|^2 ((2M_0 + \delta) |z_2|)^{\nu-1} |z_1|^{m-\nu} \leq L |z_2|^{\nu+1},$$

where $L := CM(2M_0 + \delta) 2^{m-\nu}$. Similarly we get

$$|F(z_1, z_2, \lambda)| = \prod_{j=1}^{\nu} |\zeta + (b_l - b_j) z_2 + g_j(z_1, z_2)|$$

(5.13)

$$\leq (\delta |z_2| + M |z_2|^2)((\delta + 2M_0) |z_2|)^{\nu-1}$$

$$= (\delta + M |z_2|)(\delta + 2M_0)^{\nu-1} |z_2|^\nu.$$

To estimate $U(z_1, z_2, \lambda) - z_1^{m-\nu}$, recall from the proof of Lemma 5.3 that for fixed $(z_1, z_2) \in B((1,0); \eta, \eta)$ we have, by the homogeneity of P_m,

$$U(z_1, z_2, \zeta) = \prod_{j=\nu+1}^{m} (\zeta - \beta_j(z_1, z_2))$$

$$= \prod_{j=\nu+1}^{m} (\zeta - z_1(\beta_j(1, \frac{z_2}{z_1}))).$$

Since $\beta_j(1, 0) = a_j$, $\nu + 1 \leq j \leq m$, and since the β_j depend continuously on z_1, z_2, for each $\varepsilon > 0$ there exists $\sigma(\varepsilon) > 0$ such that $|\beta_j(1, z) - a_j| \leq \varepsilon$ for $|z| \leq \sigma(\varepsilon)$ and $\nu + 1 \leq j \leq m$. Since P_m is homogeneous and since $\prod_{j=\nu+1}^{m}(-a_j) = 1$, we get from Lemma 5.5

$$\left| U(z_1, z_2, \lambda) - z_1^{m-\nu} \right|$$

$$= \left| \prod_{j=\nu+1}^{m} \left(\lambda - a_j z_1 + \left(a_j - \beta_j \left(1, \frac{z_2}{z_1} \right) \right) z_1 \right) - \prod_{j=\nu+1}^{m} (-a_j z_1) \right|$$

$$\leq C \max_{\nu+1 \leq k \leq m} \left(|\lambda| + |z_1| \left| a_k - \beta_k \left(1, \frac{z_2}{z_1} \right) \right| \right) \cdot \left| \prod_{j \neq k} (-a_j z_1) \right|$$

$$\leq C \left((M_0 + \delta)|z_2| + \varepsilon \right) |z_1|^{m-\nu-1}$$

$$\leq C 2^{m-\nu-1} \left((M_0 + \delta)|z_2| + \varepsilon \right),$$

if $\eta < \sigma(\varepsilon)/2$. From this and (5.13) we get

$$(5.14) \qquad \begin{aligned} |(F(U - Q_1))(z_1, z_2, \lambda)| &\leq (\delta + M|z_2|)(\delta + 2M_0)^{\nu-1} \\ &\quad \times C 2^{m-\nu-1}((M_0 + \delta)|z_2| + \varepsilon)|z_2|^{\nu} \\ &\leq \frac{1}{16} \left(\frac{\kappa}{2} \right)^{\nu-1} \delta |z_2|^{\nu} \end{aligned}$$

if ε and consequently η is small enough. From (5.12) we get

$$(5.15) \quad |((F - Q_0)Q_1)(z_1, z_2, \lambda)| \leq L|z_2|^{\nu+1} \leq \frac{1}{16} \left(\frac{\kappa}{2} \right)^{\nu-1} \delta |z_2|^{\nu},$$

if $\eta < \frac{1}{16L} \left(\frac{\kappa}{2} \right)^{\nu-1} \delta$. Combining (5.14) and (5.15) with (5.11) it follows from (5.10) that we can apply Rouché's theorem to conclude that there exist $\rho_0 > 0$, $R_1 > 0$, and $0 < r < \sigma/(\delta + M_0)$ such that the following holds: Whenever $t \geq R_1$ and $(z_1, z_2) \in \mathbb{C}^2$ satisfies $|z_1| < r$ and $t^{-m/\nu} \rho_0 < |z_2| < r$, to each solution $z = (z_1, z_2, b_l z_2)$ of the equation $Q(\zeta) = 0$ there corresponds a unique solution $(z_1, z_2, z_3^l(z_1, z))$ of the equation $P_m(\zeta) = t^{-m}$ satisfying

$$|z_3^l(z_1, z_2) - b_l z_2| < \delta |z_2|.$$

This implies

$$|z_3^l(z_1, z_2)| < \delta|z_2| + |b_l z_2| \leq (\delta + M_0)|z_2| = D|z_2| \leq Dr = s < \sigma$$

if we let $D := \delta + M_0$ and $s := Dr$. When z_1, z_2 are real, we get for $\zeta := (z_1, z_2, z_3^l(z_1, z_2))$:

$$t^{-m} = \overline{P_m(\zeta)} = P_m(\bar{\zeta}) \quad \text{and} \quad |\bar{\zeta} - b_l z_2| = |\bar{\zeta} - \overline{b_l z_2}| < \delta|z_2|,$$

since P_m has real coefficients and since b_l is real. By the uniqueness of the solution, this implies that $z_3^l(z_1, z_2)$ is real for real z_1, z_2. Since the equation $Q(z_1, z_2, \tau) = 0$ has ν distinct roots $b_1 z_2, \ldots, b_\nu z_2$, which are further apart from each other than $\delta|z_2|$, the lemma now follows from an obvious scaling argument. $\qquad\square$

To apply Lemma 5.6, we need the estimates of the following lemma.

Lemma 5.7. *For each $z, w \in \mathbb{C}$ and $\varepsilon > 0$ the following inequalities hold*:

(i) $|\operatorname{Im} \sqrt{z^2 + w^2}| \leq |\operatorname{Im} z| + |\operatorname{Im} w|$,
(ii) $|\operatorname{Im} z| \leq |\operatorname{Im} \sqrt{z^2 - \varepsilon^2}|$.

If, furthermore, $|z| \geq \lambda\varepsilon$ for some $\lambda > 1$, then

(iii) $|\operatorname{Im} \sqrt{z^2 - \varepsilon^2}| \leq \dfrac{\lambda}{\sqrt{\lambda^2 - 1}}|\operatorname{Im} z|$.

Proof. Let $z = x + iy$, $w = u + iv$, and $\sqrt{z^2 + w^2} = A + iB$. Then a short calculation shows that

$$AB = xy + uv \quad \text{and} \quad A^2 - B^2 = (x^2 + u^2) - (y^2 + v^2).$$

Solving for A from the first of these equations and substituting into the second one gives a quadratic equation for $B^2 \geq 0$ whose solution is

$$B^2 = \frac{1}{2}\left\{(y^2 + v^2) - (x^2 + u^2) + \left([(x^2 + u^2) - (y^2 + v^2)]^2\right.\right.$$
$$\text{(5.16)} \qquad \left.\left. +4(xy + uv)^2\right)^{\frac{1}{2}}\right\}.$$

By the Schwarz inequality, $(xy + uv)^2 \leq (x^2 + u^2)(y^2 + v^2)$, the term under the square-root is not larger than $[(x^2 + u^2) + (y^2 + v^2)]^2$, which implies

$$|\operatorname{Im} \sqrt{z^2 + w^2}| = |B| \leq \sqrt{y^2 + v^2} \leq |y| + |v| = |\operatorname{Im} z| + |\operatorname{Im} w|.$$

Hence (i) holds. To prove (ii), set $u = 0, v = \varepsilon$ in (5.16) and note that then the term under the square-root is equal to $(x^2 + y^2 - \varepsilon^2)^2 + 4y^2\varepsilon^2$. If we let $C := 2y\varepsilon/(x^2 + y^2 - \varepsilon^2)$, then we get

$$(5.17) \qquad B^2 = \frac{1}{2}(y^2 + \varepsilon^2 - x^2 + |x^2 + y^2 - \varepsilon^2|\sqrt{1 + C^2}).$$

If $x^2 + y^2 - \varepsilon^2 \geq 0$ this implies

$$|\operatorname{Im} \sqrt{z^2 - \varepsilon^2}| = |B| \geq |y| = |\operatorname{Im} z|,$$

since $\sqrt{1 + C^2} \geq 1$. If $x^2 + y^2 - \varepsilon^2 \leq 0$ then $y^2 + \varepsilon^2 - x^2 \geq 2y^2$ and hence $B^2 \geq y^2$. From this, (ii) follows also in this case.

For the inequality (iii), use the estimate $\sqrt{1 + C^2} \leq 1 + C^2/2$ in (5.17). Since $|z|^2 = x^2 + y^2 \geq \lambda^2\varepsilon^2$ we obtain

$$B^2 \leq y^2 + y^2 \frac{\varepsilon^2}{x^2 + y^2 - \varepsilon^2} \leq y^2 \left(1 + \frac{1}{\lambda^2 - 1}\right) = \frac{\lambda^2 y^2}{\lambda^2 - 1}.$$

Obviously, this implies (iii). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In the following statement, $|\cdot|$ denotes the maximum norm in $\mathbb{C}^n$. Everywhere else, the choice of the norm does not matter.

Lemma 5.8. *Suppose $0 \leq \varepsilon \leq \frac{1}{2}$ and that $v(z', z_n)$ is plurisubharmonic for $|z'| < 1$, $|z_n| < 1$ and satisfies*

(i) $v(z', z_n) \leq 1$, $\quad |z'| < 1$, $|z_n| < 1$,
(ii) $v(x', x_n) = 0$ $\quad$ *if x', x_n are real and $|x_n| \geq \varepsilon$.*

Then for each $\lambda < 1$ there is a constant C_λ such that

(iii) $v(z', z_n) \leq C_\lambda(|\operatorname{Im} z'| + |\operatorname{Im} \sqrt{z_n^2 - \varepsilon^2}|)$, $\quad |z'| \leq \lambda$, $|z_n| \leq \lambda$.

Proof. This follows from standard estimates for harmonic measure, but is perhaps most easily seen using the following explicit formulas. Let $h(\zeta)$ denote the harmonic measure of the real axis in the unit disk. That is, the subharmonic function on $|\zeta| < 1$ that is equal to 0 on the real axis, equal to 1 on $|\zeta| = 1$, and harmonic for $|\zeta| < 1$, $\operatorname{Im} \zeta \neq 0$. Well-known and easily verified formulas for $h(\zeta)$, $\zeta = x + y$, are $h(\zeta) = \frac{1}{\pi}[\arg(\zeta + 1) + \pi - \arg(\zeta - 1)] = \frac{2}{\pi}\arctan\frac{|y|}{1-(x^2+y^2)}$. Similarly, if $k_\varepsilon(\zeta)$ denotes the function harmonic in the unit disk with the real intervals $[-1, \varepsilon]$, $[\varepsilon, 1]$ removed, and with boundary values 1 on $|\zeta| = 1$ and 0 on $[-1, \varepsilon] \cup [\varepsilon, 1]$, then

$$k_\varepsilon(\zeta) = h\left(\sqrt{\frac{\zeta^2 - \varepsilon^2}{1 - \varepsilon^2\zeta^2}}\right).$$

This is easily seen by considering the sequence of analytic maps $\omega = \zeta^2, \tau = \frac{\omega - \varepsilon^2}{1 - \varepsilon^2 \omega}$, $\sigma = \sqrt{\tau}$ which represents $\{|\zeta| < 1\} \setminus ([-1, \varepsilon] \cup [\varepsilon, 1])$ as a 2-1 analytic cover of the upper half disk $\{|\sigma| < 1, \operatorname{Im} \sigma > 0\}$ with $|\zeta| = 1$ mapping to $|\sigma| = 1$ and the intervals $[-1, -\varepsilon], [\varepsilon, 1]$ to the negative and positive real σ-axis, respectively.

For $z = (z', z_n) \in \mathbb{C}^n$ with $|z'| < 1$ and $|z_n| < 1$, the maximum principle easily shows $v(z) \leq h(z_1) + \cdots + h(z_{n-1}) + k_\epsilon(z_n)$ for every plurisubharmonic function v satisfying (i) and (ii). Since $h(\zeta) \leq C_\lambda |\operatorname{Im} \zeta|$ and $k_\varepsilon(\zeta) \leq C_\lambda |\operatorname{Im} \sqrt{\zeta^2 - \varepsilon^2}|$ for $|\zeta| \leq \lambda < 1$, it follows that (iii) also holds. $\qquad\square$

Lemma 5.9. *Assume that P_m and $\xi = (1, 0, 0)$ satisfy the hypotheses of Lemma 5.6 and let $\rho_0 \geq 1, r, s > 0$, and $R_1 \geq 1$ be the constants which exist by that lemma. Then there exist constants $A_1, A_2, A_3 \geq 1$ such that for each $t \geq R_1$ the following holds: Whenever $u \in \operatorname{PSH}(V_\pm(P_m))$ satisfies the conditions (α) and (β) of 2.2(a), then for $z \in V_\pm(P_m) \cap B(t\xi; rt/2, rt/2, st/2)$ we have the estimates*

(a) $u(z_1, z_2, z_3) \leq A_1 |\operatorname{Im} z_1| + A_2 t^{1 - m/\nu}$ *if* $|z_2| \leq 4\rho_0 t^{1 - m/\nu}$,
(b) $u(z_1, z_2, z_3) \leq A_1 |\operatorname{Im} z_1| + A_3 |\operatorname{Im} z_2|$ *if* $|z_2| > 2\rho_0 t^{1 - m/\nu}$.

Proof. Since the arguments for $V_+(P_m)$ and $V_-(P_m)$ are the same, we only treat $V_+(P_m)$. Fix $u \in \operatorname{PSH}(V_+(P_m))$ satisfying (α) and (β) of 2.2(a). Since $0 \notin V_+(P_m)$ it follows from Theorem 5.2 that there exists $A_0' \geq 1$, not depending on u, such that

(5.18) $$ u(z) \leq A_0' |z|, \quad z \in V_+(P_m). $$

It is no restriction to assume in the sequel $r \leq \frac{1}{2}$ and $R_1 \geq 16\rho_0/r$. Next fix $t \geq R_1$ and define $v : B((t, 0); rt) \to [-\infty, \infty[$ by

$$ v(z_1, z_2) := \max\{u(z_1, z_2, z_3) : P_m(z_1, z_2, z_3) = 1, \ |z_3| < st\}. $$

Then v is plurisubharmonic and by Lemma 5.6 it satisfies

(5.19) $\quad v(z_1, z_2) \leq 0 \quad$ if $\quad z_1, z_2 \in \mathbb{R} \quad$ and $\quad \rho_0 t^{1 - m/\nu} \leq |z_2| < rt$.

Further, we get from Lemma 5.6 and (5.18)

$$
\begin{aligned}
v(z_1, z_2) &\leq A_0' \left(|(z_1, z_2)| + \max_{1 \leq j \leq \nu} |z_3^j(z_1, z_2)| \right) \\
&\leq A_0'(1 + D)|(z_1, z_2)| \leq A_0 |(z_1, z_2)| \\
&\leq A_0(1 + 2r)t \leq 3A_0 t \quad \text{for} \quad (z_1, z_2) \in B((t, 0); rt),
\end{aligned}
$$
(5.20)

where $A_0 := A_0'(1 + D)$. Now let

$$ \varphi(\zeta', \zeta_2) := \frac{1}{3A_0 t} \, v(rt(\zeta' + 1), r, t\zeta_2), \ (\zeta', \zeta) \in B(0; 1). $$

The properties of v imply that φ is plurisubharmonic on $B(0; 1)$, bounded by 1, and equal to 0 if ζ', ζ_2 are real and $|\zeta_2| \geq \varepsilon := \rho_0/(rt^{m/\nu})$. Hence we can apply Lemma 5.8 to φ to conclude that there is a constant A_1, independent of v, such that

$$v(z_1, z_2) \leq A_1 \left(|\operatorname{Im} z_1| + |\operatorname{Im} \sqrt{z_2^2 - (\rho_0 t^{1-m/\nu})^2}| \right),$$

(5.21) $(z_1, z_2) \in B((t, 0); rt/2)$.

The estimates (a) and (b) are now easy consequences of (5.21), part (a) by applying Lemma 5.7(i) and part (b) from Lemma 5.7(iii). $\square$

To formulate our next lemma, we introduce the following notation for cones with truncated tip. If $\xi \in \mathbb{C}^n$, $|\xi| = 1, \delta > 0$, and $R \geq 0$ are given we let

$$\Gamma(\xi, \delta, R) := \left\{ z \in \mathbb{C}^n : \left| \frac{z}{|z|} - \xi \right| < \delta, \quad |z| > R \right\}.$$

Lemma 5.10. *Let $P_m \in \mathbb{R}[z_1, z_2, z_3]$ be homogeneous of degree $m \geq 2$ and assume that P_m satisfies condition (4) of Theorem 5.1. Then for each $\xi \in V(P_m) \cap \mathbb{R}^3, |\xi| = 1$, there exist $\delta_\xi > 0$, $R_\xi \geq 1$, and $A_\xi \geq 1$ such that each $u \in \mathrm{PSH}(V_\pm(P_m))$ which satisfies the conditions (α) and (β) of 2.2(a), also satisfies*

$$u(z) \leq A_\xi |\operatorname{Im} z| \quad \text{for } z \in V_\pm(P_m) \cap \Gamma(\xi, \delta_\xi, R_\xi).$$

Proof. By hypothesis, we can apply Lemma 5.3. In particular, we can assume $\xi = (1, 0, 0)$. Then we apply Lemma 5.6 and Lemma 5.9 to get $\rho_0 \geq 1$, $\frac{1}{2} \geq r > 0$, $s > 0$, $R_1 \geq 1$, and constants A_1, A_2, A_3 such that the conclusion of Lemma 5.9 holds. To prove the present lemma it suffices to show that there exist $\delta > 0$, $R_2 \geq R_1$, and $A \geq 1$ such that each $u \in \mathrm{PSH}(V_\pm(P_m))$ which satisfies the conditions (α) and (β) of 2.2(a) also satisfies

(5.22) $u(z) \leq A |\operatorname{Im} z| \quad \text{for} \quad z \in \bigcup_{t > R_2} tB(\xi; \delta).$

Note that by Lemma 5.9(b), the estimate (5.22) holds with $A = \max(A_1, A_3)$ on the set $\bigcup_{t > R_1} \{z \in tB(\xi; r/2, r/2, \sigma/2) : |z_2| > 2\rho_0 t^{-m/\nu}\}$. To prove (5.22) on $\bigcup_{t > R_2} \{z \in tB(\xi; \delta) : |z_2| < 4\rho_0 t^{-m/\nu}\}$, for suitable $\delta > 0$ and $R_2 \geq R_1$, we will use the notation introduced in the proof of Lemma 5.6. Then we have for $z \in B(\xi; \eta, \eta, \sigma)$

$$P_m(z) = F(z) U(z), \quad F(z) = \prod_{j=1}^{\nu} (z_3 - b_j z_2 - g_j(z_1, z_2)),$$

$$U(z) = \frac{P_m(z)}{F(z)}.$$

As in the proof of Lemma 5.6 we assume without restriction that $U(1,0,0) = \prod_{j=\nu+1}^{m}(-a_j) = 1$. Then it is easy to check that for the localization at ξ we have

$$(P_m)_\xi = (F)_\xi (U)_\xi = (F)_\xi = Q_0, \quad \text{where}$$

$$Q_0(z_2, z_3) = \prod_{j=1}^{\nu}(z_3 - b_j z_2).$$

Subsequently we will only treat the case $V_+(P_m)$, since the same arguments apply to $V_-(P_m)$. To do so note that by Lemma 3.10,

$$S(\tau, z) := P_m\left(\frac{z_1}{\tau^\nu}, z_2\tau^{m-\nu}, z_3\tau^{m-\nu}\right) - 1$$

is a polynomial in (τ, z_1, z_2, z_3) which satisfies

$$S(0, z) = z_1^{m-\nu}(P_m)_\xi(z_2, z_3) = z_1^{m-\nu}Q_0(z_2, z_3).$$

Since the real numbers $b_1, \ldots, b_\nu$ are distinct, $\operatorname{grad} Q_0(z_2, z_3) \neq 0$ for $(z_2, z_3) \neq 0$. Hence

$$\frac{\partial S}{\partial z_2}(0, z) \neq 0 \quad \text{or} \quad \frac{\partial S}{\partial z_3}(0, z) \neq 0$$
$$\text{if} \quad z_1 \neq 0 \quad \text{and} \quad (z_2, z_3) \neq 0.$$

Next let $0 < r \leq \eta$, $0 < s \leq \sigma$ be the numbers which exist by Lemma 5.6 and note that

$$K := \{(0, z) \in \mathbb{R}^4 : z \in \overline{B(\xi; r/2, r/2, s/2)}, S(0, z) = 0\}$$

is compact. Since $(0,0) \notin K$, for each $(0, z) \in K$ we have $\frac{\partial S}{\partial z_2}(0, z) \neq 0$ or $\frac{\partial S}{\partial z_3}(0, z) \neq 0$. We let

$$M_j := \{z : (0, z) \in K, \frac{\partial S}{\partial z_j}(0, z) \neq 0\}, \quad j = 2, 3.$$

Then for each $z \in M_3$ the implicit function theorem implies the existence of $\delta(z) = (\delta_0(z), \ldots, \delta_3(z)) > 0$ and of a holomorphic function $h_3^z : B((0, z_1, z_2); \delta'(z)) \to B(z_3; \delta_3(z))$ such that

$$V(S) \cap B((0, z); \delta(z)) = \{(\tau, \zeta_1, \zeta_2, h_3^z(\tau, \zeta_1, \zeta_2)) : (\tau, \zeta_1, \zeta_2)$$
$$\in B((0, z_1, z_2); \delta'(z))\}.$$

Since $(0, z) \in \mathbb{R}^4 \cap V(S)$, the real implicit function theorem implies that $h_3^z(\tau, \zeta_1, \zeta_2)$ is real when (τ, ζ_1, ζ_2) is real. The same arguments apply to the points $z \in M_2$. Now note that

$$\left\{ B\left((0, z); \frac{\delta_0(z)}{2}, \frac{\delta_1(z)}{2}, \delta_2(z), \frac{\delta_3(z)}{2}\right) : z \in M_2 \right\} \cup$$
$$\left\{ B\left((0, z); \frac{\delta_0(z)}{2}, \frac{\delta_1(z)}{2}, \frac{\delta_2(z)}{2}, \delta_3(z)\right) : z \in M_3 \right\}$$

is an open cover of K. Hence there exist $z_1, \ldots, z_k \in M_2$ and $z_{k+1}, \ldots, z_l \in M_3$ such that the elements of the cover with centers $(0, z_1), \ldots, (0, z_l)$ cover K. Let $\delta_0 := \min\{\delta_0(z_j) : 1 \le j \le l\}$. Then

$$U := \bigcup_{j=1}^{k} B\left((0, z_j); \frac{\delta_0}{2}, \frac{\delta_1(z_j)}{2}, \delta_2(z_j), \frac{\delta_3(z_j)}{2}\right)$$
$$\cup \bigcup_{j=k+1}^{l} B\left((0, z_j); \frac{\delta_0}{2}, \frac{\delta_1(z_j)}{2}, \frac{\delta_2(z_j)}{2}, \delta_3(z_j)\right)$$

is a neighborhood of K. Hence

$$L_0 := \left(\overline{B((0, \xi); 1, r/2, r/2, s/2)} \setminus U \right) \cap V(S) \cap \mathbb{R}^4$$

is compact and so is its projection in the τ-space. Therefore

$$\varepsilon_1 := \inf\{\tau : (\tau, z_1, z_2, z_3) \in L_0\}$$

is obtained at some point $(\tau_0, w_1, w_2, w_3) \in L_0$. If we assume $\tau_0 = 0$, then $(0, w_1, w_2, w_3)$ is in K, in contradiction to the definition of L_0. This proves $\varepsilon_1 > 0$.

Next note that also

$$L_1 := \left(\overline{B((0, \xi); 1, r/2, r/2, s/2)} \setminus U \right) \cap V(S) \cap ([0, \varepsilon_1/2] \times \mathbb{C}^3)$$

is compact. Hence

$$\varepsilon_2 := \inf\{|\operatorname{Im} z| : (\tau, z) \in L_1\}$$

is obtained at some point $(\tau_1, \zeta) \in L_1$. If we assume $\varepsilon_2 = 0$, then (τ_1, ζ) is real and hence in L_0. By the definition of ε_1, this implies $\tau_1 \ge \varepsilon_1$, in contradiction to $\tau_1 \in [0, \varepsilon_1/2]$. This proves $\varepsilon_2 > 0$.

Now choose $R_2 \ge \max((2/\varepsilon_1)^\nu, R_1)$ so large that $\max_{1 \le j \le l} 4/(\pi\delta_1(z_j)$ $(A_1 + A_2)) \le R_2^{m/\nu}$, and fix $\tau \ge R_2$ and $\zeta \in V_+(P_m) \cap tB(\xi, r/4, r/4, s/4)$ satisfying $|\zeta_2| \le 4\rho_0 t^{1-m/\nu}$. Then let

$$\tau := \frac{1}{t^{1/\nu}}, \ z_1 := \zeta_1 \tau^\nu, \ z_2 := \zeta_2 \tau^{\nu-m}, \ z_3 := \zeta_3 \tau^{\nu-m}.$$

It is easy to check that $(\tau, z) \in V(S) \cap B((0, \xi); \varepsilon_1/2, r/4, r/4, s/4)$. Now we fix $u \in \mathrm{PSH}(V_+(P_m))$ satisfying the conditions (α) and (β) of 2.2(a). As we have seen in the proof of Lemma 5.9, there exists $A_0' \geq 1$ such that (5.18) holds. Now we distinguish the following cases:

Case $(\tau, z) \notin U$ Since $\tau \in [0, \varepsilon_1/2]$, the point (τ, z) is in L_1. By the definition of ε_2, this implies $|\mathrm{Im}\, z| \geq \varepsilon_2$.

    Case $(\tau, z) \notin U$**,** $|\mathrm{Im}\, z_1| \geq \varepsilon_2/3$ Note first that $\frac{1}{2}t \leq |\zeta_1| \leq 2t$, since $0 < r \leq \frac{1}{2}$. Hence the present hypothesis implies

$$|\mathrm{Im}\, \zeta_1| = \tau^{-\nu} |\mathrm{Im}\, z_1| \geq \frac{\varepsilon_2}{3} t \geq \frac{\varepsilon_2}{6} |\zeta_1|.$$

From this and (5.18) we get

$$u(\zeta) \leq A_0' |\zeta| \leq A_0' \left(1 + \frac{r}{2} + \frac{s}{4}\right) t$$

(5.23)
$$\leq 12 A_0' \left(1 + \frac{r}{2} + \frac{s}{4}\right) \frac{1}{\varepsilon_2} |\mathrm{Im}\, \zeta|.$$

    Case $(\tau, z) \notin U$**,** $|\mathrm{Im}(z_2, z_3)| \geq \varepsilon_2/3$ Then we have

$$|\mathrm{Im}(\zeta_2, \zeta_3)| = \tau^{m-\nu} |\mathrm{Im}(z_2, z_3)| \geq \frac{\varepsilon_2}{3} t^{1-m/\nu}.$$

From this and Lemma 5.9(a) we get

$$u(\zeta) \leq A_1 |\mathrm{Im}\, \zeta_1| + A_2 \frac{3}{\varepsilon_2} |\mathrm{Im}(\zeta_2, \zeta_3)|$$

(5.24)
$$\leq \left(A_1 + \frac{3A_2}{\varepsilon_2}\right) |\mathrm{Im}\, \zeta|.$$

Case $(\tau, z) \in U$ Without restriction we assume that there exists $w \in \{z_{k+1}, \ldots, z_l\}$ such that $(\tau, z) \in V(S) \cap B((0, w); \frac{\delta_0}{2}, \frac{\delta_1(w)}{2}, \frac{\delta_2(w)}{2}, \delta_3(w))$. By the choice of $\delta(w)$, we know that $V(S) \cap B((0, w); \delta(w))$ is the graph of a holomorphic function $h : B((0, w_1, w_2); \delta_0(w), \delta_1(w), \delta_2(w)) \to \mathbb{C}$ which is real when $(v, y_1, y_2) \in B((0, w_1, w_2); \delta_0(w), \delta_1(w), \delta_2(w)) \cap \mathbb{R}^3$. Now note that the definition of S implies that for $(y_1, y_2) \in B((w_1, w_2); \delta_1(w), \delta_2(w))$ the point $(\tau^{-\nu} y_1, \tau^{m-\nu} y_2, \tau^{m-\nu} h(\tau, y_1, y_2))$ is in $V_+(P_m)$. Therefore, we can define a plurisubharmonic function φ on $B((w_1, w_2); \delta_1(w), \delta_2(w))$ by

$$\varphi(y_1, y_2) = u(\tau^{-\nu} y_1, \tau^{m-\nu} y_2, t^{m-\nu} h(\tau, y_1, y_2)).$$

By Lemma 5.9(a) and the properties of h we get

$$\varphi(y_1, y_2) \leq (A_1 + A_2)(\tau^{-\nu} |\mathrm{Im}\, y_1| + \tau^{m-\nu}) \quad \text{and}$$
$$\varphi(y_1, y_2) \leq 0 \text{ if } (y_1, y_2) \in \mathbb{R}^2.$$

Now fix y_2 real and consider for $\operatorname{Im} y_1 \geq 0$ the function

$$\psi : y_1 \mapsto \varphi(y_1, y_2) - (A_1 + A_2)\tau^{-\nu} \operatorname{Im} y_1.$$

Then we have

$$\psi(y_1) \leq (A_1 + A_2)\tau^{m-\nu} \quad \text{and} \quad \varphi(y_1) \leq 0 \text{ if } y_1 \in \mathbb{R}.$$

From this and an application of the standard estimate for the harmonic measure of the real line in the unit disk it follows as in the proof of Lemma 5.8 that

$$\psi(y_1) \leq \frac{4}{\pi}(A_1 + A_2)\frac{2}{\delta_1(w)}\tau^{m-\nu}|\operatorname{Im} y_1|, \quad y_1 \in B(w_1; \delta_1(w)/2).$$

Arguing similarly for $\operatorname{Im} y_1 \leq 0$, we conclude for y_2 real:

$$\varphi(y_1, y_2) \leq (A_1 + A_2)\tau^{-\nu}|\operatorname{Im} y_1|\left(1 + \frac{8\tau^m}{\pi\delta_1(w)}\right)$$
$$\leq 2(A_1 + A_2)\tau^{-\nu}|\operatorname{Im} y_1|,$$

by our choice of R_2. Next we fix y_1 and argue in the same way to get

$$\varphi(y_1, y_2) \leq 2(A_1 + A_2)\tau^{-\nu}|\operatorname{Im} y_1|$$
$$+ (A_1 + A_2)\frac{8}{\pi\delta_2(w)}\tau^{m-\nu}|\operatorname{Im} y_2|$$

for $(y_1, y_2) \in B((w_1, w_2); \delta_1(w)/2, \delta_2(w)/2)$. In particular, we get

(5.25) $u(\zeta_1, \zeta_2, \zeta_3) = \varphi(z_1, z_2) \leq 2(A_1 + A_2)(|\operatorname{Im} \zeta_1| + |\operatorname{Im} \zeta_2|),$

provided that R_2 is large enough.

Now it follows from (5.25), (5.26), and (5.27) that (5.24) holds provided that we choose $\delta > 0$ small and R_2 and $A \geq 1$ large enough.

As we have noted at the beginning, this completes the proof. ☐

After all these preparations we can finally prove Theorem 5.1.

Proof of Theorem 5.1. (1) $\Rightarrow$ (2): By Lemma 2.5, the hypothesis implies that $V(P_m)$ has $\mathrm{PL}(\mathbb{R}^3, \log)$. Hence P_m is real up to a complex constant by Meise, Taylor, and Vogt [13], Lemma 2. If we assume $P_m = \lambda Q_m$ for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $Q_m \in \mathbb{R}[z_1, z_2, z_3]$, then $V_+(P_m) = \{z \in \mathbb{C}^3 : Q_m(z) = 1/\lambda\}$. Since Q_m has real coefficients and $1/\lambda \notin \mathbb{R}$, $V_+(P_m) \cap \mathbb{R}^3 = \emptyset$. Obviously, this implies that $V_+(P_m)$ fails (SPL) in contradiction to the present hypothesis. Hence P_m has real coefficients. Therefore (2) follows from (1) by the result of Meise and Taylor [9], Theorem 3.4, which we recalled in Theorem 2.7.

(2) $\Rightarrow$ (3): This holds by Lemma 2.5(a).

(3) $\Rightarrow$ (4): Let $\widetilde{P}_m$ denote the polynomial P_m, regarded as an element of $\mathbb{C}[z_1, \ldots, z_4]$, and note that $\widetilde{P}_m$ is the principal part of P. By Meise, Taylor, and Vogt [15], Theorem 4.1, $V(\widetilde{P}_m)$ has $\mathrm{PL}(\mathbb{R}^4, \log)$. Since $V(\widetilde{P}_m) = V(P_m) \times \mathbb{C}$, this implies that $V(P_m)$ satisfies $\mathrm{PL}(\mathbb{R}^3, \log)$ and hence (SPL) by Lemma 2.5. As in the proof of (1) $\Rightarrow$ (2), P_m must be real up to a complex constant. However, by Meise and Taylor [9], Lemma 2.1, a polynomial of the form $P_m(z_1, \ldots, z_n) + \lambda z_{n+1}$ where P_m has real coefficients cannot satisfy $\mathrm{PL}(\mathbb{R}^4, \log)$ unless λ is also real. Therefore P_m must have real coefficients. Then the last condition in (4) follows from (3) by Theorem 4.1.

(4) $\Rightarrow$ (1): We will only prove that $V_+(P_m)$ satisfies (SPL), since one can argue in the same way for $V_-(P_m)$. To do so, Proposition 4.5 of Meise and Taylor [9] will be applied. This result shows that $V_+(P_m)$ satisfies (SPL) if and only if the following three conditions hold:

(i) $V_+(P_m)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(\xi)$ at each $\xi \in V_+(P_m) \cap \mathbb{R}^3$,
(ii) there are constants $A, B > 0$ such that $u(z) \le A|z| + B$ for all plurisubharmonic functions u on $V_+(P_m)$ satisfying $u(z) \le |z| + o(|z|)$ on V and $u(z) \le 0$ on $V_+(P_m) \cap \mathbb{R}^3$,
(iii) for each $\xi \in V(P_m) \cap \mathbb{R}^n, |\xi| = 1$, there exist A_ξ, δ_ξ, and $R_\xi > 0$ such that each $u \in \mathrm{PSH}(V_+(P_m))$ which satisfies the conditions (α) and (β) of 2.2(a), also satisfies $u(z) \le A_\xi |\mathrm{Im}\, z|$ for $z \in V_+(P_m) \cap \Gamma(\xi, \delta_\xi, R_\xi)$.

To check these conditions, note first that $V_+(P_m)$ is a manifold at each point by Euler's rule. That is,

$$\sum_{j=1}^{3} z_j \frac{\partial P_m}{\partial z_j}(z) = m P_m(z) = m, \quad z \in V_+(P_m),$$

so $\mathrm{grad}\, P_m(z) \ne 0$ for each $z \in V_+(P_m)$. Hence $\mathrm{PL}_{\mathrm{loc}}(\xi)$ holds at each point of $V_+(P_m)$ as we have noted in Remark 3.2. Thus condition (i) holds. Next note that the hypotheses in (4) obviously imply that P_m is square-free. Hence condition (ii) follows from Theorem 5.2 applied to $Q(z) = -1$. Since condition (iii) holds by Lemma 5.10, we conclude that $V_+(P_m)$ satisfies (SPL). $\square$

Remark. Let $P_m \in \mathbb{R}[z_1, \ldots, z_n]$ be homogeneous of degree m. By Meise and Taylor [9], Theorem 4.8, $V_\pm(P_m)$ satisfies (SPL) whenever P_m is of principal type (i.e., $\sum_{j=1}^{n} |\frac{\partial P_m}{\partial z_j}(\xi)|^2 \ne 0$ for $\xi \in \mathbb{R}^n \setminus \{0\}$) and each irreducible factor of P_m has at least one nontrivial real zero. For $n = 2$ this sufficient condition is also necessary and equivalent to P_m being strictly hyperbolic. Note that in several steps of the proof of Theorem 5.1 we have used that $(P_m)_\theta$ is hyperbolic at each $\theta \in V(P_m) \cap \mathbb{R}^3, |\theta| = 1$. For $n = 3$, Theorem 5.1 shows that the condition that P_m is of principal type is not

necessary, since $V(P_m) \cap \mathbb{R}^3$ may contain non-zero singular points, as the following examples show.

Example 5.11. (a) For $P_3(x, y, z) := y^2 z - x(x - z)^2$ the variety $V(P_3)$ is singular exactly at the points $\{(\lambda, 0, \lambda) : \lambda \in \mathbb{C}\}$ and the varieties $V_{\pm}(P_3)$ satisfy (SPL).
To prove this, note first that

$$\operatorname{grad} P_3(x, y, z) = (-3x^2 + 4xz - z^2, 2yz, y^2 + 2x^2 - 2xz).$$

From this it follows easily that

$$\{\xi \in \mathbb{C}^3 : P_3(\xi) = 0 \text{ and } \operatorname{grad} P_3(\xi) = 0\} = \{(\lambda, 0, \lambda) : \lambda \in \mathbb{C}\}.$$

This proves the first assertion. To prove the second one we show that the conditions in 5.1(4) are satisfied. Obviously, P_3 has real coefficients. To show that $V(P_3)$ satisfies $\mathrm{PL}(\mathbb{R}^3, \log)$ and hence (SPL), note first that a simple computation shows for $\theta = (1, 0, 1)$

$$(P_3)_\theta(x, y, z) = y^2 - (x - z)^2.$$

Hence $(P_3)_\theta$ is the product of two distinct real linear forms and therefore square-free. Moreover, P_3 is locally hyperbolic in the sense of Andersson (see Hörmander [8], 6.4). Thus it follows from Hörmander [8], Theorem 6.5, that $P_3(D)$ is a surjective differential operator from the space of all analytic functions on $\mathbb{R}^3$ into itself. Since P_3 is irreducible and not elliptic, by Meise, Taylor, and Vogt [15], Corollary 3.14, and Lemma 2.5 it follows that $V(P_3)$ has (SPL). Therefore, Theorem 5.1 implies that $V_+(P_3)$ and $V_-(P_3)$ satisfy (SPL).
Note that $\{(x, y) \in \mathbb{R}^2 : P_3(x, y, 1) = 0\}$ is a well-known algebraic curve, namely Newton's knot.

(b) For $P_3(x, y, z) = x^3 + y^3 - 3xyz$ the variety $V(P_3)$ is singular exactly at the points $\{(0, 0, \lambda) : \lambda \in \mathbb{C}\}$ and the varieties $V_{\pm}(P_m)$ both satisfy (SPL).
This can be shown similarly as in part (a). Note that $\{(x, y) \in \mathbb{R}^2 : P_3(x, y, 1) = 0\}$ is the Cartesian leaf.

The following example shows that the Phragmén-Lindelöf property is not inherited by quasihomogeneous principal parts.

Example 5.12. Set $P(z_1, \ldots, z_4) = z_1 z_2(z_3^2 - z_1^2) - z_4$. By Theorem 5.1, its variety $V(P)$ satisfies $\mathrm{PL}(\mathbb{R}^4, \log)$. Define the weight $d = (1, 1, 2, 6)$, and denote by $Q(z) = z_1 z_2 z_3^2 - z_4$ the d-quasihomogeneous principal part of P. The principal part Q_4 of Q is not square-free, hence by Corollary 4.6, the variety $V(Q)$ does not satisfy $\mathrm{PL}(\mathbb{R}^4, \log)$. This shows that $\mathrm{PL}(\mathbb{R}^n, \omega)$ is not inherited by quasihomogeneous principal parts. Note that the classical principal part P_m of P satisfies $\mathrm{PL}(\mathbb{R}^n, \omega)$ whenever P does, as Meise, Taylor, and Vogt have shown in [15], 4.1.

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