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EXACT SOLUTIONS FOR ANISOTROPIC, MULTIPLE SCATTERING BY PARALLEL PLANE DISPERSIONS

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NOMENCLATURE

a_i	angular distribution coefficients, $i = 0, 1, \ldots, N$
A_i	coefficients in expansion of $\cos m\varphi = A_0 + \sum_{i=1}^{m} A_i \cos^i \varphi$
$b_i(\tau_1,\mu,\mu_o)$	coefficients in expansion I-21 of angular dependence of specific intensity of reflected radiation. Defined in Appendix A-1
B	fraction of radiation scattered backward. Defined by I-47
B_6	six-flux representation of backward scattering component
BP	height of backward peak of phase function, p(-1.0)
$c_i(\tau_1,\mu,\mu_o)$	coefficients in expansion I-22 of angular dependence of diffuse portion of transmitted radiation. Defined in Appendix A-1
D	diameter of scattering particle
$E_n(Z)$	defined by $\int_Z^\infty \frac{e^{-x}}{x^n} dx$
$f(\Theta)$	angular distribution function, same as $p(\cos\Theta)/4\pi$
$f(x)$	any function of x
$f^*(x)$	any tabular function of x
$f^{*(n)}(x)$	approximation to solution of I-39 after n iterations
F	fraction of radiation scattered forward, defined by I-46
$F(\mu)$	any function of μ
F_6	six-flux representation of forward scattering component
$F_k^m(\tau_1,\mu,\mu')$	function defined by I-3
FP	height of forward peak of phase function, p(1.0)
$G_k^m(\tau_1,\mu,\mu')$	function defined by I-4
$H(\mu)$	Chandrasekhar's H function, the same as $\psi_o^*(\infty,\mu)$

$I_1, I_2, ..$	discrete components of integrated radiant flux
$I(Z,\mu,\varphi)$	specific intensity of radiation at position Z in the solid angle defined by μ and ϕ
$I_0(-\mu_0, \varphi_0)$	intensity of a parallel beam incident on plane parallel dispersion in the direction $(-\mu_0, \varphi)$
m	index of refraction of scattering particle
m_i	constants derived in equation (III-29, 30) from the approximation of the isotropic scattering integral equation
n	number of particles per unit volume
n_i	coefficients of the approximate kernal derived in equation (III-18)
N	number of terms in the phase function
p	number of ordinates (excluding the zeroth ordinate) used in numerical integration with I=35
$p(\mu)$	phase function
P	peakedness of the phase function in the forward direction defined by I-49
P_B	peakedness of the phase function in the backward direction defined by I-50
$P_1(\mu)$	Legendre polynomial
$P_1^m(\mu)$	associated Legendre polynomials
r_i	i-th root of $P_n(x)$
R	integrated reflectance
$S(\vec{r})$	Virtual source function
$S(\tau_1; \mu, \varphi; \mu_0, \varphi_0)$	scattering function defined by I-11
$S^{(m)}(\tau_1; \mu, \mu_0)$	function defined by I-13

t	thickness of dispersion
T	total integrated transmission
T_D	diffuse portion of integrated transmission
$\mathcal{T}(\tau_1; \mu, \varphi; \mu_0, \varphi_0)$	transmission defined by I-16
$\mathcal{T}^{(m)}(\tau_1; \mu, \varphi_0)$	function defined by I-18
W_i	weighting coefficients for use in numerical integration by I-33, $i = 0, \ldots, p$
x_i	specific values of ordinates to be used in numerical integration by I-33, $i = 0, \ldots, p$
x	μ_0-interpolation factor for integrated reflectance
y	μ_0-interpolation factor for total integrated transmission
X	six-flux representation of sidewise scattering component
$X(\mu)$	Chandrasekhar's X function--the same as $\psi_0^0(\tau_1, \mu)$
$Y(\mu)$	Chandrasekhar's Y function--the same as $\phi_0^0(\tau_1, \mu)$
z	distance into the dispersion in optical units
α	$\pi D/\lambda$, circumference of particle expressed in wavelength
α_i	positive roots of equation (III-26)
β_i	positive roots of the equation $\beta^4 - \frac{b}{c}\beta^2 + \frac{1}{c} = 0$, where $a = \frac{5}{21}$, $b = \frac{5}{7}$, $c = \frac{4}{105}$
δ_{nm}	Kronecker delta ($\delta_{nm} = 0$ when $m \neq n$; $\delta_{nm} = 1.0$ when $m = n$)
ε	error in numerical integration with I-35
θ	azimuthal angle of radiation
θ_0	azimuthal angle of incident radiation

λ	wavelength of incident radiation
μ	cos Θ direction of radiation
μ_o	cos Θ_o direction of incident radiation
$\rho(\vec{r})$	energy density of isotropic scattering at a position $\vec{r}$
τ_1	optical thickness of dispersion
σ	Total cross-section of a particle (scattering and absorption)
$\phi_l^m(\tau_1,\mu)$	functions defined by I-2
φ	bearing angle of radiation
φ_o	bearing angle of incident radiation
$\psi_l^m(\tau_1,\mu)$	functions defined by I-1
ω_o	albedo for single scattering, ratio of scattered to intercepted radiation
ω_l	coefficients in Legendre polynomial expansion of phase function
ω_l^m	$\omega_l \frac{(l-m)!}{(l+m)!}$

ABSTRACT

The effect of anisotropic scattering on radiative transfer was investigated theoretically. Exact solutions were obtained for a plane source obliquely incident on a multiply scattering, parallel plane dispersion by numerical integration. The results are utilized to evaluate the various approximate models which have been proposed for radiative transfer and neutron diffusion. A new variable-order, diffusion-type model is proposed as an improved approximation.

INTRODUCTION

The long range objective of this research is to develop methods for predicting the transport of thermal radiation through the atmosphere.

The scattering of radiation by dispersed material presents a problem of considerable mathematical difficulty. Exact solutions have been completed only for highly idealized conditions: isotropic or Rayleigh scattering in a one-dimensional configuration (1). Accordingly many approximate models have been developed (2). Examples are the diffusion model and the various discrete flux models.

In the initial phase of this research a discrete flux model was used to obtain approximate results for a finite, spherical source both above and inside a haze (3). The work was then redirected to the development of exact solutions for particular conditions as discussed below.

The lack of exact solutions for general conditions makes it difficult to compare and evaluate the various approximate models which have been proposed. As a consequence of a general conference on radiant transport (4) and subsequent discussions with Dr. S. Chandrasekhar and AFSWP personnel in which this situation was stressed, the immediate objective of this research was shifted to the development of exact solutions for idealized but significant conditions. Particular attention has been given to the development of exact solutions for anisotropic scattering. Solutions were first obtained for parallel-plane radiation, obliquely incident on a semi-infinite dispersion. These results are presented in the annual report for the previous year (5). The present report presents similar results for parallel-plane radiation obliquely incident on parallel-plane dispersions of finite thickness. The method of solution follows the lead of Chandrasekhar (1) and involves the reiterative numerical solution of integral equations on a high speed computer. (The University of Michigan IBM-704). The formulation of the method of solution and the numerical results are presented in Part I of the report. In Part II numerical values obtained from various approximate models are compared with the exact values presented in Part I. Suggested formulas for interpolation and extrapolation of the exact values are also included. None of the approximate models which had previously been proposed proved to be entirely satisfactory and a new variable-order, diffusion-type model is developed in Part III.

During the period while this work was being done an approximate but exceedingly detailed incremental model for one-dimensional radiant transport with anisotropic scattering was developed by the Internuclear Corporation (6). Their final results have not been available for comparison with the values obtained in this investigation but comparison of preliminary values for a few cases indicates that their model may be quite accurate.

In the continuation of this research it is planned 1) to refine the exact method of solution to expedite the numerical calculations 2) to extend the new variable order diffusion-type model for anistropic scattering and 3) to develop exact or nearly exact solutions for two-dimensional problems utilizing numerical methods and/or the model.

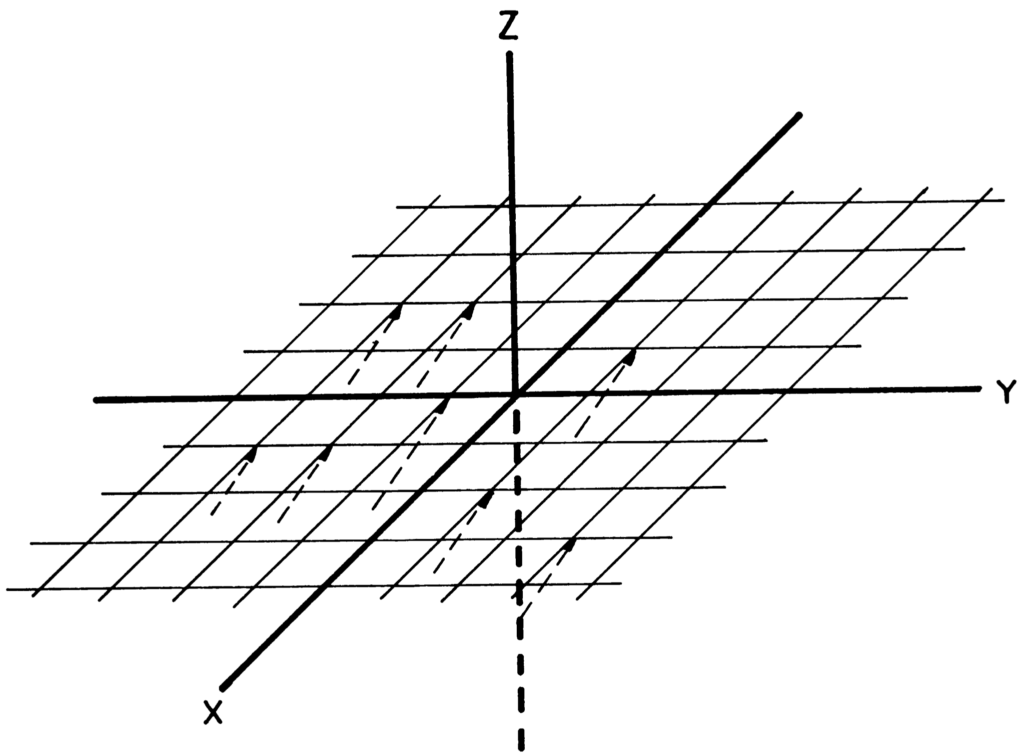
PART I - NUMERICAL SOLUTION OF THE TRANSPORT EQUATION

A. Mathematical Formulation

1. Geometry and Coordinate System.

This investigation is concerned with the irradiation of a semi-infinite slab by a uniform parallel flux as indicated in Figure I-1. The Z = 0 plane represents the interface between region (1) consisting of free space containing the source and region (2) consisting of a uniform dispersion of multiply scattering and absorbing particles each characterized by the same absorption cross-section, scattering cross-section and phase function, $p(\cos \Theta)$, for single scattering. The slab has an optical thickness τ_1 and the $Z = \tau_1$ plane represents the interface between region (2) and region (3) consisting of free space. The objective of this study is to determine the intensities of transmitted and reflected radiation as a function of position, direction, and the characteristics of the source and dispersion.

The coordinate system used in the analysis is shown in Figure I-2. An incident flux of intensity I_0 strikes the interface at an angle Θ_0 with respect to the normal (Z axis). The direction at Z = 0 of the reflected intensity leaving the dispersion is defined by the azimuthal angle Θ, measured from the normal, and the bearing angle φ measured from the Y axis. For simplicity the coordinate system is



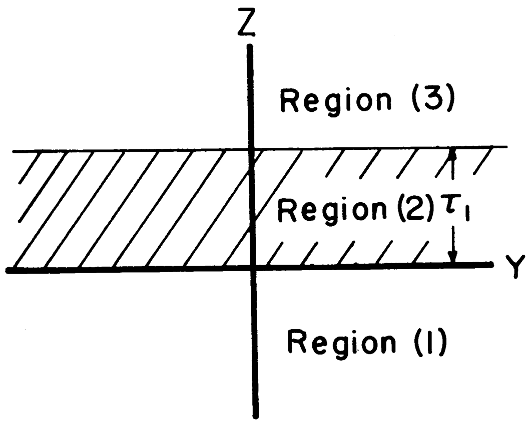
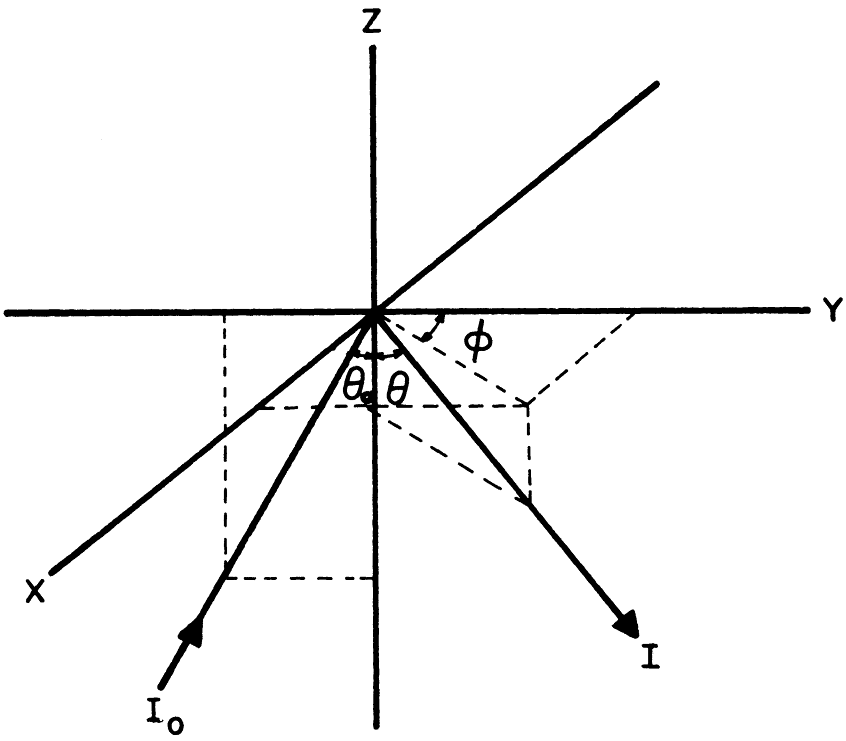


Figure I-1 Geometry of the Problem



XY plane: media interphase

I_o: incident flux

I : emergent flux under consideration

$\theta_o = \cos^{-1} \mu_o, \theta = \cos^{-1} \mu, \phi$ = Bearing angle

Figure I-2 Coordinate system

oriented so that $\varphi_0 = 0$, i.e., I_0 lies in the YZ plane.

2. The ψ and ϕ Functions.

The following integral equations for the two basic functions ψ_l^m and ϕ_l^m have been derived by Chandrasekhar[(1)]*.

$$\psi_l^m(\tau_1,\mu) = P_l^m(\mu) + \frac{1}{2}\mu\sum_{k=m}^{N}(-1)^{l+k}\,\omega_k^m\int_0^1\left[\psi_k^m(\tau_1,\mu)\psi_k^m(\tau_1,\mu') - \phi_k^m(\tau_1,\mu)\phi_k^m(\tau_1,\mu')\right]P_l^m(\mu')\frac{d\mu'}{\mu+\mu'} \qquad \text{I-1}$$

$$\phi_l^m(\tau_1,\mu) = e^{-\tau_1/\mu}P_l^m(\mu) + \frac{1}{2}\mu\sum_{k=m}^{N}\omega_k^m\int_0^1\left[\phi_k^m(\tau_1,\mu)\psi_k^m(\tau_1,\mu') - \psi_k^m(\tau_1,\mu)\phi_k^m(\tau_1,\mu')\right]P_l^m(\mu')\frac{d\mu'}{\mu-\mu'} \qquad \text{I-2}$$

$m = 0, 1, \cdots, N$

$l = m, m+1, \cdots, N$

where,

μ = $\cos\theta$

τ_1 = optical thickness of media (number of mean free paths for scattering) = $n\sigma^t$

$P_l^m(\mu)$ = associated Legendre polynomials

N = number of terms in the expansion of the phase function

$$p(\mu) = \sum_{l=0}^{N}\omega_l P_l(\mu)$$

ω_l = coefficients of phase function expansion

ω_l^m = $\omega_l \dfrac{(l-m)!}{(l+m)!}$

* Hereafter referred to as R. T.

The equations may be written in more convenient form by defining

$$F_k^m(\tau_1,\mu,\mu') = \frac{\omega_k^m}{\mu+\mu'}\left[\psi_k^m(\tau_1,\mu)\psi_k^m(\tau_1,\mu') - \phi_k^m(\tau_1,\mu)\phi_k^m(\tau_1,\mu')\right] \qquad \text{I-3}$$

$$G_k^m(\tau_1,\mu,\mu') = \frac{\omega_k^m}{\mu-\mu'}\left[\phi_k^m(\tau_1,\mu)\psi_k^m(\tau_1,\mu') - \psi_k^m(\tau_1,\mu)\phi_k^m(\tau_1,\mu')\right] \qquad \text{I-4}$$

The equations then become

$$\psi_l^m(\tau_1,\mu) = P_l^m(\mu) + \frac{1}{2}\mu\sum_{k=m}^{N}(-1)^{l+k}\int_0^1 F_k^m(\tau_1,\mu,\mu')P_l^m(\mu')d\mu' \qquad \text{I-5}$$

$$\phi_l^m(\tau_1,\mu) = e^{-\tau_1/\mu}P_l^m(\mu) + \frac{1}{2}\mu\sum_{k=m}^{N}\int_0^1 G_k^m(\tau_1,\mu,\mu')P_l^m(\mu')d\mu' \qquad \text{I-6}$$

$m = 0, 1, \cdots, N$

$l = m, m+1, \cdots, N$

For any value of N, equations I-1 and I-2 yield N + 1 systems of simultaneous integral equations. Each system consists of 2(N+1), 2N,..., 2 simultaneous equations corresponding respectively to $m = N, N-1,\ldots 0$. As an example, using the notation $F_1^m = F_1^m(\tau_1,\mu,\mu')$ and $G_1^m = G_1^m(\tau_1,\mu,\mu')$, the systems of integral equations for N = 2 are

<u>for m = 0</u>

$$\psi_0^0(\tau_1,\mu) = 1 + \frac{1}{2}\mu\int_0^1 [F_0^0 - F_1^0 + F_2^0]\,d\mu' \qquad \text{I-7a}$$

$$\psi_1^0(\tau_1,\mu) = \mu - \frac{1}{2}\mu\int_0^1 [F_0^0 - F_1^0 + F_2^0]\,\mu' d\mu' \qquad \text{I-7b}$$

$$\psi_2^0(\tau_1,\mu) = \frac{1}{2}(3\mu^2 - 1) + \frac{1}{2}\mu\int_0^1 [F_0^0 - F_1^0 + F_2^0]\,\frac{1}{2}(3\mu'^2 - 1)\,d\mu' \qquad \text{I-7c}$$

$$\phi_0^0(\tau_1,\mu) = e^{-\tau_1/\mu} + \frac{1}{2}\mu\int_0^1 [G_0^0 + G_1^0 + G_2^0]\,d\mu' \qquad \text{I-7d}$$

$$\phi_1^\circ(\tau_1,\mu) = \mu e^{-\tau_1/\mu} + \frac{1}{2}\mu\int_0^1 [G_0^\circ + G_1^\circ + G_2^\circ]\mu' d\mu' \tag{I-7e}$$

$$\phi_2^\circ(\tau_1,\mu) = \frac{1}{2}(3\mu^2-1)e^{-\tau_1/\mu} + \frac{1}{2}\mu\int_0^1 [G_0^\circ + G_1^\circ + G_2^\circ]\frac{1}{2}(3\mu'^2-1)d\mu' \tag{I-7f}$$

<u>for m = 1</u>

$$\psi_1^1(\tau_1,\mu) = (1-\mu^2)^{1/2} + \frac{1}{2}\mu\int_0^1 [F_1^1 - F_2^1](1-\mu'^2)^{1/2} d\mu' \tag{I-8a}$$

$$\psi_2^1(\tau_1,\mu) = 3(1-\mu^2)^{1/2}\mu - \frac{1}{2}\mu\int_0^1 [F_1^1 - F_2^1]3(1-\mu'^2)^{1/2}\mu' d\mu' \tag{I-8b}$$

$$\phi_1^1(\tau_1,\mu) = (1-\mu^2)^{1/2}e^{-\tau_1/\mu} + \frac{1}{2}\mu\int_0^1 [G_1^1 + G_2^1](1-\mu'^2)^{1/2} d\mu' \tag{I-8c}$$

$$\phi_2^1(\tau_1,\mu) = 3(1-\mu^2)^{1/2}\mu e^{-\tau_1/\mu} + \frac{1}{2}\mu\int_0^1 [G_1^1 + G_2^1]3(1-\mu'^2)^{1/2}\mu' d\mu' \tag{I-8d}$$

<u>m = 2</u>

$$\psi_2^2(\tau_1,\mu) = 3(1-\mu^2) + \frac{1}{2}\mu\int_0^1 (F_2^2)3(1-\mu'^2)d\mu' \tag{I-9a}$$

$$\phi_2^2(\tau_1,\mu) = 3(1-\mu^2)e^{-\tau_1/\mu} + \frac{1}{2}\mu\int_0^1 (G_2^2)3(1-\mu'^2)d\mu' \tag{I-9b}$$

3. The Intensity of Reflected and Transmitted Radiation

The intensity of radiation reflected diffusely by a semi-infinite slab at the plane Z = 0 in the direction (μ, φ) may be expressed in terms of a scattering function*

$$I(0, \mu, \varphi) = \frac{I_0(-\mu_0, \varphi_0)}{4\pi\mu} S(\tau_1; \mu, \varphi; \mu_0, \varphi_0) \tag{I-10}$$

where $I_0(-\mu_0, \varphi_0)$ = incident flux intensity in the direction $(-\mu_0, \varphi_0)$.

$I(0, \mu, \varphi)$ = intensity of flux reflected in the direction (μ, φ) at Z = 0.

S = Scattering function

The scattering function is**

$$S(\tau_1; \mu, \varphi; \mu_0, \varphi_0) = \sum_{m=0}^{N} S_1^{(m)}(\tau_1, \mu, \mu_0) \cos m(\varphi_0 - \varphi) \tag{I-11}$$

or for $\varphi_0 = 0$

$$S(\tau_1; \mu, \varphi; \mu_0, \varphi_0) = \sum_{m=0}^{N} S_1^{(m)}(\tau_1, \mu, \mu_0) \cos m\varphi \tag{I-12}$$

where,

$$S_1^{(m)}(\tau_1; \mu, \mu_0) = (2 - \delta_{0,m}) \mu\mu_0 \sum_{l=m}^{N} (-1)^{l+m} F_l^m(\tau_1, \mu, \mu_0) \tag{I-13}$$

* R. T. Page 161
** R. T. Pages 180 and 177

The diffuse reflected intensity may then be written as

$$\frac{I(0,\mu,\varphi)}{I_o(-\mu_o,0)} = \frac{\mu_o}{4\pi}\sum_{m=0}^{N} \cos m\varphi \,(2-\delta_{o,m})\sum_{l=m}^{N}(-1)^{m+l}\,F_l^m(\tau_1,\mu,\mu_o) \qquad \text{I-14}$$

In a similar manner the intensity of radiation transmitted diffusely through the slab at the $Z = \tau_1$ plane may be expressed in terms of a transmission function*

$$I(\tau_1,-\mu,\varphi) = \frac{I_o(-\mu_o,\varphi_o)}{4\pi\mu}\,\mathcal{J}(\tau_1;\mu,\varphi;\mu_o,\varphi_o) \qquad \text{I-15}$$

where, $I_o(-\mu_o,\varphi_o)$ = incident flux intensity in the direction $(-\mu_o,\varphi_o)$

$I(\tau_1,-\mu,\varphi)$ = intensity of flux at $Z = \tau_1$ transmitted in the direction $(-\mu, \varphi)$

$\mathcal{J}$ = transmission function

The transmission function is**

$$\mathcal{J}(\tau_1;\mu,\varphi;\mu_o,\varphi_o) = \sum_{m=0}^{N}\mathcal{J}^{(m)}(\tau_1,\mu,\mu_o)\cos m(\varphi_o-\varphi) \qquad \text{I-16}$$

or for $\varphi_o = 0$

$$\mathcal{J}(\tau_1;\mu,\phi;\mu_o,\varphi_o) = \sum_{m=0}^{N}\mathcal{J}^{(m)}(\tau_1,\mu,\mu_o)\cos m\varphi \qquad \text{I-17}$$

where,

$$\mathcal{J}^{(m)}(\tau_1,\mu,\mu_o) = \mu\mu_o(2-\delta_{o,m})\sum_{l=m}^{N}G_l^m(\tau_1,\mu,\mu_o) \qquad \text{I-18}$$

The diffuse portion of the transmitted intensity may then be written as

$$\frac{I(\tau_1,-\mu,\varphi)}{I_o(-\mu_o,0)} = \frac{\mu_o}{4\pi}\sum_{m=0}^{N}(2-\delta_{o,m})\cos m\varphi\sum_{l=m}^{N}G_l^m(\tau_1,\mu,\mu_o) \qquad \text{I-19}$$

* R. T. Page 161
** R. T. Pages 180 and 177

Equations I-14 and I-19 may be written in a slightly more convenient form by noting that

$$\cos m\varphi = A_0 + \sum_{i=1}^{m} A_i \cos^i \varphi \qquad \text{I-20}$$

where, for example, for m equal to 2, $A_0 = -1$, $A_1 = 0$, and $A_2 = 2$.

Thus, new expressions can be derived for the reflected and the diffuse portion of the transmitted intensities of the following form

$$\frac{I(0,\mu,\varphi)}{I_0(-\mu_0,0)} = b_0(\tau_1,\mu,\mu_0) + \sum_{m=1}^{N} b_m(\tau_1,\mu,\mu_0)\cos^m\varphi \qquad \text{I-21}$$

and

$$\frac{I(\tau_1,-\mu,\varphi)}{I_0(-\mu_0,0)} = c_0(\tau_1,\mu,\mu_0) + \sum_{m=1}^{N} c_m(\tau_1,\mu,\mu_0)\cos^m\varphi \qquad \text{I-22}$$

This means that for a N term phase function and a given combination of τ_1, μ, and μ_0, the intensities need be calculated at only N + 1 values of φ in order to obtain the complete angular distributions. The explicit relations for $b_m(\tau_1,\mu,\mu_0)$ and $c_m(\tau_1,\mu,\mu_0)$ in terms of $\psi_1^m(\tau_1,\mu,\mu_0)$ and $\phi_1^m(\tau_1,\mu,\mu_0)$ for N = 2 are derived in Appendix A-1.

4. Integrated Reflectance and Transmission

The integrated reflectance is defined as the ratio of reflected to incident power passing through a unit area of the interface. The flux, I_0, is defined as the radiant energy passing through a unit area normal to the direction of propagation per unit time. Therefore, the incident power striking a unit area of the interface is $I_0 \times \mu_0$. Similarly, the reflected power passing out of a unit area of the interface is $\int_0^1 \int_0^{2\pi} I(0,\mu,\varphi)\mu\, d\mu\, d\varphi$. The integrated reflectance is then expressed as

$$R = \frac{1}{I(-\mu_0)\mu_0} \int_0^1 \int_0^{2\pi} I(0,\mu,\varphi)\mu\, d\mu\, d\varphi \qquad \text{I-23a}$$

Replacing $I(0,\mu,\varphi)$ by I-14 and noting that

$$\int_0^{2\pi} F(\mu)\cos m\varphi\, d\varphi = 2\pi F(\mu) \text{ or } 0 \quad \text{for} \quad m = 0 \text{ or } m \neq 0 \text{ respectively} \qquad \text{I-24}$$

$$R = \frac{1}{2}\sum_{l=0}^{N} (-1)^l \int_0^1 \mu F_l^{\circ}(\tau_1,\mu,\mu_0)\, d\mu \qquad \text{I-23b}$$

In a similar manner the integrated diffuse transmission T_D is defined as the ratio of transmitted to incident power passing through a unit area of interface.

Thus

$$T_D = \frac{1}{I_0(-\mu_0)\mu_0} \int_0^1 \int_0^{2\pi} I(\tau_1,-\mu,\varphi)\mu\, d\mu\, d\varphi \qquad \text{I-25a}$$

and substituting I-19 for $I(\tau_1,-\mu,\varphi)$

$$T_D = \frac{1}{2}\sum_{l=0}^{N} \int_0^1 \mu G_l^{\circ}(\tau_1,\mu,\mu_0)\, d\mu \qquad \text{I-25b}$$

The total transmission T is the sum of the direct transmission and the diffuse transmission.

Thus $$T = e^{-\tau_1/\mu_0} + T_D \qquad \text{I-26}$$

It has been shown[(5)] that, since,

$$\psi_1^{\circ}(\tau_1,\mu) = \mu - \frac{1}{2}\mu \sum_{k=0}^{N} (-1)^k \int_0^1 F_k^{\circ}(\tau_1,\mu,\mu')\mu'\, d\mu' \qquad \text{I-27}$$

then

$$1 - \frac{\psi_1^\circ(\tau_1,\mu)}{\mu} = \frac{1}{2}\sum_{k=0}^{N}(-1)^k\int_0^1 F_k^\circ(\tau_1,\mu,\mu')\mu'd\mu' \qquad \text{I-28}$$

and

$$R = 1 - \frac{\psi_1^\circ(\tau_1,\mu_0)}{\mu_0} \qquad \text{for} \quad N \geq 1, \ \mu_0 > 0 \qquad \text{I-29}$$

Similarly it can be shown that, since

$$\phi_1^\circ(\tau_1,\mu) = \mu e^{-\tau_1/\mu} + \frac{1}{2}\mu\sum_{k=0}^{N}\int_0^1 G_k^\circ(\tau_1,\mu,\mu')\mu'd\mu' \qquad \text{I-30}$$

Then

$$\frac{\phi_1^\circ(\tau_1,\mu)}{\mu} - e^{-\tau_1/\mu} = \frac{1}{2}\sum_{k=0}^{N}\int_0^1 G_k^\circ(\tau_1,\mu,\mu')\mu'd\mu' \qquad \text{I-31}$$

and

$$T_D = \frac{\phi_1^\circ(\tau_1,\mu_0)}{\mu_0} - e^{-\tau_1/\mu_0} \qquad \text{I-32}$$

For

$$N \geq 1, \ \mu_0 > 0$$

The total transmission is simply

$$T = \frac{\phi_1^\circ(\tau_1,\mu_0)}{\mu_0} \qquad \text{I-33}$$

B. Numerical Procedure

1. The General Numerical Problem.

As shown in the previous section, the intensities of reflected and transmitted radiation for a particular phase function for single scattering can be expressed in terms of the solutions of sets of simultaneous, non-linear integral equations. The numerical problem, then, is to solve these sets in order to obtain values for the intensity, I, in terms of its parameters.

The most straightforward method of obtaining numerical solutions of integral equations is simple iteration using the original equations. For example, in the case of a single integral equation of the type

$$f(x) = g(x) + \lambda \int_0^1 f(x)f(x')h(x')dx' \qquad \text{I-34}$$

where f(x) is the function to be found

g(x) and h(x) are known functions of x

and λ is some known constant,

the method involves the following:

1. Replacing the continuous function, f(x), by a tabular function, f*(x) where f*(x) consists of a set of p values of $f(x_i)$, $\{f_o, f_1, \ldots f_p\}$, corresponding to a set of values of x_i, $\{x_o, x_1, \ldots x_p\}$, where p is the number of points.
2. Choosing an initial $f^{*(0)}(x)$.
3. Evaluating the right side of I-34 by some mechanical quadrature method--thus, obtaining a new $f^{*(1)}(x)$.

4. Testing to see if $f^{*(1)}(x)$ agrees with $f^{*(0)}(x)$ everywhere within acceptable limits and, if not,

5. Repeating the process using $f^{*(1)}(x)$ in place of $f^{*(0)}(x)$.

6. The process is continued until the sequence: $f^{*(0)}(x)$, $f^{*(1)}(x), \ldots, f^{*(n)}(x)$ converges to some limiting value.

Advancing one step in the iteration procedure to obtain $f^{*(n+1)}(x)$ from $f^{*(n)}(x)$ requires making p+1 numerical integrations.

Integration by mechanical quadrature consists of replacing the integral by a sum, or in other words

$$\int_0^1 f(x)\,dx = \sum_{i=0}^{p} W_i f(x_i) + \epsilon \qquad \text{I-35}$$

where ϵ is the error. Thus, selecting a specific method of numerical integration involves selecting the set of values of x, $\{x_0, x, x_p\}$, and the set of weighting coefficients, $\{W_0, W_1, \ldots W_p\}$.

For integration by the trapezoidal rule

$$\begin{aligned} x_i &= i/p \\ W_0 = W_p &= 1/2p \\ W_i &= 1/p \quad (i = 1, \ldots, p-1) \end{aligned} \qquad \text{I-36}$$

For integration by Simpson's rule

$$\begin{aligned} p &= 2k \quad \text{where k is an anteger} \\ x_i &= i/p \\ W_0 = W_p &= 1/3p \\ W_i &= 4/3p \quad i = 1, 3, \ldots, p-1 \\ W_i &= 2/3p \quad i = 2, 4, \ldots, p-2 \end{aligned} \qquad \text{I-37}$$

For integration using n-point Gaussian quadrature

$$x_i = (r_{i+1} + 1)/2 \qquad \text{I-38}$$

where r_i is the i-th root of $P_n(x)$

$$W_i = \frac{1}{2\,P_n'(x)} \int_{-1}^{1} \frac{P_n(x)\,dx}{x - x_i}$$

Thus, the definition of a specific numerical procedure involves:

1. Selecting the number of points, p+1, $\{x_i\}$ and $\{W_i\}$.
2. Selecting the initial function $f^{*(0)}(x)$.
3. Selecting a test to determine when the sequence $f^{*(0)}(x)$, $f^{*(1)}(x), \ldots, f^{*(n)}(x)$ has converged.

It should be noted that the fact that the function $f^*(x)$ has converged to some limiting value merely assures that a solution has been found to the following equation:

$$f^*(x_j) = g^*(x_j) + \lambda \sum_{i=0}^{p} W_i\, f^*(x_j)\, f^*(x_i)\, h^*(x_i) \qquad \text{I-39}$$

for j = 0, 1,..., p. This does not, however, guarantee that $f^*(x) = f(x)$. Convergence in this latter sense may be tested by comparison of the results with (a) results obtained by other methods (b) results obtained using more accurate methods of numerical integration, and (c) solutions for idealized and limiting cases.

Extension of the preceeding comments to the case of N+1 simultaneous integral equations is straightforward. The general form of the equations is

$$f_l(x) = g_l(x) + \sum_{k=0}^{N} \lambda_{k,l} \int_0^1 f_k(x)\, f_k(x')\, h_l(x')\, dx', \quad (l = 0, 1, \cdots, N) \qquad \text{I-40}$$

Advancing one step in the iteration procedure involves obtaining

$$f_0^{*(n+1)} \ldots\ldots f_N^{*(n+1)} \quad \text{from } f_0^{*(n)} \ldots f_N^{*(n)} .$$

2. Specific Numerical Techniques Used in Computation.

A computer program was written to solve equations I-1 and I-2 in their general form by the method of simple iteration. The programs were written in a form suitable for use on the IBM 704 or 709 and are described in detail in Appendix C. All calculations, however, were performed on the IBM 704.

Every effort was made to develop general techniques suitable for solution of equations with an arbitrary value of N; arbitrary set of constants in the phase function; an arbitrary method of numerical integration; any thickness, τ_1; any albedo for single scattering, ω_0 ; and any choice of starting functions. The same procedure was used for all values of m and N to obtain all of the functions presented in this report. The only limitation on the size of the problem that can be solved is that of computer time required and number of storage locations available.

In practice, almost all of the results were obtained using five point Gausian quadrature. For this case, using numerical values given by Lowan, Davids, and Levenson[7]

$$\begin{aligned} W_0 &= 0.11846 & \mu_0 &= 0.04691 \\ W_1 &= 0.23931 & \mu_1 &= 0.23077 \\ W_2 &= 0.28444 & \mu_2 &= 0.50000 \\ W_3 &= 0.23931 & \mu_3 &= 0.76923 \\ W_4 &= 0.11846 & \mu_4 &= 0.95309 \end{aligned}$$

In addition some values were obtained for comparison using a 10-point Simpson's Rule technique.

The starting functions used for all calculations were

$$\psi_l^m(\tau_1,\mu)^{(0)} = P_l^m(\mu) \qquad \text{I-41a}$$

$$\phi_l^m(\tau_1,\mu)^{(0)} = e^{-\tau_1/\mu}\, P_l^m(\mu) \qquad \text{I-41b}$$

After considerable experimenting, it was found that for $m \neq 0$ a simple numerical test was satisfactory to determine convergence. In other words, when

$$\underset{i}{\text{Max}}\left| f^{*(n+1)}(x_i) - f^{*(n)}(x_i)\right| < \text{Tolerance} \qquad \text{I-42a}$$

the iteration was stopped. For all cases the tolerance was chosen as 0.001.

For $m = 0$, however, it was found that a better test was to stop the iteration whenever

$$\left[f^{*(n+1)}(x_i) - f^{*(n)}(x_i)\right]\left[f^{*(n)}(x_i) - f^{*(n-1)}(x_i)\right] < 0 \qquad \text{I-42b}$$

for ϕ_0^0 or ψ_0^0 or whenever I-42a was satisfied. This effectively stops the iteration as soon as the incremental change in any function reverses sign. This gave the best agreement with known solutions for special cases (i.e. the $X(\mu)$ and $Y(\mu)$ functions of Chandrasekhar[8] which are the same as $\phi_0^0(\mu)$ and $\psi_0^0(\mu)$ in this study).

Some difficulty occurs when evaluating $G_l^m(\tau_1, \mu, \mu')$ for the case $\mu = \mu'$

$$\frac{G_l^m(\tau_1,\mu,\mu)}{\omega_l^m} = \lim_{\mu \to \mu'} \frac{\psi_l^m(\mu)\phi_l^m(\mu') - \phi_l^m(\mu)\psi_l^m(\mu')}{\mu - \mu'} \qquad \text{I-43}$$

However, by L'Hospital's rule the limit is

$$\frac{G_l^m(\tau_1,\mu,\mu)}{\omega_l^m} = \psi_l^m(\mu)\frac{d\phi_l^m(\mu)}{d\mu'} - \phi_l^m(\mu)\frac{d\psi_l^m(\mu)}{d\mu'} \qquad \text{I-44}$$

In practice, $\frac{d\phi_l^m(\mu)}{d\mu'}$ and $\frac{d\psi_l^m(\mu)}{d\mu'}$ were evaluated at a given value of μ by fitting a second order polynomial to the three nearest values of the function, and differentiating analytically.

In addition, a program was written for calculating the integrated diffuse reflection and transmission for any set of functions using equations I-23b and I-25b.

A discussion of the effectiveness of the numerical procedures will be deferred to section C-5 of this report after the presentation of numerical results.

C. Solutions to Specific Problems

1. Choice of Independent Variables.

The choice of specific values of the independent variables was based on the following:

1. To indicate quantitatively the effect of the independent variables for representative conditions.

2. To gain experience in techniques of numerical solution of the problems which would be useful in solving further problems.

3. To obtain solutions for specific problems which can be used as a standard for evaluating approximate techniques.

In order to completely specify the problem the following parameters must be chosen: 1) number of terms in the phase function, N; 2) coefficients of the angular distribution function, a_0, a_1,...aN; 3) albedo for single scattering, ω_0 ; and 4) the optical thickness of the dispersion, τ_1.

In order to obtain a reasonable picture of the effect of ω_0 and τ_1 a minimum of three values of each variable must be used, making a total of at least nine solutions for each phase function. There is an almost endless number of possible phase functions. For instance, for only a two term phase function ($a_0 = 1$) using 10 values of a_1 and a_2 would require 900 solutions. For a five term phase function, 900,000 solutions would be required.

In addition, for a given value of m, the effort in solving one set of integral equations increases with the square of the number of equations in a set. Also, the number of sets of equations increases with N. For example, in comparable units of computer time, the effort in solving the equations for N = 0, 1, 2, 3, 4, and 5 is

N = 0

$(1)^2$ = 1 unit

N = 1

$(1)^2 + (2)^2$ = 5 units

N = 2

$(1)^2 + (2)^2 + (3)^2$ = 14 units

N = 3

$(1)^2 + (2)^2 + (3)^2 + (4)^2$ = 30 units

N = 4

$(1)^2 + (2)^2 + (3)^2 + (4)^2 + (5)^2$ = 55 units

N = 5

$(1)^2 + (2)^2 + (3)^2 + (4)^2 + (5)^2 + (6)^2$ = 91 units

(Note: in the preceeding it is assumed that all coefficients in the phase function are non-zero; in estimating the labor required to solve a set of equations N may be taken as the number of non-zero terms in the phase function)

Thus, it is apparent that the selection of representative phase functions is very important and that the number of non-zero terms in the phase function must be kept small in order to be able to vary ω_0 and τ_1 sufficiently to achieve the previously stated objectives.

a) Choice of the Phase Function

The construction of a phase function for scattering by spherical particles depends on the wave length λ of the radiation, the diameter of the scattering particle D and the index of refraction m. The problem is fully described by Chu and Churchill[9]. In summary, the phase function, $p(\cos\Theta)$ or $p(\mu)$, can be written

$$p(\mu) = 1 + \sum_{n=1}^{\infty} a_n P_n(\mu) \qquad \text{I-45}$$

where $P(\mu)/4\pi$ is the fraction of randomly polarized radiation scattered by a spherical particle into a unit solid angle in the direction Θ.

Θ is $\cos^{-1}(\mu)$

$a_n(\alpha, m)$ angular distribution coefficients

$P_n(\mu)$ are Legendre polynomials

$\alpha = \pi D/\lambda$

The phase function (I-45) is for non-aborbing particles ($\omega_0 = 1.0$). For partially absorbing particles with complex indices of refraction, the angular distribution coefficients must be recomputed for each value of ω_0. As a good approximation, however, the phase function may be considered to be $\omega_0 p(\mu)$ where $p(\mu)$ is the phase function of a non-absorbing particle.

Probably the most important single characteristic of any given phase function is the fraction scattered forward and backward, F and B, where

$$F = 2\pi \int_0^1 p(\mu)\, d\mu \qquad \text{I-46}$$

$$B = 2\pi \int_{-1}^0 p(\mu)\, d\mu \qquad \text{I-47}$$

$$B = 1 - F, \quad \text{since} \quad 2\pi \int_{-1}^1 p(\mu) d\mu = 1 \qquad \text{I-48}$$

A good measure of how "peaked" is the phase function is the ratio of the second moment to the zeroth moment in the forward and backward direction

$$P = 2\pi \int_0^1 p(\mu)\mu^2 d\mu \Big/ F \qquad \text{I-49}$$

and

$$P_B = 2\pi \int_{-1}^0 p(\mu)\mu^2 d\mu \Big/ B \qquad \text{I-50}$$

where P = Peakedness in Forward Direction.

P_B = Peakedness in Backward Direction.

In addition, other characteristics are

FP = Height of the Forward Peak, i.e. p(1.0)

BP = Height of the Backward Peak, i.e. p(-1.0)

The equations for these parameters in terms of the distribution coefficients, $a_0, \ldots, a_N$ are

$$F = \frac{1}{2} + \frac{a_1}{4} - \frac{a_3}{16} + \frac{a_5}{32} - \frac{5a_7}{256} + \frac{7a_9}{512} - \frac{21a_{11}}{2048} + \frac{33a_{13}}{4096} - \cdots \qquad \text{I-51}$$

$$B = 1 - F \qquad \text{I-48}$$

$$P = \left[\frac{1}{6} + \frac{a_1}{8} + \frac{a_2}{15} + \frac{a_3}{48} - \frac{a_5}{384} + \frac{a_7}{1280} - \frac{a_9}{3072} + \frac{a_{11}}{6144} - \frac{3a_{13}}{32716} + \cdots\right] \Big/ F \qquad \text{I-52}$$

$$P_B = \left[\frac{1}{6} - \frac{a_1}{8} + \frac{a_2}{15} - \frac{a_3}{48} + \frac{a_5}{384} - \frac{a_7}{1280} + \frac{a_9}{3072} - \frac{a_{11}}{6144} + \frac{3a_{13}}{32716} - \cdots\right] \Big/ B \qquad \text{I-53}$$

$$FP = 1 + a_1 + a_2 + a_3 + a_4 + \cdots \qquad \text{I-54}$$

$$BP = 1 - a_1 + a_2 - a_3 + a_4 - \cdots \qquad \text{I-55}$$

For N = 0 the phase function has only one term, $a_0 = 1$. Thus, all quantities are fixed

$F = 1/2$

$B = 1/2$

$P = 1/3$

$P_B = 1/3$

$FP = BP = 1$

For N = 1 the phase function is completely specified by choosing any one characteristic such as F. For n = 2, the phase function is completely specified by choosing F and P (or any other pair of characteristics). As more terms are added, more characteristics of the function may be choosen independently. In order to consider a significant range of ω_0 and τ_1 with the computer time available a two term phase function was used for all computations including a few special cases in which some of the coefficients were zero.

A careful study was made of all possible two term phase functions satisfying the conditions that $p(\mu)$ be positive for all values of μ and that FP be greater than BP. It was found that the phase function

$$p(\mu) = \omega_0 \left[1 + a_1 P_1(\mu) + a_2 P_2(\mu) \right]$$

gave the greatest range of F and P while satisfying the conditions that $p(\mu) \geq 0$ for $0 \leq \mu \leq 1$ and $FP \geq BP$. Figure I-3 is a plot of FP vs F with P and BP as parameters. The shaded area represents all values of F and FP for which $p(\mu)$ was always positive.

From consideration of Figure I-3, the following phase functions were selected for computation.

TABLE I-1 Phase Functions Used in Computation

a_1	a_2	F	P	Remarks
0.00000	0.00000	0.50000	0.33333	Isotropic Scattering
0.00000	0.50000	0.50000	0.40000	Rayleigh Scattering
1.00000	1.81650	0.75000	0.5504	Arbitrary: same F, different P
1.00000	1.04904	0.75000	0.4821	
1.00000	0.00000	0.75000	0.3899	
1.73205	1.00000	0.9330	0.4821	Largest F for N = 2
1.73458	2.24153	0.9336	0.5821	Same Phase Function as Reference (5).

The last phase function listed in Table I-1 is the same as that in reference (5). Computations with it were made only for $\tau_1 = \infty$, since $p(\mu)$ was not always positive for this function. The results were used as a check on the accuracy of the numerical techniques.

All of the phase functions are plotted in figures 4a, 4b, and 4 c. In these figures the function $f(\Theta)$ is plotted which is equivalent to $p(\cos\Theta)/4\pi$.

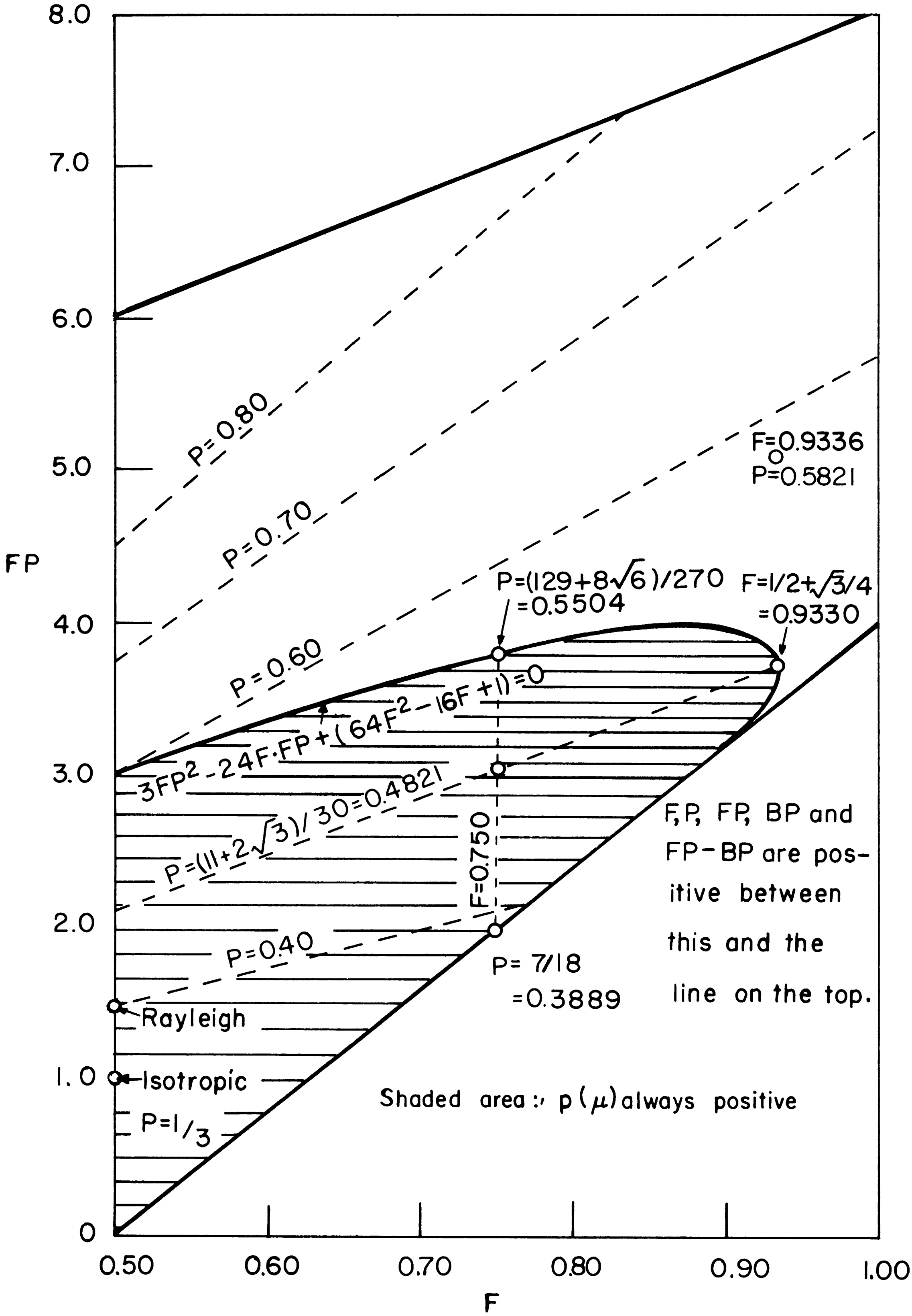


Figure I-3 Characteristics of a two term phase function

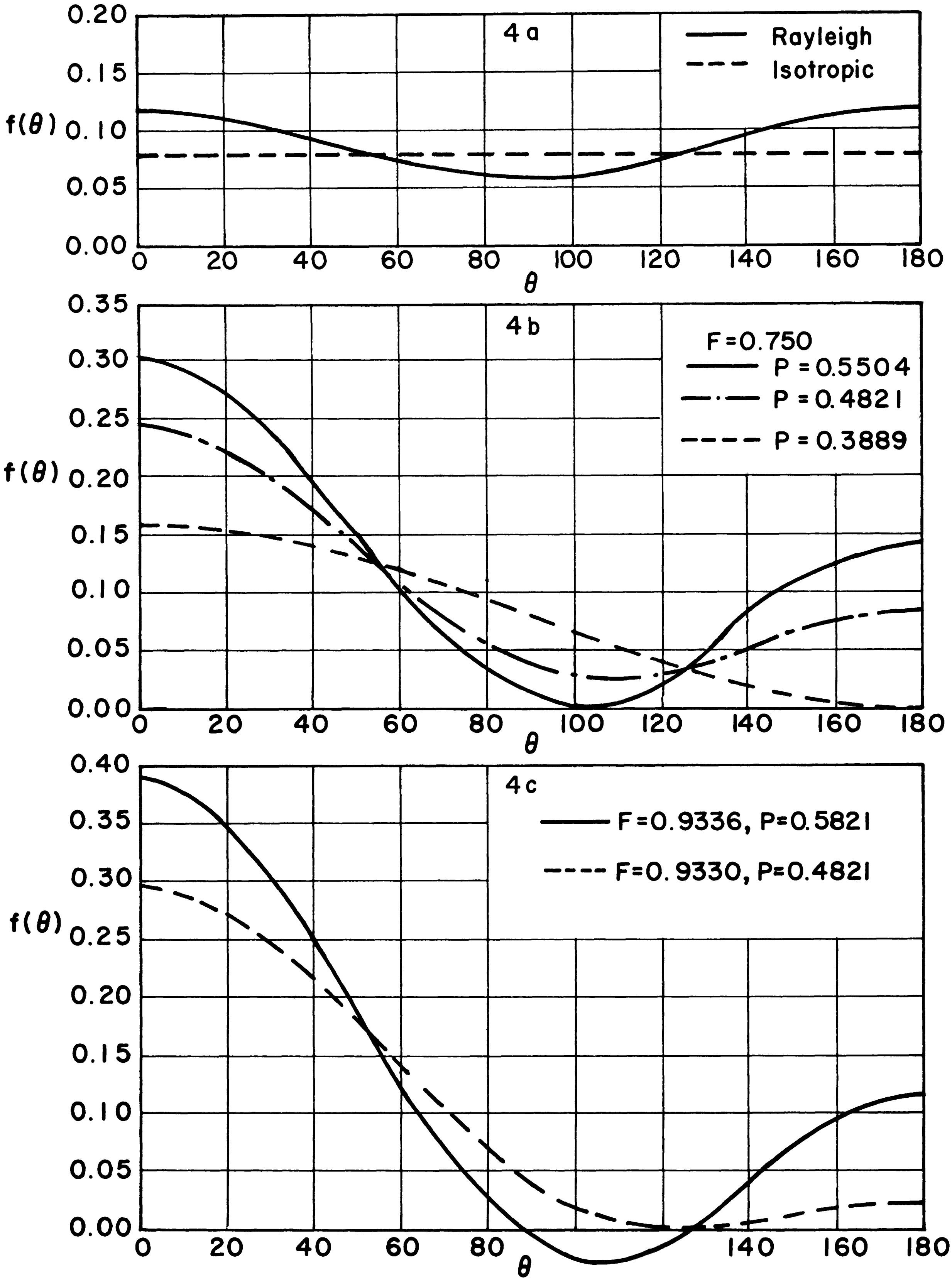


Figure I-4 Phase functions used in calculations, F = 0.50, F = 0.75, and F = 0.933, Parameters of P

b) Choice of Other Independent Variables

The independent variables other than the phase function (i.e. ω_0 and τ_1) were selected to give as good an indication of their effect as possible.

For most of the calculations ω_0 was chosen as 0.90. This was done to be able to compare results for $\tau_1 = \infty$ with previous solutions. For every phase function results were obtained at least for $\tau_1 = 1.00$. For isotropic scattering, Rayleigh scattering and for the function corresponding to F = 0.9330 and P = 0.4821, results were obtained at several values of τ_1.

A total of 36 problems were solved; the values of the parameters for each are listed in Table I-2.

Table I-2 Values of the Parameters Used in Computation

Problem Number	F	P	τ_1	ω_0
1	0.500	0.333	0.05	0.90
2	0.500	0.333	0.25	0.90
3	0.500	0.333	0.50	0.90
4	0.500	0.333	1.00	0.90
5	0.500	0.333	1.50	0.90
6	0.500	0.333	2.00	0.90
7	0.500	0.333	00	0.90
8	0.500	0.333	0.05	0.60
9	0.500	0.333	0.50	0.60
10	0.500	0.333	00	0.60
11	0.500	0.333	0.05	0.30
12	0.500	0.333	0.50	0.30
13	0.500	0.333	00	0.30
14	0.500	0.400	0.05	0.90
15	0.500	0.400	0.25	0.90
16	0.500	0.400	0.50	0.90
17	0.500	0.400	1.00	0.90
18	0.500	0.400	1.50	0.90
19	0.500	0.400	2.00	0.90
20	0.500	0.400	00	0.90
21	0.750	0.5504	1.00	0.90
22	0.750	0.4921	1.00	0.90
23	0.750	0.3889	1.00	0.90
24	0.9330	0.4821	0.05	0.90
25	0.9330	0.4921	0.25	0.90
26	0.9330	0.4921	0.50	0.90
27	0.9330	0.4821	1.00	0.90
28	0.9330	0.4821	2.00	0.90
29	0.9330	0.4821	00	0.90
30	0.9330	0.4821	0.05	0.60
31	0.9330	0.4821	0.50	0.60
32	0.9330	0.4821	00	0.60
33	0.9330	0.4821	0.05	0.30
34	0.9330	0.4821	0.50	0.30
35	0.9330	0.4821	00	0.30
36	0.9336	0.5821	00	0.90

2. The ψ and ϕ Functions

For each set of parameters as indicated in Table I-2 the complete set of ψ_l^m and ϕ_l^m functions are calculated at the Gaussian quadrature points. These functions are tabulated in Appendix B. The number of iterations required to evaluate the functions for each value of m is given. In every case except Problem 36 for m = 0, the starting functions are those given by Equations I-41a and I-41b.

3. Sample Values of the Intensity of Reflected and Transmitted Radiation

The normalized intensity of reflected radiation and the diffuse portion of the transmitted radiation are, in general, a function of two angles ϕ and θ and four parameters: θ_0, ω_0, τ_1, and the phase function.

The effect of the variables is illustrated by considering a sample problem with

1) $\theta_0 = \cos^{-1} 0.5$ (i.e. $\mu_o = 0.50$)

2) $\theta = \cos^{-1} 0.76923$ (i.e. $\mu = 0.76923$)

3) $\omega_o = 0.90$

4) $\tau_1 = 1.00$

5) A two term phase function such that F = 0.9330 and P = 0.4821

The angular distribution of the specific intensity is obtained as a function of the bearing angle φ as well as these six parameters. The effect of each variable is indicated by varying θ_0, θ, ω_o, τ_1, and the phase function, one at a time, while holding the remaining variables at the above, "standard values". (There are exceptions which are noted.)

The specific intensities for each case were calculated on a desk calculator using I-21 or I-22 and the equations in Appendix A-1. The apparently arbitrary values of μ and μ_0 which were used correspond to the Gaussian points for the five-point Gaussian quadrature utilized for integration. The values of the coefficients b_0, b_1, b_2, c_0, c_1, and c_2 which appear in equations I-21 and I-22 are tabulated in Table I-3.

The effect of θ_0 (or μ_0) is shown in Figures I-5a and I-5b. In this case μ was held at 0.50 and four different values of μ_0 were used. It is observed that there is a rather complicated dependence of the intensity of reflected radiation on μ_0, with some of the curves overlapping. At $\varphi = 180^\circ$ the maximum intensity of reflected radiation occurs for μ_0 between 0.23079 and 0.76923 which is not surprising since the cosine of the angle of incidence is 0.50.

The effect of θ (or μ) is shown in Figures I-6a and I-6b with all other variables at the "standard" values. In this case the dependence of the intensity of both reflected and transmitted radiation on μ produce overlapplng curves. The maximum intensity of reflected radiation at $\varphi = 180^\circ$ apparently occurs at a viewing angle almost the same as the angle of incidence.

The effect of ω_0 is shown in Figures I-7a and I-7b. In this case τ_1 is 0.50. It is apparent that decreasing ω_0 simply decreases the intensity of the reflected and diffuse portion of the transmitted radiation without affecting the nature of the dependence on φ.

The effect of τ_1 is shown in Figures I-8a and I-8b. The effect of decreasing τ_1 is similar to that of decreasing ω_0: the intensity is decreased but the dependence on φ is not changed very much.

Table I-3 Values of the Coefficients Used in Equations I-21 and I-22 to Calculate the Specific Intensities of Reflected and Transmitted Radiation.

F	P	ω_0	τ_1	μ_0	μ	b_0	b_1	b_2	c_0	c_1	c_2
0.500	0.333	0.90	1.0	0.5000	0.0469	0.1033	0.0000	0.0000	0.0326	0.0000	0.0000
"	"	"	"	"	0.7692	0.0560	0.0000	0.0000	0.0429	0.0000	0.0000
"	"	"	0.25	"	"	0.0219	0.0000	0.0000	0.0174	0.0000	0.0000
"	"	0.60	0.50	"	"	0.0205	0.0000	0.0000	0.0183	0.0000	0.0000
"	0.400	0.90	1.0	"	0.0469	0.0691	0.0020	0.0765	0.0268	0.0035	0.0130
"	"	"	"	"	0.7692	0.0485	0.0091	0.0133	0.0388	0.0061	0.0091
0.750	0.550	"	"	"	"	0.0078	0.0186	0.0581	0.0348	0.0443	0.0420
"	0.482	"	"	"	"	0.0247	0.0014	0.0300	0.0459	0.0285	0.0209
"	0.389	"	"	"	"	0.0423	0.0195	0.0000	0.0571	0.0131	0.0000
0.933	0.482	"	"	"	"	0.0367	0.0225	0.0297	0.0000	0.0000	0.0000
"	"	"	1.0	"	0.0469	0.0214	0.1189	0.1597	0.0304	0.0329	0.0283
"	"	"	"	"	0.2308	0.0230	0.0845	0.1163	0.0432	0.0502	0.0407
"	"	"	"	"	0.7692	0.0132	0.0201	0.0284	0.0601	0.0467	0.0197
"	"	"	"	"	0.9531	0.0126	0.0063	0.0055	0.0719	0.0245	0.0040
"	"	"	"	0.0469	0.5000	0.0020	0.0112	0.0150	0.0028	0.0031	0.0027
"	"	"	"	0.2308	"	0.0106	0.0390	0.0537	0.0197	0.0232	0.0188
"	"	"	"	0.7692	"	0.0204	0.0310	0.0436	0.0924	0.0718	0.0304
"	"	"	"	0.9531	"	0.0241	0.0120	0.0105	0.1371	0.0467	0.0077
"	"	"	0.50	0.5000	0.7692	0.0044	0.0142	0.0232	0.0392	0.0425	0.0210
"	"	"	0.25	"	"	0.0003	0.0076	0.0157	0.0245	0.0298	0.0150
"	"	0.60	0.50	"	"	0.0007	0.0076	0.0149	0.0218	0.0254	0.0135
"	"	0.30	"	"	"	0.0006	0.0030	0.0072	0.0093	0.0117	0.0065

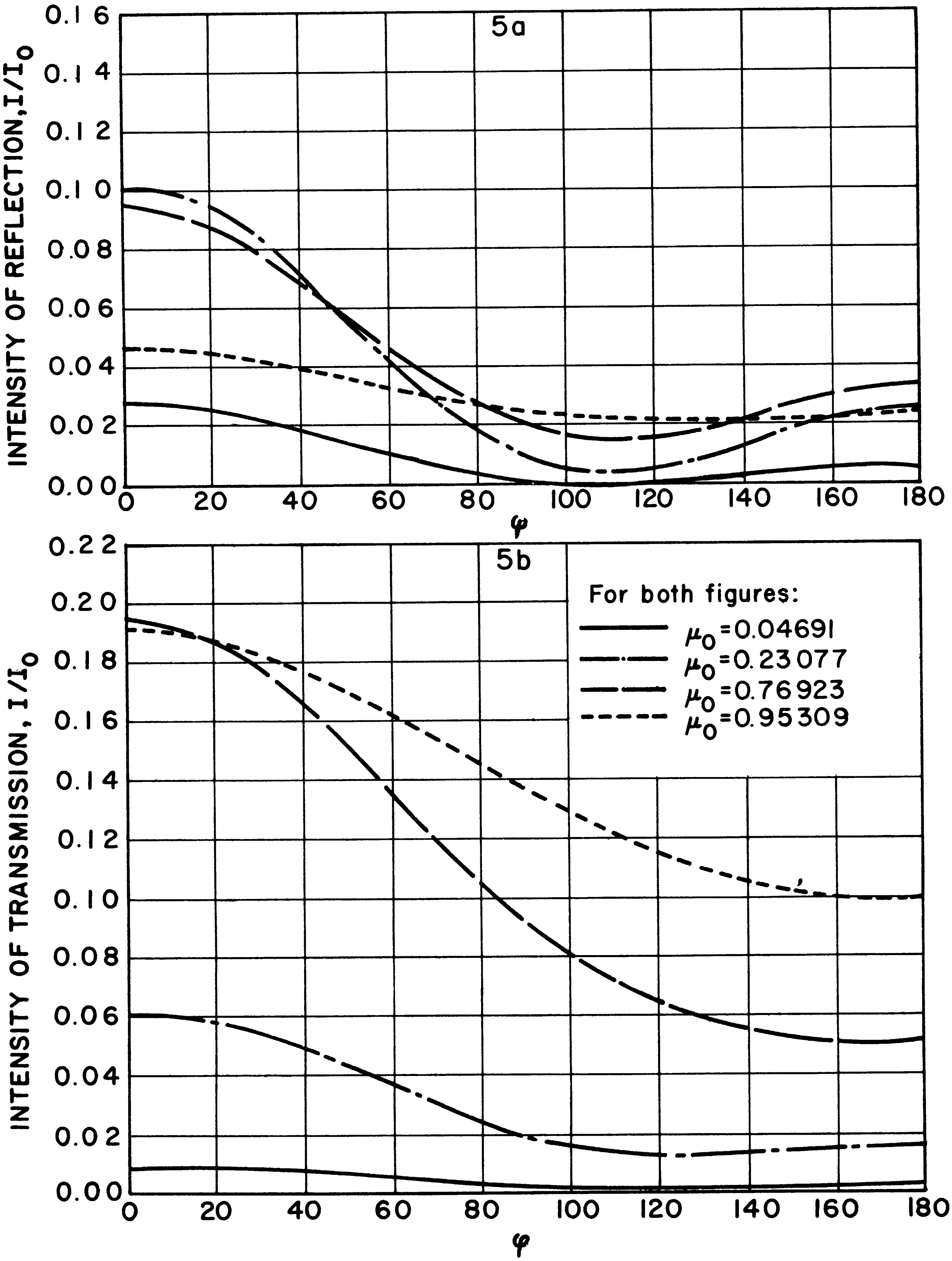


Figure I-5 Intensity of reflected and transmitted radiation as function of bearing angle, I/I_0 vs. φ , parameters of μ_0, $\mu = 0.50$, $\omega_0 = 0.90$, $\tau_1 = 1.00$, $F = 0.9330$, and $P = 0.4821$

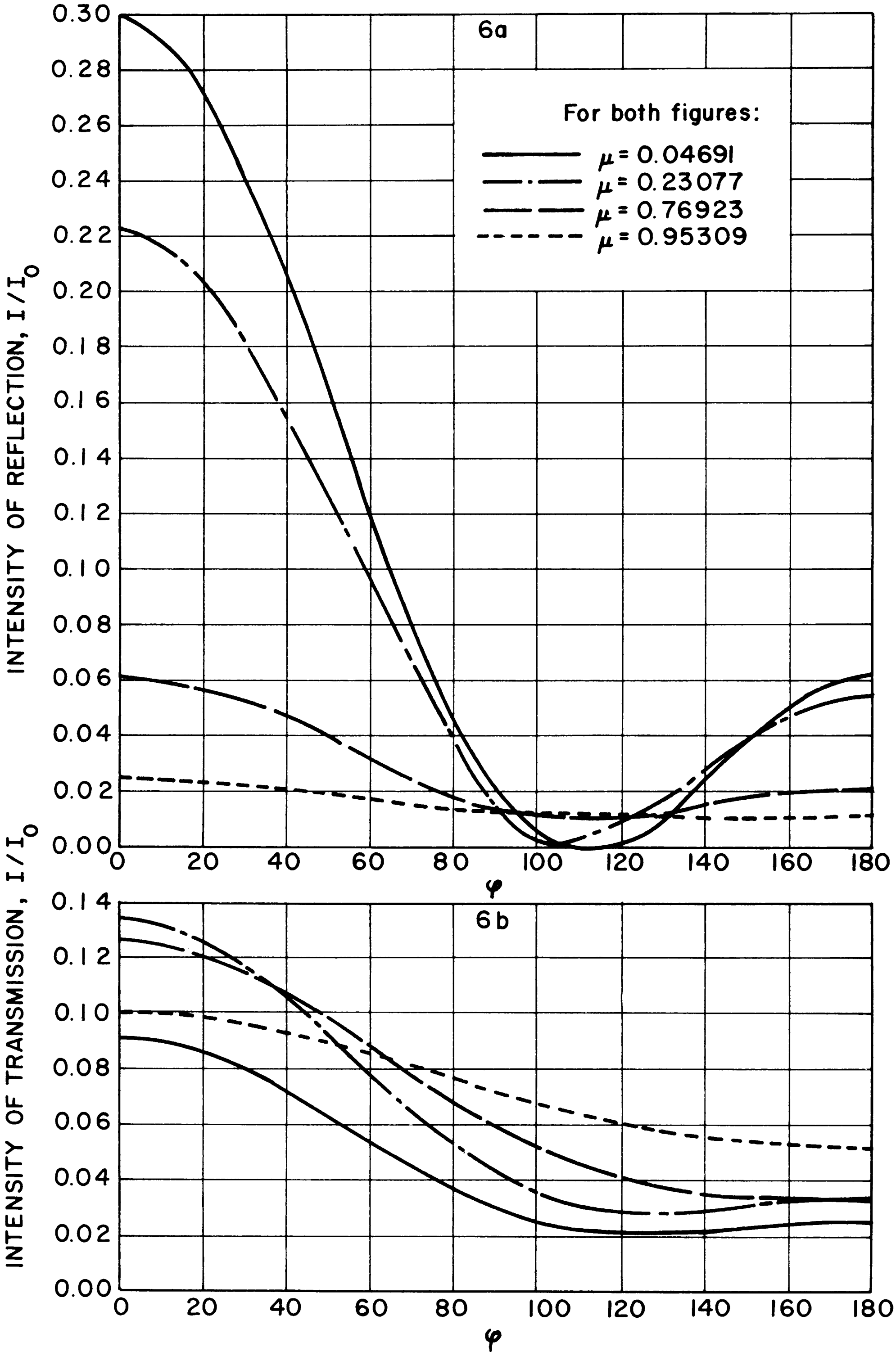


Figure I-6 Intensity of reflected and transmitted radiation as function of bearing angle, I/I_0 vs. φ, parameters of μ, $\mu_0 = 0.50$, $\omega_0 = 0.90$, $\tau_1 = 1.00$, $F = 0.9330$, and $P = 0.4821$

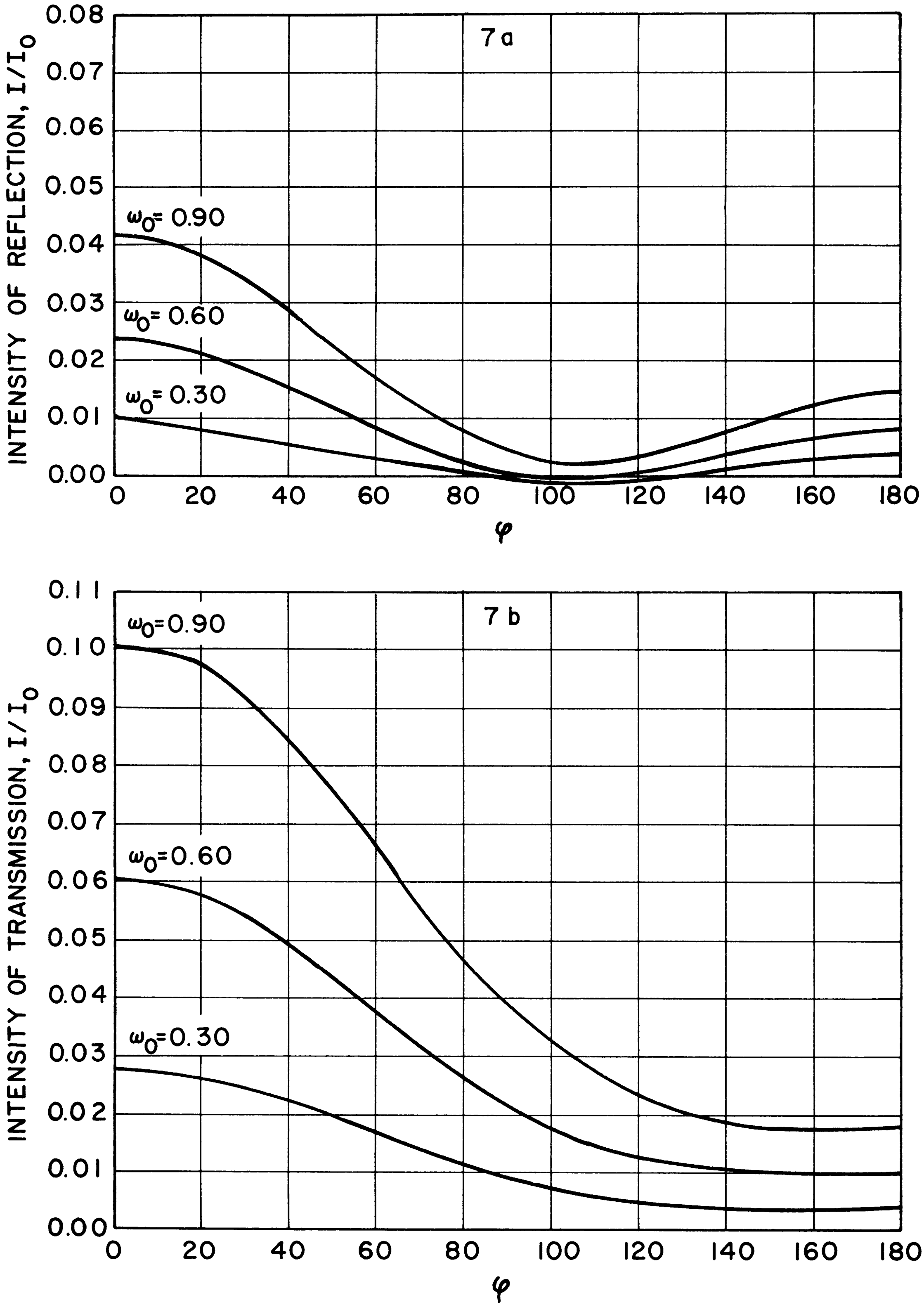
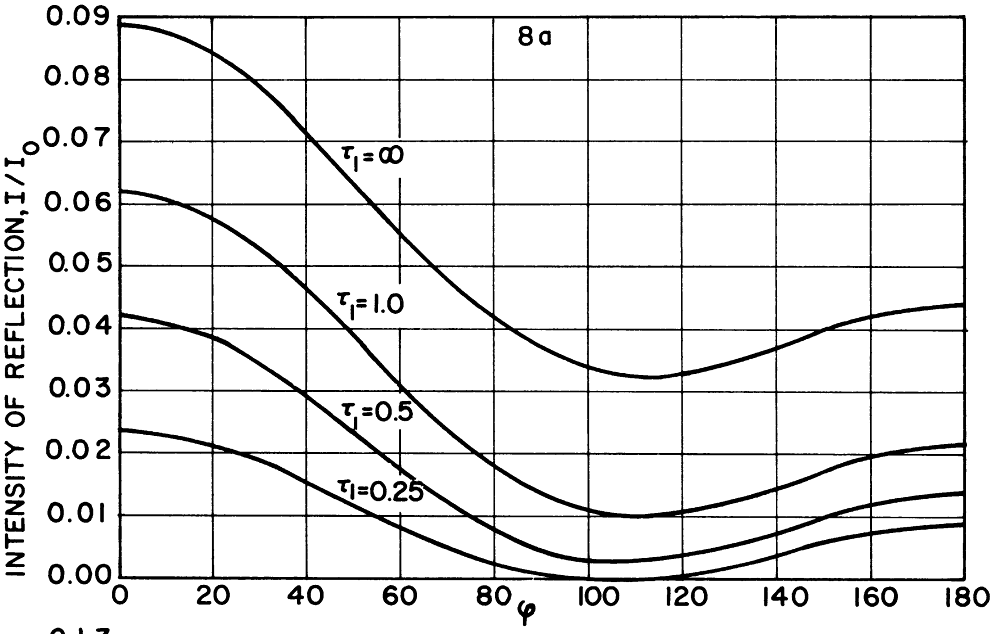


Figure I-7 Intensity of reflected and transmitted radiation as function of bearing angle, I/I_0 vs. φ , parameters of ω_0
$\mu_0 = 0.50$, $\mu = 0.76923$, $\tau_1 = 0.50$, $F = 0.9330$, $P = 0.4821$



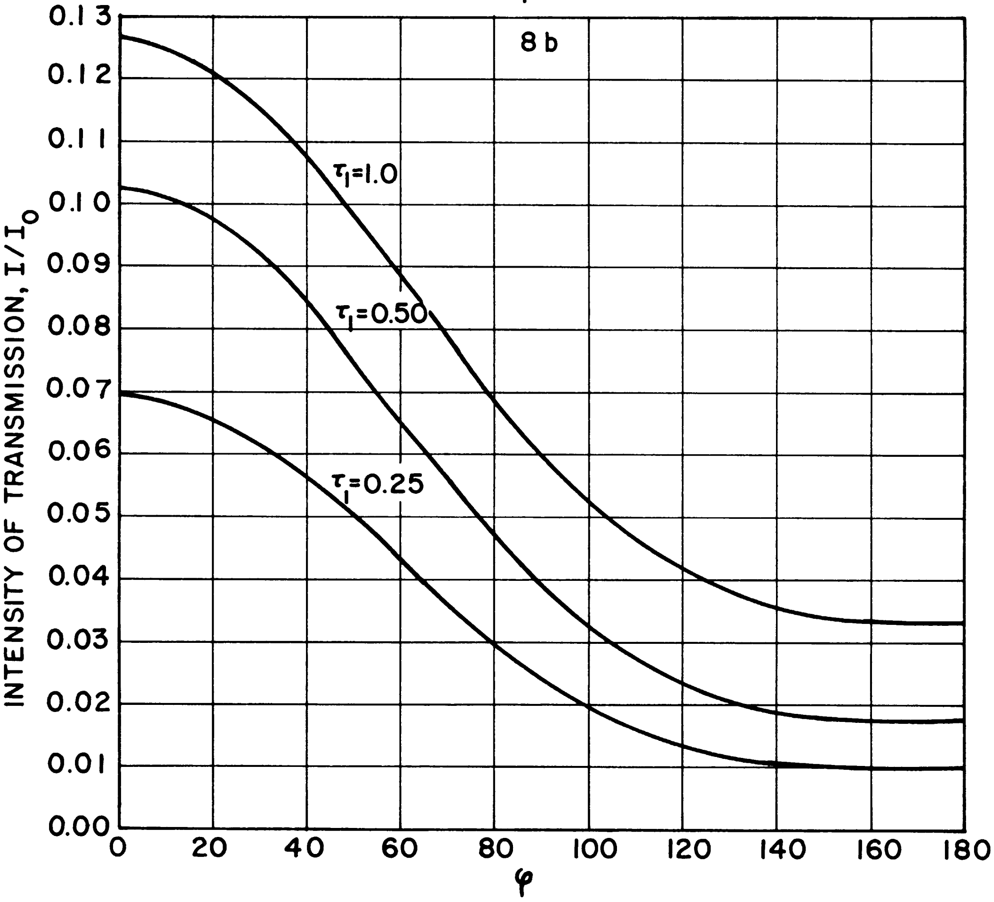


Figure I-8 Intensity of reflected and transmitted radiation as function of bearing angle, I/I_0 vs. φ , parameters of τ_1, $\mu_0 = 0.50$, $\mu = 0.76923$, $\omega_0 = 0.90$, $F = 0.9330$, and $P = 0.4821$

The effect of phase function is indicated in Figures I-9a and I-9b. Results are given for isotropic and Rayleigh scattering in which F is the same (0.50), but P is different (0.3333 and 0.40). An additional set of results was obtained for $\mu = 0.04691$. It can be seen that the two phase functions produce approximately the same angular distribution of intensity except for reflected radiation with a near grazing viewing angle ($\mu = 0.04691$) in which the "dumbell" characteristic of the Rayleigh phase function is accentuated.

Another illustration of the effect of phase function is provided in the polar plots of Figures I-10a and I-10b. In this case the three phase functions for F = 0.750 and P = 0.5504, 0.4821, and 0.3899 are used with the "standard" values of the other variables. It can be seen that the different values of P at the same value of F produce noticeably different angular distributions for the reflected and transmitted intensity.

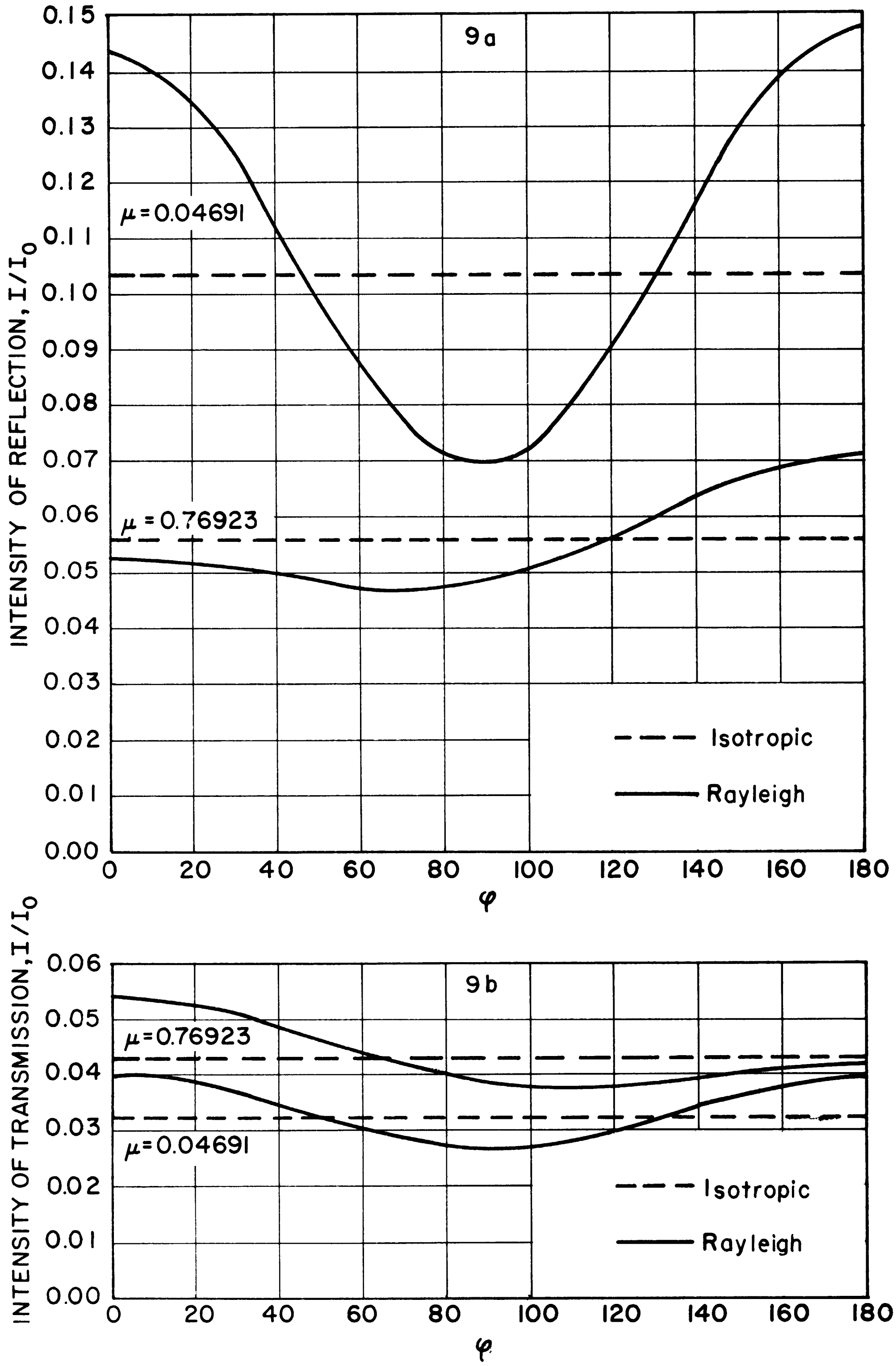


Figure I-9 Intensity of reflected and transmitted radiation as function of bearing angle, I/I_0 vs. φ, isotropic and Rayleigh scattering, $\mu = 0.04691$ and 0.76923, $\mu_0 = 0.50$, $\omega_0 = 0.90$, $\tau_1 = 1.00$

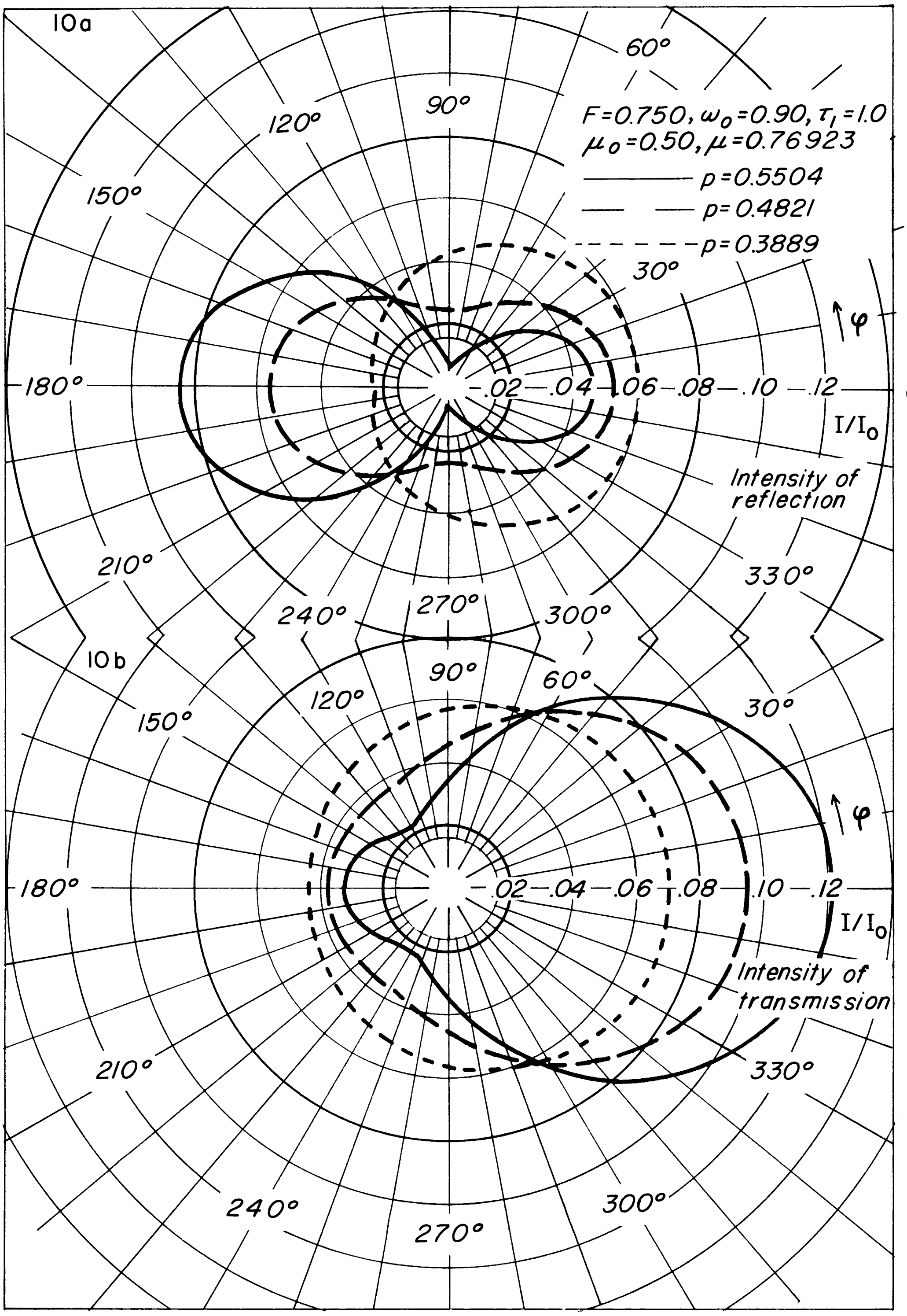


Figure I-10 Intensity of reflected and transmitted radiation as function of bearing angle, I/I_0 vs. φ , parameters of P, $\mu_0 = 0.50$, $\mu = 0.76923$, $\omega_0 = 0.90$, $\tau_1 = 1.00$, F = 0.750

4. The Integrated Reflection and Transmission

The integrated reflectance, the diffuse portion of the integrated transmission, and the total transmission were calculated for each set of conditions and the results are tabulated with the ψ_l^m and ϕ_l^m functions in Appendix B. In addition, these values are presented graphically in Figures I-11 to I-25.

The integrated reflectance for isotropic scattering is shown as a function of angle of incidence in Figures I-11, I-12a, and I-12b for ω_0 of 0.90, 0.60, and 0.30 with τ_1 as a parameter. Similar plots are presented for anisotropic scattering with F = 0.9330 and P = 0.4821 in Figures I-13, I-14a, and I-14b.

The total integrated transmission, including both diffuse and direct components, for isotropic scattering is plotted vs. μ_0 in Figure I-15 for $\omega_0 = 0.90$. In Figures I-16 the total transmission for isotropic scattering is presented for $\tau_1 = 0.05$ and 0.50 with parameters of ω_0. Similar plots are given for anisotropic scattering in Figures I-17 and I-18.

Values of the integrated reflectance and transmission were also obtained for Rayleigh scattering for which F is the same (0.50) as for isotropic scattering, but P is different (0.40 compared with 0.333). The results as shown in Figures I-11 and I-15 are almost indistinguishable from those for isotropic scattering. Integrated reflectances for

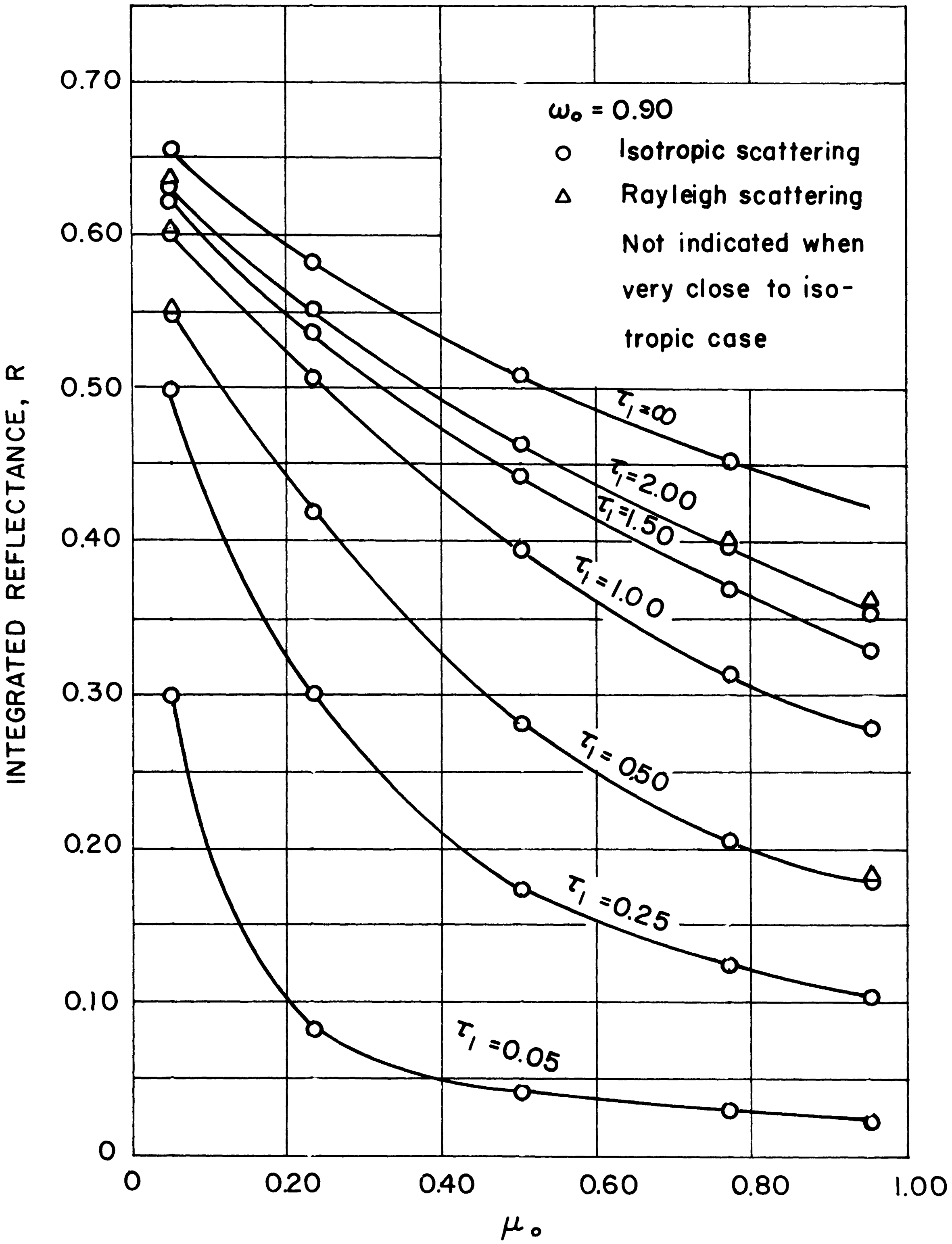


Figure I-11 Integrated reflectance as function of angle of incidence for isotropic and Rayleigh scattering, R vs. μ_o, parameters of τ_1, $\omega_o = 0.90$

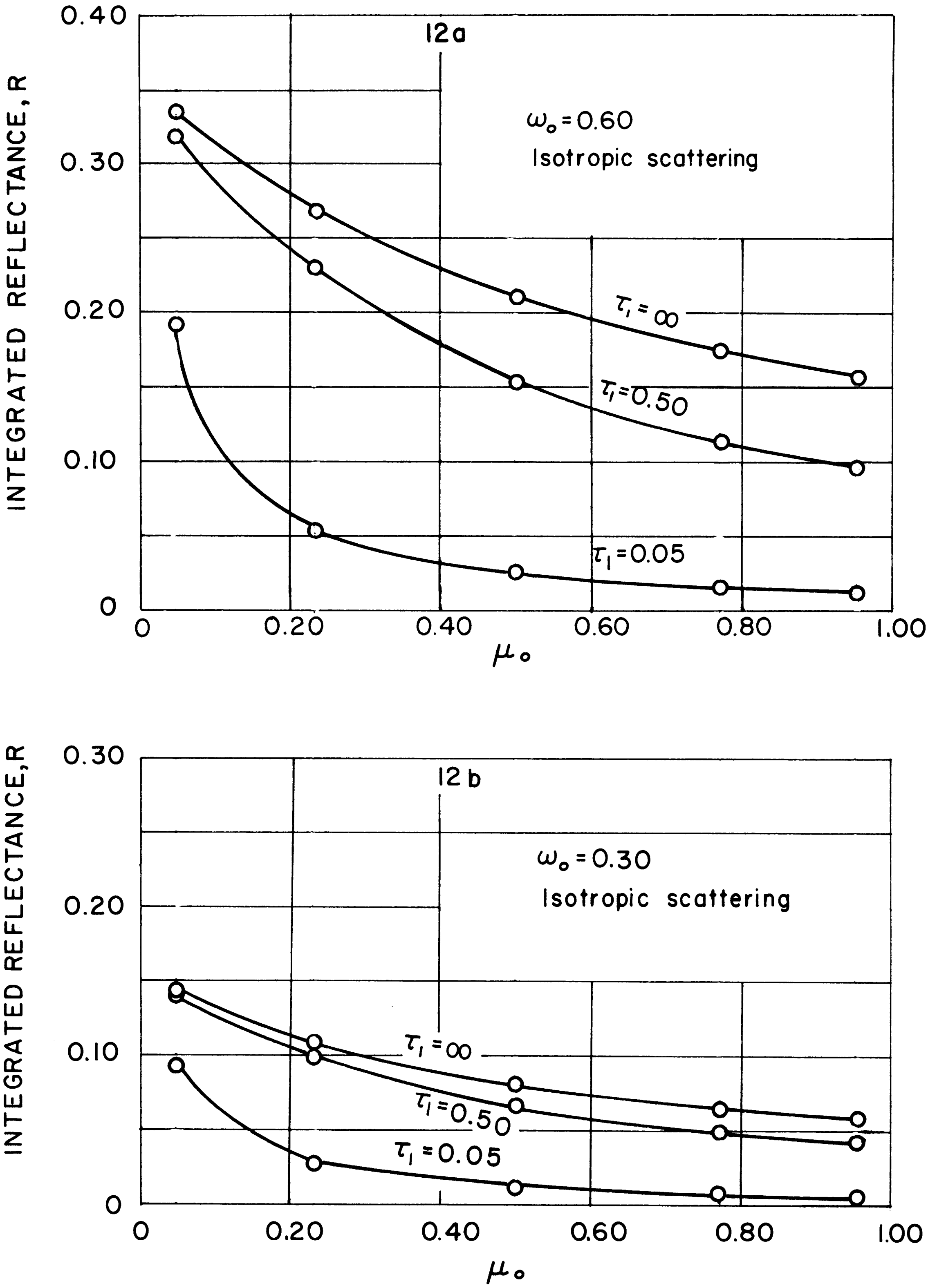


Figure I-12 Integrated reflectance as function of angle of incidence for isotropic scattering, R vs. μ_o, parameters of τ_1, ω_o = 0.60 and 0.30

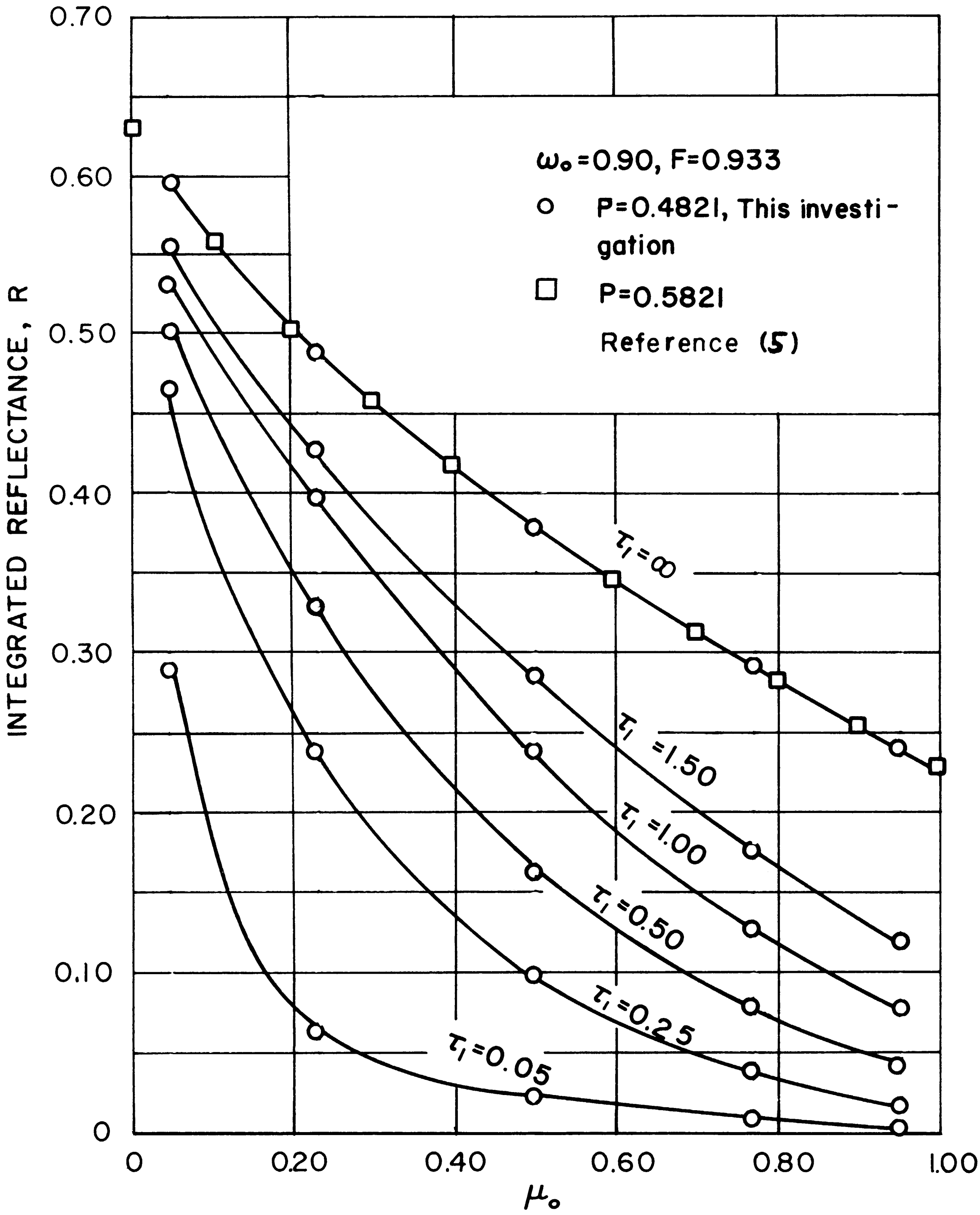


Figure I-13 Integrated reflectance as function of angle of incidence for anisotropic scattering, R vs. μ_o, parameters of τ_1, ω_o = 0.90, F = 0.9330, P = 0.4821

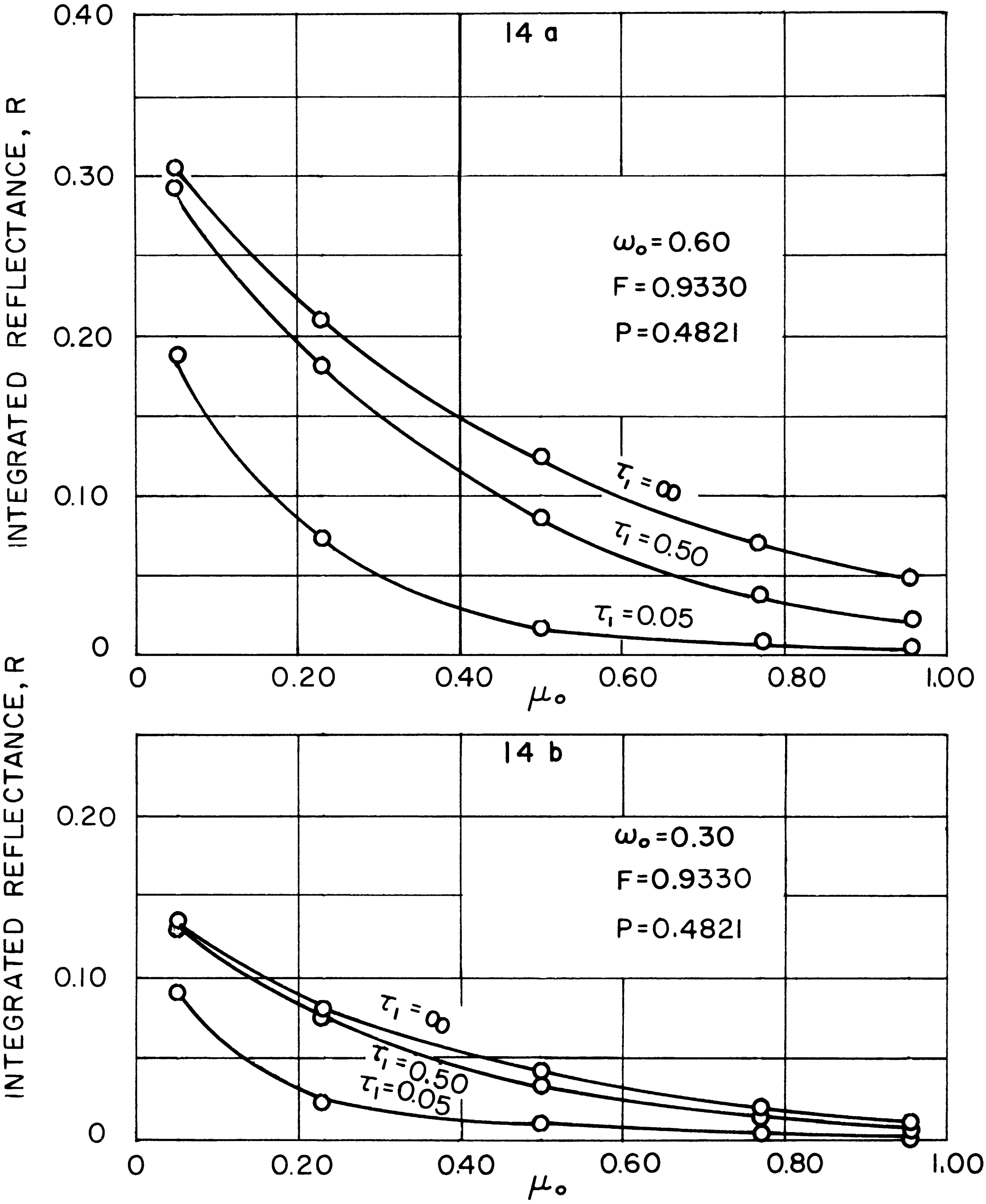


Figure I-14 Integrated reflectance as function of angle of incidence for anisotropic scattering, R vs. μ_o, parameters of τ_1, ω_o = 0.60 and 0.30, F = 0.9330, P = 0.4821

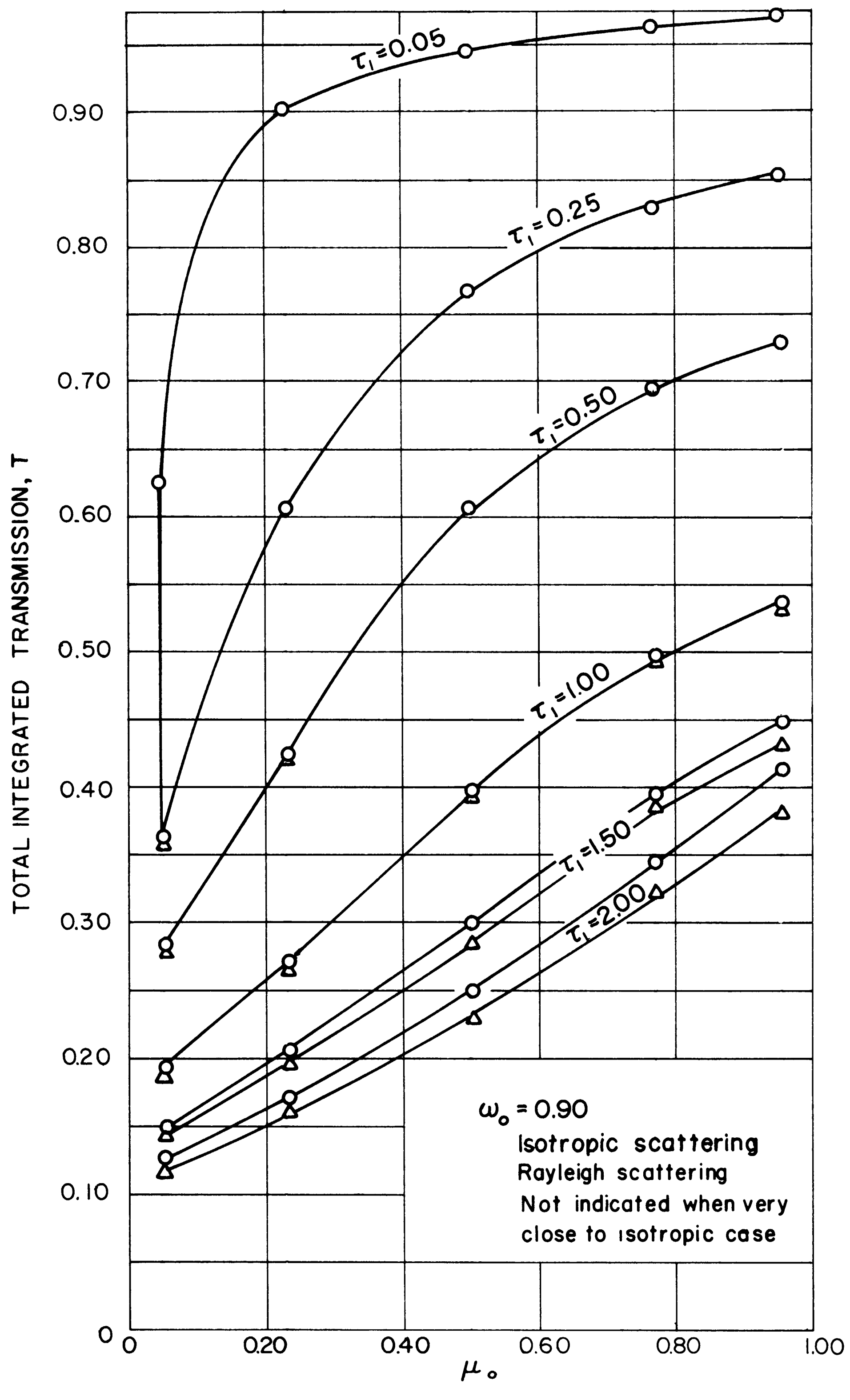


Figure I-15 Total integrated transmission as function of angle of incidence for isotropic and Rayleigh scattering, T vs. μ_o, parameters of τ_1, ω_o= 0.90

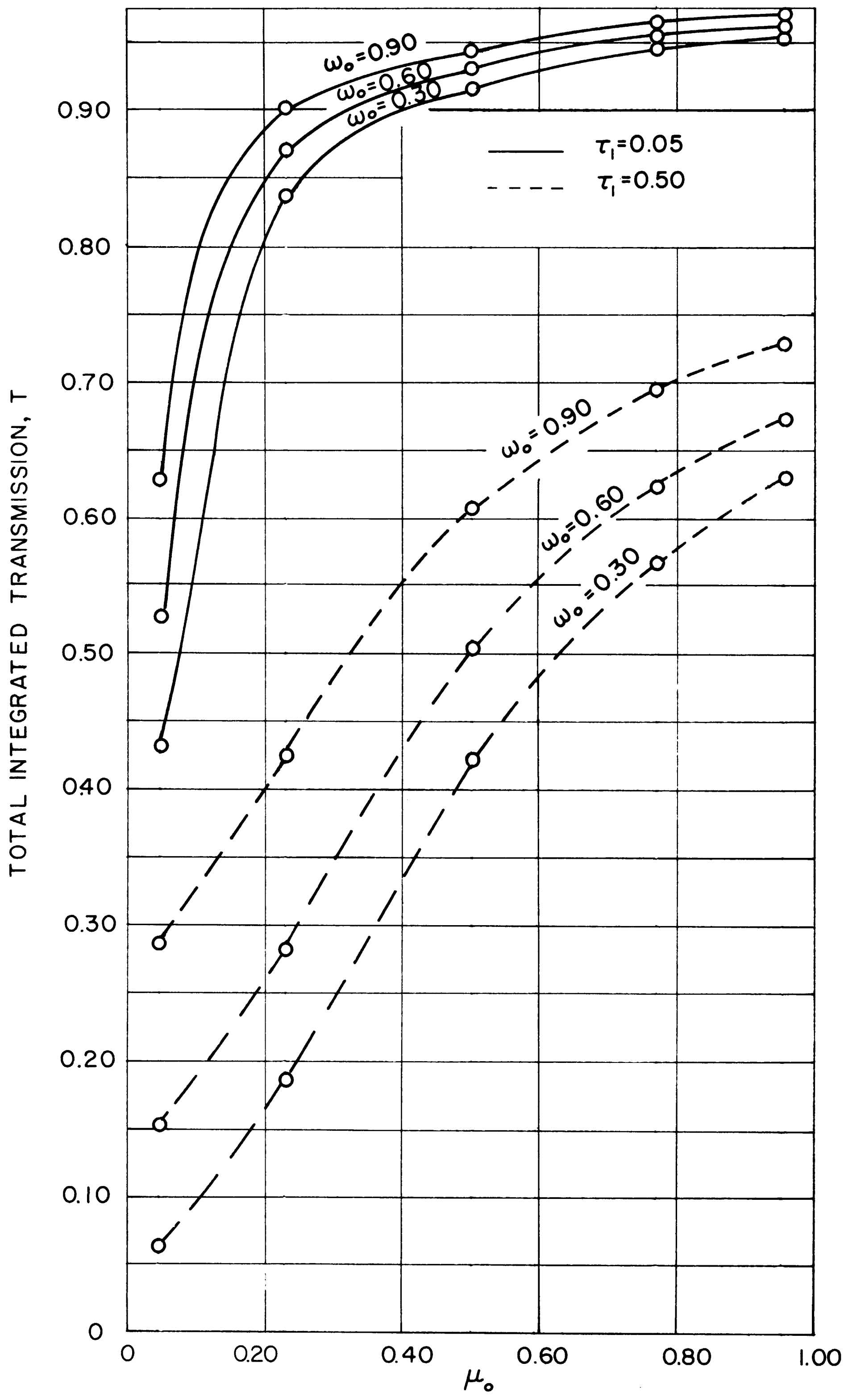


Figure I-16 Total integrated transmission as function of angle of incidence for isotropic scattering, T vs. μ_o, parameters of ω_o, τ_1 = 0.05 and 0.50

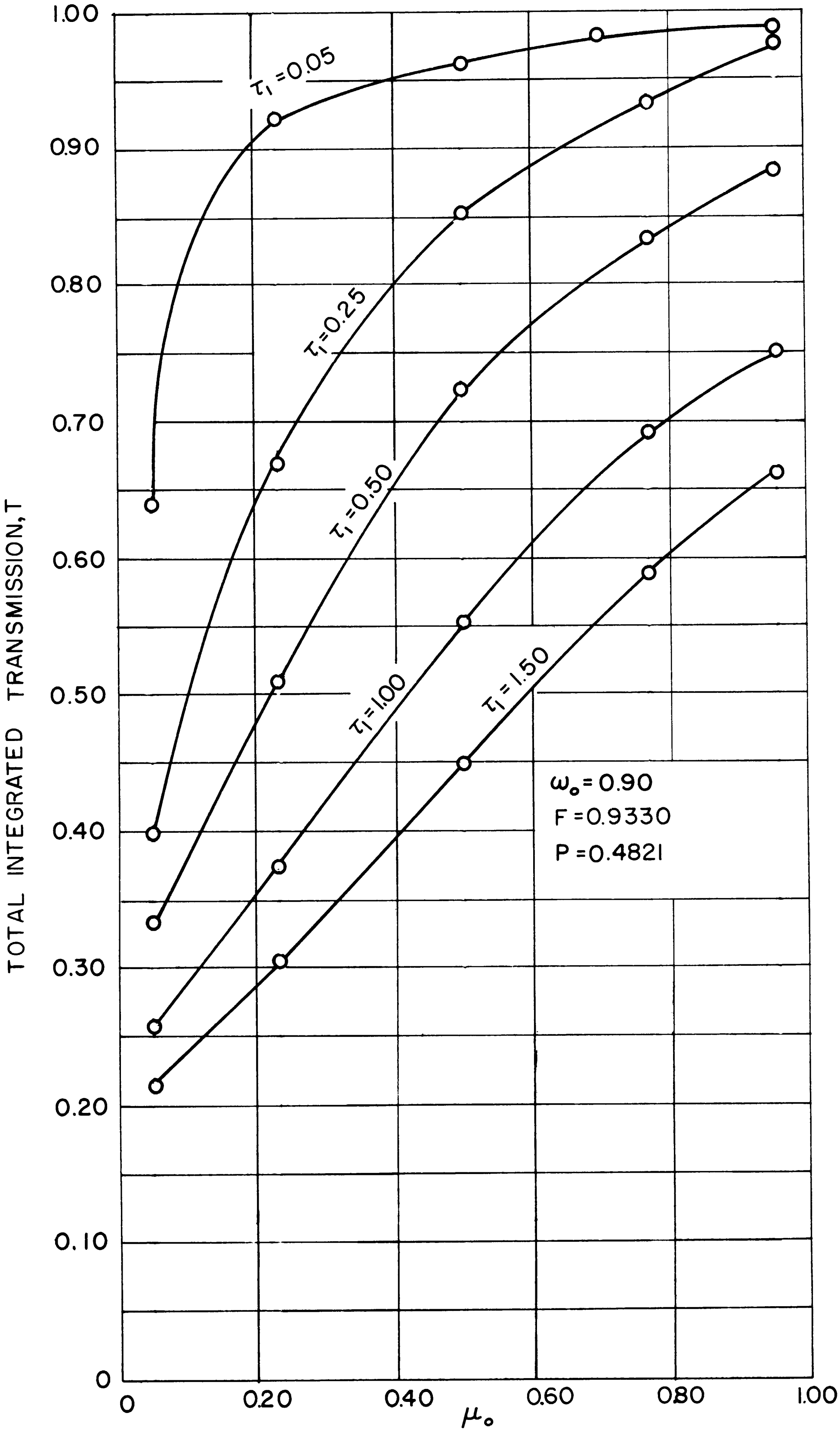


Figure I-17 Total integrated transmission as function of angle of incidence for anisotropic scattering, T vs. μ_o, parameters of τ_1, ω_o = 0.90, F = 0.9330, P = 0.4821

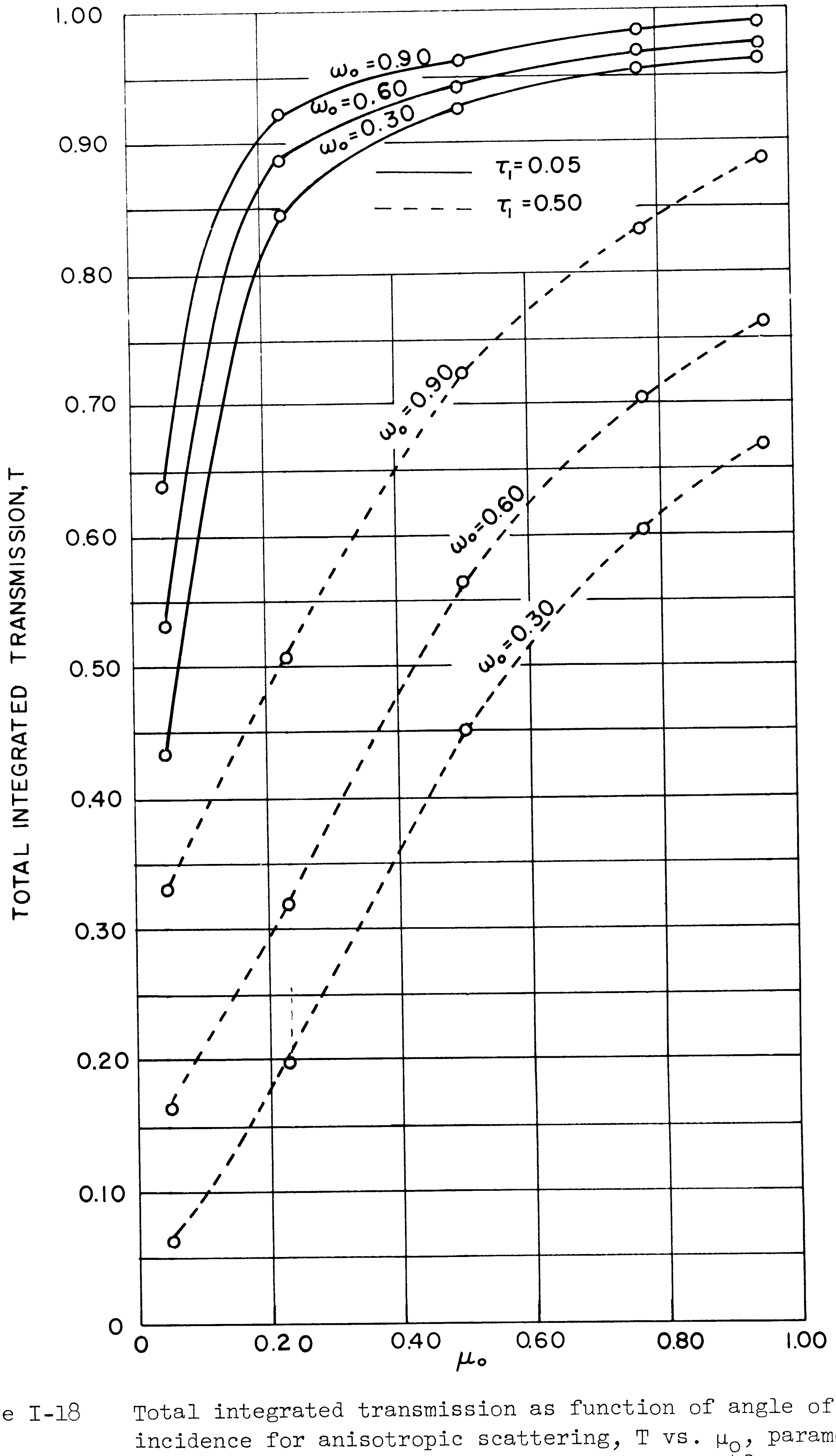


Figure I-18 Total integrated transmission as function of angle of incidence for anisotropic scattering, T vs. μ_0, parameters of ω_o, τ_1 = 0.05 and 0.50, F = 0.9330, P = 0.4821

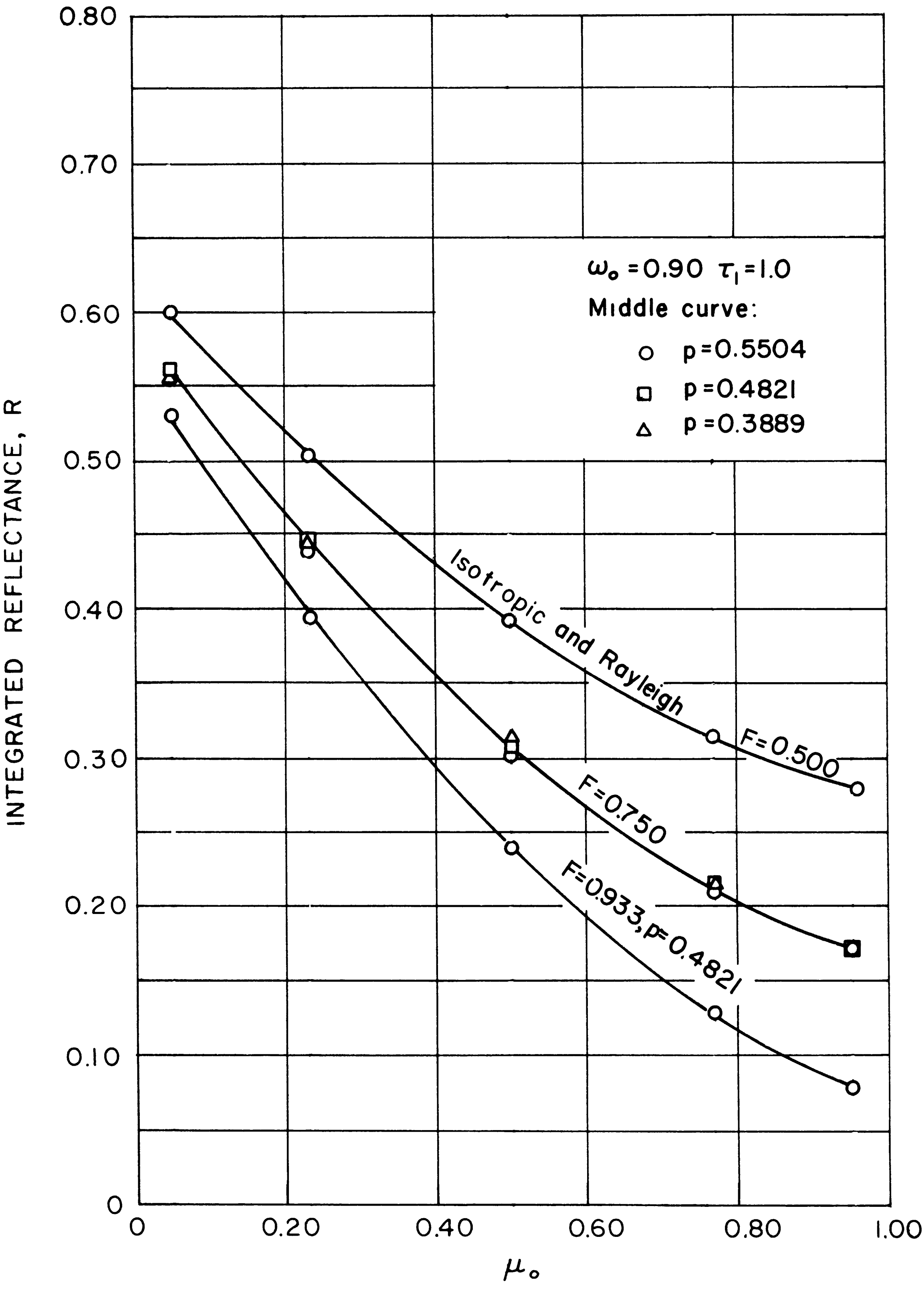


Figure I-19 Integrated reflectance as function of angle of incidence, R vs. μ_0, parameters of F, $\omega_0 = 0.90$ and $\tau_1 = 1.00$

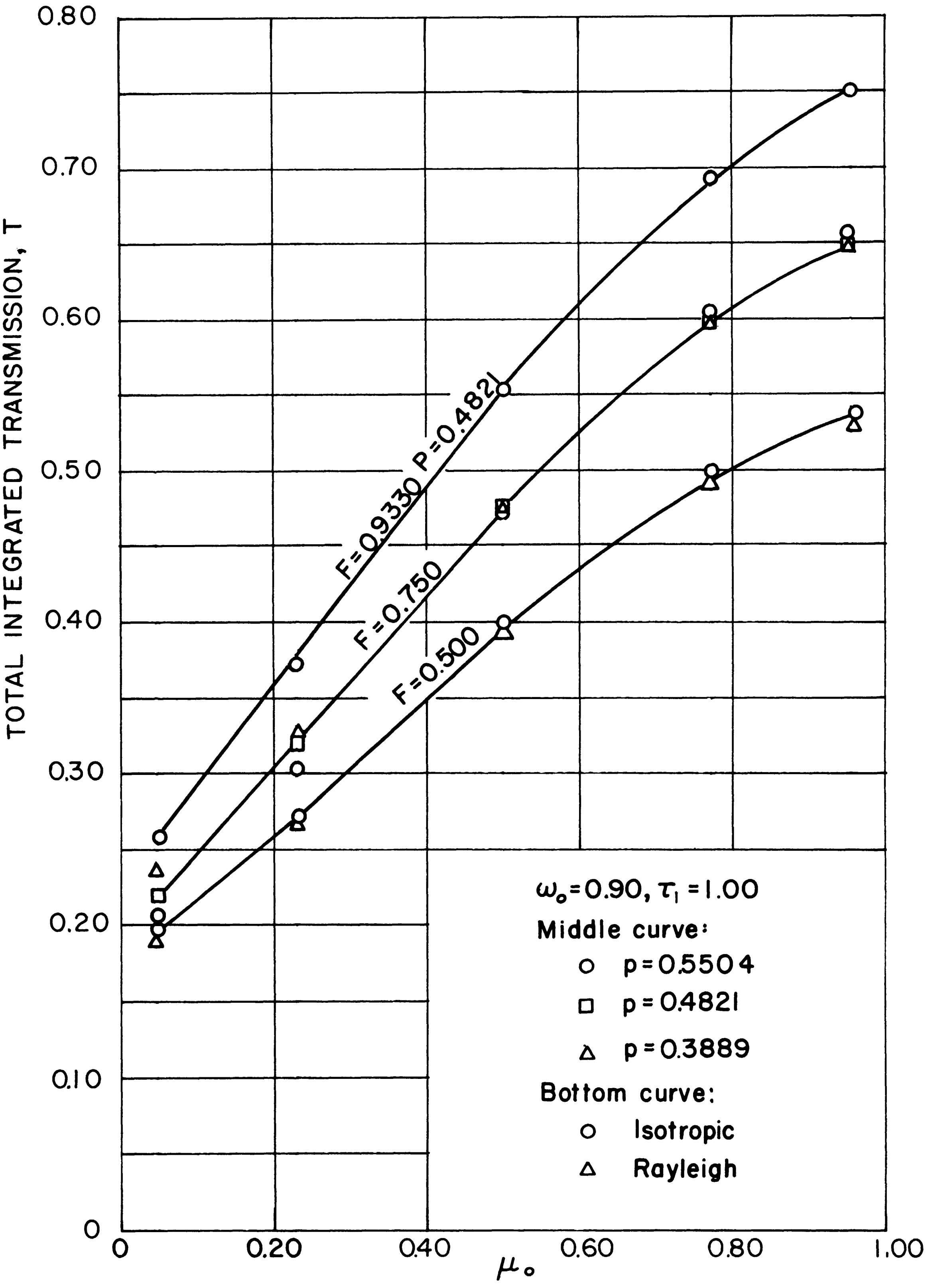


Figure I-20 Total integrated transmission as function of angle of incidence, T vs. μ_0, parameters of F, ω_0= 0.90 and τ_1 = 1.00

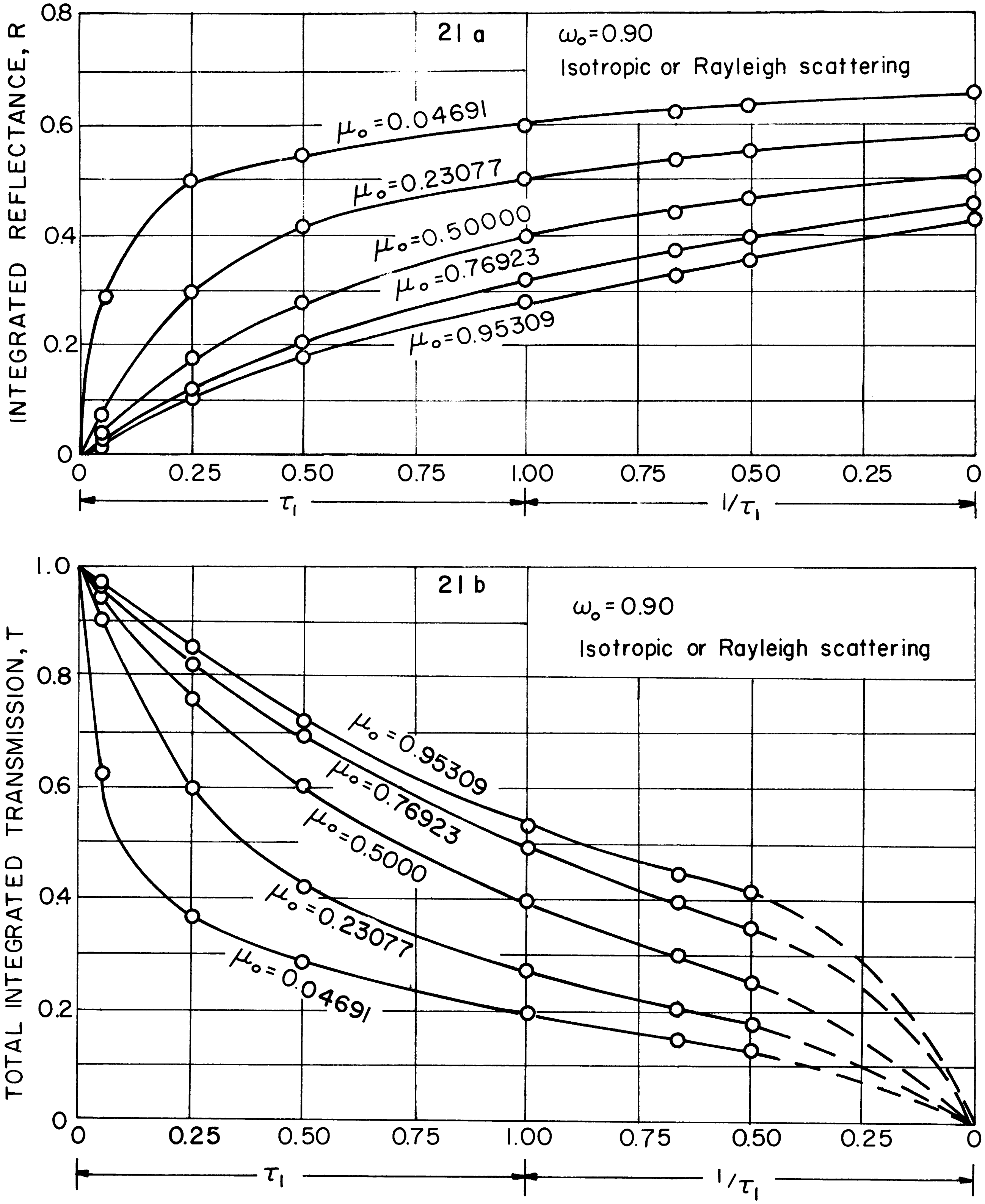


Figure I-21 Integrated reflectance and total integrated transmission as function of optical thickness of dispersion, R and T vs. τ_1, parameters of μ_o isotropic or Rayleigh scattering F = 0.50

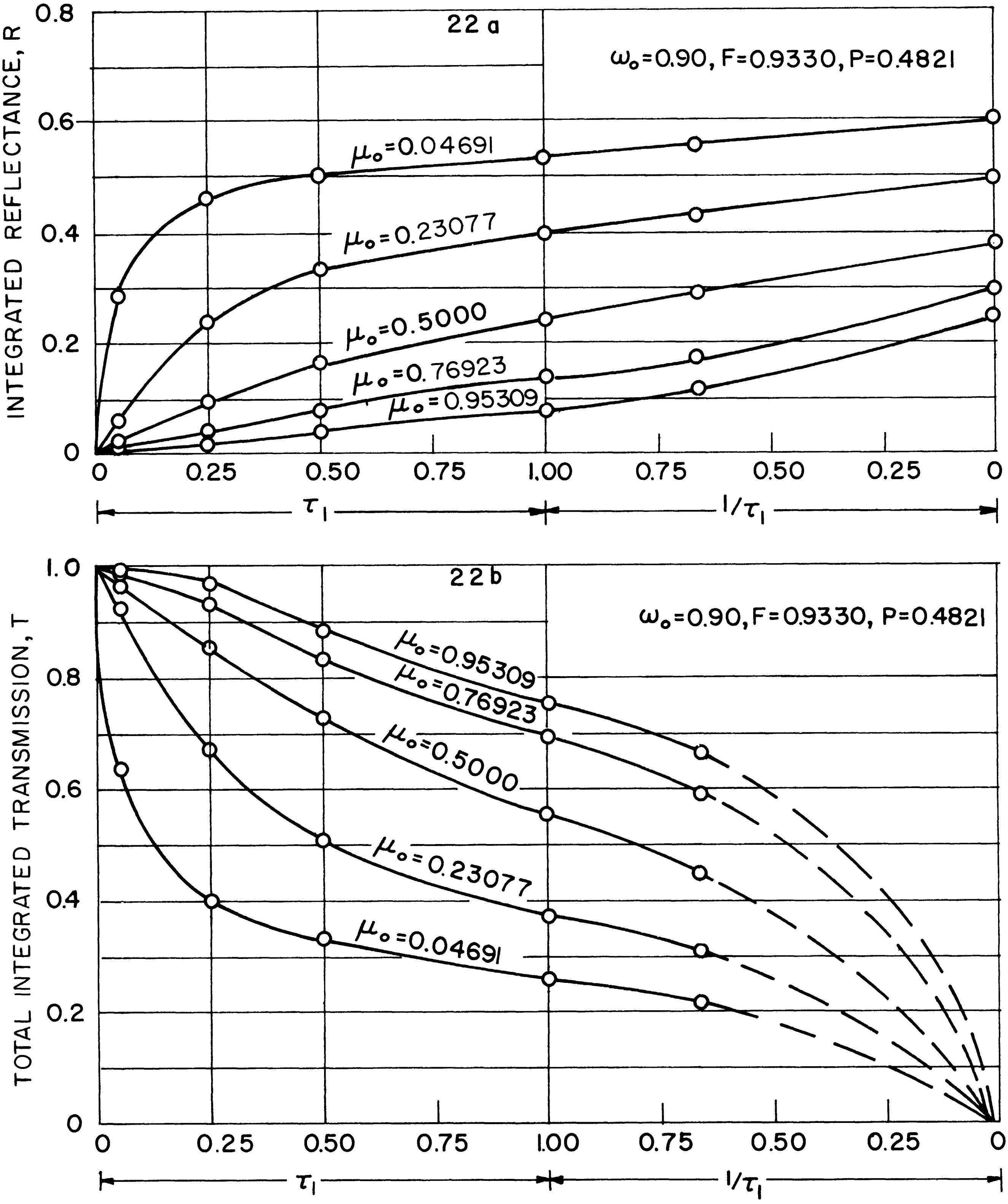


Figure I-22 Integrated reflectance and total integrated transmission as function of optical thickness of dispersion, R and T vs. τ_1, parameters of μ_o, anisotropic scattering F = 0.9330 and P = 0.4821

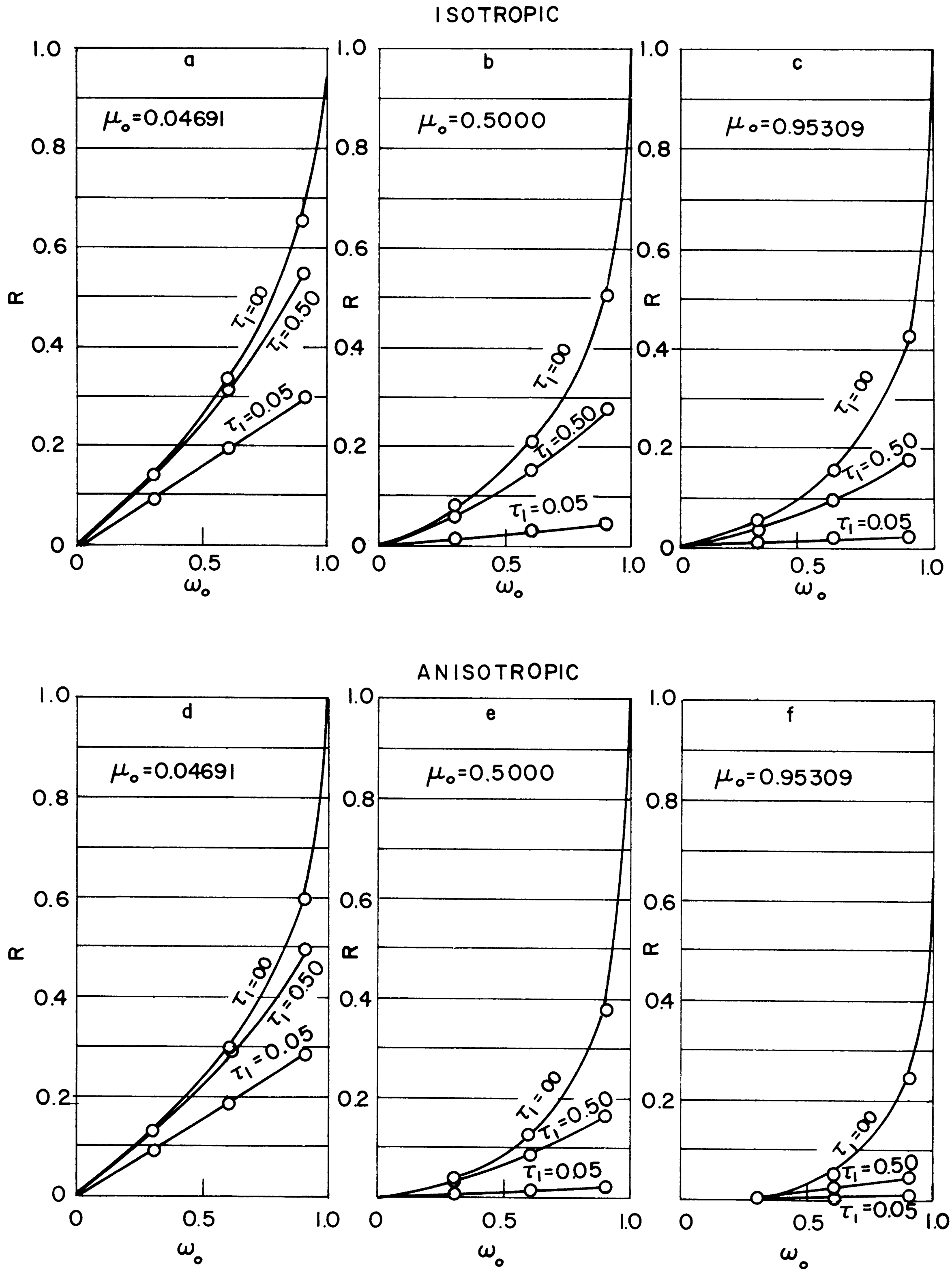


Figure I-23 Integrated reflectance as function of albedo for single scattering, R vs. ω_o, parameters of F, μ_o, and τ_1.

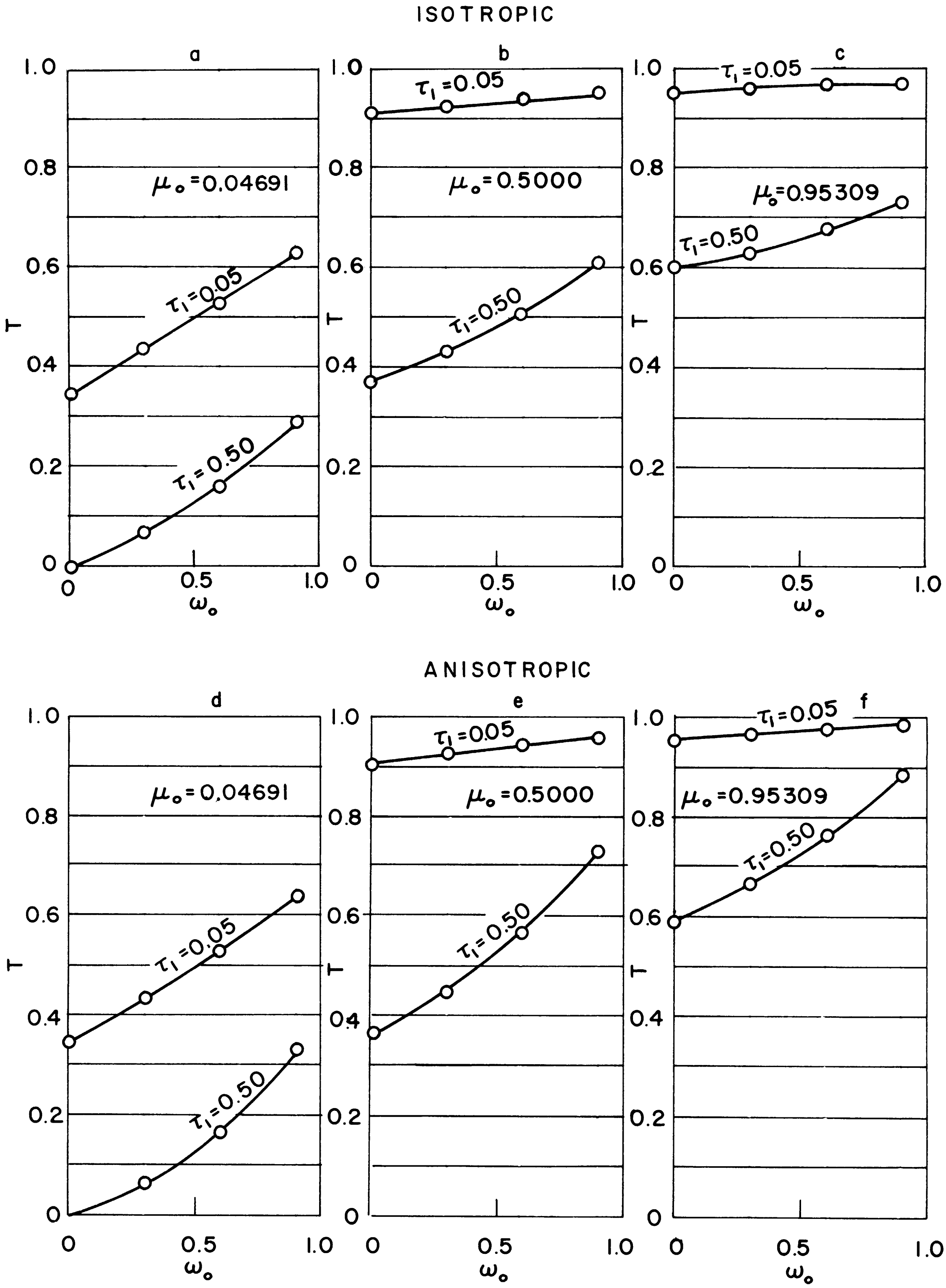
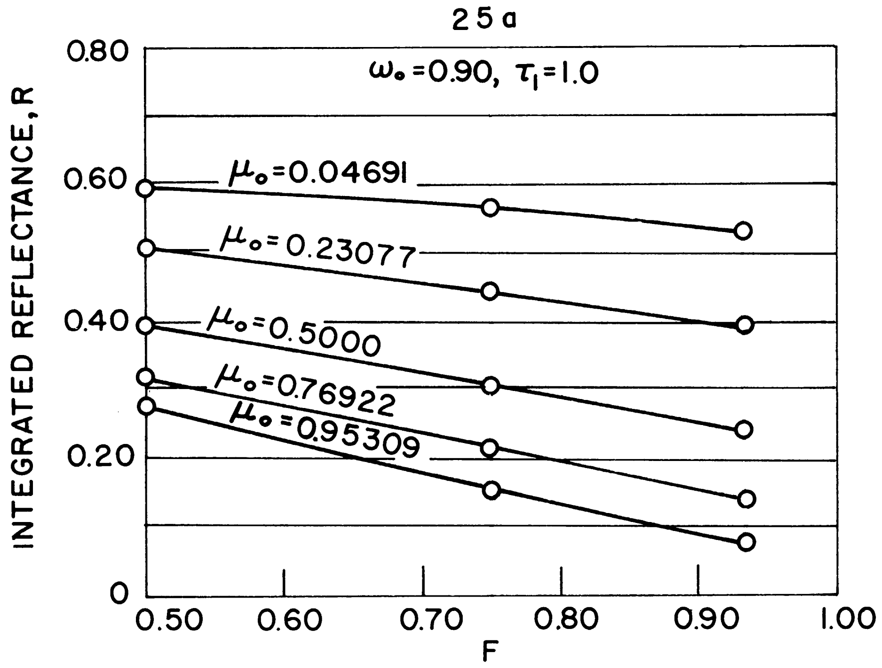


Figure I-24 Total integrated transmission as function of albedo for single scattering, T vs. ω_o, parameters of F, μ_o, τ_1.



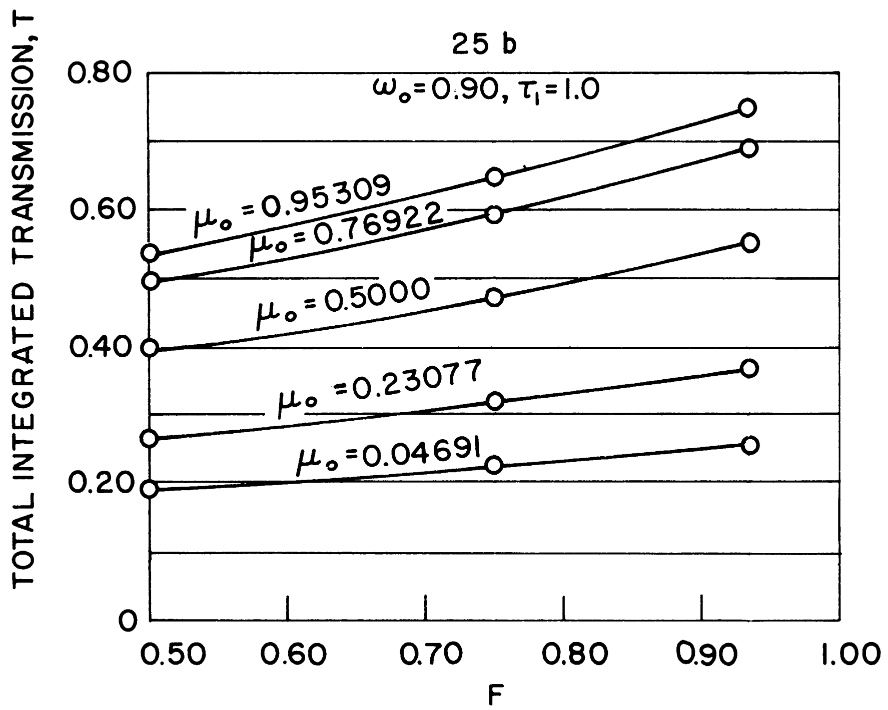


Figure I-25 Integrated reflectance and total integrated transmission as function of phase function, R and T vs. F, parameters of μ_0, $\omega_0 = 0.90$, $\tau_1 = 1.00$

the anisotropic phase function of reference (5) for $\tau_1 = \infty$ in which F was 0.9336 fall on the same curve as the results for this investigation in which F is almost the same (0.9330), but P is different (0.4821 compared with 0.5821).

The effect of anisotropy is best indicated in Figures I-19 and I-20 where the integrated reflectance and the total transmission are plotted versus angle of incidence with $\tau_1 = 1.00$ for each of the five phase functions considered in this investigation. Again, it is noted that the results for the three different phase functions with F = 0.750, but with different values of P are almost identical.

The integrated reflectance and total transmission are cross-plotted versus τ_1 with μ_o as a parameter for isotropic scattering in Figures I-21a and I-21b and for anisotropic scattering in Figures I-22a and I-22b. This should provide a satisfactory method of interpolating for different values of τ_1. As is expected, the dependence on τ_1 is almost linear for small values of τ_1.

The effect of ω_o on the integrated reflectance is indicated in Figures I-23a and I-23f for isotropic and anisotropic scattering. The effect of ω_o on the total transmission is shown in the same manner in Figures I-24a and I-24f. The dependence on ω_o is almost linear for small values of ω_o. For small values of τ_1 it is linear over the whole range of ω_o.

The effect of phase function—in particular, the effect of F—is illustrated by cross plots of R and T versus F in Figures I-25a and I-25b where μ_o is a parameter. This indicates the approximate error of neglecting anisotropy and provides a means of estimating the reflection and transmission for an arbitrary dispersion on the basis of the value of F for the phase function. As can be seen from Figures I-19 and I-20, the effect of P is not large. In fact, except for the integrated transmission at the two largest angles of incidence, the complete variation of reflectance and transmission with the range of P considered in this study is indicated by the size of the circles used to plot the points in Figures I-25a and I-25b.

5. Discussion of Numerical Technique

It is very difficult to estimate the error involved in the numerical solution of the integral equations. Probably the best indication is comparison with previous solutions and with other methods of solution. The following solutions are available for comparison.

1. Chandrasekhar's H functions[1] for $\omega_0 = 0.90$, 0.60, and 0.30. The $H(\mu)$ functions are the same as the $\psi_0^0(\mu)$ functions of this report for $\tau_1 = \infty$ and isotropic scattering. The comparison is given in Table I-4. The values generally agree to within 0.1%.

2. Chandrasekhar's X and Y functions [8] for $\omega_0 = 0.90$ and $\tau_1 = 0.05$, 0.25, 0.50, and 1.00. The $X(\mu)$ and $Y(\mu)$ are the same as the $\psi_0^0(\mu)$ and $\phi_0^0(\mu)$ functions of this report for isotropic scattering. The comparison is given in Table I-5. The values generally agree to within a few percent.

3. The $\psi_0^0(\mu)$, $\psi_1^0(\mu)$, $\psi_2^0(\mu)$, $\psi_1^1(\mu)$, $\psi_2^1(\mu)$ and $\psi_2^2(\mu)$ functions calculated by Churchill, et.al.[5] for $a_0 = 1.00000$, $a_1 = 1.73458$, $a_2 = 2.24153$, $\omega_0 = 0.90$ and $\tau_1 = \infty$ with iteration using a 20-point Simpson's rule technique. The comparison is given in Table I-6. The values generally agree to within 1%.

4. The integrated reflectance calculated for Rayleigh scattering, $\omega_0 = 0.90$ and $\tau_1 = \infty$ using the method described in Appendix A-2. The comparison is given in Table I-7. The agreement is very good.

TABLE I-4 Comparison of $\psi_0^0(\mu)$ Functions for Isotropic Scattering and with Chandrasekhar's H Functions.

ω_0	0.90		0.60		0.30	
μ	$H(\mu)$	$\phi_0^0(\mu)$	$H(\mu)$	$\phi_0^0(\mu)$	$H(\mu)$	$\phi_0^0(\mu)$
0.04691	1.0937	1.0936	1.0520	1.0520	1.0233	1.0233
0.23077	1.3233	1.3236	1.1584	1.1587	1.0661	1.0663
0.50000	1.5560	1.5554	1.2485	1.2456	1.0976	1.0975
0.76923	1.7299	1.7289	1.3011	1.3009	1.1160	1.1159
0.95309	1.8273	1.8259	1.3291	1.3288	1.1249	1.1248

Table I-5 Comparison of the $\psi_0^0(\mu)$ and $\phi_0^0(\mu)$ Functions for isotropic Scattering and $\omega_0 = 0.90$ with Chandrasekhar's X and Y Functions.

$T_1 = 0.05$

μ	X	ψ_0^0	Y	ϕ_0^0
0.04691	1.0560	1.0546	0.3914	0.3838
0.23077	1.0778	1.0726	0.8805	0.9049
0.50000	1.0814	1.0783	0.9852	0.9824
0.76923	1.0826	1.0800	1.0190	1.0138
0.95309	1.0831	1.0806	1.0314	1.0252

$T_1 = 0.25$

μ	X	ψ_0^0	Y	ϕ_0^0
0.04691	1.0810	1.0797	0.0441	0.0467
0.23077	1.1967	1.1993	0.5017	0.4990
0.50000	1.2440	1.2413	0.8285	0.8317
0.76923	1.2610	1.2611	0.9672	0.9501
0.95309	1.2668	1.2694	1.0228	1.0001

$T_1 = 0.50$

μ	X	ψ_0^0	Y	ϕ_0^0
0.04691	1.0861	1.0842	0.0269	0.0275
0.23077	1.2553	1.2530	0.2769	0.2636
0.50000	0.3528	1.3465	0.6506	0.6497
0.76923	1.3948	1.3896	0.8636	0.8462
0.95309	1.4122	1.4089	0.9583	0.9249

$T_1 = 1.00$

μ	X	ψ_0^0	Y	ϕ_0^0
0.04691	1.0902	1.0888	0.0171	0.0165
0.23077	1.2933	1.2915	0.1888	0.1231
0.50000	1.4574	1.4541	0.4120	0.4012
0.76923	1.5472	1.5427	0.6646	0.6476
0.95309	1.5880	1.5852	0.8000	0.7553

Table I-6 Comparison of the ψ and ϕ Functions of this Report with Similar Functions of Reference (4) Calculated Using a 20-point Simpson's Rule Technique for $a_1 = 1.73458$, $a_2 = 2.24153$, $\omega_o = 0.90$ and $\tau_1 = \infty$.

μ	$\psi_o^o(\mu)$ Ref.	$\psi_o^o(\mu)$ This Rept.	ψ_1^o Ref.	ψ_1^o This Rept.
0.04691	1.1186	1.1185	0.0187	0.0185
0.23077	1.3538	1.3550	0.1182	0.1179
0.50000	1.4865	1.4868	0.3121	0.3121
0.76923	1.4693	1.4699	0.5413	0.5412
0.95309	1.3946	1.3948	0.7094	0.7094

μ	ψ_2^o Ref.	ψ_2^o This Rept.	ψ_1^1 Ref.	ψ_1^1 This Rept.
0.04691	0.5368	0.5370	1.0515	1.0516
0.23077	0.5110	0.5124	1.0888	1.0896
0.50000	0.2078	0.2079	0.9564	9.0561
0.76923	0.3782	0.3771	0.6578	0.6583
0.95309	0.9347	0.9348	0.2822	0.2905

μ	ψ_2^1 Ref.	ψ_2^1 This Rept.	ψ_2^2 Ref.	ψ_2^2 This Rept.
0.04691	0.1084	0.1080	3.1399	3.1376
0.23077	0.5908	0.5911	3.1955	3.1955
0.50000	1.2591	1.2596	2.6409	2.6406
0.76923	1.5187	1.5179	1.4697	1.4695
0.95309	0.8840	0.9173	0.3330	0.3331

Table I-7 Comparison of Integrated Reflectance for Rayleigh Scattering, $\tau_1 = \infty$ and $\omega_o = 0.90$ with Values Calculated using Method Described in Appendix A-2.

μ_o	Appendix	This Report
0.04691	0.660	0.656
0.23077	0.581	0.580
0.50000	0.507	0.507
0.76923	0.453	0.453
0.95309	0.424	0.424

It must be noted that in all the preceeding comparisons (except for the case of $\mu = 0.50$) the results from other investigators were linearly interpolated to get values for comparison with the results at the Gaussian quadrature points of this report. From the comparisons, however, it is estimated that the functions are generally accurate to within ± 0.02 which in most cases is an error of less than two to three percent. The intensities of reflected and transmitted radiation as well as the integrated reflectance and transmittance are of the same general accuracy as the functions. This is evident from equations I-15, I-19, I-29 and I-32.

The use of Gaussian quadrature seems preferable to Simpson's Rule or to the trapezoidal rule in that five point quadrature afford the same accuracy as the use of ten equally spaced ordinates. In addition, a better approximation may be had by adding only one additional point where with Simpson's rule, in order to obtain a better approximation, it is necessary to double the number of points. The disadvantage of the use of quadrature is that results are obtained at odd values of μ. On the other hand, if in using the results they are to be integrated again, this may be an advantage, since quadrature can be used to perform the integration.

The time in seconds required for one iteration on the IBM 704 is

$$\text{Time} = 0.025\ (\text{no. Points})^2 x (\text{no. equations})^2 x (\text{no. iterations}) \quad \text{I-56}$$

For example, using five-point quadrature; for m=0, N=2; performing four iterations

$$\text{Time} = 0.025\,(5)^2(3)^2(4) \qquad \text{I-57}$$

$$= 22.5 \text{ seconds}$$

Considering the cost of 704 time at \$5.00 per minute

$$\text{Cost} = \$2.08 \times 10^{-3}\,(\text{no. points})^2 \times (\text{no. equations})^2 \times (\text{no. iterations}) \qquad \text{I-58}$$

To scale the cost to some other machine, simply multiply the factor \$2.08 x 10^{-3} by (cost per operation on new machine)/(cost per operation on the IBM 704).

The 704 performs 40,000 operations on the average per second so its cost per operation is

$$\$5/(60 \times 40{,}000) = \$2.08 \times 10^{-6} \text{ per operation}$$

It is estimated that the \$2.08 x 10^{-3} factor in I-58 could be reduced by a factor of five through the use of efficient hand-programming.

PART II EVALUATION OF APPROXIMATE MODELS

Exact solutions in closed form are not attainable by analytical methods for problems of multiple scattering. Exact solutions are attainable by numerical methods, as in Part I, but the required calculations are lengthy and expensive, and the results are for specific cases. Simple, approximate solutions therefore have considerable value if reliable. In this section the reflectance and transmission obtained from various approximate models are compared with the "exact" values obtained in Part I.

A. Approximate Models

1. Two-flux Model(11,12)

The two-flux formulation of the slab problem with an obliquely incident plane-parallel source is

$$\mu_0 \frac{dI_1}{dz} = -(1-\omega_0 F) I_1 + \omega_0 B I_2 \qquad \text{II-1a}$$

$$\mu_0 \frac{dI_2}{dz} = (1-\omega_0 F) I_2 - \omega_0 B I_1 \qquad \text{II-1b}$$

Where I_1 is the integrated radiant flux in the forward direction and I_2 the integrated radiant flux in the backward direction, and F and B are the forward and the backward components respectively of the phase function for single scattering.

Equations II-1a and II-1b can be solved with the boundary conditions $I_1 = 1$ at $Z = o$, and $I_2 = o$ at $Z = \tau_1$ to give

$$R = \frac{(I_2)_{z=0}\mu_o}{(I_1)_{z=0}\mu_o} = \frac{(1 - \omega_o F - p)(1 - e^{-p\tau_1/\mu_o})}{\omega_o B\left(1 - \frac{1-\omega_o F - p}{1-\omega_o F + p} e^{-2p\tau_1/\mu_o}\right)} \qquad \text{II-2}$$

and

$$T = \frac{(I_2)_{z=\tau_1}\mu_o}{(I_1)_{z=0}\mu_o} = \frac{2p e^{-p\tau_1/\mu_o}}{1 - \omega_o F + p - (1 - \omega_o F - p)e^{-2p\tau_1/\mu_o}} \qquad \text{II-3}$$

where

$$p = \sqrt{(1 - \omega_o F)^2 - (\omega_o B)^2} \qquad \text{II-4}$$

2. Six-flux Model (11,12)

The six-flux formulation of the slab problem with an obliquely incident parallel-plane source is

$$\frac{dI_1}{dz} = -C_1 \sec\theta_o I_1 + C_2 \sec\theta_o I_2 + C_3 \sec\theta_o (I_3 + I_4) \qquad \text{II-5a}$$

$$-\frac{dI_2}{dz} = -C_1 \sec\theta_o I_2 + C_2 \sec\theta_o I_1 + C_3 \sec\theta_o (I_3 + I_4) \qquad \text{II-5b}$$

$$\frac{dI_3}{dz} = -C_1 \csc\theta_o I_3 + C_2 \csc\theta_o I_4 + C_3 \csc\theta_o (I_1 + I_2) \qquad \text{II-5c}$$

$$-\frac{dI_4}{dz} = -C_1 \csc\theta_o I_4 + C_2 \csc\theta_o I_3 + C_3 \csc\theta_o (I_1 + I_2) \qquad \text{II-5d}$$

$$I_5 = I_6 = \frac{\omega_o X}{1 - \omega_o F_6 - \omega_o B_6}(I_1 + I_2 + I_3 + I_4) \qquad \text{II-6}$$

Where I_1, I_2, I_3, I_4, I_5 and I_6 are the components of the radiant flux in the forward, backward and sidewise directions, and

$$C_1 = 1 - \omega_o F_6 - \frac{2\omega_o X^2}{1 - \omega_o F_6 - \omega_o B_6} \tag{II-7a}$$

$$C_2 = \omega_o B_6 + \frac{2\omega_o X^2}{1 - \omega_o F_6 - \omega_o B_6} \tag{II-7b}$$

$$C_3 = \omega_o X + \frac{2\omega_o X^2}{1 - \omega_o F_6 - \omega_o B_6} \tag{II-7c}$$

where F_6, B_6 and X are the six-flux representations of forward, backward and sidewise scattering components. $F_6 = (F)(P)$, $B_6 = (B)(P)$ $X = (1 - F_6 - B_6)/4$.

The solution for Equations II-5a to II-6 with the boundary conditions $I_1 = 1$ and $I_3 = o$ at $Z = o$, and $I_2 = I_4 = o$ at $Z = \tau_1$, is reproduced in Appendix A-3.

3. Richard's Modified Diffusion Theory (13)

The transport equation is approximated as

$$\frac{d^2 q}{dz^2} - K^2 q = -(1 - K^2)\rho_d \tag{II-8}$$

where z is normal distance in mean free paths for scattering, $K = \sqrt{3(1-\omega_o)/(3-2\omega_o)}$, ρ_d is the density of unscattered photons, ρ is the photon density and q is defined by

$$q = (1 - \tfrac{2}{3}\omega_o)\rho - \rho_d \tag{II-9}$$

The source strength is $\rho_d = e^{-z/\mu_o}$.

Equation II-8 can be solved for the conditions $\omega_0 = 0$ in vacuum, ρ finite, and q and $\frac{dq}{dz}$ both continuous at $\tau_1 = o$ and τ_1.

Giving

$$R = \frac{1}{\mu_0}\frac{dq}{dz}\bigg|_{z=0} = \frac{-2K(1-\mu_0)e^{-\tau_1/\mu_0} + (1+K)(1-K\mu_0)e^{K\tau_1} - (1-K)(1+K\mu_0)e^{-K\tau_1}}{(1+K)^2 e^{K\tau_1} - (1-K)^2 e^{-K\tau_1}} \frac{(1-K^2)\mu_0}{1-K^2\mu_0^2} \qquad \text{II-10}$$

and

$$T = -\frac{1}{\mu_0}\frac{dq}{dz}\bigg|_{z=\tau_1} + e^{-\tau_1/\mu_0}$$

$$= \frac{2K(1+\mu_0) - (1+K)(1+K\mu_0)e^{K\tau_1 - \tau_1/\mu_0} + (1-K)(1-K\mu_0)e^{-K\tau_1 - \tau_1/\mu_0}}{(1+K)^2 e^{K\tau_1} - (1-K)^2 e^{-K\tau_1}} \frac{1-K^2}{1-K^2\mu_0^2} + e^{-\tau_1/\mu_0} \qquad \text{II-11}$$

The above two equations are for $\omega_0 < 1$. The solutions for the case $\omega_0 = 1$ can be found easily and are omitted here.

4. Numerical Results

Numerical values were computed from the above approximate solutions for comparison with the "exact" results obtained in Part I. The computed values are given in Tables B-1 to B-5 in Appendix B and are illustrated in Figures II-1 through II-5.

Table B-1 gives reflectances for isotropic scattering as obtained from Richard's model, the two-flux model and the six-flux model. The exact values are included for comparison. The six-flux results for Rayleigh scattering are also included. The exact values for Rayleigh scattering did not vary significantly from the values for isotropic scattering and hence are not reproduced here. The two-flux model

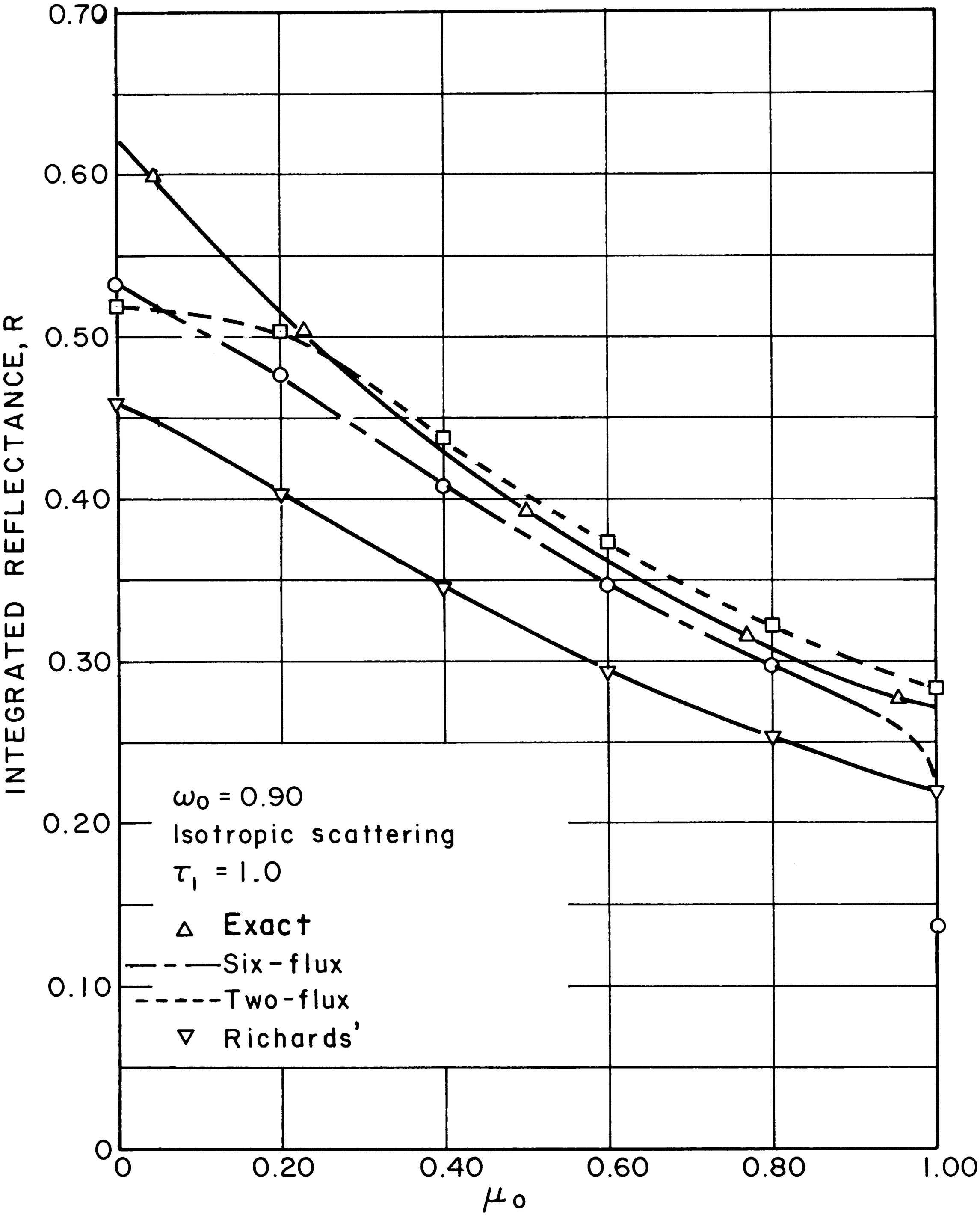


Figure II-1 Integrated reflectance for various solutions as functions of angle of incidence for isotropic scattering, R vs. μ_o, $\omega_o = 0.90$, $\tau_1 = 1.0$

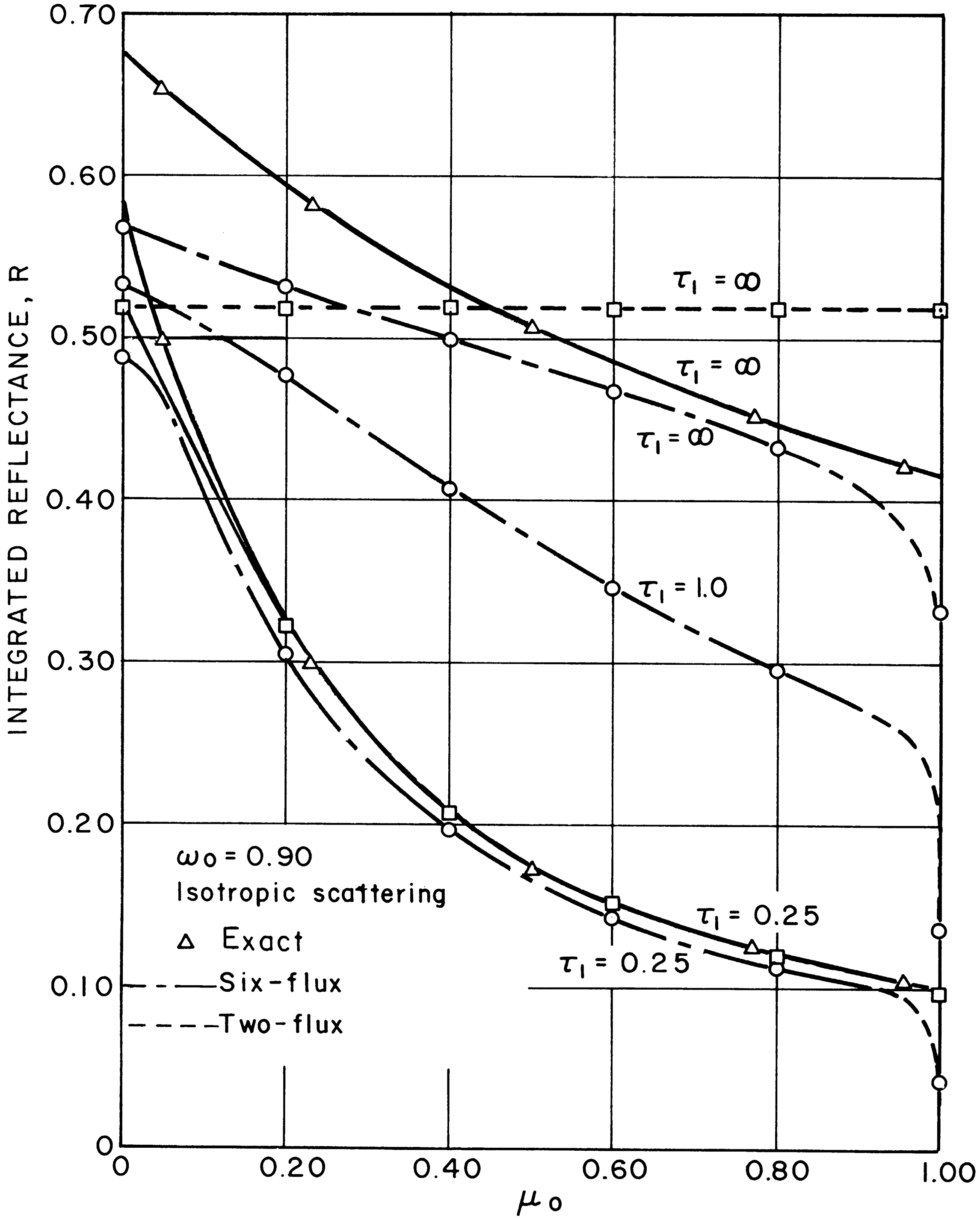


Figure II-2 Integrated reflectance for various solutions as function of angle of incidence for isotropic scattering, R vs. μ_o, parameter of τ_1, $\omega_o = 0.90$

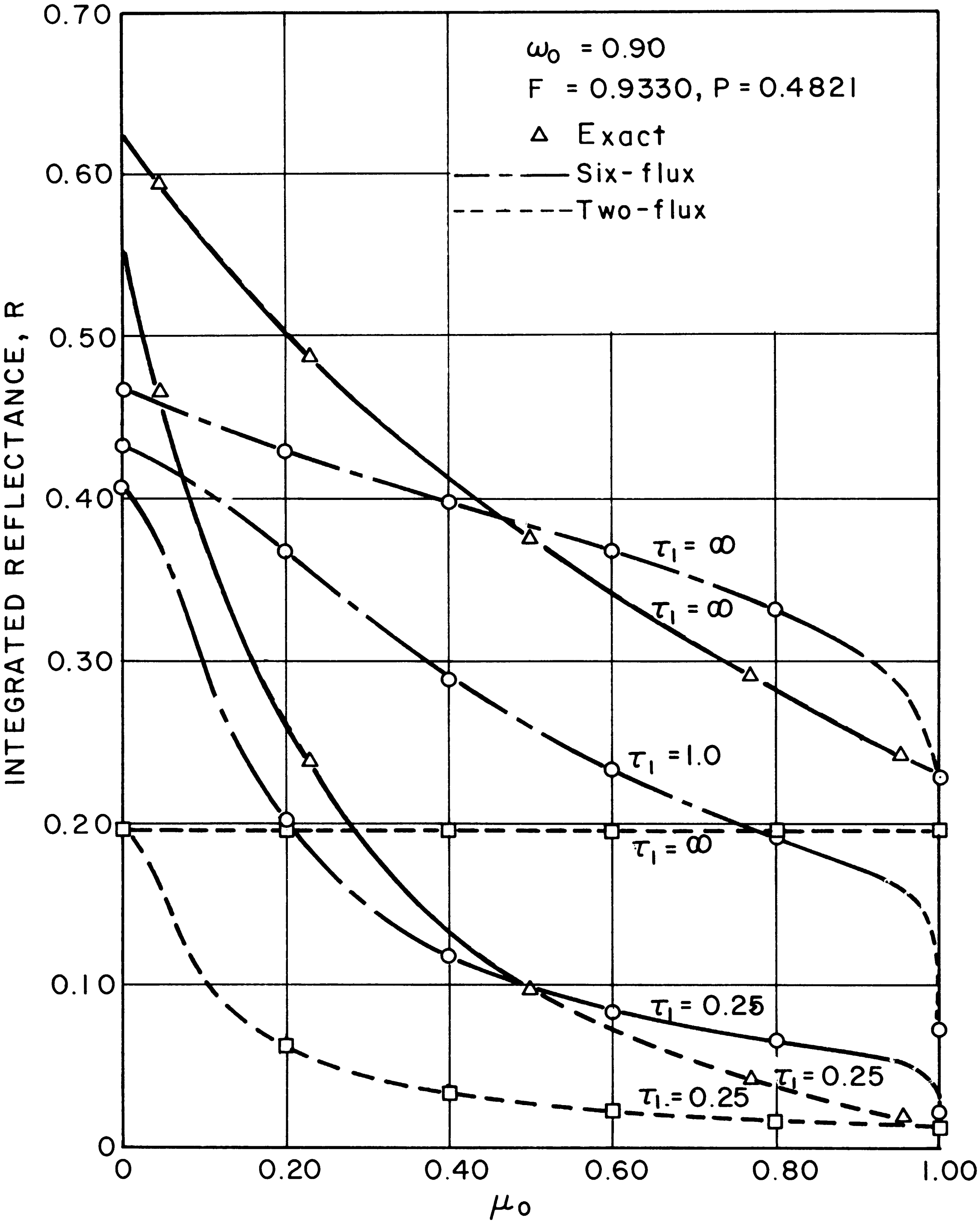


Figure II-3 Integrated reflectance for various solutions as function of angle of incidence for anisotropic scattering, R. vs. μ_o, parameter of τ_1, $\omega_o = 0.90$, F = 0.9330, P = 0.4821

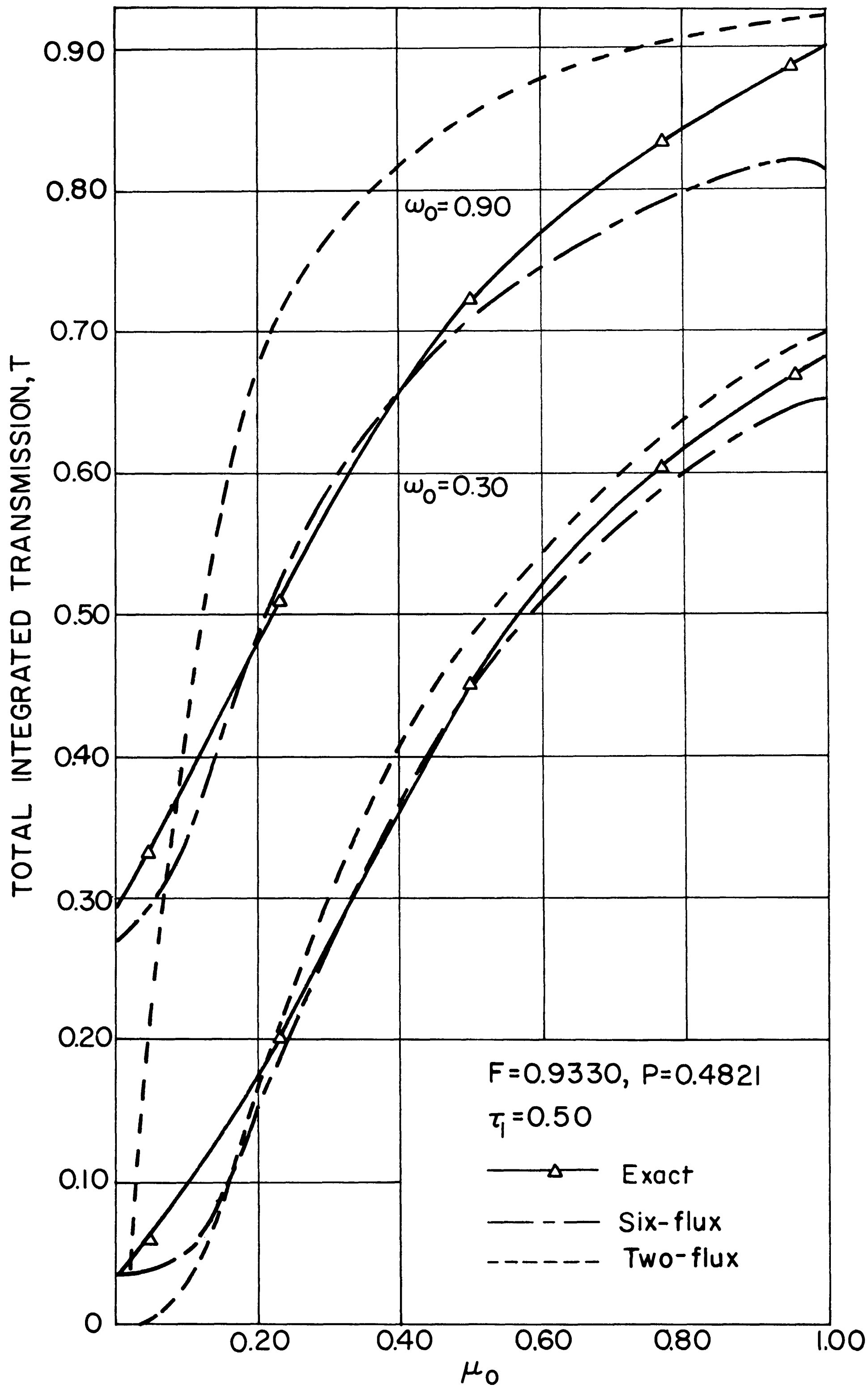


Figure II-4 Total integrated transmission, various solutions as function of angle of incidence for anisotropic scattering, R vs. μ_o, parameter of ω_o, $\tau_1 = 0.50$, $F = 0.9330$, $P = 0.4821$

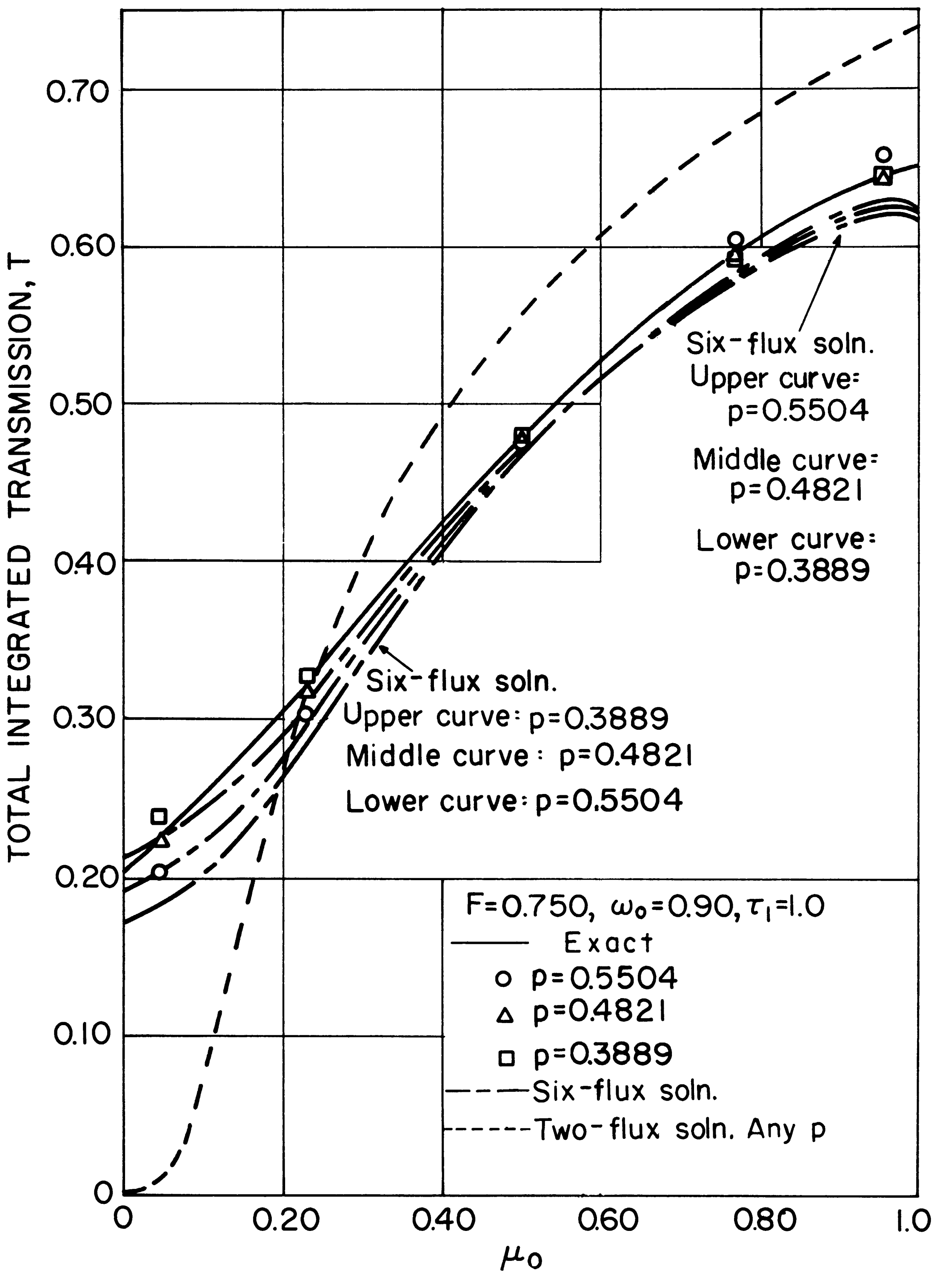


Figure II-5 Total integrated transmission, various solutions as function of angle of incidence for anisotropic scattering, R vs. μ_o, parameter of P, ω_o= 0.90, τ_1 = 1.0, F = 0.750

obviously gives identical values for Rayleigh and isotropic scattering. Representative values are plotted in Figures II-1 and II-2. From the tabulated and plotted values it can be observed that the six-flux solution is generally better than the two-flux solution which is in turn generally better than the Richard's solution. The simple diffusion solution which is not given is poorer than Richard's solution. Exceptions are for normal incidence where the six-flux solution falls off and for infinite thickness for which the two-flux reflectance is invariant with angle of incidence and inferior to Richard's solution. All of the approximations improve as the albedo, ω_0, increases and as the optical thickness, τ_1, decreases. This behavior can be attributed to the decreased contribution of multiple scattering to the reflectance.

Table B-2 gives transmissions for the same conditions and models as in Table B-1, except for the omission of the infinitely thick medium. Again the values for Rayleigh scattering are included for the six-flux model, are identical for the two-flux model and are essentially equal for the exact method. These values are not illustrated graphically but study of the tabulated values indicates clearly that 1) the six-flux values are better than Richard's values which are this time better than the two-flux values. The two flux values are two low for small μ_o but improve for $\mu_o > 0.5$. Richard's values and the six-flux values are remarkably equal for normal incidence. The greater the albedo, ω_0, the better are the approximate values. No clear cut trend in accuracy is apparent with thickness. The predictions of transmission by Richard's and the six-flux model are better than the predictions of reflectance but the opposite is the case with the two-flux model.

Table B-3 and B-4 give reflectances and transmissions respectively for anisotropic scattering with F = 0.933 and P = 0.4821 as computed from the six-flux and two-flux models. The exact values are also reproduced for comparison. Representative values are plotted in Figures II-3 and II-4. It can be observed that the six-flux model is more accurate than the two-flux model for both reflection and transmission. The six-flux model is most reliable near $\mu_o = 0.5$ and for large ω_o and appears to give consistently low reflectances for normal incidence.

Table B-5 gives reflectances and transmissions for anisotropic scattering and with F = 0.750 and P = 0.5821, 0.4821 and 0.3889 as computed exactly and from the six-flux and two-flux models. The computed transmissions are illustrated in Figure II-5. The six-flux model yields a dependence or P which is in accordance with the exact values. The two-flux model of course indicates no dependence on P.

B. Interpolation Factors

Examination of the approximate solutions and the trend of the exact values provided the basis for the development of several simple relationships for approximate interpolation and extrapolation of the computed values.

The integrated reflectance and the total transmission are observed to be strong functions of μ_o, ω_o, F and τ_1 but only weakly dependent on P. Interpolation equations for μ_o and τ_1 were developed for some conditions as follows.

For isotropic scattering by a half-infinite medium simple diffusion theory suggests the interpolation formula

$$R = \frac{R_o}{1 + \left(\frac{R_o}{R_1} - 1\right)\mu_o} \qquad \text{II-12}$$

where R_o and R_1 are the integrated reflectance for grazing ($\mu_o = 0$) and normal ($\mu_o = 1$) incidence, respectively. This equation is found to be satisfactory for interpolation of R with respect to μ_o, even though the absolute values of R indicated by diffusion theory are seriously in error. The success of this interpolation is indicated by the top curve (for $\tau_1 = \infty$) in Figures II-6 and II-7, and in Table II-1.

Richard's model does not give accurate values of R for isotropic scattering by finite slabs but the indicated dependence of R on τ_1, is

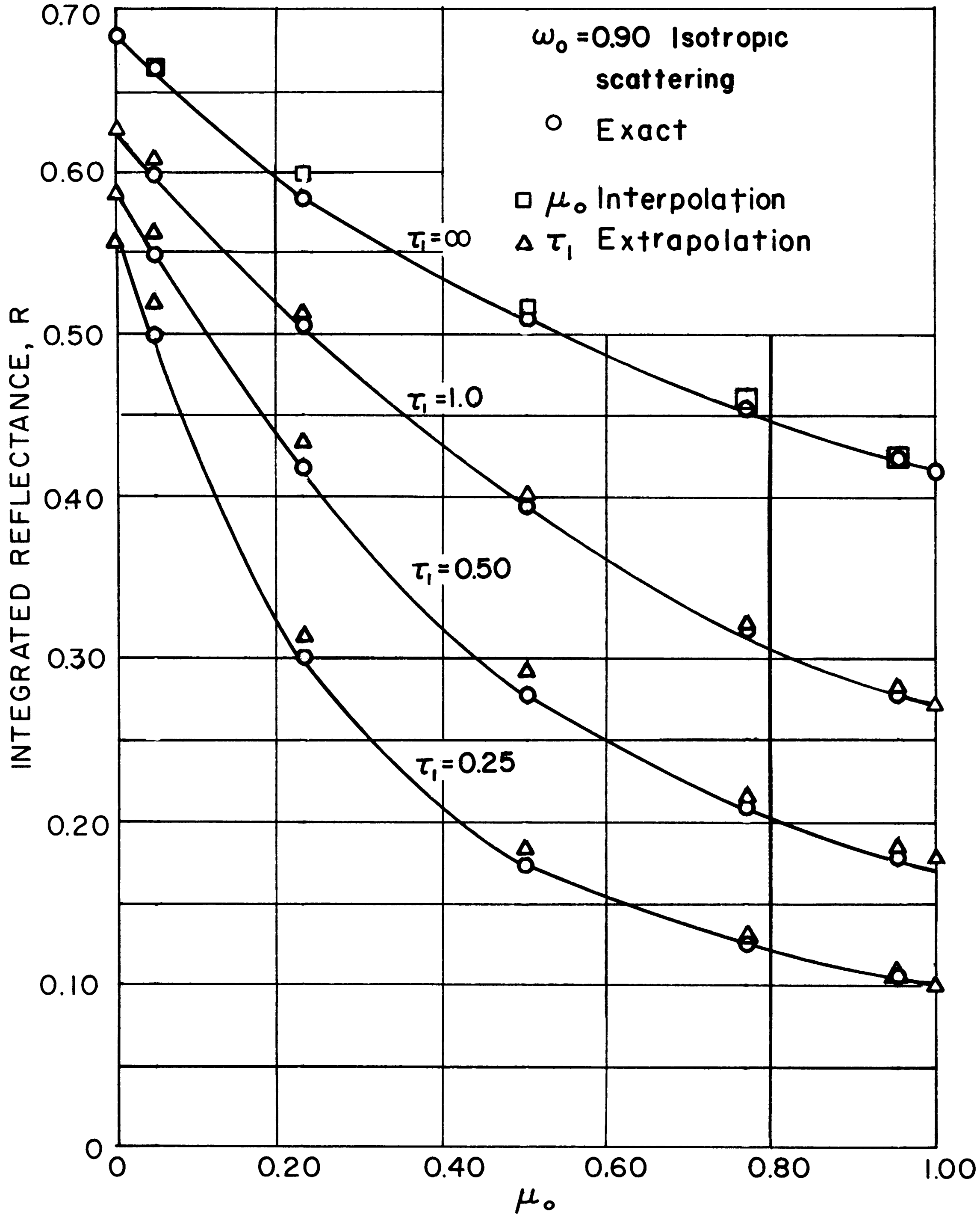


Figure II-6 Integrated reflectance, exact, interpolated and extrapolated as function of angle of incidence for isotropic scattering, R vs. μ_o, parameter of τ_1, ω_o = 0.90

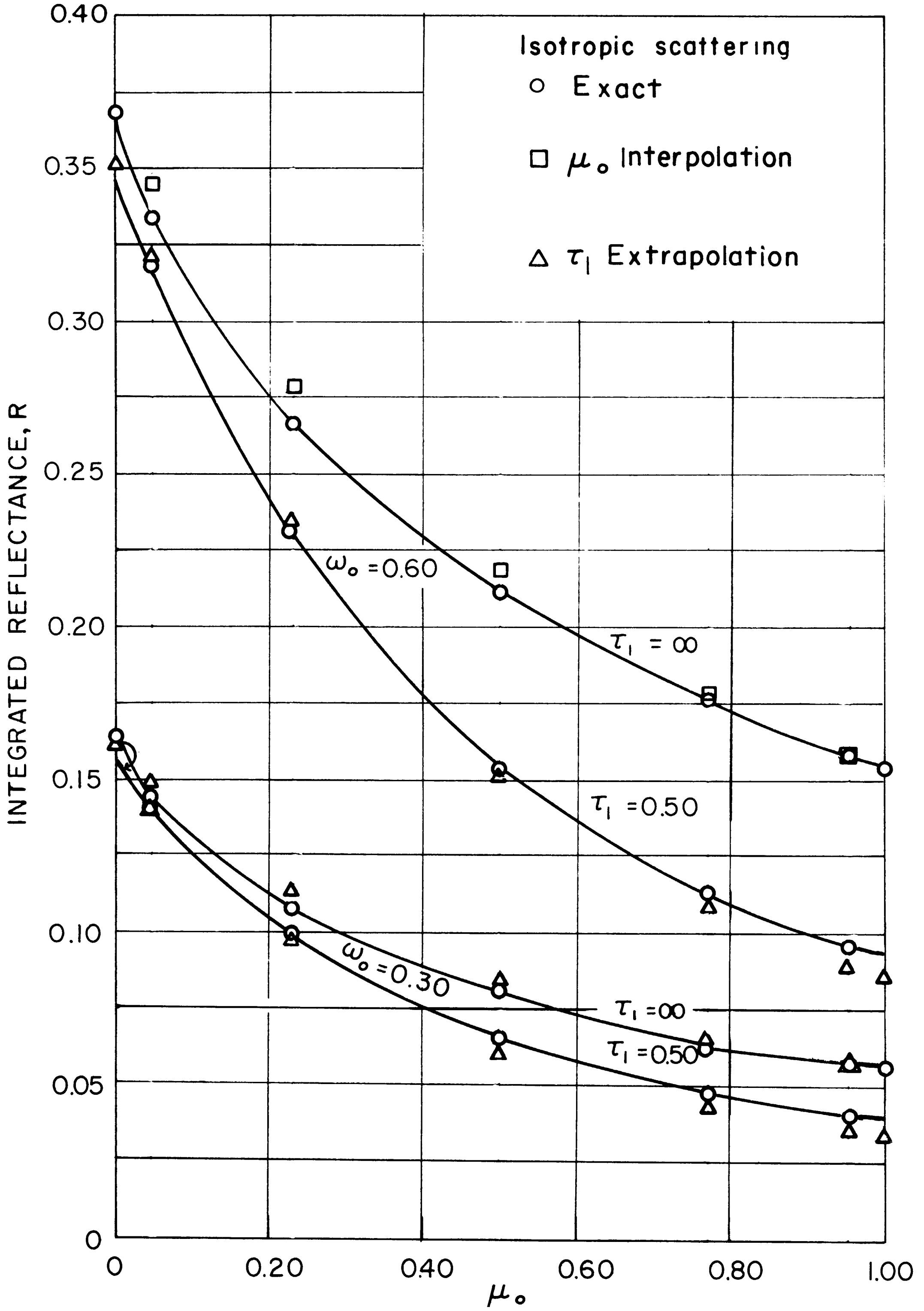


Figure II-7 Integrated reflectance, exact, interpolated and extrapolated as function of angle of incidence for isotropic scattering, R vs. μ_o, parameters of ω_o and τ_1

Table II-1 Interpolation and Extrapolation of R for Isotropic Scattering with Respect to μ_o and τ_1

$\omega_o = 0.90$:

	$\tau_1 = \infty$		$\tau_1 = 1.0$		$\tau_1 = 0.5$		$\tau_1 = 0.25$	
μ_o	Exact*	Interp.	Exact*	Extrap.	Exact*	Extrap.	Exact*	Extrap.
0.00000	0.684	0.684	0.624	0.625	0.584	0.586	0.571	0.556
0.04691	0.664	0.664	0.598	0.606	0.548	0.561	0.498	0.516
0.23077	0.582	0.595	0.504	0.511	0.417	0.433	0.300	0.316
0.50000	0.508	0.517	0.394	0.401	0.279	0.293	0.173	0.185
0.76923	0.452	0.456	0.316	0.322	0.209	0.215	0.125	0.129
0.95309	0.423	0.423	0.279	0.284	0.179	0.184	0.105	0.105
1.00000	0.415	0.415	0.271	0.274	0.170	0.176	0.100	0.100

	$\omega_o = 0.60$: $\tau_1 = \infty$		$\tau_1 = 0.50$		$\omega_o = 0.30$: $\tau_1 = \infty$		$\tau_1 = 0.50$	
μ_o	Exact*	Interp.	Exact*	Extrap.	Exact*	Extrap.	Exact*	Extrap.
0.00000	0.368	0.368	0.345	0.352	0.163	0.163	0.156	0.161
0.04691	0.334	0.346	0.318	0.321	0.144	0.150	0.142	0.141
0.23077	0.267	0.279	0.232	0.236	0.108	0.114	0.100	0.098
0.50000	0.212	0.218	0.154	0.152	0.082	0.085	0.066	0.061
0.76923	0.177	0.179	0.114	0.109	0.066	0.067	0.048	0.045
0.95309	0.159	0.159	0.097	0.091	0.059	0.059	0.041	0.036
1.00000	0.155	0.155	0.093	0.087	0.057	0.057	0.040	0.035

* Extrapolated by Lagrange's formula for $\mu_o = 0$ and $\mu_o = 1.0$

excellent suggesting the equation

$$R_{\mu_0,\tau_1} = R_{\mu_0,\infty}\left(\frac{R_{\mu_0,\tau_1}}{R_{\mu_0,\infty}}\right)_{Richards} \qquad \text{II-13}$$

for the extrapolation of exact values of R for $\tau_1 = \infty$ and any μ_o and ω_0 to finite values of τ_1. The success of this formula is indicated in Figures II-6 and II-7 and in Table II-1 for the finite values of τ_1. The ratios of R for Richard's solution were taken from Tables B-1 and B-2. The success of the interpolation equations indicated that exact calculations for isotropic scattering could be limited to normal and grazing incidence on a half-infinite medium.

For anisotropic scattering the expression

$$R = xR_1 + (1-x)R_0 \qquad \text{II-14}$$

where x is a function of ω_0, τ_1 and μ_o but not of F, can be used to interpolate for μ_o. Values of x can therefore be determined from computed values of R for isotropic scattering. A plot of x as a function of μ_o for several values of τ_1 and ω_0 are shown in Figures II-8 and II-9. These figures were prepared from the isotropic values. (The six-flux solution erroneously indicates independence of x from τ_1 and ω_0 as well as from F and P.) Values of R for intermediate values of μ_o were computed from the exact values of R_o and R_1 and the values of x in Figures II-8 and II-9 using Equation II-14 and are given in Table II-2 and in Figures II-10 and II-11 (triangles). The agreement with the exact values is seen to be reasonably good.

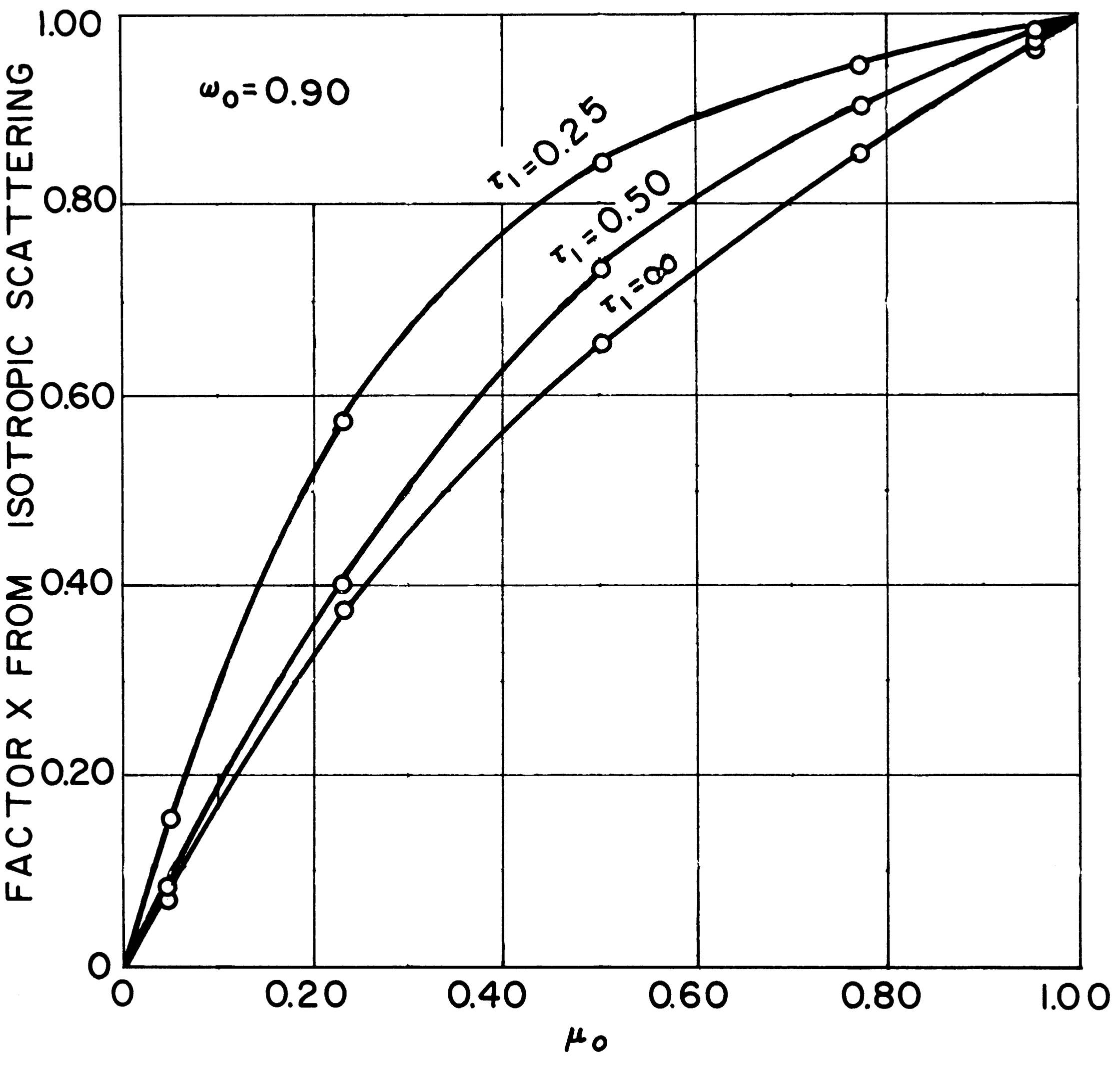


Figure II-8 Factor x in equation II-14 calculated from isotropic R_0 and R_1, as function of angle of incidence, x vs. μ_0, parameter of ω_0

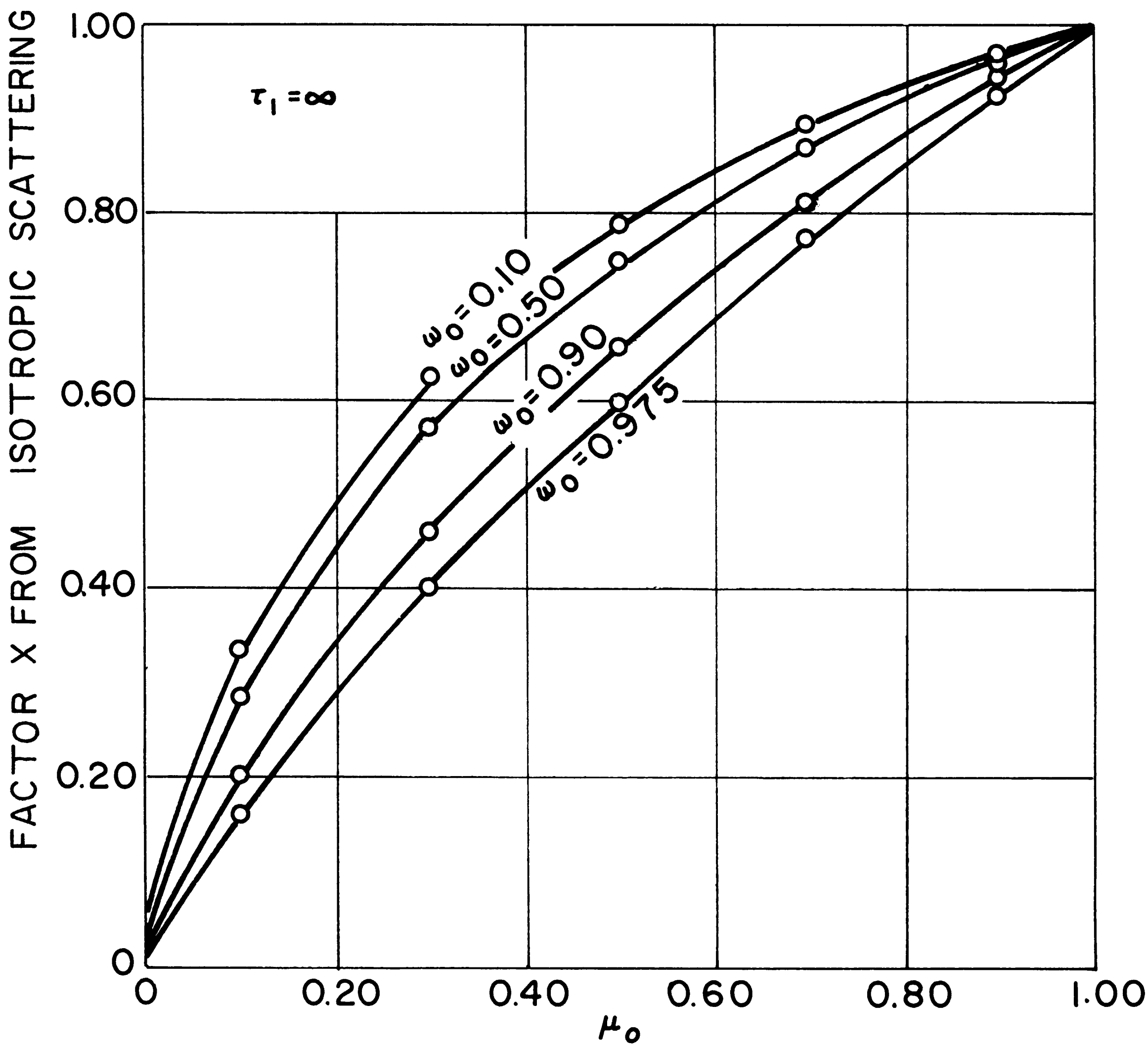


Figure II-9 Factor x in equation II-14 calculated from isotropic R_o and R_1 as function of angle of incidence, x vs. μ_o, parameters of τ_1

Table II-2 Interpolation of R with Respect to μ_o for Anisotropic Scattering

F = 0.9330, P = 0.4821,

ω_o = 0.90:

	$\tau_1 = \infty$		τ_1 = 1.0		τ_1 = 0.5		τ_1 = 0.25	
μ_o	Exact*	Interp.	Exact*	Interp.	Exact*	Interp.	Exact*	Interp.
0.00000	0.628	0.628	0.567	0.567	0.547	0.547	0.552	0.552
0.04691	0.595	0.599	0.530	0.530	0.500	0.502	0.466	0.468
0.23077	0.489	0.478	0.396	0.397	0.333	0.338	0.238	0.241
0.50000	0.377	0.370	0.239	0.241	0.163	0.166	0.097	0.096
0.76923	0.292	0.299	0.131	0.131	0.078	0.079	0.043	0.041
0.95309	0.244	0.245	0.079	0.078	0.041	0.041	0.018	0.019
1.00000	0.233	0.233	0.067	0.067	0.030	0.030	0.013	0.013

F = 0.9330, P = 0.4821,

	ω_o = 0.60:				ω_o = 0.30:			
	$\tau_1 = \infty$		τ_1 = 0.50		$\tau_1 = \infty$		τ_1 = 0.50	
μ_o	Exact*	Interp.	Exact*	Interp.	Exact*	Interp.	Exact*	Interp.
0.00000	0.336	0.336	0.330	0.330	0.157	0.157	0.151	0.151
0.04691	0.303	0.289	0.293	0.296	0.144	0.139	0.131	0.134
0.23077	0.206	0.197	0.180	0.189	0.108	0.105	0.075	0.082
0.50000	0.124	0.121	0.085	0.092	0.082	0.081	0.034	0.040
0.76923	0.073	0.073	0.040	0.042	0.066	0.066	0.015	0.018
0.95309	0.048	0.049	0.021	0.021	0.059	0.059	0.008	0.009
1.00000	0.043	0.043	0.016	0.016	0.058	0.058	0.007	0.007

F = 0.7500, P = 0.4821,

ω_o = 0.90;

	τ_1 = 1.0	
μ_o	Exact*	Interp.
0.00000	0.598	0.598
0.04691	0.564	0.566
0.23077	0.445	0.451
0.50000	0.310	0.317
0.76923	0.218	0.221
0.95309	0.175	0.176
1.00000	0.166	0.166

* Extrapolated by Lagrange's formula for μ_o = 0 and μ_o = 1.0

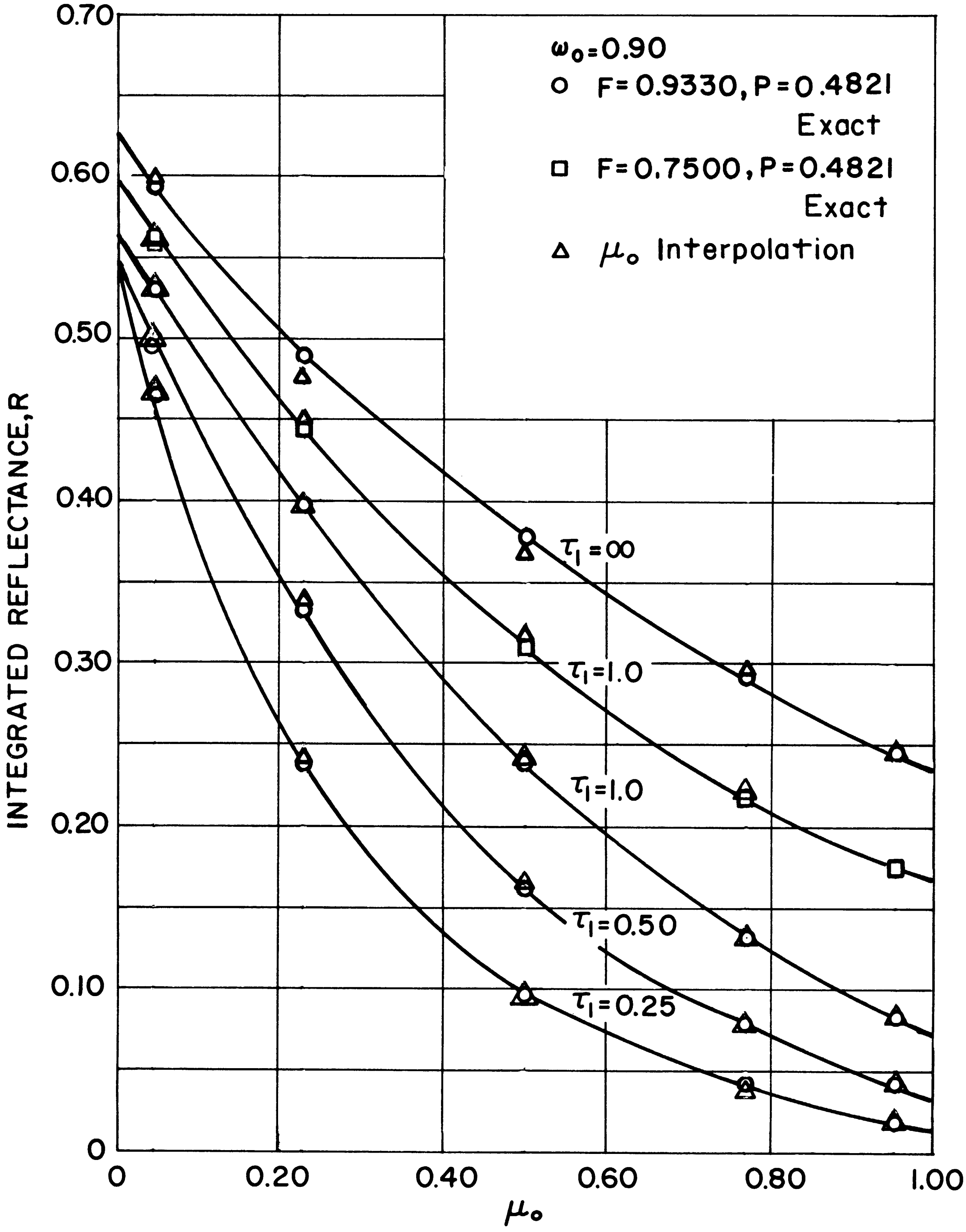


Figure II-10 Integrated reflectance, exact and interpolated as function of angle of incidence for anisotropic scattering, R vs. μ_o, parameter of τ_1, ω_o= 0.90, F = 0.9330, P = 0.4821

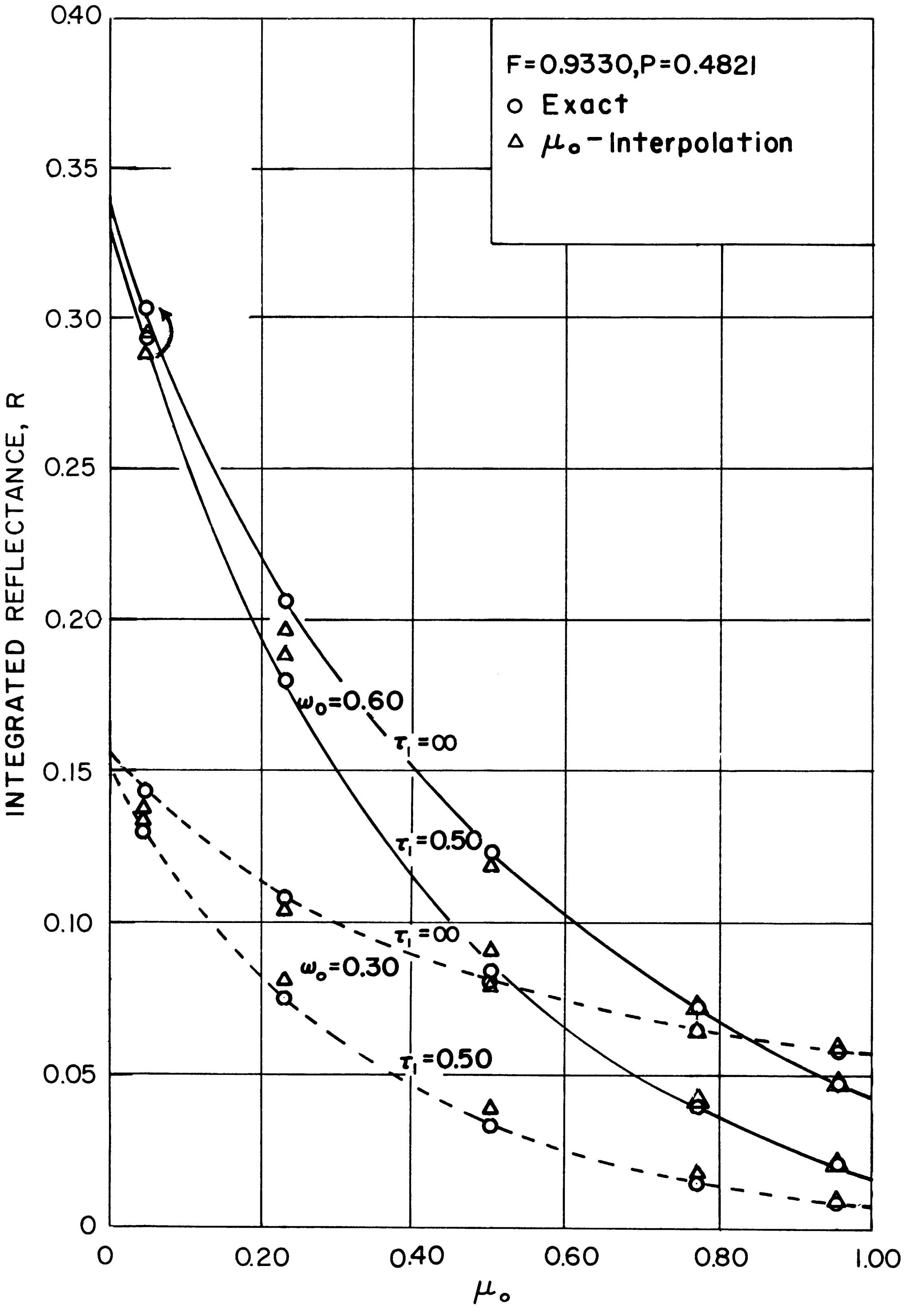


Figure II-11 Integrated reflectance, exact and interpolated as function of angle of incidence for anisotropic scattering, R vs. μ_o, parameters of ω_o & τ_1

The same type of interpolation, i.e.,

$$T = y T_0 + (1 - y) T_1 \qquad \text{II-15}$$

was also successful for the total transmission as indicated in Table II-3.

Table II-3 Interpolation of T with Respect to μ_o for Anisotropic Scattering

	$F = 0.9330,\ P = 0.4821,\ \omega_o = 0.90:\ \tau_1 = 1.0$		$\tau_1 = 0.50$		$\tau_1 = 0.25$	
μ_o	Exact*	Interp.	Exact*	Interp.	Exact*	Interp.
0.00000	0.761	0.761	0.904	0.904	0.991	0.991
0.04691	0.750	0.754	0.885	0.886	0.979	0.983
0.23077	0.692	0.698	0.833	0.841	0.933	0.951
0.50000	0.553	0.551	0.723	0.733	0.853	0.881
0.76923	0.372	0.365	0.507	0.505	0.669	0.689
0.95309	0.259	0.256	0.331	0.328	0.396	0.402
1.00000	0.236	0.236	0.291	0.291	0.297	0.297

	$F = 0.9330,\ P = 0.4821,\ \omega_o = 0.60:\ \tau_1 = 0.50$		$\omega_o = 0.30:\ \tau_1 = 0.50$		$F = 0.7500,\ P = 0.4821,\ \omega_o = 0.90:\ \tau_1 = 1.0$	
μ_o	Exact*	Interp.	Exact*	Interp.	Exact*	Interp.
0.00000	0.783	0.783	0.692	0.692	0.652	0.652
0.04691	0.761	0.761	0.668	0.665	0.645	0.646
0.23077	0.701	0.701	0.601	0.600	0.598	0.598
0.50000	0.565	0.561	0.452	0.433	0.476	0.473
0.76923	0.319	0.309	0.198	0.189	0.318	0.313
0.95309	0.165	0.161	0.063	0.059	0.222	0.221
1.00000	0.143	0.143	0.052	0.052	0.203	0.203

* Extrapolated by Lagrange's formula for $\mu_o = 0$ and $\mu_o = 1.0$

PART III - DEVELOPMENT OF IMPROVED APPROXIMATE MODELS

A. Introduction

It is well known in classical physics that multiple scattering processes can be described approximately by means of the classical diffusion equation. The major difficulty in the application of classical diffusion equation to the problems of multiple scattering however, is in the specification of the correct boundary condition. Various authors have investigated the use of the simple diffusion equation in multiple scattering problems, and formulated modified versions of the diffusion equation [13,14,15]. In this investigation it is shown that the various modified diffusion equations can be derived in a unified manner from the integral equation of transport, and therefore can be not only refined indefinitely, but also, in principle, can have appropriate boundary conditions derived corresponding to each step of refinement. On this basis a new modified diffusion equation is proposed and solutions compared numerically with known exact solutions. Better agreement is found than with previous models. The boundary conditions are obtained from purely mathematical reasoning and a good physical interpretation is yet to be found. To simplify the mathematical formulation, isotropic scattering has been considered as an example, but the extension to anisotropic scattering is trivial, and will be carried out in the continuation of this work.

B. Approximate Differential Equations

For isotropic scattering, the integral equation of transport for the photon density $\rho(\vec{r})$ is

$$\rho(\vec{r}) = S(\vec{r}) + \omega_0 \iiint_{\substack{\text{volume of}\\ \text{dispersion}}} \frac{e^{-|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|^2} \rho(\vec{r}')\, dv' \qquad \text{III-1}$$

where

$S(\vec{r})$ is the source term,
ω_0 is the albedo of single scattering, and
$\vec{r}$ represents the space coordinates measured in mean free paths.

Equation (1) is a linear integral equation of the Fredholm type in three dimensions.

$$\rho(\vec{r}) = S(\vec{r}) + \omega_0 \iiint K(\vec{r},\vec{r}')\, \rho(\vec{r}')\, dv' \qquad \text{III-2}$$

For some special forms of the kernal $K(\vec{r},\vec{r}\,)$ this equation can be reduced to a simple differential equations. In this section, it is shown that the various forms of diffusion equation can be obtained by such reduction after approximating the kernal

$$K(\vec{r},\vec{r}') = \frac{e^{-|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|^2}$$

by comparatively simple forms.

Consider the class of $K(\vec{r},\vec{r}\,)$ whose Fourier transform can be represented by ratio of two polynomials*:

$$\frac{N(k^2)}{D(k^2)} = \frac{a_0 + a_1 k^2 + a_2 k^4 + \cdots\cdots + a_n k^{2n}}{b_0 + b_1 k^2 + b_2 k^4 + \cdots\cdots + b_m k^{2m}}$$

* The same derivation applied to a two dimensional case was given in Ref. (16)

so that,

$$K(\vec{r},\vec{r}\,') = \frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty} \frac{N(k^2)}{D(k^2)}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}\, dk_x\, dk_y\, dk_z \qquad \text{III-3}$$

Then, since

$$\nabla^2 e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')} = -k^2 e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')} \qquad \text{III-4}$$

Equation (3) may be reduced to:

$$K(|\vec{r}-\vec{r}\,'|) = N(-\nabla^2)\frac{1}{(2\pi)^{3/2}}\iiint_{-\infty}^{\infty} \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{D(k^2)}\, dk_x\, dk_y\, dk_z \qquad \text{III-5}$$

Substituting Equation (5) into (2) yields

$$\rho(\vec{r}) = S(\vec{r}) + \omega_0 N(-\nabla^2)\iiint_{\substack{\text{volume of}\\ \text{dispersion}}} \rho(\vec{r}\,')\, dv'\, \frac{1}{(2\pi)^{3/2}}\iiint_{-\infty}^{\infty} \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}}{D(k^2)}\, dk_x\, dk_y\, dk_z \qquad \text{III-6}$$

Now, operating on both sides of (6) by $D(-\nabla^2)$,

$$D(-\nabla^2)\rho(\vec{r}) = D(-\nabla^2)S(\vec{r}) + \omega_0 N(-\nabla^2)\iiint_{\substack{\text{volume}\\ \text{of dispersion}}} \rho(\vec{r}\,')\frac{dv'}{(2\pi)^{3/2}}\iiint_{-\infty}^{\infty} e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}\, dk_x\, dk_y\, dk_z \qquad \text{III-7}$$

It is recognized that

$$\frac{1}{(2\pi)^{3/2}}\iiint e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}\, dk_x\, dk_y\, dk_z = \delta(\vec{r},\vec{r}\,')$$

the three dimensional δ-function. Since Equation (7) may be reduced to:

$$\left[D(-\nabla^2) - \omega_0 N(-\nabla^2)\right]\rho(\vec{r}) = D(-\nabla^2)S(\vec{r}) \quad , \qquad \text{III-8}$$

Equation (8) cannot be used directly for Equation (1), because the kernal in Equation (1) cannot be exactly expressed in the form given by Equation (3). In fact, it is easy to see that

$$\frac{e^{-|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|^2} = \frac{1}{(2\pi)^{3/2}}\iiint_{-\infty}^{\infty} e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}\frac{\tan^{-1}k}{k}\, dk_x\, dk_y\, dk_z$$

Thus a whole class of approximate solutions of Equation (1) can be obtained from approximations of the function

$$F(k) = \frac{\tan^{-1} k}{k} = 1 - \frac{1}{3}k^2 + \frac{1}{5}k^4 + \dots \qquad \text{III-9}$$

in terms of ratio of two polynomials.

It is interesting to note that although the classical diffusion equation [14], the modified diffusion equation proposed by Richards[13], and of the modified diffusion equation proposed by Grosjean[15], were obtained from various physical and/or mathematical arguments, they can also be obtained in a unified manner by different approximations of Equation (9). These are respectively

$$\text{(i)} \quad F(k) \cong \frac{1}{1+\frac{1}{3}k^2} = 1 - \frac{1}{3}k^2 + \frac{1}{9}k^4 + \dots \qquad \text{III-10}$$

so that Equation (8) takes the form

$$\nabla^2 \rho(\vec{r}) - 3(1-\omega_0)\rho(\vec{r}) = -(3-\nabla^2) S(\vec{r}) \qquad \text{III-11}$$

$$\text{(ii)} \quad F(k) \cong \frac{1+\frac{2}{3}k^2}{1+k^2} = 1 - \frac{1}{3}k^2 + \frac{1}{3}k^4 + \dots \qquad \text{III-12}$$

so that Equation (8) takes the form:

$$\nabla^2 \rho(\vec{r}) - \left(\frac{1-\omega_0}{1-\frac{2}{3}\omega_0}\right)\rho(\vec{r}) = \frac{-(1-\nabla^2)}{1-\frac{2}{3}\omega_0} S(\vec{r}) \qquad \text{III-13}$$

$$\text{(iii)} \quad F(k) \cong \frac{1+\frac{1}{3}k^2}{1+\frac{2}{3}k^2} = 1 - \frac{1}{3}k^2 + \frac{2}{9}k^4 + \dots \qquad \text{III-14}$$

so that Equation (8) takes the form:

$$\nabla^2\rho(\vec{r}) - \frac{3(1-\omega_0)}{2-\omega_0}\rho(\vec{r}) = -\frac{(3-2\nabla^2)}{2-\omega_0} S(\vec{r}) \qquad \text{III-15}$$

The left hand sides of Equations (11), (13) and (15) are the equivalent terms of the classical diffusion equation, the modified diffusion equation proposed by Richards and the modified diffusion equation proposed by Grosjean respectively.

It is mathematically difficult to estimate the errors involved in these approximations, perhaps the only valid criterion is to compare the results obtained in various simple problems with exact solutions. Intuitively, however, it seems to be reasonable to require any approximation of F(k) to be as close as possible to the exact function at least in the limiting case of k = 0, and $k \to \infty$. Inspection of Equation (9), (10), (12) and (14) shows that as $k \to 0$ all three approximations are exact to the coefficient k^2 of the expansion of F(k). However, as $k \to \infty$,

$$\lim_{k \to \infty} \frac{\tan^{-1}k}{k} = 0$$

and only the simple diffusion equation has the correct limit.

Based upon the above observations it is clear that a class of differential equations based on diffusion type of approximation can be systematically obtained as follows:

(i) First approximation:

$$F(k) \cong \frac{1}{1+\frac{1}{3}k^2} = 1 - \frac{1}{3}k^2 + \frac{1}{9}k^4 - \dots$$

For which, the approximated kernal is,

$$K(\vec{r},\vec{r}') = \frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty} \frac{1}{1+\frac{1}{3}k^2}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}\, dk_x\, dk_y\, dk_z = \frac{3}{4\pi|\vec{r}-\vec{r}'|}\, e^{-\sqrt{3}|\vec{r}-\vec{r}'|} \qquad \text{III-16}$$

and the resulting differential Equation is:

$$\nabla^2\rho(\vec{r}) - 3(1-\omega_0)\rho(\vec{r}) = (\nabla^2-3)\,S(\vec{r})$$

(ii) Second approximation

$$F(k) \cong \frac{1+ak^2}{1+bk^2+ck^4}$$

For best approximation at small values of k,

$$a = \frac{8}{21}$$

$$b = \frac{5}{7}$$

and

$$c = \frac{4}{105}$$

so that

$$F(k) \cong \frac{1+\frac{8}{21}k^2}{1+\frac{5}{7}k^2+\frac{4}{105}k^4} = 1-\frac{1}{3}k^2+\frac{1}{5}k^4-\frac{1}{7}k^6+\frac{2}{21}k^8+\ldots \qquad \text{III-17}$$

corresponding to this approximation.

$$\begin{aligned} K(\vec{r},\vec{r}') &= \frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty} \frac{1+ak^2}{1+bk^2+ck^4}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}\, dk_x\, dk_y\, dk_z \\ &= \frac{1}{4\pi|\vec{r}-\vec{r}'|}\left[n_1 e^{-\beta_1|\vec{r}-\vec{r}'|} + n_2 e^{-\beta_2|\vec{r}-\vec{r}'|}\right] \end{aligned} \qquad \text{III-18}$$

where

$$\beta_1 = 1.2344$$
$$\beta_2 = 4.1505$$

are the positive roots of the equation

$$c\beta^4 - b\beta^2 + 1 = 0$$

while

$$n_1 = \frac{\beta_1^2 a - 1}{c(\beta_1^2 - \beta_2^2)} = 0.7013$$

and

$$n_2 = \frac{\beta_2^2 a - 1}{c(\beta_2^2 - \beta_1^2)} = 9.2987$$

The resulting differential equation is:

$$\left[\nabla^4 + \frac{a\omega_0 - b}{c}\nabla^2 + \frac{1-\omega_0}{c}\right]\rho(\vec{r}) = \left[\nabla^4 - \frac{b}{c}\nabla^2 + \frac{1}{c}\right]S(\vec{r}) \qquad \text{III-19}$$

In principle, the approximations can be extended indefinitely. But numerical calculations given in the subsequent sections indicate that with the correct form for the boundary conditions, the first approximation yields very good results for the photon density, and the second approximation is almost indistinguishable from the corresponding exact solution.

C. Point Source in Infinite Medium

For the problem of multiple scattering involving sources in an infinite medium, reduction of integral Equation (1) into differential Equations (11) and (19) enables one to find the approximate solution by use of Green's functions. It is well known that the solution of the equation

$$\nabla^2 G_1(\vec{r},\vec{r}\,') - K^2 G_1(\vec{r},\vec{r}\,') = -\delta(\vec{r},\vec{r}\,') \qquad \text{III-20}$$

subject to the condition that $G_1(\vec{r},\vec{r}\,')$ is zero as $r \rightarrow \infty$ is,

$$G_1(\vec{r},\vec{r}\,') = \frac{1}{4\pi} \frac{e^{-K|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \qquad \text{III-21}$$

This Green's function can be used to solve the equation of the 1st approximation

$$\nabla^2 \rho(\vec{r}) - 3(1-\omega_0)\rho(\vec{r}) = -(3-\nabla^2)S(\vec{r})$$

Let

$$K = \sqrt{3(1-\omega_0)}$$

multiply Equation (20) by $-(3-\nabla'^2)S(\vec{r}\,')$* and integrate. It is easy to see that the solution of (11) is

$$\rho(\vec{r}) = \iiint_{\text{all space}} G(\vec{r},\vec{r}\,')\,[3-\nabla'^2]\,S(\vec{r}\,')\,dv'$$

* means the appropriate differentiation with respect to the primed coordinate variables

This yields*

$$\rho(\vec{r}) = S(\vec{r}) + (3-K^2)\iiint G(\vec{r},\vec{r}')\, S(\vec{r}')\, dv' \qquad \text{III-22}$$

or more explicitly:

$$\rho(\vec{r}) = S(\vec{r}) + \frac{3\omega_0}{4\pi}\iiint \frac{e^{-\sqrt{3(1-\omega_0)}\,|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\, S(\vec{r}')\, dv' \qquad \text{III-23}$$

Similarly, the solution of the equation

$$\left[\nabla^4 + \frac{a\omega_0 - b}{c}\nabla^2 + \frac{1-\omega_0}{c}\right] G_2(\vec{r},\vec{r}') = -\delta(\vec{r},\vec{r}') \qquad \text{III-24}$$

subject to the condition that $G_2(\vec{r},\vec{r}')\to 0$ as $r\to\infty$ is,

$$G_2(\vec{r},\vec{r}') = \frac{1}{4\pi|\vec{r}-\vec{r}'|}\,\frac{1}{(\alpha_1^2-\alpha_2^2)}\left[e^{-\alpha_1|\vec{r}-\vec{r}'|} - e^{-\alpha_2|\vec{r}-\vec{r}'|}\right] \qquad \text{III-25}$$

where α_1, α_2 are the positive roots of the equation**

$$\alpha^4 + \left(\frac{a\omega_0 - b}{c}\right)\alpha^2 + \frac{1-\omega_0}{c} = 0 \qquad \text{III-26}$$

* Equation (22) is obtained by using the second Green's identity, so that

$$-\iiint G(\vec{r},\vec{r}')\,\nabla'^2 S(\vec{r}')\, dv' = -\iiint S(\vec{r}')\,\nabla'^2 G(\vec{r},\vec{r}')\, dv'$$

But in view of Equation (19)

$$-\iiint S(\vec{r}')\nabla'^2 G(\vec{r},\vec{r}')\, dv' = \iiint S(\vec{r}')\left[\delta(\vec{r},\vec{r}') - K^2 G(\vec{r},\vec{r}')\right] dv'$$

$$= S(\vec{r}) - K^2\iiint S(\vec{r}')\, G(\vec{r},\vec{r}')\, dv'$$

** For the present problem, $\omega_0 \leq 1$, α_1, α_2 exist, and are given in Table III-1.

Using the Green function $G_2(\vec{r},\vec{r}')$ it is possible to write down formally the solution of the equation in an infinite medium.

$$\left[\nabla^4 + \frac{a\omega_0 - b}{c}\nabla^2 + \frac{1-\omega_0}{c}\right]\rho(\vec{r}) = \left[\nabla^4 - \frac{b}{c}\nabla^2 + \frac{1}{c}\right]S(\vec{r}) \qquad \text{III-27}$$

The result is:

$$\rho(\vec{r}) = S(\vec{r}) + \frac{\omega_0}{4\pi}\iiint S(\vec{r}')\left[\frac{m_1 e^{-\alpha_1|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} + \frac{m_2 e^{-\alpha_2|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\right]dv' \qquad \text{III-28}$$

where

$$m_1 = \frac{1-a\alpha_1^2}{(\alpha_2^2-\alpha_1^2)c} \qquad \text{III-29}$$

and

$$m_2 = \frac{1-a\alpha_2^2}{(\alpha_1^2-\alpha_2^2)c} \qquad \text{III-30}$$

Table III-1 The explicit values of α_1, α_2, m_1, m_2 as a function of

ω_0	α_1	α_2	m_1	m_2
0	1.2344	4.1505	1.7013	9.2987
0.1	1.2039	4.0374	0.7916	9.2084
0.2	1.1683	3.9224	0.8987	9.1013
0.3	1.1264	3.8054	1.0264	8.9736
0.4	1.0765	3.6866	1.1793	8.8207
0.5	1.0159	3.5662	1.3632	8.6368
0.6	0.9408	3.4446	1.5847	8.4153
0.7	0.8447	3.3221	1.8517	8.1483
0.8	0.7161	3.1996	2.1721	7.8280
0.9	0.5264	3.0778	2.5532	7.4468
1.0	0	2.9580	3.0000	7.0000

Equation (23) and (28) can be directly applied to the simple problem of an isotropic point source of unit strength at the origin in an infinite dispersion. The source term may be taken as the unscattered radiation,

$$S(\vec{r}) = \frac{e^{-r}}{4\pi r^2} \qquad \text{III-31}$$

Substituting Equation (31) into Equation (23) and (28) yields, respectively, for the first approximation:

$$\rho(\vec{r}) = \frac{e^{-r}}{4\pi r^2} + \frac{\omega_0}{8\pi r}\frac{3}{\sqrt{3(1-\omega_0)}}\Big\{ e^{\sqrt{3(1-\omega_0)}\,r} E_1[(1+\sqrt{3(1-\omega_0)})r] + e^{-\sqrt{3(1-\omega_0)}\,r} E_1[(1-\sqrt{3(1-\omega_0)})r] + e^{-\sqrt{3(1-\omega_0)}\,r} \ln\frac{1+\sqrt{3(1-\omega_0)}}{1-\sqrt{3(1-\omega_0)}}\Big\} \qquad \text{III-32}$$

and the second approximation,

$$\rho(r) = \frac{e^{-r}}{4\pi r^2} + \frac{\omega_0}{8\pi r}\Big\{\frac{m_1}{\alpha_1}\Big[e^{\alpha_1 r}E_1[(1+\alpha_1)r] - e^{-\alpha_1 r}E_1[(1-\alpha_1)r] + e^{-\alpha_1 r}\ln\frac{1+\alpha_1}{1-\alpha_1}\Big] + \frac{m_2}{\alpha_2}\Big[e^{\alpha_2 r}E_1[(1+\alpha_2)r] - e^{-\alpha_2 r}E_1[(1-\alpha_2)r] + e^{-\alpha_2 r}\ln\frac{1+\alpha_2}{1-\alpha_2}\Big]\Big\} \qquad \text{III-33}$$

where $E_1(x)$ is the exponential integral defined by

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt \qquad \text{III-34}$$

Some numerical results calculated from Equation (32) and Equation (33) are compared with the known exact solution in Figure III-1 for $\omega_0 = 0.3$. Results of both the first and the second approximations are very close to the exact solution whereas the solution of the simple diffusion differs significantly.

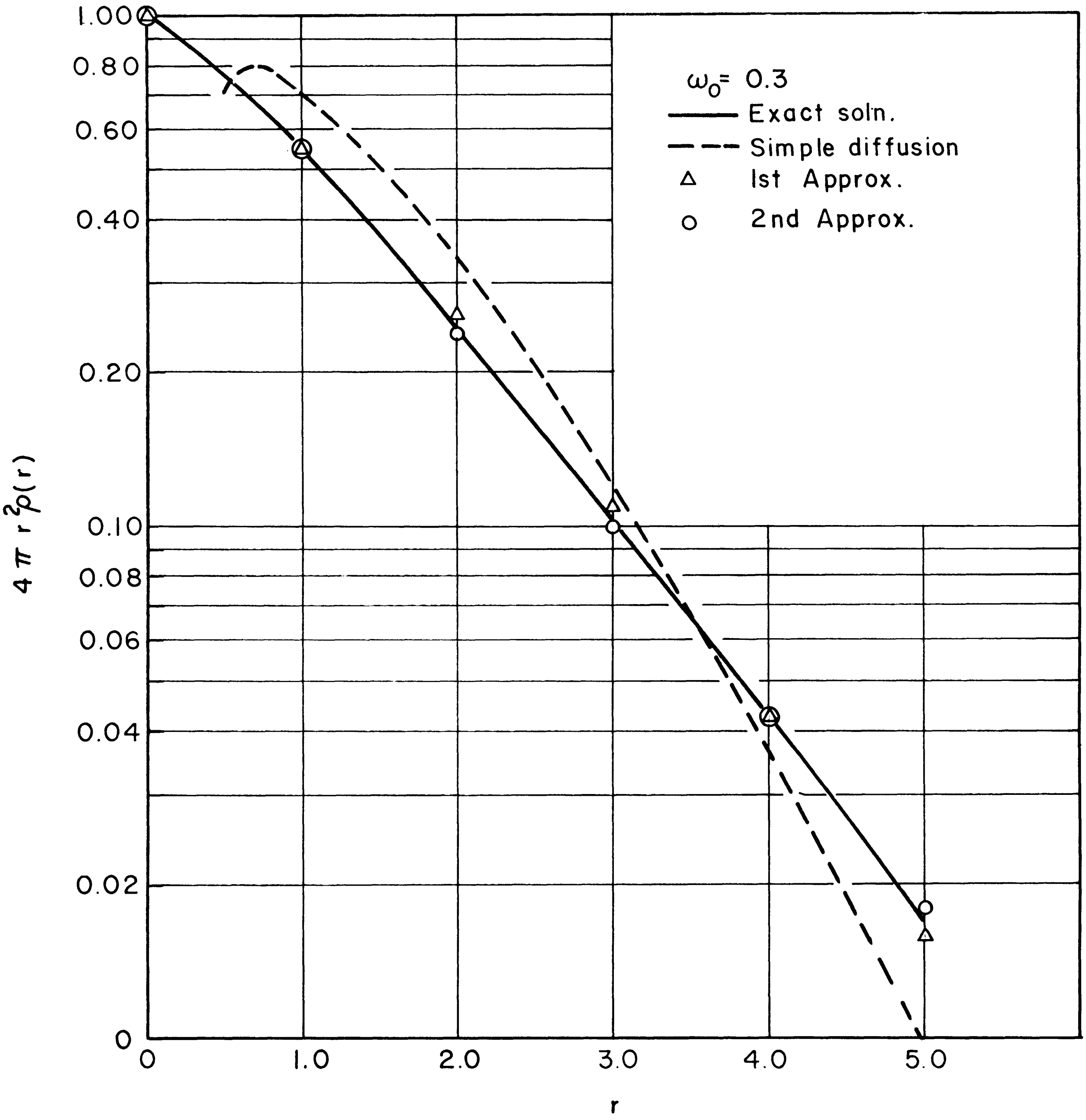


Figure III-1 $4\pi r^2\rho(r)$ as a function of r for an infinite medium with point source

D. Boundary Conditions

The excellent numerical results obtained from the approximate solutions in the case of an infinite medium suggests the possibility of applying the same approximation to a finite region of dispersion. It is well known that the classical diffusion approximation yields very poor results near the boundaries of a dispersion. This inaccuracy probably results from poor representation of the actual boundary conditions. In this section, instead of using physical arguments, a rigorous mathematical derivation for the boundary conditions appropriate to each order of approximation is derived. It will then be shown that the resulting boundary conditions yield results that are very close to the exact solutions.

Mathematically, for a finite region, the integral equation to be solved is

$$\rho(\vec{r}) = S(\vec{r}) + \omega_0 \iiint_V K(|\vec{r}-\vec{r}'|)\rho(\vec{r}')\, dv' \qquad \text{III-35}$$

where V is the volume of the dispersion.

In one dimensional cases such equations are known as Weiner -Hopf equations and has various application in electrical engineering problems(17)(18). For the present problem, the kernal is of the exponential type, and may be represented by

$$K(|\vec{r}-\vec{r}'|) = \frac{1}{4\pi}\frac{1}{|\vec{r}-\vec{r}'|}\sum_i n_i e^{-\beta_i|\vec{r}-\vec{r}'|} = \frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty}\frac{1+ak^2}{1+bk^2+ck^4}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}\, dk_x\, dk_y\, dk_z$$

III-36

The integral Equation (2) may be transformed to the differential Equation

$$\left[c\nabla^4+(a\omega_0-b)\nabla^2+(1-\omega_0)\right]\rho(\vec{r}) = \left(c\nabla^4-b\nabla^2+1\right)S(\vec{r}) \qquad \text{III-14}$$

Note that if c = a = o, and b = 1/3, Equation (36) and (19) represents the first approximation, so that the discussion that follows is applicable to both the first and second approximation. The complete solution of Equation (19) is in general made up of a particular solution depending on $S(\vec{r})$ and a complementary function which is the general solution of the homogeneous Equation, i.e. Equation (19) with $S(\vec{r})=0$. The complimentary function would of course involve arbitrary constants, which are to be determined by the boundary conditions. Since Equation (19) is actually derived from the integral Equation (36), the boundary conditions can be derived directly by mathematical manipulation without resorting to physical arguments.

Let V be the volume of the dispersion, Γ the boundary of V (assume convex) and $\hat{n}$ be the outward normal to Γ from V, then the value of $\rho(v)$ at the boundary can be directly obtained by use of Green's identity. Values of $\rho(v)$ at the boundary Γ must satisfy the condition

$$
\begin{aligned}
& b \iint_{\Gamma} \{ [\rho(\vec{r}) - S(\vec{r})] \nabla K(|\vec{r}-\vec{r}'|) - K(|\vec{r}-\vec{r}'|) \nabla [\rho(\vec{r}) - S(\vec{r})] \} \cdot \hat{n} \, da \\
& + c \iint_{\Gamma} \{ \nabla [\rho(\vec{r}) - S(\vec{r})] \nabla^2 K(|\vec{r}-\vec{r}'|) - \nabla^2 [\rho(\vec{r}) - S(\vec{r})] \nabla K(|\vec{r}-\vec{r}'|) \} \cdot \hat{n} \, da \\
& + c \iint_{\Gamma} \{ K(|\vec{r}-\vec{r}'|) \nabla \nabla^2 [\rho(\vec{r}) - S(\vec{r})] - \nabla^2 [\rho(\vec{r}) - S(\vec{r})] \nabla \nabla^2 K(|\vec{r}-\vec{r}'|) \} \cdot \hat{n} \, da \\
& + a \omega_0 \iint_{\Gamma} [K(|\vec{r}-\vec{r}'|) \nabla \rho(\vec{r}) - \rho(\vec{r}) \nabla K(|\vec{r}-\vec{r}'|)] \cdot \hat{n} \, da \\
& = 0 \qquad \text{III-37}
\end{aligned}
$$

for all $\vec{r}'$ inside V.

The proof of Equation (37) is straight forward, and is outlined in the following steps:

(i) Multiply Equation (19) by $K(|\vec{r}-\vec{r}'|)$ and integrate over v yielding:

$$
\begin{aligned}
& \iiint_{v} K(|\vec{r}-\vec{r}'|) [1 - b\nabla^2 + c\nabla^4] \rho(\vec{r}) \, dv \\
& - \iiint_{v} K(|\vec{r}-\vec{r}'|) [1 - b\nabla^2 + c\nabla^4] S(\vec{r}) \, dv \\
& - \omega_0 \iiint_{v} K(|\vec{r}-\vec{r}'|) [1 - a\nabla^2] \rho(\vec{r}) \, dv = 0 \qquad \text{III-38}
\end{aligned}
$$

(ii) Using Green's identity, Equation (38) is reduced to:

$$
\begin{aligned}
&\iiint_V \rho(\vec{r})\left[1-b\nabla^2+c\nabla^4\right]K(|\vec{r}-\vec{r}'|)\,dv \\
&-\iiint_V S(\vec{r})\left[1-b\nabla^2+c\nabla^4\right]K(|\vec{r}-\vec{r}'|)\,dv \\
&-\omega_0\iiint_V \rho(\vec{r})\left[1-a\nabla^2\right]K(|\vec{r}-\vec{r}'|)\,dv \\
&= -b\iint_\Gamma \left[K(|\vec{r}-\vec{r}'|)\nabla\rho(\vec{r})-\rho(\vec{r})\nabla K(|\vec{r}-\vec{r}'|)\right]\cdot\hat{n}\,da \\
&\quad +c\iint_\Gamma \left[\nabla^2K(|\vec{r}-\vec{r}'|)\nabla\rho(\vec{r})-\rho(\vec{r})\nabla\nabla^2K(|\vec{r}-\vec{r}'|)\right]\cdot\hat{n}\,da \\
&\quad +c\iint_\Gamma \left[K(|\vec{r}-\vec{r}'|)\nabla\nabla^2\rho(\vec{r})-\nabla^2\rho(\vec{r})\nabla K(|\vec{r}-\vec{r}'|)\right]\cdot\hat{n}\,da \\
&\quad +b\iint_\Gamma \left[K(|\vec{r}-\vec{r}'|)\nabla S(\vec{r})-S(\vec{r})\nabla K(|\vec{r}-\vec{r}'|)\right]\cdot\hat{n}\,da \\
&\quad -c\iint_\Gamma \left[\nabla S(\vec{r})\nabla^2K(|\vec{r}-\vec{r}'|)-\nabla^2S(\vec{r})\nabla K(|\vec{r}-\vec{r}'|)\right]\cdot\hat{n}\,da \\
&\quad -c\iint_\Gamma \left[K(|\vec{r}-\vec{r}'|)\nabla\nabla^2S(\vec{r})-S(\vec{r})\nabla\nabla^2K(|\vec{r}-\vec{r}'|)\right]\cdot\hat{n}\,da \\
&\quad +a\omega_0\iint_\Gamma \left[K(|\vec{r}-\vec{r}'|)\nabla\rho(\vec{r})-\rho(\vec{r})\nabla K(|\vec{r}-\vec{r}'|)\right]\cdot\hat{n}\,da\,. \qquad \text{III-39}
\end{aligned}
$$

(iii) From the expression of $K(|\vec{r}-\vec{r}'|)$ given in Equation (36), it can be shown that the left hand side of Equation (39) is identically zero. Collecting the terms given in the right hand side of Equation (39) produces the boundary condition given by Equation (37).

Equation (39) is the general boundary condition that can be derived. It can, in principle, be applied to any geometry and source distribution. However, in applying it to simple problems some simplified version will be considered.

Before considering the simplified version of the boundary conditions, it is to be noted that Equation (35) is a linear integral equation of Fredholm type, and therefore can be solved in terms of orthogonal expansion without resorting to boundary conditions[19]. In principle, for the particular type of $K(|\vec{r}-\vec{r}'|)$ it is possible to find a complete set of eigenfunctions $\phi_i(\vec{r})$ corresponding to eigenvalues λ_i , satisfying the monogeneous equation

$$\phi_i(\vec{r}) = \lambda_i \iiint_V K(|\vec{r}-\vec{r}'|)\, \phi_i(\vec{r}')\, d\sigma' \qquad \text{III-40}$$

If such a set can be found, then, it can be shown that the orthogonal to each other in the sense that

$$\iiint_V \phi_i(\vec{r})\, \phi_j(\vec{r})\, d\sigma = 0 \qquad i \neq j$$

This immediately leads to the solution of Equation (35) in a series form and the result is:

$$\rho(\vec{r}) = \sum_i A_i \phi_i(\vec{r}) \qquad \text{III-41}$$

where

$$A_i = \frac{B_i \lambda_i}{\lambda_i - \omega_0} \qquad \text{III-42}$$

and B_i is related to the source terms by

$$B_i = \frac{\iiint_V S(\vec{r})\, \phi_i(\vec{r})\, d\sigma}{\iiint_V \phi_i^2(\vec{r})\, d\sigma} \qquad \text{III-43}$$

The approximated kernal given by Equation (36) enables one to find λ_i and $\phi_i(\vec{r})$ in a systematic way. Since $\phi_i(\vec{r})$ must satisfy the homogeneous differential equation

$$\left[c\nabla^4 + (a\lambda_i - b)\nabla^2 + (1-\lambda_i)\right]\phi_i(\vec{r}) = 0 \qquad \text{III-44}$$

the forms of $\phi_i(\vec{r})$ corresponding to each λ_i , are known in terms of arbitrary constants and λ_i . Substituting the expression for $\phi_i(\vec{r})$

thus obtained into Equation (40) yields the characteristic equation for λ_i, and the constants for $\phi_i(\vec{r})$. In this one dimensional case, such a procedure has frequently been used e.g. (18). The extension to three dimensional case is straight forward.

In the present work, the primary purpose is to test the accuracy of the approximation, so that only simple problems such as parallel dispersion of finite and infinite thickness are considered. For such one dimensional cases, both the integral Equation and the boundary condition can be greatly simplified. The boundary condition used in the present work therefore combines the idea of the above two methods to minimize the numerical calculations. In the next section, such simplified boundary conditions are illustrated.

E. Solution of One-Dimensional Problem

1. Second Approximation

a) Medium bounded by parallel planes

To test the accuracy of the proposed approximation consider the problem of a parallel plane flux of unit intensity obliquely incident on a parallel dispersion bounded by two planes Z = 0 and $Z = \tau_1$. If the direction of the incidence is inclined at an angle $\Theta = \cos^{-1}\mu_0$ from the Z-axis, then $\rho(\vec{r})$ is a function of Z only so that Equation (19) is reduced to:

$$\left[c\frac{d^4}{dz^4} + (a\omega_0 - b)\frac{d^2}{dz^2} + (1-\omega_0)\right]\rho(z) = \left[c\frac{d^4}{dz^4} - b\frac{d^2}{dz^2} + 1\right] S(z)$$

where $S(z)$ is obviously of the form

$$S(z) = e^{-z/\mu_0}$$

The general solution of Equation (19) can be written in the form:

$$\rho(z) = A_1 e^{-\alpha_1 z} + B_1 e^{-\alpha_2 z} + A_2 e^{\alpha_1 z} + B_2 e^{\alpha_2 z} + (1+D)e^{-z/\mu_0} \qquad \text{III-45}$$

where

$$D = \frac{\omega_0\left(1-\frac{a}{\mu_0^2}\right)}{(1-\omega_0) + (a\omega_0 - b)\frac{1}{\mu_0^2} + c/\mu_0^4} \qquad \text{III-46}$$

The constants A_1 , A_2, B_1, B_2, can be obtained from the original integral equation. In the present one dimensional case, this integral equation takes the form:

$$\rho(z) = e^{-z/\mu_0} + \omega_0 \iiint_V \rho(z')\, K(|\vec{r}-\vec{r}'|)\, dv' \qquad \text{III-47}$$

where $K(|\vec{r}-\vec{r}'|)$ is given by Equation (18) as

$$K(|\vec{r}-\vec{r}'|) = \frac{1}{4\pi|\vec{r}-\vec{r}'|}\left[n_1 e^{-\beta_1|\vec{r}-\vec{r}'|} + n_2 e^{-\beta_2|\vec{r}-\vec{r}'|}\right]$$

By carrying out the integrations that are independent of Z, Equation (47) is reduced to the form:

$$\rho(z) = e^{-z/\mu_0} + \frac{\omega_0}{2}\int_0^{\tau_1}\left[\frac{n_1}{\beta_1} e^{-\beta_1|z-z'|} + \frac{n_2}{\beta_2} e^{-\beta_2|z-z'|}\right]\rho(z')\,dz' \qquad \text{III-48}$$

Now, substituting Equation (45) into Equation (48), and collecting terms involving the same exponential forms of Z, the following boundary conditions for A and B are obtained

(i) For $\omega_0 \neq 1$

$$\frac{A_1}{-\alpha_1+\beta_1} + \frac{A_2}{\alpha_1+\beta_1} + \frac{B_1}{-\alpha_2+\beta_1} + \frac{B_2}{\alpha_2+\beta_1} + \frac{1+D}{-\frac{1}{\mu_0}+\beta_1} = 0$$

$$\frac{A_1}{-\alpha_1+\beta_2} + \frac{A_2}{\alpha_1+\beta_2} + \frac{B_1}{-\alpha_2+\beta_2} + \frac{B_2}{\alpha_2+\beta_2} + \frac{1+D}{-\frac{1}{\mu_0}+\beta_2} = 0$$

$$\frac{A_1 e^{-\alpha_1\tau_1}}{\alpha_1+\beta_1} + \frac{A_2 e^{\alpha_1\tau_1}}{-\alpha_1+\beta_1} + \frac{B_1 e^{-\alpha_2\tau_1}}{\alpha_2+\beta_1} + \frac{B_2 e^{\alpha_2\tau_1}}{-\alpha_2+\beta_1} + \frac{(1+D)e^{-\tau_1/\mu_0}}{\frac{1}{\mu_0}+\beta_1} = 0$$

$$\frac{A_1 e^{-\alpha_1\tau_1}}{\alpha_1+\beta_2} + \frac{A_2 e^{\alpha_1\tau_1}}{-\alpha_1+\beta_2} + \frac{B_1 e^{-\alpha_2\tau_1}}{\alpha_2+\beta_2} + \frac{B_2 e^{\alpha_2\tau_1}}{-\alpha_2+\beta_2} + \frac{(1+D)e^{-\tau_1/\mu_0}}{\frac{1}{\mu_0}+\beta_2} = 0$$

The set of equations can be solved for the four constants, so that the values of $\rho(Z)$ are completely determined.

The above equations are valid for $\omega_0 \neq 1$. (ii) For $\omega_0 = 1$,

(See table 1)

$\alpha_1 = 0 \qquad \alpha_2 = 2.9580$

So that the solution of Equation (19) is

$$\rho(z) = A_1 + A_2 z + B_1 e^{-\alpha_2 z} + B_2 e^{\alpha_2 z} + (1+D) e^{-z/\mu_0} \qquad \text{III-49}$$

The "boundary conditions" for determining A_1, A_2, B_1, B_2 are slightly modified to:

$$\frac{A_1}{\beta_1} - \frac{A_2}{\beta_1^2} + \frac{B_1}{-\alpha_2+\beta_1} + \frac{B_2}{\alpha_2+\beta_1} + \frac{1+D}{-\frac{1}{\mu_0}+\beta_1} = 0$$

$$\frac{A_1}{\beta_2} - \frac{A_2}{\beta_2^2} + \frac{B_1}{-\alpha_2+\beta_2} + \frac{B_2}{\alpha_2+\beta_2} + \frac{1+D}{-\frac{1}{\mu_0}+\beta_2} = 0$$

$$\frac{A_1}{\beta_1} + \frac{(1+\beta_1\tau_1)}{\beta_1^2}A_2 + \frac{B_1 e^{-\alpha_2\tau_1}}{\alpha_2+\beta_1} + \frac{B_2 e^{\alpha_2\tau_1}}{-\alpha_2+\beta_1} + \frac{(1+D)e^{-\tau_1/\mu_0}}{\frac{1}{\mu_0}+\beta_1} = 0$$

$$\frac{A_1}{\beta_2} + \frac{(1+\beta_2\tau_1)}{\beta_2^2}A_2 + \frac{B_1 e^{-\alpha_2\tau_1}}{\alpha_2+\beta_2} + \frac{B_2 e^{\alpha_2\tau_1}}{-\alpha_2+\beta_2} + \frac{(1+D)e^{-\tau_1/\mu_0}}{\frac{1}{\mu_0}+\beta_2} = 0$$

b) Semi-infinite medium

For a semi-infinite medium, $\tau_1 \to \infty$ the results can be simplified considerably. Following the same procedure as in the finite case,

(i) For $\omega_0 \neq 1$

$$\rho(z) = A_1 e^{-\alpha_1 z} + B_1 e^{-\alpha_2 z} + (1+D) e^{-z/\mu_0} \qquad \text{III-50}$$

where D is given by Equation (46) and A_1 and B_1 are determined from:

$$\frac{A_1}{-\alpha_1+\beta_1} + \frac{B_1}{-\alpha_2+\beta_1} + \frac{1+D}{-\frac{1}{\mu_0}+\beta_1} = 0$$

$$\frac{A_1}{-\alpha_1+\beta_2} + \frac{B_1}{-\alpha_2+\beta_2} + \frac{1+D}{-\frac{1}{\mu_0}+\beta_2} = 0$$

(ii) For $\omega_0 = 1$

$$\rho(z) = A_1 + B_1 e^{-\alpha_2 z} + (1+D) e^{-z/\mu_0} \qquad \text{III-51}$$

where A_1 and B_1 are determined from:

$$\frac{A_1}{\beta_1} + \frac{B_1}{-\alpha_2+\beta_1} + \frac{1+D}{-\frac{1}{\mu_0}+\beta_1} = 0$$

$$\frac{A_1}{\beta_2} + \frac{B_1}{-\alpha_2+\beta_2} + \frac{1+D}{-\frac{1}{\mu_0}+\beta_2} = 0$$

2. First Approximation

The formula above were derived for the second approximation. The corresponding formulas for the first approximation can simply be obtained by letting a = c = o and b = 1/3. The explicit results for the one dimensional problem, using the first approximation are given below:

(i) Integral Equation

$$\rho(z) = e^{-z/\mu_0} + \frac{\sqrt{3}}{2}\omega_0 \int_0^{\tau_1} e^{-\sqrt{3}|z-z'|}\rho(z')\,dz' \qquad \text{III-48'}$$

(ii) Differential Equation

$$\left[\frac{1}{3}\frac{d^2}{dz^2} - (1-\omega_0)\right]\rho(z) = \left(\frac{1}{3\mu_0^2} - 1\right) e^{-z/\mu_0} \qquad \text{III-19'}$$

(iii) General solution of the differential equation

$$\rho(z) = A_1 e^{-\alpha_1 z} + A_2 e^{\alpha_1 z} + (1+D)\, e^{-z/\mu_0} \qquad \text{III-45'}$$

where

$$D = \frac{\omega_0}{(1-\omega_0) - \frac{1}{3\mu_0}} \qquad \text{III-46'}$$

and $\alpha_1 = [3(1-\omega_0)]^{1/2}$

(iv) Semi-infinite dispersion, $\omega_0 \neq 1$

$$\rho(z) = A_1 e^{-\alpha_1 z} + (1+D)\, e^{-z/\mu_0} \qquad \text{III-50'}$$

where

$$A_1 = \frac{-(1+D)}{-\frac{1}{\mu_0}+\sqrt{3}}\,(\sqrt{3}-\alpha_1) \qquad \text{III-52}$$

(v) Semi-infinite dispersion, $\omega_0 = 1$

$$\rho(z) = A_1 + (1+D)\, e^{-z/\mu_0} \qquad \text{III-51'}$$

$$A_1 = \frac{-(1+D)}{\left(\sqrt{3}-\frac{1}{\mu_0}\right)}\sqrt{3} \qquad \text{III-53}$$

(vi) Parallel plane dispersion $\omega_0 \neq 1$

$$\rho(z) = A_1 e^{-\alpha_1 z} + A_2 e^{\alpha_1 z} + (1+D)\, e^{-z/\mu_0} \qquad \text{III-45'}$$

where A_1 and A_2 are determined from,

$$\frac{A_1}{-\alpha_1+\sqrt{3}} + \frac{A_2}{\alpha_1+\sqrt{3}} + \frac{(1+D)}{-\frac{1}{\mu_0}+\sqrt{3}} = 0$$

$$\frac{A_1 e^{-\alpha_1\tau}}{\alpha_1+\sqrt{3}} + \frac{A_2 e^{\alpha_1\tau}}{\alpha_1+\sqrt{3}} + \frac{(1+D)e^{-\tau/\mu_0}}{\frac{1}{\mu_0}+\sqrt{3}} = 0$$

(vii) Parallel plane dispersion, $\omega_0 = 1$

$$\rho(z) = A_1 + A_2 z + (1+D)\, e^{-z/\mu_0} \qquad \text{III-49'}$$

where A_1 and A_2 are determined by

$$\frac{A_1}{\sqrt{3}} - \frac{A_2}{3} + \frac{1+D}{-\frac{1}{\mu_0}+\sqrt{3}} = 0$$

$$\frac{A_1}{\sqrt{3}} + \frac{(1+\sqrt{3}\,\tau_1)}{3} A_2 + \frac{(1+D)\, e^{-\tau_1/\mu_0}}{\sqrt{3}+\frac{1}{\mu_0}} = 0$$

F. Calculation of Albedo

Available results on the exact solutions of the parallel plane problem are generally expressed in terms of the reflectance of a half space, and the transmittance T or reflectance R of a parallel plane dispersion. One may calculate these parameters by direct integration of the expression for $\rho(Z)$. The resulting expressions are given below:

1. First Approximation

(i) Semi-infinite medium, $\omega_0 \neq 1$ III-50'

$$\rho(z) = A_1 e^{-\alpha_1 z} + (1+D) e^{-z/\mu_0}$$

$$R = \frac{\omega_0 A_1}{2\alpha_1 \mu_0}\left\{1 - \frac{1}{\alpha_1}\ln(1+\alpha_1)\right\} + \frac{\omega_0(1+D)}{2}\left\{1 - \mu_0 \ln\left(1+\frac{1}{\mu_0}\right)\right\} \qquad \text{III-54'}$$

(ii) Semi-infinite medium, $\omega_0 = 1$

$$\rho(z) = A_1 + (1+D)e^{-z/\mu_0} \qquad \text{III-51'}$$

$$R = \frac{\omega_0 A_1}{4\mu_0} + \frac{\omega_0(1+D)}{2}\left\{1 - \mu_0 \ln\left(1+\frac{1}{\mu_0}\right)\right\} \qquad \text{III-55'}$$

(iii) Parallel plane dispersion, $\omega_0 \neq 1$

$$\rho(z) = A_1 e^{-\alpha_1 z} + A_2 e^{\alpha_1 z} + (1+D)e^{-z/\mu_0} \qquad \text{III-45'}$$

Then

$$R = A_1 R_{\alpha_1} + A_2 R_{-\alpha_1} + (1+D) R_{\frac{1}{\mu_0}} \qquad \text{III-56'}$$

and

$$T = A_1 T_{\alpha_1} + A_2 T_{-\alpha_1} + (1+D) T_{\frac{1}{\mu_0}} \qquad \text{III-57'}$$

where

$$R_{\alpha_i} = \frac{\omega_0}{2\alpha_i \mu_0}\left\{1 - \frac{1}{\alpha_i}\ln|1+\alpha_i| + e^{-\tau_1\alpha_i}\frac{1}{\alpha_i}E_1(\tau_1) - e^{-\tau_1\alpha_i}E_2(\tau_1) - \frac{E[\tau_1(1+\alpha_i)]}{\alpha_i}\right\}$$ III-58

and

$$T_{\alpha_i} = \frac{-\omega_0}{2\alpha_i \mu_0}\left\{e^{-\alpha_i\tau_1} + \frac{e^{-\alpha_i\tau_1}}{\alpha_i}\ln|1-\alpha_i| - \frac{1}{\alpha_i}E_1(\tau_1) - E_2(\tau_1) + \frac{e^{-\alpha_i\tau_1}E_1[\tau_1(1-\alpha_i)]}{\alpha_i}\right\}$$ III-59

and

$$E_n = \int_T^\infty \frac{e^{-x}}{x^n}\,dx$$

(iv) Parallel plane dispersion, $\omega_0 = 1$

$$\rho(z) = A_1 + A_2 z + (1+D)e^{-z/\mu_0}$$ III-49'

$$R = A_1\frac{\omega_0}{2\mu_0}\left[\tfrac{1}{2} - E_3(\tau_1)\right] + A_2\frac{\omega_0}{2\mu_0}\left[\tfrac{1}{3} - E_4(\tau_1) - \tau_1 E_3(\tau_1)\right] + (1+D)R_{\frac{1}{\mu_0}}$$ III-60'

$$T = A_1\frac{\omega_0}{2\mu_0}\left[\tfrac{1}{2} - E_3(\tau_1)\right] + A_2\frac{\omega_0}{2\mu_0}\left[-\tfrac{1}{3} + \tfrac{1}{2}\tau_1 + E_4(\tau_1)\right] + (1+D)T_{\frac{1}{\mu_0}}$$ III-61'

2. Second Approximation

(i) Semi-infinite medium $\omega_0 \neq 1$

$$\rho(z) = A_1 e^{-\alpha_1 z} + B_1 e^{-\alpha_2 z} + (1+D)e^{-z/\mu_0}$$ III50

$$R = \frac{\omega_0 A_1}{2\alpha_1\mu_0}\left\{1 - \frac{1}{\alpha_1}\ln(1+\alpha_1)\right\} + \frac{\omega_0 B_1}{2\alpha_2\mu_0}\left\{1 - \frac{1}{\alpha_2}\ln(1+\alpha_2)\right\} + \frac{\omega_0(1+D)}{2}\left\{1 - \mu_0\ln\left(1+\frac{1}{\mu_0}\right)\right\}$$ III-54

(ii) Semi-infinite medium $\omega_0 = 1$

$$\rho(z) = A_1 + B_1 e^{-\alpha_2 z} + (1+D)e^{-z/\mu_0}$$ III-51

$$R = \frac{\omega_0 B_1}{2\alpha_2\mu_0}\left\{1 - \frac{1}{\alpha_2}\ln(1+\alpha_2)\right\} + \frac{\omega_0 A_1}{4\mu_0} + \frac{\omega_0(1+D)}{2}\left\{1 - \mu_0\ln\left(1+\frac{1}{\mu_0}\right)\right\}$$ III-55

(iii) Parallel plane medium

$$\rho(z) = A_1 e^{-\alpha_1 z} + A_2 e^{\alpha_1 z} + B_1 e^{-\alpha_2 z} + B_2 e^{\alpha_2 z} + (1+D) e^{-z/\mu_0} \qquad \text{III-45}$$

$$R = A_1 R_{\alpha_1} + A_2 R_{-\alpha_1} + B_1 R_{\alpha_2} + B_2 R_{-\alpha_2} + (1+D) R_{\frac{1}{\mu_0}} \qquad \text{III-56}$$

$$T = A_1 T_{\alpha_1} + A_2 T_{-\alpha_1} + B_1 T_{\alpha_2} + B_2 T_{-\alpha_2} + (1+D) T_{\frac{1}{\mu_0}} \qquad \text{III-57}$$

where R_{α_i} and T_{α_i} are defined as Equations (58) and (59) respectfully.

(iv) Parallel plane dispersion $\quad \omega_0 = 1$

$$\rho(z) = A_1 + A_2 z + B_1 e^{-\alpha_2 z} + B_2 e^{\alpha_2 z} + (1+D) e^{-z/\mu_0} \qquad \text{III-49}$$

$$R = B_1 R_{\alpha_1} + B_2 R_{-\alpha_1} + \frac{\omega_0}{2\mu_0}\left[\tfrac{1}{2} - E_3(\tau_1)\right] A_1 + \frac{\omega_0}{2\mu_0}\left[\tfrac{1}{3} - E_4(\tau_1) - \tau_1 E_3(\tau_1)\right] A_2 + (1+D) R_{\frac{1}{\mu_0}} \qquad \text{III-60}$$

$$T = B_1 T_{\alpha_1} + B_2 T_{-\alpha_1} + \frac{\omega_0}{2\mu_0}\left[\tfrac{1}{2} - E_3(\tau_1)\right] A_1 + \frac{\omega_0}{2\mu_0}\left[-\tfrac{1}{3} + \tfrac{1}{2}\tau_1 + E_4(\tau_1)\right] A_2 + (1+D) T_{\frac{1}{\mu_0}} \qquad \text{III-61}$$

G. Numerical Results

Some typical numerical results calculated by using the appropriate approximate formulas derived in the previous section are plotted in Figures III 2-8.

In Figure III-2, the integrated reflectance R for a semi-infinite medium is plotted versus μ_0 for $\omega_0 = 0.9$ and 0.5. The results of the second approximation are indistinguishable from the exact values. In Figures III 3-5 the integrated reflectance R is plotted versus ω_0 for several values of μ_0. The numerical values were calculated from the first and second approximations. Again, the second approximations are indistinguishable from the exact values and the first approximation are acceptably accurate.

In Figure III-6, the reflectance R for a finite medium is plotted against μ_0 for several ω_0. Again the results of the second approximation are very good. The deviation of the first approximation from the exact solution becomes bigger as ω_0 increases. In Figures III-7 and 8, the diffusive transmission and the integrated transmission are plotted against μ_0 for several ω_0. Only second approximations are compared with the exact solutions.

From all these figures, it is obvious that the results of the second approximation are practically indistinguishable from the exact solution for all conditions, and the simple diffusion equation with correct boundary conditions (i.e. the first approximation) is an acceptable approximation.

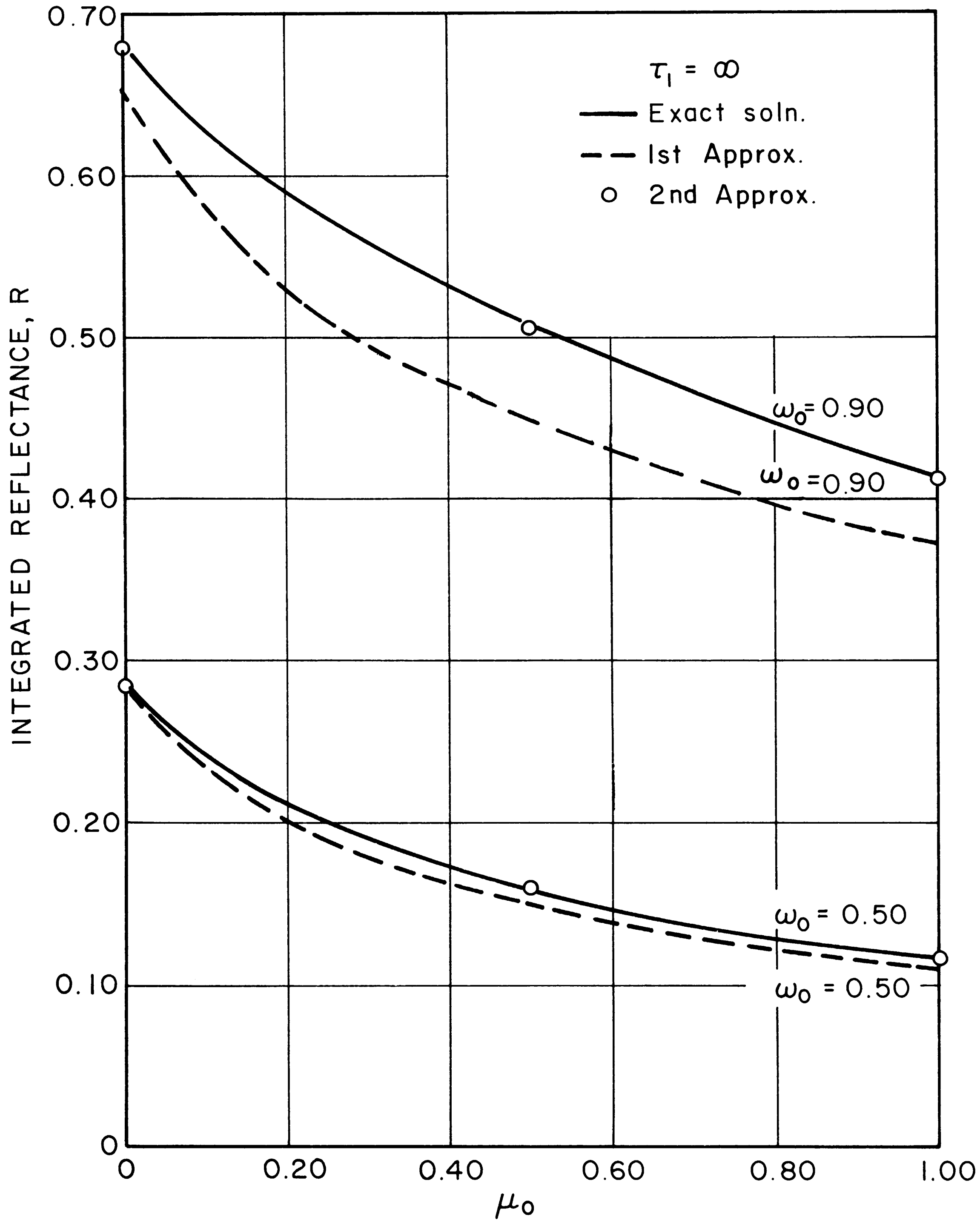


Figure III-2 Integrated reflectance as a function of incident angle, R vs μ_o, parameters of ω_o, and $\tau_1 = \infty$

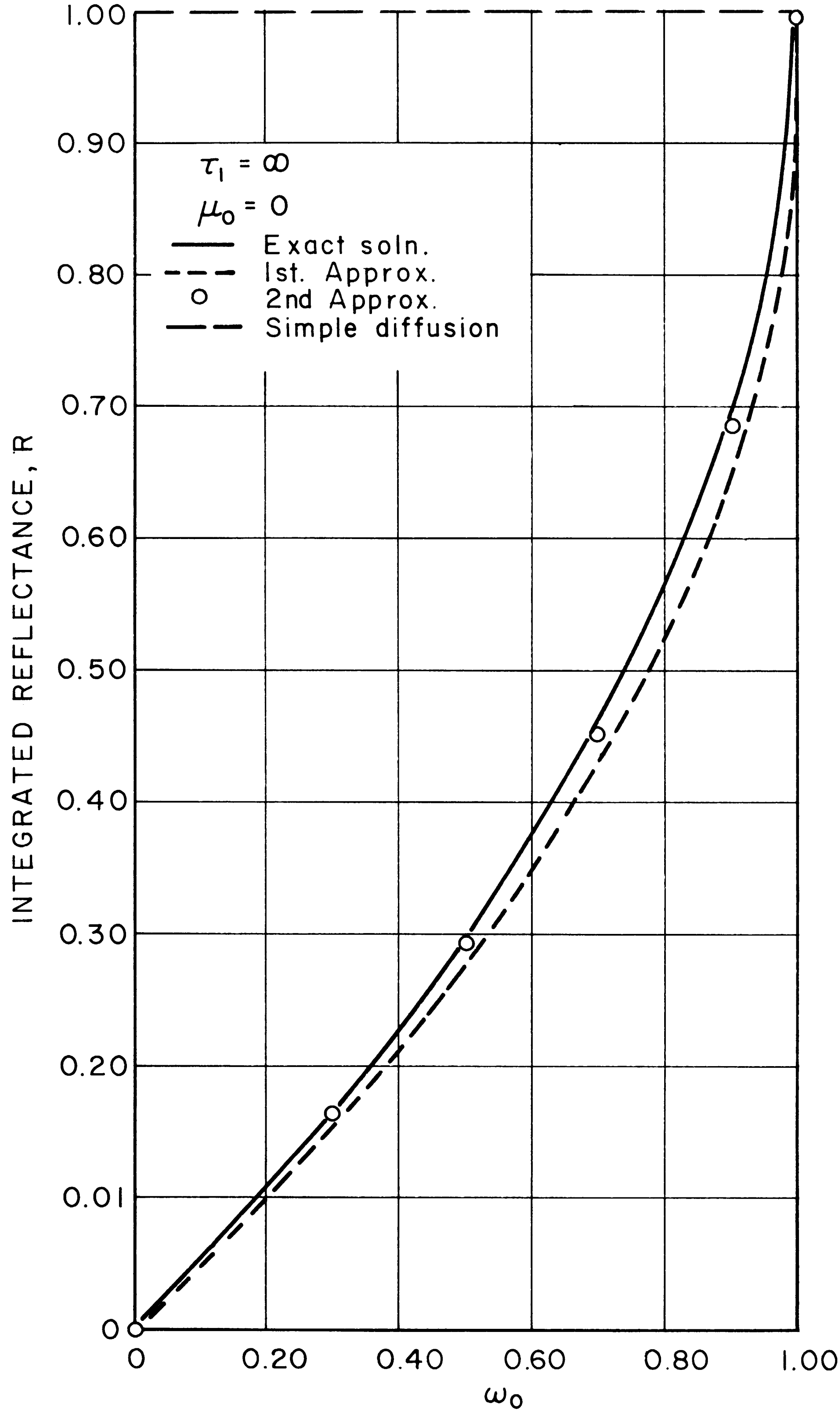


Figure III-3 Integrated reflectance as a function of albedo for single scattering, R vs ω_0, $\mu_0 = 0$, and $\tau_1 = \infty$

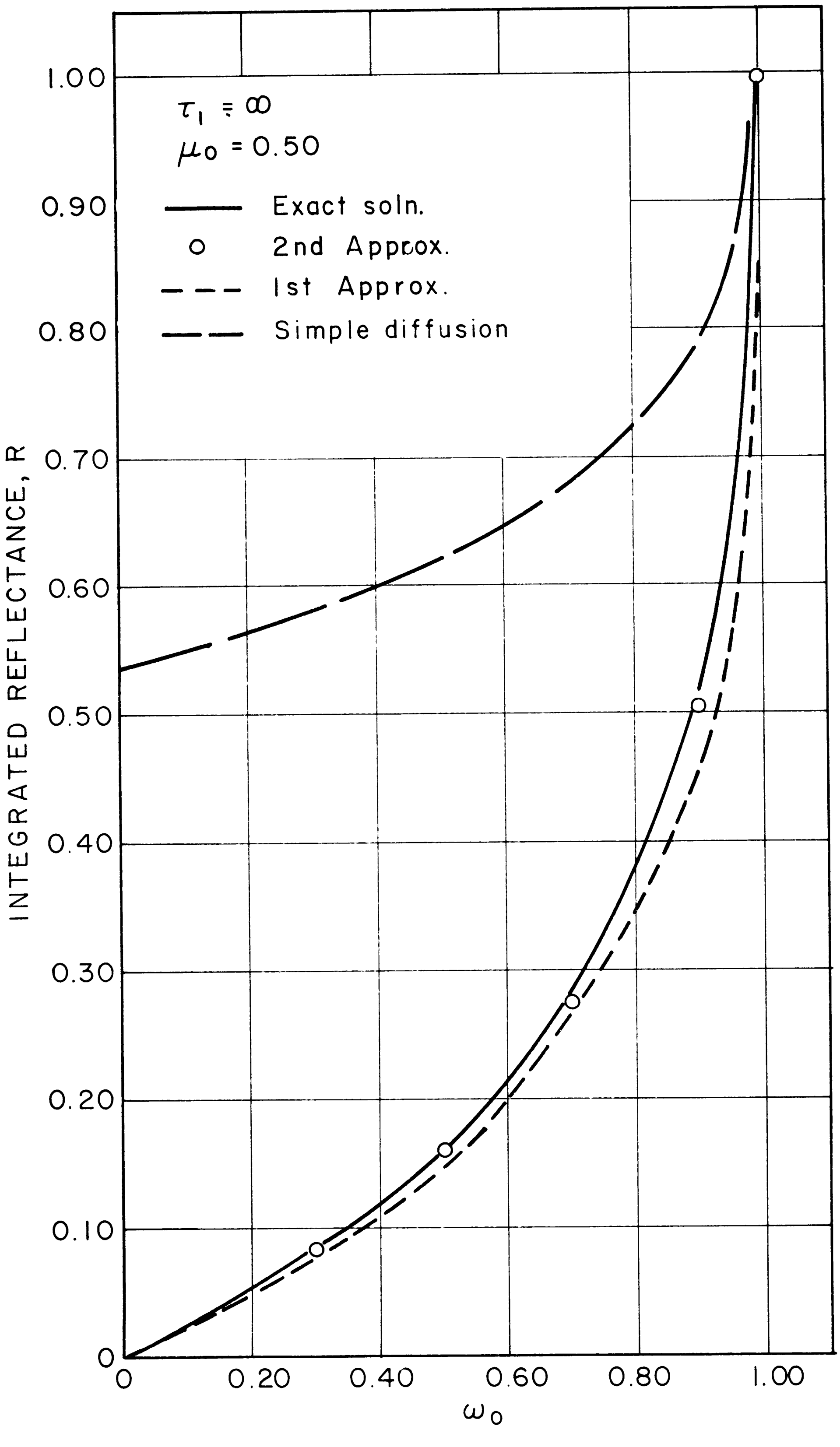


Figure III-4 Integrated reflectance as a function of albedo for single scattering, R vs. ω_0 , μ_o = 0.5, and τ_1 = oo

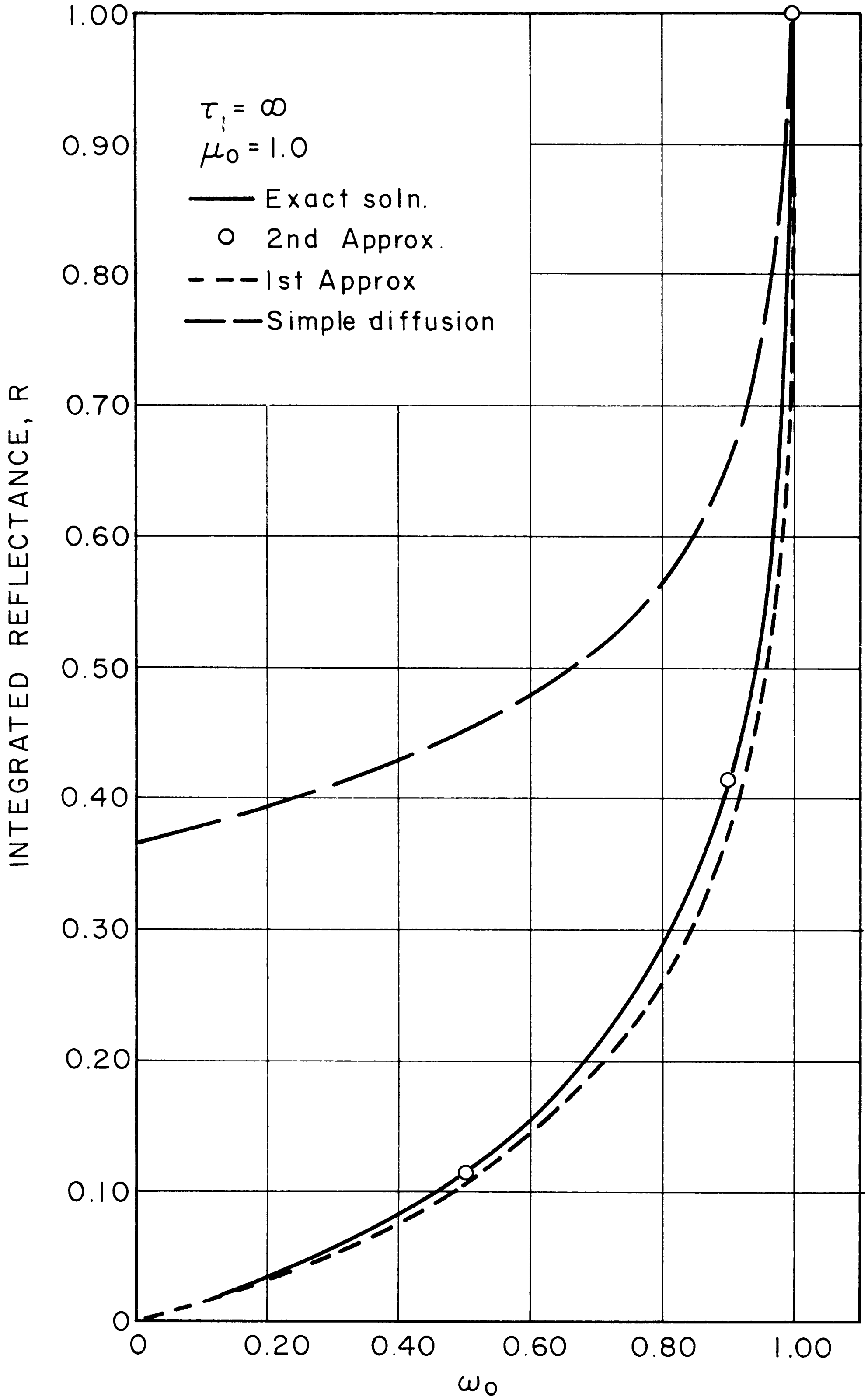


Figure III-5 Integrated reflectance as a function of albedo for single scattering, R vs. ω_o μ_o = 1.0, and $\tau_1 = \infty$

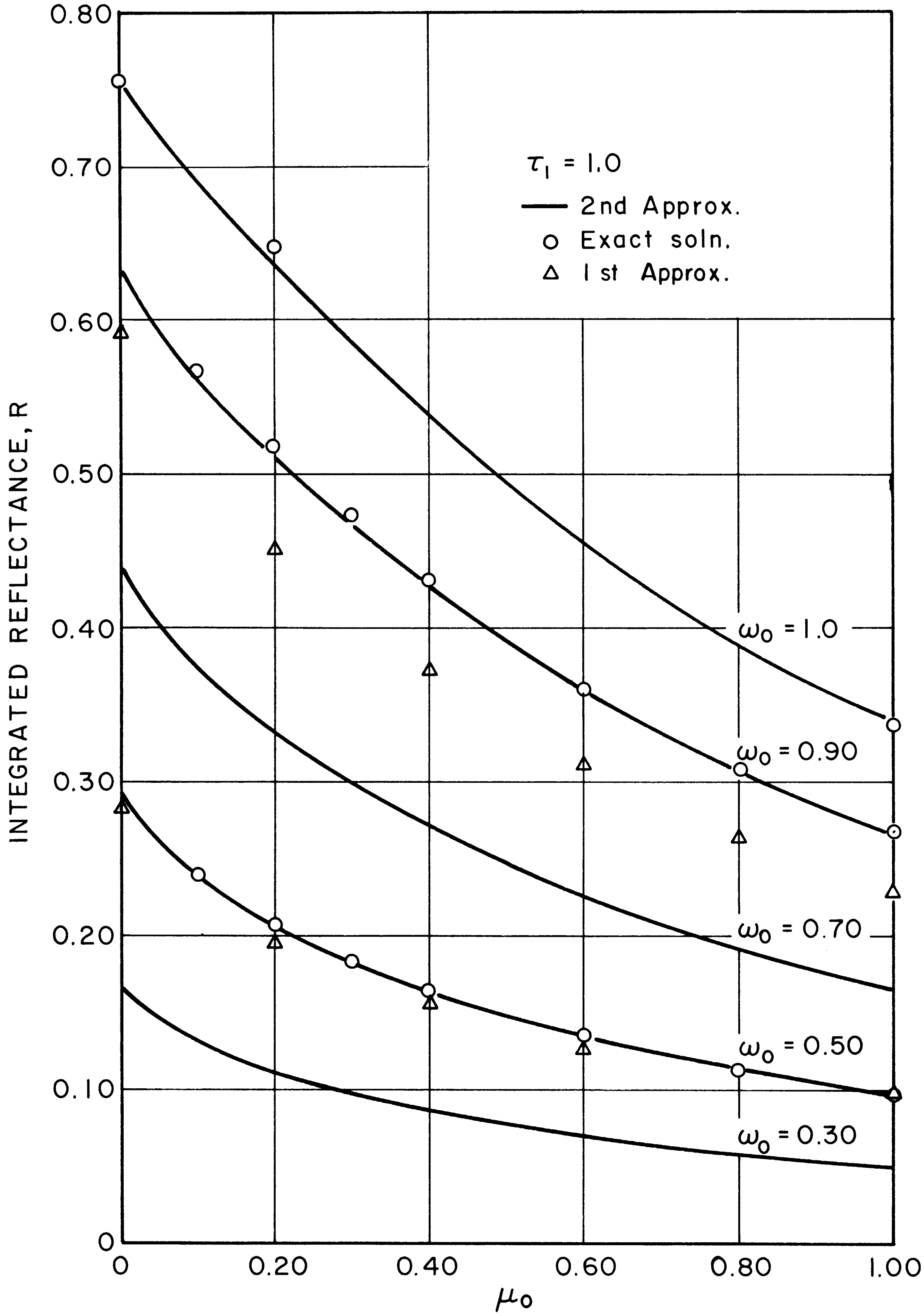


Figure III-6 Integrated reflectance as a function of incident angle, R vs. μ_o, parameters of ω_o, and $\tau_1 = \infty$

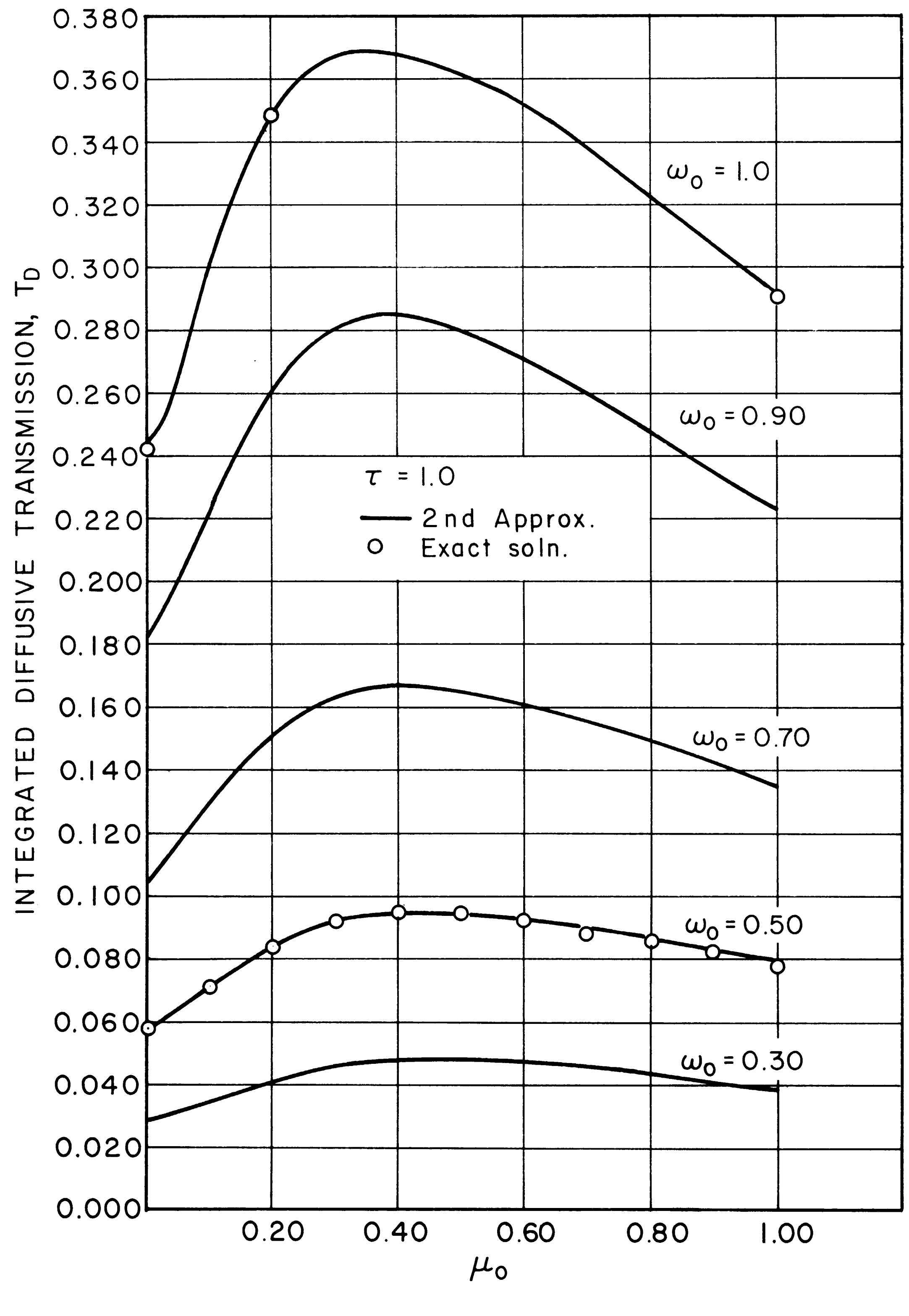


Figure III-7 Integrated diffusive transmission as a function of incident angle, T_D vs. μ_0, parameters of ω_0, and $\tau_1 = 1$

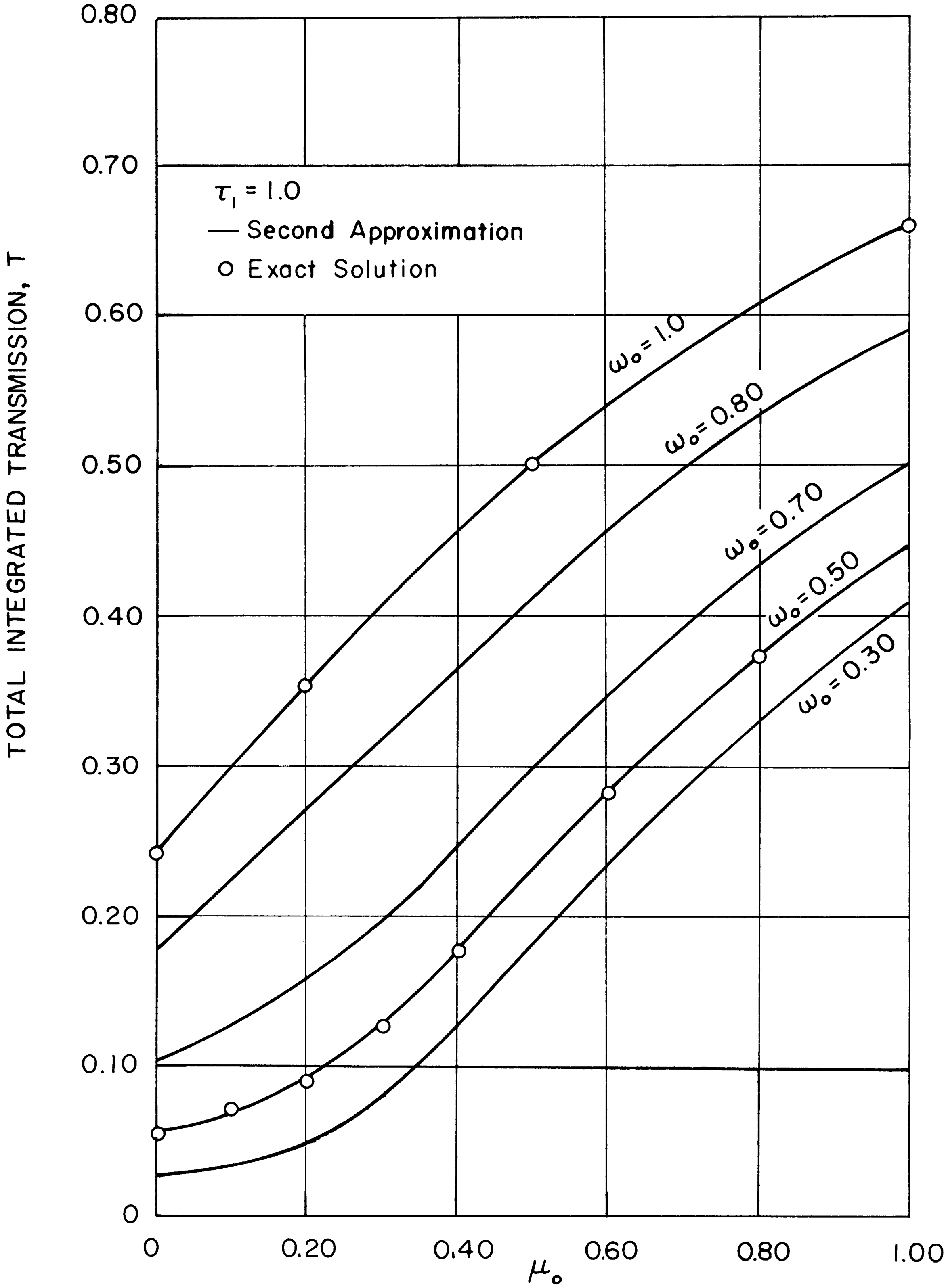


Figure III-8 Total integrated transmission as a function of incident angle, T vs μ_o, parameters of ω_o , and τ_1 = 1

H. Conclusions

From the excellent numerical check with the exact solutions of one dimensional problems with isotropic scattering, it is reasonable to believe that the diffusion-type approximation, if used with correct boundary conditions, can yield very good results which are adequate for all practical purposes. In the present work a systematic approach to the diffusion-type approximation has been developed. A method of determining the correct boundary conditions appropriate to each order of approximation has also been developed.

Since the exact solutions of the multiple scattering problem are generally difficult or impossible to obtain, the need for a comparatively simple, yet reasonably accurate approximate method of solution is obvious. It therefore seems worthwhile, to exploit further the possibilities of using the diffusion-type approximation. This extended investigation should proceed along the following two directions: (i) use of the suggested method to find approximate solutions for geometries where exact solutions do not exist, such as a point source in front of a parallel dispersion, (ii) extension of the method to anisotropic scattering, for which some exact solutions are now available for comparison.

SUMMARY AND CONCLUSIONS

1. A satisfactory method was developed for numerical integration of the transport equation for parallel plane radiation obliquely incident on a parallel-plane dispersion. The important advance is in the allowance for strongly anisotropic scattering. Previous results were limited to isotropic or Rayleigh (nearly isotropic) scattering.

2. The solutions are exact except for errors in the numerical procedure itself. These errors arise from numerical roundoff, from termination of the iterative procedure at an arbitrary point and from integration by quadrature. The error due to the use of quadrature arises from the use of a finite number of terms, i.e., from division of the integration into a finite number of intervals. The numerical results agree very well with previous results for special cases (isotropic and Rayleigh scattering, and anisotropic scattering by semi-infinite dispersions) indicating that the net numerical errors are not serious.

3. The numerical and graphical results illustrate the effects of

   a) the phase function for single scattering
   b) the optical thickness of the dispersion
   c) the angle of incidence
   d) the albedo for single scattering (ratio of scattered to scattered plus absorbed radiation)

4. The numerical results are limited to finite absorption since the iterative procedure did not converge satisfactorily for zero absorption. It appears feasible but unnecessary to develop a separate numerical procedure for zero absorption since the results for small finite absorption can readily be extrapolated to zero absorption.

5. The reflectance and transmission for strongly anisotropic scattering was found to differ decisively from that for isotropic and Rayleigh scattering. A similar result was previously found for semi-infinite dispersions. It is therefore concluded that solutions for isotropic scattering do not provide a reliable guide for the reflectance or transmission for dispersions of liquid or solid particles.

6. The transmission and reflectance were found to be dependent primarily on the fraction of the radiation scattered into the backward (or forward) hemisphere. Only slight dependence was found on the other characteristics of the phase function. This suggests that the complicated phase functions associated with large circumference to wavelength ratios can be approximated by simple functions without serious error if the back-scattered fraction is preserved. This result is important from a computational point-of-view since the extent and cost of the computations goes up rapidly with the number of terms used to represent the phase function.

7. Many approximate models have been proposed for multiple scattering. The solutions based on these models were compared with the "exact" values for both isotropic and anisotropic scattering. The six-flux model was found to be the best of these models but to be a fair approximation, Richard's modified diffusion model and the two-flux model were poorer in all cases and the simple diffusion model was seriously in error in all cases.

8. Despite their inability to provide good absolute values for the transmission and reflectance the above approximate models were successfully utilized to provide simple expressions for interpolation and extrapolation of the exact values.

9. A new variable-order diffusion-type model was developed which yields very accurate results for isotropic scattering. It is proposed to develop this model further for anisotropic scattering.

10. All accurate solutions for multiple scattering are currently limited to one-dimensional radiant transport. However both the numerical method and the variable order diffusion-type model hold promise for the computation of two-dimensional radiant transport.

APPENDIX A

Derivation and List of Equations

1. Derivation of Explicit Relations for $b_m(\tau_1, \mu, \mu_0)$ and $c_m(\tau_1, \mu, \mu_0)$.

The diffuse reflected intensity is

$$\frac{I(0,\mu,\varphi)}{I_0(-\mu)} = \frac{\mu_0}{4\pi}\sum_{m=0}^{N}(2-\delta_{0,m})\cos m\varphi \sum_{l=m}^{N}(-1)^{l+m} F_l^m(\tau_1,\mu,\mu_0) \qquad \text{I-14}$$

substituting relation I-3 for $F_l^m(\tau_1,\mu,\mu_0)$ and letting

$$\omega_l^m = \omega_0 A_l \frac{(l-m)!}{(l+m)!} \qquad \text{A1-1}$$

gives

$$\frac{I(0,\mu,\varphi)}{I_0(-\mu)} = \frac{\mu_0\omega_0}{4\pi(\mu+\mu_0)}\Bigg\{\sum_{l=0}^{N}(-1)^l A_l\left[\psi_l^0(\mu)\psi_l^0(\mu_0) - \phi_l^0(\mu)\phi_l^0(\mu_0)\right]$$
$$+2\sum_{m=1}^{N}\cos m\varphi\sum_{l=m}^{N}(-1)^{l+m}A_l\frac{(l-m)!}{(l+m)!}\left[\psi_l^m(\mu_0)\psi_l^m(\mu) - \phi_l^m(\mu_0)\phi_l^m(\mu)\right]\Bigg\} \qquad \text{A1-2}$$

and for N = 2

$$\cos 2\varphi = 2\cos^2\varphi - 1$$

$$A_0 \equiv 1$$

$$\frac{I(0,\mu,\varphi)}{I_0(-\mu)} = \frac{\mu_0\omega_0}{4\pi(\mu+\mu_0)}\Big\{\left[\psi_0^0(\mu_0)\psi_0^0(\mu) - \phi_0^0(\mu_0)\phi_0^0(\mu)\right] - A_1\left[\psi_1^0(\mu_0)\psi_1^0(\mu) - \phi_1^0(\mu_0)\phi_1^0(\mu)\right]$$
$$+A_2\left[\psi_2^0(\mu_0)\psi_2^0(\mu) - \phi_2^0(\mu_0)\phi_2^0(\mu)\right] + A_1\cos\varphi\left[\psi_1^1(\mu_0)\psi_1^1(\mu) - \phi_1^1(\mu_0)\phi_1^1(\mu)\right]$$
$$-A_2\frac{\cos\varphi}{3}\left[\psi_2^1(\mu_0)\psi_2^1(\mu) - \phi_2^1(\mu_0)\phi_2^1(\mu)\right]$$
$$+A_2\frac{2\cos^2\varphi - 1}{6}\left[\psi_2^2(\mu_0)\psi_2^2(\mu) - \phi_2^2(\mu_0)\phi_2^2(\mu)\right]\Big\} \qquad \text{A1-3}$$

Therefore, let

$$K = \frac{\mu_o \omega_o}{4\pi(\mu+\mu_o)} \qquad \text{A1-4}$$

$$Q_o = [\psi_o^o(\mu_o)\psi_o^o(\mu) - \phi_o^o(\mu_o)\phi_o^o(\mu)] - A_1[\psi_1^o(\mu_o)\psi_1^o(\mu) - \phi_1^o(\mu_o)\phi_1^o(\mu)]$$

$$+ A_2[\psi_2^o(\mu_o)\psi_2^o(\mu) - \phi_2^o(\mu_o)\phi_2^o(\mu)] \qquad \text{A1-5}$$

$$Q_1 = A_1[\psi_1^1(\mu_o)\psi_1^1(\mu) - \phi_1^1(\mu_o)\phi_1^1(\mu)] - \frac{A_2}{3}[\psi_2^1(\mu_o)\psi_2^1(\mu) - \phi_2^1(\mu_o)\phi_2^1(\mu)] \qquad \text{A1-6}$$

$$Q_2 = \frac{A_2}{6}[\psi_2^2(\mu_o)\psi_2^2(\mu) - \phi_2^2(\mu_o)\phi_2^2(\mu)] \qquad \text{A1-7}$$

and the expressions for $b_m(\tau_1, \mu, \mu_o)$ become

$$b_o = K(Q_o - Q_2) \qquad \text{A1-8a}$$

$$b_1 = KQ_1 \qquad \text{A1-8b}$$

$$b_2 = 2KQ_2 \qquad \text{A1-8c}$$

The diffuse reflected intensity may be written as

$$\frac{I(0,\mu,\mu_o)}{I_o(-\mu_o)} = b_o + b_1\cos\varphi + b_2\cos^2\varphi \qquad \text{A1-9}$$

Proceeding in the same manner to derive the relations for $c_m(\tau_1, \mu, \mu_o)$, the expression for the diffuse transmitted intensity can be written as

$$\frac{I(\tau_1,-\mu,\varphi)}{I_o(-\mu)} = \frac{\mu_o}{4\pi}\sum_{m=0}^{N}(2-\delta_{o,m})\cos m\varphi \sum_{l=m}^{N} G_l^m(\tau_1,\mu,\mu_o) \qquad \text{I-19}$$

Equations I-14 and I-29 are of the same form with the term $(-1)^{l+m} F_l^m(\tau_1,\mu,\mu_o)$ in I-14 replaced by $G_l^m(\tau_1,\mu,\mu_o)$ in I-19. The only difference between F_l^m and G_l^m is that the positions of $\psi_l^m(\mu)$ and $\phi_l^m(\mu)$ are interchanged and the term $\mu+\mu_o$ in F_l^m is replaced by $\mu-\mu_o$ in G_l^m.

Therefore, if

1. All signs are made positive in front of the A_l' s

2. $\mu + \mu_o$ is replaced in K by $\mu - \mu_o$

3. $\psi_l^m(\mu)$ and $\phi_l^m(\mu)$ are interchanged

The expression for the diffuse transmitted intensity may be written directly.

$$K' = \frac{\mu_o \omega_o}{4\pi(\mu - \mu_o)} \qquad \text{A1-10}$$

$$Q_o' = [\psi_o^o(\mu_o)\phi_o^o(\mu) - \phi_o^o(\mu_o)\psi_o^o(\mu)] + A_1[\psi_1^o(\mu_o)\phi_1^o(\mu) - \phi_1^o(\mu_o)\psi_1^o(\mu)]$$

$$+ A_2[\psi_2^o(\mu_o)\phi_2^o(\mu) - \phi_2^o(\mu_o)\psi_2^o(\mu)] \qquad \text{A1-11}$$

$$Q_1' = A_1[\psi_1^1(\mu_o)\phi_1^1(\mu) - \phi_1^1(\mu_o)\psi_1^1(\mu)] + \frac{A_2}{3}[\psi_2^1(\mu_o)\phi_2^1(\mu) - \phi_2^1(\mu_o)\psi_2^1(\mu)] \qquad \text{A1-12}$$

$$Q_2' = \frac{A_2}{6}[\psi_2^2(\mu_o)\phi_2^2(\mu) - \phi_2^2(\mu_o)\psi_2^2(\mu)] \qquad \text{A1-13}$$

with the expressions for c_m

$$c_o = K'[Q_o' - Q_2'] \quad , \qquad \text{A1-14a}$$

$$c_1 = K'Q_1' \qquad \text{A1-14b}$$

$$c_2 = 2K'Q_2' \qquad \text{A1-14c}$$

and the diffuse transmitted intensity may be written as

$$\frac{I(\tau_1, -\mu, \varphi)}{I_o(-\mu_o)} = c_o + c_1 \cos\varphi + c_2 \cos^2\varphi \qquad \text{A1-15}$$

2. Integrated Reflectance of Half Space for Rayleigh Scattering

There is a section on integral formulations in reference (4). For Rayleigh scattering $(\omega_1 = 0)$, Equation I-151 in that report then reduces to, for k = 0 and 1,

$$\left(\frac{1}{2}\omega_0 A_{00} - 1\right)\mu U_0 + U_1 + \frac{1}{2}\omega_2 A_{20}\mu U_2 = 0 \qquad \text{A2-1}$$

$$\left(1 + \frac{3}{2}\omega_0 \mu A_{01}\right)U_0 - 3\mu U_1 + \left(2 + \frac{3}{2}\omega_2 A_{21}\mu\right)U_2 = 0 \qquad \text{A2-2}$$

where A and U are defined as

$$\psi_k(\mu) = U_k(\mu) K(\mu) \qquad \text{A2-3}$$

$$A_{rk} = \int_0^1 U_r(\mu') P(-\mu') K(\mu') d\mu' \qquad \text{A2-4}$$

K is equivalent to the H function in Chandrasekhar's Radiative Transfer (1) and P is the Legendre polynomial. According to Equation I-157 in reference (5).

$$U_k(\mu) = P_k(\mu) \quad \text{when} \quad \mu = 0 \qquad \text{A2-5}$$

By this additional condition Equations A2-1 and A2-2 can be solved for μ_0, μ_1 and μ_2.

$$U_0 = 1 + \frac{3}{4}\omega_2 A_{21}\mu + \frac{3}{4}\omega_2 A_{20}\mu^2 \qquad \text{A2-6}$$

$$U_1 = \left(1 - \frac{1}{2}\omega_0 A_{00} + \frac{1}{4}\omega_2 A_{20}\right)\mu + \left[\frac{3}{4}\left(1 - \frac{1}{2}\omega_0 A_{00}\right)\omega_2 A_{21} + \frac{3}{8}\omega_0\omega_2 A_{01} A_{20}\right]\mu^2 \qquad \text{A2-7}$$

$$U_2 = -\frac{1}{2} - \frac{3}{4}\omega_0 A_{01}\mu + \frac{3}{4}\left(2 - \omega_0 A_{00}\right)\mu^2 \qquad \text{A2-8}$$

The K function is

$$\frac{1}{K(\mu)} = 1 - \frac{\mu}{2}\int_0^1 \frac{\Psi(\mu') K(\mu')}{\mu + \mu'} d\mu', \qquad \text{A2-9}$$

and

$$\Psi(\mu') = \sum_{r=0}^{N} (-1)^r \omega_r U_r(\mu') U_r(-\mu'). \qquad \text{A2-10}$$

The latter can be written in terms of A's using Equations A2-6, A2-7 and A2-8

$$\Psi(\mu') = \omega_0 + \frac{1}{4}\omega_2 + \Big(-\frac{3}{2}\omega_2 + \frac{3}{4}\omega_0\omega_2 A_{00} - \frac{9}{16}\omega_0^2\omega_2 A_{01}^2 + \frac{3}{2}\omega_0\omega_2 A_{20} - \frac{9}{16}\omega_0\omega_2^2 A_{21}^2\Big)\mu'^2 + \Big(\frac{9}{4}\omega_2 - \frac{9}{4}\omega_0\omega_2 A_{00} + \frac{9}{16}\omega_0\omega_2^2 A_{20}^2\Big)\mu'^4 \qquad \text{A2-11}$$

The A's can be cancelled out by the use of Equations (I-149) and (I-152) in the reference (5).

$$U_k(\mu)K(\mu) = P_k(\mu) + \frac{1}{2}\mu\sum_{r=0}^{N}(-1)^r\omega_r\psi_r(\mu)\int_0^1\frac{P_k(-\mu')\psi_r(\mu')d\mu'}{\mu+\mu'}, \quad k=0,1,\cdots,N \qquad \text{A2-12}$$

For k = 0, multiplying A2-12 by $P_0(\mu)$ then integrating from 0 to 1 gives

$$A_{00} = 1 + \frac{1}{4}\omega_0 A_{00}^2 - \frac{1}{4}\omega_2 A_{21}^2 \qquad \text{A2-13}$$

Similarly for k = 1 and 2.

$$A_{11} = \frac{1}{3} - \frac{1}{4}\omega_0 A_{01}^2 - \frac{1}{4}\omega_2 A_{21}^2 \qquad \text{A2-14}$$

$$A_{20} = -\frac{1}{8}(\omega_0 A_{00}^2 + \omega_2 A_{20}^2) + \frac{3}{8}(\omega_0 A_{01}^2 + \omega_2 A_{21}^2) \qquad \text{A2-15}$$

Then A2-11 can be simplified with A2-13, A2-14, and A2-15 to give

$$\Psi(\mu') = \omega_0 + \frac{1}{4}\omega_2 - \frac{3}{4}(2-\omega_0)\omega_2\mu'^2 + \frac{9}{4}(1-\omega_0)\omega_2\mu'^4 \qquad \text{A2-16}$$

The numerical solution of R was carried out as follows. The K function was first evaluated by the computer using an iterative procedure. Then the moments of K are computed.

$$\alpha_n = \int_0^1 K(\mu)\mu^n d\mu \qquad \text{A2-17}$$

Then A's are computed from the values of α's and A2-6, A2-7, and A2-8. However it appears more convenient to use A2-6 and A2-8 to derive the following.

$$A_{00} = \alpha_0 + \frac{3}{4}\omega_2\alpha_1 A_{21} + \frac{3}{4}\omega_2\alpha_2 A_{20} \qquad \text{A2-18}$$

$$-A_{01} = \alpha_1 + \frac{3}{4}\omega_2\alpha_2 A_{21} + \frac{3}{4}\omega_2\alpha_3 A_{20} \qquad \text{A2-19}$$

$$A_{20} = -\frac{1}{2}\alpha_0 - \frac{3}{4}\omega_0\alpha_1 A_{01} + \frac{3}{4}(2 - \omega_0 A_{00})\alpha_2 \qquad \text{A2-20}$$

$$-A_{21} = -\frac{1}{2}\alpha_1 - \frac{3}{4}\omega_0\alpha_2 A_{01} + \frac{3}{4}(2 - \omega_0 A_{00})\alpha_3 \qquad \text{A2-21}$$

Finally R can be found by

$$\begin{aligned} R &= 1 - \frac{1}{\mu_0}\psi_1(\mu_0) \\ &= 1 - \frac{1}{\mu_0} U_1(\mu_0) K(\mu_0) \\ &= 1 - \Big[1 - \frac{1}{2}\omega_0 A_{00} + \frac{1}{4}\omega_2 A_{20} + \Big(\frac{3}{4}\omega_2 A_{21} + \frac{3}{8}\omega_0\omega_2 A_{01} A_{20} \\ &\quad - \frac{3}{8}\omega_0\omega_2 A_{00} A_{21} \Big)\mu_0 \Big] K(\mu_0) \end{aligned} \qquad \text{A2-22}$$

Sample values of the results for $\omega_0 = 0.9$, 0.6, and 0.3 are tabulated in Table A-1.

Table A-1 The K-functions and Related Numerical Values for Computing the Integrated Reflectance of Half Space, Rayleigh Scattering

	$\omega_0 = 0.90$		$\omega_0 = 0.60$		$\omega_0 = 0.30$	
μ_0	$K(\mu_0)$	R	$K(\mu_0)$	R	$K(\mu_0)$	R
0.00	1.0000	0.687	1.0000	0.369	1.0000	0.164
0.10	1.1863	0.630	1.0996	0.308	1.0436	0.128
0.20	1.3144	0.591	1.1578	0.273	1.0667	0.110
0.30	1.4214	0.559	1.2018	0.247	1.0832	0.098
0.40	1.5147	0.531	1.2371	0.227	1.0960	0.088
0.50	1.5976	0.507	1.2665	0.211	1.1063	0.081
0.60	1.6722	0.485	1.2915	0.197	1.1148	0.075
0.70	1.7400	0.466	1.3131	0.186	1.1220	0.071
0.80	1.8020	0.449	1.3319	0.176	1.1282	0.067
0.90	1.8591	0.433	1.3486	0.168	1.1335	0.064
1.00	1.9117	0.418	1.3634	0.161	1.1382	0.061

ω_0	α_0	α_1	α_2	α_3	A_{00}	$-A_{01}$	$-A_{20}$	$-A_{21}$
0.90	1.5582	0.8493	0.5868	0.4486	1.5198	0.8223	0.0295	0.1138
0.60	1.2442	0.6481	0.4391	0.3321	1.2252	0.6351	0.0203	0.1165
0.30	1.0969	0.5579	0.3744	0.2817	1.0889	0.5525	0.0093	0.1211

3. **Solution for the Six-flux Formulation of the Slab Problem**

The following solution of equations II-7a to II-8 subject to the boundary conditions $I_1 = 1$ and $I_3 = 0$ at $Z = 0$ and $I_2 = I_4 = 0$ at $Z = \tau_1$ is reproduced directly from a previous report to avoid errors in transcription. For this reason the symbolism is slightly different. In equations A3-1 to A3-31 T is used in place of τ_1 for the optical thickness of the dispersion, θ is used in place of θ_0 for the angle the incident radiation makes with the normal to the slab, but as in this report Z refers to the normal optical distance through the slab.

All of the other symbols such as the A's, a's, b's, Δ , p, etc. are defined as used: <u>the definitions of these symbols apply only to equations A3-1 to A3-31 and have no relation to the definition of the symbols in the NOMENCLATURE or in the rest of this report.</u>

$$I_1 = \frac{A_{11}}{\Delta} e^{-p_1 Z} + \frac{A_{12}}{\Delta} e^{-p_1(T-Z)} + \frac{A_{13}}{\Delta} e^{-p_2 Z} + \frac{A_{14}}{\Delta} e^{-p_2(T-Z)} \quad \text{A3-1}$$

$$I_2 = \frac{A_{21}}{\Delta} e^{-p_1 Z} + \frac{A_{22}}{\Delta} e^{-p_1(T-Z)} + \frac{A_{23}}{\Delta} e^{-p_2 Z} + \frac{A_{24}}{\Delta} e^{-p_2(T-Z)} \quad \text{A3-2}$$

$$I_3 = \frac{A_{31}}{\Delta} e^{-p_1 Z} + \frac{A_{32}}{\Delta} e^{-p_1(T-Z)} + \frac{A_{33}}{\Delta} e^{-p_2 Z} + \frac{A_{34}}{\Delta} e^{-p_2(T-Z)} \quad \text{A3-3}$$

and

$$I_4 = \frac{A_{41}}{\Delta} e^{-p_1 Z} + \frac{A_{42}}{\Delta} e^{-p_1(T-Z)} + \frac{A_{43}}{\Delta} e^{-p_2 Z} + \frac{A_{44}}{\Delta} e^{-p_2(T-Z)} \quad \text{A3-4}$$

where:

$$\Delta = (a_1b_3-a_3b_1)^2-(a_2b_3-a_3b_2)^2 e^{-2p_1T}-(a_1b_4-a_4b_1)^2 e^{-2p_2T} + 2(a_1b_2-a_2b_1)(a_3b_4-a_4b_3)e^{-(p_1+p_2)T}+(a_2b_4-a_4b_2)^2 e^{-2(p_1+p_2)T} \quad \text{A3-5}$$

$$A_{11} = a_1b_3(a_1b_3-a_3b_1)-a_1b_4(a_1b_4-a_4b_1)e^{-2p_2T}+a_1b_2(a_3b_4-a_4b_3)e^{-(p_1+p_2)T} \quad \text{A3-6}$$

$$A_{21} = a_2b_3(a_1b_3-a_3b_1)-a_2b_4(a_1b_4-a_4b_1)e^{-2p_2T}+a_2b_2(a_3b_4-a_4b_3)e^{-(p_1+p_2)T} \quad \text{A3-7}$$

$$A_{31} = b_1b_3(a_1b_3-a_3b_1)-b_1b_4(a_1b_4-a_4b_1)e^{-2p_2T}+b_1b_2(a_3b_4-a_4b_3)e^{-(p_1+p_2)T} \quad \text{A3-8}$$

$$A_{41} = b_2b_3(a_1b_3-a_3b_1)-b_2b_4(a_1b_4-a_4b_1)e^{-2p_2T}+b_2^2(a_3b_4-a_4b_3)e^{-(p_1+p_2)T} \quad \text{A3-9}$$

$$A_{12} = -a_2b_3(a_2b_3-a_3b_2)e^{-p_1T}-a_2b_1(a_3b_4-a_4b_3)e^{-p_2T}+a_2b_4(a_2b_4-a_4b_2)e^{-(p_1+2p_2)T} \quad \text{A3-10}$$

$$A_{22} = -a_1b_3(a_2b_3-a_3b_2)e^{-p_1T}-a_1b_1(a_3b_4-a_4b_3)e^{-p_2T}+a_1b_4(a_2b_4-a_4b_2)e^{-(p_1+2p_2)T} \quad \text{A3-11}$$

$$A_{32} = -b_2b_3(a_2b_3-a_3b_2)e^{-p_1T}-b_2b_1(a_3b_4-a_4b_3)e^{-p_2T}+b_2b_4(a_2b_4-a_4b_2)e^{-(p_1+2p_2)T} \quad \text{A3-12}$$

$$A_{42} = -b_1b_3(a_2b_3-a_3b_2)e^{-p_1T}-b_1^2(a_3b_4-a_4b_3)e^{-p_2T}+b_1b_4(a_2b_4-a_4b_2)e^{-(p_1+2p_2)T} \quad \text{A3-13}$$

$$A_{13} = -a_3b_1(a_1b_3-a_3b_1)+a_3b_2(a_2b_3-a_3b_2)e^{-2p_1T}+a_3b_4(a_1b_2-a_2b_1)e^{-(p_1+p_2)T} \quad \text{A3-14}$$

$$A_{23} = -a_4b_1(a_1b_3-a_3b_1)+a_4b_2(a_2b_3-a_3b_2)e^{-2p_1T}+a_4b_4(a_1b_2-a_2b_1)e^{-(p_1+p_2)T} \quad \text{A3-15}$$

$$A_{33} = -b_3b_1(a_1b_3-a_3b_1)+b_3b_2(a_2b_3-a_3b_2)e^{-2p_1T}+b_3b_4(a_1b_2-a_2b_1)e^{-(p_1+p_2)T} \quad \text{A3-16}$$

$$A_{43} = -b_4b_1(a_1b_3-a_3b_1)+b_4b_2(a_2b_3-a_3b_2)e^{-2p_1T}+b_4^2(a_1b_2-a_2b_1)e^{-(p_1+p_2)T} \quad \text{A3-17}$$

$$A_{14} = -a_4b_3(a_1b_2-a_2b_1)e^{-p_1T}+a_4b_1(a_1b_4-a_4b_1)e^{-p_2T}-a_4b_2(a_2b_4-a_4b_2)e^{-(2p_1+p_2)T} \quad \text{A3-18}$$

$$A_{24} = -a_3b_3(a_1b_2-a_2b_1)e^{-p_1T}+a_3b_1(a_1b_4-a_4b_1)e^{-p_2T}-a_3b_2(a_2b_4-a_4b_2)e^{-(2p_1+p_2)T} \quad \text{A3-19}$$

$$A_{34} = -b_4b_3(a_1b_2-a_2b_1)e^{-p_1T}+b_4b_1(a_1b_4-a_4b_1)e^{-p_2T}-b_4b_2(a_2b_4-a_4b_2)e^{-(2p_1+p_2)T} \quad \text{A3-20}$$

$$A_{44} = -b_3^2(a_1b_2-a_2b_1)e^{-p_1T}+b_3b_1(a_1b_4-a_4b_1)e^{-p_2T}-b_3b_2(a_2b_4-a_4b_2)e^{-(2p_1+p_2)T} \quad \text{A3-21}$$

in which,

$$p_1 = \left\{ \frac{1}{2} \left[1- \sqrt{1-\sin^2 2\theta \left[\frac{(C_1-C_2)^2-4C_3^2}{(C_1-C_2)^2} \right]} \right] \right\}^{1/2} \sqrt{C_1^2-C_2^2} \sec\theta \csc\theta \quad \text{A3-22}$$

$$p_2 = \left\{ \frac{1}{2} \left[1 + \sqrt{1 - \sin^2 \; 2\,\theta \left[\frac{(C_1-C_2)^2 - 4C_3^2}{(C_1-C_2)^2} \right]} \right] \right\}^{1/2} \sqrt{C_1^2 - C_2^2} \;\; \sec\theta \csc\theta \qquad \text{A3-23}$$

$$a_1 = \frac{(C_1+C_2)\ \sec\theta + p_1}{(C_1+C_2)\ \sec\theta} \qquad \text{A3-24}$$

$$a_2 = \frac{(C_1+C_2)\ \sec\theta - p_1}{(C_1+C_2)\ \sec\theta} \qquad \text{A3-25}$$

$$a_3 = \frac{(C_1+C_2)\ \sec\theta + p_2}{(C_1+C_2)\ \sec\theta} \qquad \text{A3-26}$$

$$a_4 = \frac{(C_1+C_2)\ \sec\theta - p_2}{(C_1+C_2)\ \sec\theta} \qquad \text{A3-27}$$

$$b_1 = \frac{(C_1^2-C_2^2)\ \sec^2\theta - p_1^2}{2(C_1+C_2)\ C_3\ \sec^2\theta} \;\; \frac{(C_1+C_2)\ \csc\theta + p_1}{(C_1+C_2)\ \csc\theta} \qquad \text{A3-28}$$

$$b_2 = \frac{(C_1^2-C_2^2)\ \sec^2\theta - p_1^2}{2(C_1+C_2)\ C_3\ \sec^2\theta} \;\; \frac{(C_1+C_2)\ \csc\theta - p_1}{(C_1+C_2)\ \csc\theta} \qquad \text{A3-29}$$

$$b_3 = \frac{(C_1^2-C_2^2)\ \sec^2\theta - p_2^2}{2(C_1+C_2)\ C_3\ \sec^2\theta} \;\; \frac{(C_1+C_2)\ \csc\theta + p_2}{(C_1+C_2)\ \csc\theta} \qquad \text{A3-30}$$

and

$$b_4 = \frac{(C_1^2-C_2^2)\ \sec^2 - \theta\ p_2^2}{2(C_1+C_2)\ C_3\ \sec^2\theta} \;\; \frac{(C_1+C_2)\ \csc\theta - p_2}{(C_1+C_2)\ \csc\theta} \qquad \text{A3-31}$$

APPENDIX B

Tables of Computed Functions

The tabulated results obtained from the computer are presented in this appendix.

In the first portion of the appendix the complete set of $\psi_\ell^m(\mu)$ and $\phi_\ell^m(\mu)$ functions, the integrated reflectance R, the diffuse portion of the integrated transmission T_D, and the total integrated transmission T are given as a function of μ for each of the 36 sets of parameters used. The results are presented as one long table (Table B-0) which is subdivided into 36 problems. The values of the parameters corresponding to each problem are listed in Table I-2.

In the last portion of the appendix (Tables B-1 to B-5) tabular results are given which compare the exact solution with results obtained from approximate models.

Table B-0 The ψ_ℓ^m and ϕ_ℓ^m Functions, Integrated Reflectances, Diffuse Portion of the Integrated Transmission, and Total Integrated Transmission for 36 sets of Parameters (Listed in Table I-2).

PROBLEM 1 TAU = 0.050 ALBEDO FOR SINGLE SCATTERING = 0.90

A0 = 1.0000 A1 = 0.0000 A2 = 0.0000 F = 0.500 P = 0.3333

NUMBER OF ITERATIONS WAS 2

μ	ψ_0^0	ϕ_0^0	R	T_D	T
0.04691	1.0546	0.3838	0.299	0.284	0.628
0.23077	1.0726	0.9049	0.078	0.098	0.903
0.50000	1.0783	0.9824	0.042	0.039	0.944
0.76923	1.0800	1.0138	0.028	0.025	0.962
0.95309	1.0806	1.0252	0.023	0.019	0.968

PROBLEM 2 TAU = 0.250 ALBEDO FOR SINGLE SCATTERING = 0.90

A0 = 1.0000 A1 = 0.0000 A2 = 0.0000 F = 0.500 P = 0.3333

NUMBER OF ITERATIONS WAS 4

μ	ψ_0^0	ϕ_0^0	R	T_D	T
0.04691	1.0797	0.0467	0.498	0.360	0.365
0.23077	1.1993	0.4990	0.300	0.268	0.607
0.50000	1.2413	0.8317	0.173	0.161	0.768
0.76923	1.2611	0.9501	0.125	0.104	0.827
0.95309	1.2694	1.0001	0.105	0.085	0.854

PROBLEM 3 TAU = 0.500 ALBEDO FOR SINGLE SCATTERING = 0.90

A0 = 1.0000 A1 = 0.0000 A2 = 0.0000 F = 0.500 P = 0.3333

NUMBER OF ITERATIONS WAS 4

μ	ψ_0^0	ϕ_0^0	R	T_D	T
0.04691	1.0842	0.0275	0.548	0.286	0.286
0.23077	1.2530	0.2636	0.417	0.312	0.426
0.50000	1.3465	0.6497	0.279	0.239	0.607
0.76923	1.3896	0.8462	0.209	0.171	0.693
0.95309	1.4089	0.9249	0.179	0.136	0.728

PROBLEM 4   TAU = 1.000   ALBEDO FOR SINGLE SCATTERING = 0.90

A0 = 1.0000 A1 = 0.0000 A2 = 0.0000 F = 0.500 P = 0)3333

NUMBER OF ITERATIONS WAS 6

μ	ψ_0^0	ϕ_0^0	R	T_D	T
0.04691	1.0888	0.0165	0.598	0.197	0.197
0.23077	1.2915	0.1231	0.504	0.258	0.271
0.50000	1.4541	0.4012	0.394	0.263	0.398
0.76923	1.5427	0.6476	0.316	0.225	0.498
0.95309	1.5852	0.7553	0.279	0.186	0.536

PROBLEM 5   TAU = 1.500   ALBEDO FOR SINGLE SCATTERING = 0.90

A0 = 1.0000 A1 = 0.0000 A2 = 0.0000 F = 0.500 P = 0.3333

NUMBER OF ITERATIONS WAS 9

μ	ψ_0^0	ϕ_0^0	R	T_D	T
0.04691	1.0910	0.0123	0.622	0.152	0.152
0.23077	1.3063	0.0845	0.537	0.205	0.207
0.50000	1.5009	0.2769	0.444	0.248	0.298
0.76923	1.6202	0.5158	0.370	0.253	0.395
0.95309	1.6774	0.6565	0.330	0.242	0.449

PROBLEM 6   TAU = 2.000   ALBEDO FOR SINGLE SCATTERING = 0.90

A0 = 1.0000 A1 = 0.0000 A2 = 0.0000 F = 0.500 P = 0.3333

NUMBER OF ITERATIONS WAS 9

μ	ψ_0^0	ϕ_0^0	R	T_D	T
0.04691	1.0919	0.0098	0.632	0.129	0.129
0.23077	1.3122	0.0665	0.552	0.174	0.174
0.50000	1.5204	0.2145	0.465	0.233	0.251
0.76923	1.6567	0.4370	0.395	0.271	0.345
0.95309	1.7224	0.6056	0.355	0.290	0.413

PROBLEM 7 TAU = INFINITY ALBEDO FOR SINGLE SCATTERING = 0.90

A0 = 1.0000 A1 = 0.0000 A2 = 0.0000 F = 0.500 P = 0.3333

NUMBER OF ITERATIONS WAS 16

μ	ψ_0^0	ϕ_0^0	R	T_D	T
0.04691	1.0936	0.0000	0.654	0.000	0.000
0.23077	1.3236	0.0000	0.581	0.000	0.000
0.50000	1.5554	0.0000	0.507	0.000	0.000
0.76923	1.7289	0.0000	0.452	0.000	0.000
0.95309	1.8259	0.0000	0.422	0.000	0.000

PROBLEM 8 TAU = 0.050 ALBEDO FOR SINGLE SCATTERING = 0.60

A0 = 1.0000 A1 = 0.0000 A2 = 0.0000 F = 0.500 P = 0.3333

NUMBER OF ITERATIONS WAS 3

μ	ψ_0^0	ϕ_0^0	R	T_D	T
0.04691	1.0355	0.3696	0.1931	0.1817	0.526
0.23077	1.0477	0.8694	0.0529	0.0665	0.871
0.50000	1.0516	0.9557	0.0273	0.0271	0.932
0.76923	1.0527	0.9877	0.0182	0.0173	0.954
0.95309	1.0531	0.9991	0.0148	0.0132	0.962

PROBLEM 9 TAU = 0.500 ALBEDO FOR SINGLE SCATTERING = 0.600

A0 = 1.0000 A1 = 0.0000 A2 = 0.0000 F = 0.500 P = 0.3333

NUMBER OF ITERATIONS WAS 5

μ	ψ_0^0	ϕ_0^0	R	T_D	T
0.04691	1.0507	0.0145	0.318	0.154	0.154
0.23077	1.1448	0.1939	0.232	0.168	0.283
0.50000	1.1958	0.5221	0.154	0.134	0.502
0.76923	1.2186	0.7046	0.114	0.101	0.623
0.95309	1.2283	0.7827	0.097	0.083	0.675

PROBLEM 10 TAU = INFINITY ALBEDO FOR SINGLE SCATTERING = 0.60

A0 = 1.0000 A1 = 0.0000 A2 = 0.0000 F = 0.500 P = 0.3333

NUMBER OF ITERATIONS WAS 7

μ	ψ_o^o	ϕ_o^o	R	T_D	T
0.04691	1.0520	0.0000	0.335	0.000	0.000
0.23077	1.1587	0.0000	0.267	0.000	0.000
0.50000	1.2456	0.0000	0.212	0.000	0.000
0.76923	1.3009	0.0000	0.177	0.000	0.000
0.95309	1.3288	0.0000	0.158	0.000	0.000

PROBLEM 12 TAU = 0.500 ALBEDO FOR SINGLE SCATTERING = 0.30

A0 = 1.0000 A1 = 0.0000 A2 = 0.0000 F = 0.500 P = 0.3333

NUMBER OF ITERATIONS WAS 4

μ	ψ_o^o	ϕ_o^o	R	T_D	T
0.04691	1.0232	0.0059	0.1416	0.0636	0.0636
0.23077	1.0632	0.1469	0.0999	0.0696	0.1842
0.50000	1.0844	0.4323	0.0658	0.0569	0.4248
0.76923	1.0936	0.6000	0.0484	0.0439	0.5659
0.95309	1.0974	0.6743	0.0410	0.0366	0.6284

PROBLEM 11 TAU = 0.050 ALBEDO FOR SINGLE SCATTERING = 0.30

A0 = 1.0000 A1 = 0.0000 A2 = 0.0000 F = 0.500 P = 0.3333

NUMBER OF ITERATIONS WAS 2

μ	ψ_o^o	ϕ_o^o	R	T_D	T
0.04691	1.0172	0.3564	0.0935	0.0874	0.431
0.23077	1.0237	0.8365	0.0265	0.0337	0.839
0.50000	1.0252	0.9307	0.0132	0.0139	0.918
0.76923	1.0256	0.9624	0.0087	0.0087	0.945
0.95309	1.0258	0.9740	0.0071	0.0067	0.956

PROBLEM 13 TAU = INFINITY ALBEDO FOR SINGLE SCATTERING = 0.30

A0 = 1.0000 A1 = 0.0000 A2 = 0.0000 F = 0.500 P = 0.3333

NUMBER OF ITERATIONS WAS 5

μ	ψ_0^0	ϕ_0^0	R	T_D	T
0.04691	1.0233	0.0000	0.1438	0.0000	0.0000
0.23077	1.0663	0.0000	0.1079	0.0000	0.0000
0.50000	1.0975	0.0000	0.0817	0.0000	0.0000
0.76923	1.1159	0.0000	0.0663	0.0000	0.0000
0.95309	1.1248	0.0000	0.0588	0.0000	0.0000

PROBLEM 14 TAU = 0.050 ALBEDO FOR SINGLE SCATTERING = 0.90

A0 = 1.0000 A1 = 0.0000 A2 = 0.5000 F = 0.500 P = 0.400

NUMBER OF ITERATIONS FOR M = 0, 1, AND 2 WERE 2, 2, AND 2

μ	ψ_0^0	ψ_1^0	ψ_2^0	ψ_1^1	ψ_2^1	ψ_2^2	ϕ_0^0	ϕ_1^0
0.04691	1.0590	0.0000	-0.5158	0.0000	0.1409	3.0162	0.3864	0.0000
0.23077	1.0771	0.0000	-0.4452	0.0000	0.6760	2.8703	0.9132	0.0000
0.50000	1.0799	0.0000	-0.1493	0.0000	1.3040	2.2751	0.9835	0.0000
0.76923	1.0756	0.0000	0.3670	0.0000	1.4802	1.2387	1.0093	0.0000
0.95309	1.0705	0.0000	0.8457	0.0000	0.8688	0.2799	1.0155	0.0000

μ	ϕ_2^0	ϕ_1^1	ϕ_2^1	ϕ_2^2	R	T_D	T
0.04691	-0.1817	0.0000	0.0488	1.0452	0.299	0.283	0.627
0.23077	-0.3774	0.0000	0.5447	2.3280	0.077	0.100	0.905
0.50000	-0.1370	0.0000	1.1801	2.0618	0.042	0.038	0.943
0.76923	0.3433	0.0000	1.3871	1.1615	0.028	0.026	0.963
0.95309	0.8015	0.0000	0.8247	0.2639	0.023	0.020	0.969

PROBLEM 15 TAU = 0.250 ALBEDO FOR SINGLE SCATTERING = 0.90

A0 = 1.0000 A1 = 0.0000 A2 = 0.5000 F = 0.500 P = 0.400

NUMBER OF ITERATIONS FOR M = 0, 1, AND 2 WERE 4, 2, AND 3

μ	ψ_0^0	ψ_1^0	ψ_2^0	ψ_1^1	ψ_2^1	ψ_2^2	ϕ_0^0	ϕ_1^0
0.04691	1.0857	0.0000	-0.5214	0.0000	0.1410	3.0222	0.0487	0.0000
0.23077	1.2102	0.0000	-0.4690	0.0000	0.6807	2.8997	0.5063	0.0000
0.50000	1.2461	0.0000	-0.1759	0.0000	1.3161	2.3058	0.8379	0.0000
0.76923	1.2525	0.0000	0.3446	0.0000	1.4954	1.2569	0.9439	0.0000
0.95309	1.2471	0.0000	0.8287	0.0000	0.8781	0.2822	0.9849	0.0000

μ	ϕ_2^0	ϕ_1^1	ϕ_2^1	ϕ_2^2	R	T_D	T
0.04691	-0.0099	0.0000	0.0010	0.0257	0.500	0.356	0.361
0.23077	-0.1737	0.0000	0.2340	1.0025	0.299	0.267	0.606
0.50000	-0.1237	0.0000	0.8030	1.4149	0.173	0.163	0.770
0.76923	0.2418	0.0000	1.0843	0.9141	0.125	0.107	0.830
0.95309	0.6315	0.0000	0.6787	0.2182	0.105	0.090	0.859

PROBLEM 16 TAU = 0.500 ALBEDO FOR SINGLE SCATTERING = 0.90

A0 = 1.0000 A1 = 0.0000 A2 = 0.5000 F = 0.500 P = 0.400

NUMBER OF ITERATIONS FOR M = 0, 1, AND 2 WERE 5, 2, AND 3

μ	ψ_0^0	ψ_1^0	ψ_2^0	ψ_1^1	ψ_2^1	ψ_2^2	ϕ_0^0	ϕ_1^0
0.04691	1.0908	0.0000	-0.5217	0.0000	0.1411	3.0222	0.0284	0.0000
0.23077	1.2680	0.0000	-0.4723	0.0000	0.6822	2.9054	0.2680	0.0000
0.50000	1.3570	0.0000	-0.1809	0.0000	1.3226	2.3157	0.6554	0.0000
0.76923	1.3863	0.0000	0.3414	0.0000	1.5049	1.2638	0.8365	0.0000
0.95309	1.3909	0.0000	0.8285	0.0000	0.8841	0.2839	0.9033	0.0000

μ	ϕ_2^0	ϕ_1^1	ϕ_2^1	ϕ_2^2	R	T_D	T
0.04691	-0.0018	0.0000	0.0002	0.0049	0.553	0.280	0.280
0.23077	-0.0584	0.0000	0.0831	0.3492	0.419	0.308	0.422
0.50000	-0.0838	0.0000	0.4960	0.8722	0.281	0.236	0.604
0.76923	0.1671	0.0000	0.7948	0.6696	0.212	0.170	0.692
0.95309	0.4789	0.0000	0.5304	0.1700	0.183	0.138	0.729

PROBLEM 17 TAU = 1.000 ALBEDO FOR SINGLE SCATTERING = 0.90

A0 = 1.0000 A1 = 0.0000 A2 = 0.5000 F = 0.500 P = 0.400

NUMBER OF ITERATIONS FOR M = 0, 1, AND 2 WERE 6, 3, AND 3

μ	ψ_0^o	ψ_1^o	ψ_2^o	ψ_1^1	ψ_2^1	ψ_2^2	ϕ_0^o	ϕ_1^o
0.04691	1.0949	0.0000	-0.5215	0.0000	0.1411	3.0223	0.0162	0.0000
0.23077	1.3044	0.0000	-0.4711	0.0000	0.6826	2.9065	0.1226	0.0000
0.50000	1.4614	0.0000	-0.1772	0.0000	1.3263	2.3196	0.4017	0.0000
0.76923	1.5334	0.0000	0.3485	0.0000	1.5116	1.2671	0.6412	0.0000
0.95309	1.5594	0.0000	0.8387	0.0000	0.8888	0.2848	0.7341	0.0000

μ	ϕ_2^o	ϕ_1^1	ϕ_2^1	ϕ_2^2	R	T_D	T
0.04691	0.0010	0.0000	0.0001	0.0010	0.601	0.190	0.190
0.23077	-0.0014	0.0000	0.0122	0.0459	0.504	0.253	0.266
0.50000	-0.0205	0.0000	0.1900	0.3272	0.394	0.259	0.394
0.76923	0.0917	0.0000	0.4247	0.3550	0.317	0.221	0.493
0.95309	0.2876	0.0000	0.3220	0.1019	0.282	0.180	0.530

PROBLEM 18 TAU = 1.500 ALBEDO FOR SINGLE SCATTERING = 0.90

A0 = 1.0000 A1 = 0.0000 A2 = 0.5000 F = 0.500 P = 0.400

NUMBER OF ITERATIONS FOR M = 0, 1, AND 2 WERE 7, 3, AND 3

μ	ψ_0^o	ψ_1^o	ψ_2^o	ψ_1^1	ψ_2^1	ψ_2^2	ϕ_0^o	ϕ_1^o
0.04691	1.0968	0.0000	-0.5213	0.0000	0.1411	3.0223	0.0113	0.0000
0.23077	1.3178	0.0000	-0.4699	0.0000	0.6826	2.9066	0.0800	0.0000
0.50000	1.5056	0.0000	-0.1731	0.0000	1.3270	2.3200	0.2678	0.0000
0.76923	1.6080	0.0000	0.3556	0.0000	1.5132	1.2677	0.4952	0.0000
0.95309	1.6491	0.0000	0.8476	0.0000	0.8901	0.2850	0.6242	0.0000

μ	ϕ_2^o	ϕ_1^1	ϕ_2^1	ϕ_2^2	R	T_D	T
0.04691	0.0013	0.0000	0.0001	0.0003	0.623	0.142	0.142
0.23077	0.0066	0.0000	0.0027	0.0076	0.537	0.194	0.196
0.50000	0.0076	0.0000	0.0737	0.1223	0.443	0.238	0.288
0.76923	0.0656	0.0000	0.2260	0.1870	0.371	0.241	0.383
0.95309	0.1937	0.0000	0.1945	0.0608	0.334	0.226	0.433

PROBLEM 19 TAU = 2.000 ALBEDO FOR SINGLE SCATTERING = 0.90

A0 = 1.0000 A1 = 0.0000 A2 = 0.5000 F = 0.500 P = 0.400

NUMBER OF ITERATIONS FOR M = 0, 1, AND 2 WERE 11, 3, AND 3

μ	ψ_0^0	ψ_1^0	ψ_2^0	ψ_1^1	ψ_2^1	ψ_2^2	ϕ_0^0	ϕ_1^0
0.04691	1.0981	0.0000	-0.5212	0.0000	0.1411	3.0223	0.0092	0.0000
0.23077	1.3261	0.0000	-0.4692	0.0000	0.6826	2.9066	0.0632	0.0000
0.50000	1.5304	0.0000	-0.1709	0.0000	1.3271	2.3201	0.2048	0.0000
0.76923	1.6521	0.0000	0.3598	0.0000	1.5137	1.2678	0.4145	0.0000
0.95309	1.7034	0.0000	0.8528	0.0000	0.8905	0.2850	0.5620	0.0000

μ	ϕ_2^0	ϕ_1^1	ϕ_2^1	ϕ_2^2	R	T_D	T
0.04691	0.0012	0.0000	0.0000	0.0001	0.636	0.118	0.118
0.23077	0.0077	0.0000	0.0009	0.0018	0.554	0.161	0.162
0.50000	0.0177	0.0000	0.0291	0.0458	0.468	0.215	0.233
0.76923	0.0562	0.0000	0.1200	0.0982	0.400	0.248	0.322
0.95309	0.1438	0.0000	0.1170	0.0361	0.363	0.259	0.382

PROBLEM 20 TAU = INFINITY ALBEDO FOR SINGLE SCATTERING = 0.90

A0 = 1.0000 A1 = 0.0000 A2 = 0.5000 F = 0.500 P = 0.400

NUMBER OF ITERATIONS FOR M = 0, 1, AND 2 WERE 16, 3, AND 3

μ	ψ_0^0	ψ_1^0	ψ_2^0	ψ_1^1	ψ_2^1	ψ_2^2	ϕ_0^0	ϕ_1^0
0.04691	1.0996	0.0000	-0.5210	0.0000	0.1411	3.0223	0.0000	0.0000
0.23077	1.3362	0.0000	-0.4677	0.0000	0.6826	2.9066	0.0000	0.0000
0.50000	1.5623	0.0000	-0.1661	0.0000	1.3271	2.3207	0.0000	0.0000
0.76923	1.7192	0.0000	0.3701	0.0000	1.5138	1.2679	0.0000	0.0000
0.95309	1.7999	0.0000	0.8677	0.0000	0.8906	0.2850	0.0000	0.0000

μ	ϕ_2^0	ϕ_1^1	ϕ_2^1	ϕ_2^2	R	T_D	T
0.04691	0.0000	0.0000	0.0000	0.0000	0.656	0.000	0.000
0.23077	0.0000	0.0000	0.0000	0.0000	0.580	0.000	0.000
0.50000	0.0000	0.0000	0.0000	0.0000	0.507	0.000	0.000
0.76923	0.0000	0.0000	0.0000	0.0000	0.453	0.000	0.000
0.95309	0.0000	0.0000	0.0000	0.0000	0.424	0.000	0.000

PROBLEM 21 TAU = 1.000 ALBEDO FOR SINGLE SCATTERING = 0.90

A0 = 1.0000 A1 = 1.0000 A2 = 1.8165 F = 0.750 P = 0.5504

NUMBER OF ITERATIONS FOR M = 0, 1, AND 2 WERE 4, 11, AND 5

μ	ψ_0^0	ψ_1^0	ψ_2^0	ψ_1^1	ψ_2^1	ψ_2^2	ϕ_0^0	ϕ_1^0
0.04691	1.1075	0.0211	-0.5329	1.0266	0.1242	3.1069	0.0164	0.0089
0.23077	1.3101	0.1331	-0.5012	1.0228	0.6458	3.1140	0.1332	0.0665
0.50000	1.3954	0.3549	-0.2024	0.8819	1.3221	2.5446	0.4580	0.2319
0.76923	1.3537	0.6124	0.3586	0.6120	1.5497	1.4059	0.7536	0.4677
0.95309	1.2821	0.7956	0.8892	0.2757	0.9220	0.3175	0.8755	0.6381

μ	ϕ_2^0	ϕ_1^1	ϕ_2^1	ϕ_2^2	R	T_D	T
0.04691	0.0006	0.0027	0.0044	0.0052	0.563	0.203	0.203
0.23077	-0.0034	0.0397	0.0494	0.0788	0.438	0.290	0.303
0.50000	-0.0162	0.2019	0.2957	0.4097	0.303	0.337	0.472
0.76923	0.1277	0.3034	0.5826	0.4311	0.213	0.333	0.605
0.95309	0.3629	0.2116	0.4639	0.1220	0.173	0.307	0.657

PROBLEM 22 TAU = 1.000 ALBEDO FOR SINGLE SCATTERING = 0.90

A0 = 1.0000 A1 = 1.0000 A2 = 1.0490 F = 0.750 P = 0.4821

NUMBER OF ITERATIONS FOR M = 0, 1, AND 2 WERE 5, 5, AND 4

μ	ψ_0^0	ψ_1^0	ψ_2^0	ψ_1^1	ψ_2^1	ψ_2^2	ϕ_0^0	ϕ_1^0
0.04691	1.0984	0.0207	-0.5263	1.0276	0.1228	3.0559	0.0185	0.0102
0.23077	1.2944	0.1295	-0.4847	1.0351	0.6288	2.9865	0.1422	0.0727
0.50000	1.3958	0.3481	-0.1923	0.9181	1.2706	2.4050	0.4702	0.2396
0.76923	1.3865	0.6065	0.3426	0.6611	1.4793	1.3195	0.7712	0.4682
0.95309	1.3458	0.7926	0.8450	0.3063	0.8782	0.2971	0.9051	0.6317

μ	ϕ_2^0	ϕ_1^1	ϕ_2^1	ϕ_2^2	R	T_D	T
0.04691	0.0011	0.0023	0.0036	0.0024	0.564	0.222	0.222
0.23077	-0.0002	0.0334	0.0392	0.0574	0.445	0.305	0.318
0.50000	-0.0134	0.1777	0.2572	0.3569	0.310	0.341	0.476
0.76923	0.1145	0.2583	0.5119	0.3825	0.218	0.325	0.598
0.95309	0.3290	0.1647	0.3853	0.1092	0.175	0.295	0.645

PROBLEM 23 TAU = 1.000 ALBEDO FOR SINGLE SCATTERING = 0.90

A0 = 1.0000 A1 = 1.0000 A2 = 0.0000 F = 0.750 P = 0.3889

NUMBER OF ITERATIONS FOR M = 0 AND 1 WERE 6 AND 4

μ	ψ_0^0	ψ_1^0	ψ_2^0	ψ_1^1	ψ_2^1	ψ_2^2	ϕ_0^0	ϕ_1^0
0.04691	1.0856	0.0208	0.0000	1.0288	0.0000	0.0000	0.0198	0.0112
0.23077	1.2689	0.1282	0.0000	1.0498	0.0000	0.0000	0.1464	0.0767
0.50000	1.3856	0.3445	0.0000	0.9607	0.0000	0.0000	0.4720	0.2433
0.76923	1.4151	0.6033	0.0000	0.7185	0.0000	0.0000	0.7817	0.4675
0.95309	1.4126	0.7913	0.0000	0.3424	0.0000	0.0000	0.9396	0.6300

μ	ϕ_2^0	ϕ_1^1	ϕ_2^1	ϕ_2^2	R	T_D	T
0.04691	0.0000	0.0018	0.0000	0.0000	0.559	0.238	0.238
0.23077	0.0000	0.0269	0.0000	0.0000	0.448	0.316	0.329
0.50000	0.0000	0.1540	0.0000	0.0000	0.315	0.345	0.480
0.76923	0.0000	0.2180	0.0000	0.0000	0.220	0.324	0.596
0.95309	0.0000	0.1312	0.0000	0.0000	0.174	0.296	0.646

PROBLEM 24 TAU = 0.050 ALBEDO FOR SINGLE SCATTERING = 0.90

A0 = 1.0000 A1 = 1.7321 A2 = 1.0000 F = 0.9330 P = 0.4821

NUMBER OF ITERATIONS FOR M = 0, 1, AND 2 WERE 2, 4, AND 3

μ	ψ_0^0	ψ_1^0	ψ_2^0	ψ_1^1	ψ_2^1	ψ_2^2	ϕ_0^0	ϕ_1^0
0.04691	1.0622	0.0336	-0.5187	1.0385	0.1182	3.0397	0.3903	0.0300
0.23077	1.0745	0.2166	-0.4485	1.0213	0.6486	2.9005	0.9285	0.2142
0.50000	1.0643	0.4890	-0.1510	0.9086	1.2790	2.3066	1.0007	0.4807
0.76923	1.0438	0.7628	0.3689	0.6676	1.4635	1.2528	1.0309	0.7542
0.95309	1.0262	0.9499	0.8511	0.3151	0.8916	0.2812	1.0390	0.9415

μ	ϕ_2^0	ϕ_1^1	ϕ_2^1	ϕ_2^2	R	T_D	T
0.04691	-0.1837	0.3718	0.0703	1.0601	0.287	0.295	0.639
0.23077	-0.3821	0.8605	0.5889	2.3706	0.060	0.117	0.922
0.50000	-0.1373	0.8387	1.2127	2.0874	0.023	0.055	0.960
0.76923	0.3473	0.6440	1.4139	1.1753	0.009	0.045	0.982
0.95309	0.8097	0.3108	0.8396	0.2670	0.004	0.040	0.989

PROBLEM 25 TAU = 0.250 ALBEDO FOR SINGLE SCATTERING = 0.90

A0 = 1.0000 A1 = 1.7321 A2 = 1.0000 F = 0.9330 P = 0.4821

NUMBER OF ITERATIONS FOR M = 0, 1, AND 2 WERE 6, 10, AND 3

μ	ψ_0^0	ψ_1^0	ψ_2^0	ψ_1^1	ψ_2^1	ψ_2^2	ϕ_0^0	ϕ_1^0
0.04691	1.0890	0.0251	-0.5256	1.0522	0.1068	3.0524	0.0541	0.0186
0.23077	1.1976	0.1758	-0.4792	1.0859	0.5918	2.9633	0.5391	0.1543
0.50000	1.1860	0.4514	-0.1851	0.9716	1.2777	2.3655	0.9148	0.4269
0.76923	1.1337	0.7363	0.3424	0.7104	1.4325	1.2914	1.0700	0.7177
0.95309	1.0806	0.9348	0.8344	0.3328	0.8507	0.2901	1.1597	0.9336

μ	ϕ_2^0	ϕ_1^1	ϕ_2^1	ϕ_2^2	R	T_D	T
0.04691	-0.0116	0.0303	0.0248	0.0378	0.466	0.394	0.396
0.23077	-0.1771	0.4320	0.3258	1.0475	0.238	0.335	0.669
0.50000	-0.1199	0.6730	0.9280	1.4696	0.097	0.256	0.853
0.76923	0.2664	0.5879	1.1875	0.9455	0.043	0.226	0.933
0.95309	0.6743	0.3072	0.7458	0.2255	0.018	0.230	0.979

PROBLEM 26 TAU = 0.500 ALBEDO FOR SINGLE SCATTERING = 0.90

A0 = 1.0000 A1 = 1.7321 A2 = 1.0000 F = 0.9330 P = 0.4821

NUMBER OF ITERATIONS FOR M = 0, 1, AND 2 WERE 5, 10, AND 4

μ	ψ_0^0	ψ_1^0	ψ_2^0	ψ_1^1	ψ_2^1	ψ_2^2	ϕ_0^0	ϕ_1^0
0.04691	1.0926	0.0235	-0.5262	1.0532	0.1057	3.0527	0.0337	0.0156
0.23077	1.2466	0.1545	-0.4854	1.1044	0.5701	2.9763	0.3067	0.1173
0.50000	1.2615	0.4199	-0.1968	1.0005	1.1953	2.3878	0.7653	0.3632
0.76923	1.1958	0.7115	0.3311	0.7317	1.4104	1.3067	1.0156	0.6422
0.95309	1.1267	0.9164	0.8262	0.3421	0.8419	0.2939	1.1318	0.8434

μ	ϕ_2^0	ϕ_1^1	ϕ_2^1	ϕ_2^2	R	T_D	T
0.04691	-0.0023	0.0014	0.0160	0.0098	0.500	0.331	0.331
0.23077	-0.0588	0.1901	0.1720	0.3764	0.333	0.392	0.507
0.50000	-0.0779	0.4782	0.6520	0.9228	0.163	0.355	0.723
0.76923	0.1919	0.4925	0.9454	0.7034	0.078	0.311	0.833
0.95309	0.5234	0.2768	0.6291	0.1781	0.041	0.293	0.885

PROBLEM 27 TAU = 1.000 ALBEDO FOR SINGLE SCATTERING = 0.90

A0 = 1.0000 A1 = 1.7321 A2 = 1.0000 F = 0.9330 P = 0.4821

NUMBER OF ITERATIONS FOR M = 0, 1, AND 2 WERE 5, 10, AND 4

μ	ψ_0^0	ψ_1^0	ψ_2^0	ψ_1^1	ψ_2^1	ψ_2^2	ϕ_0^0	ϕ_1^0
0.04691	1.0951	0.0223	-0.5262	1.0537	0.1051	3.0528	0.0212	0.0119
0.23077	1.2736	0.1408	-0.4859	1.1107	0.5618	2.9790	0.1630	0.0852
0.50000	1.3347	0.3832	-0.1988	1.0178	1.1727	2.3969	0.5354	0.2784
0.76923	1.2749	0.6723	0.3281	0.7470	1.3913	1.3145	0.8972	0.5418
0.95309	1.1966	0.8816	0.8239	0.3493	0.8334	0.2959	1.0802	0.7334

μ	ϕ_2^0	ϕ_1^1	ϕ_2^1	ϕ_2^2	R	T_D	T
0.04691	0.0016	0.0053	0.0080	0.0023	0.530	0.259	0.259
0.23077	0.0028	0.0544	0.0677	0.0563	0.396	0.359	0.372
0.50000	-0.0043	0.2286	0.3185	0.3540	0.239	0.418	0.553
0.76923	0.1298	0.3207	0.5855	0.3758	0.131	0.419	0.692
0.95309	0.3496	0.2072	0.4387	0.1085	0.079	0.400	0.750

PROBLEM 28 TAU = 1.500 ALBEDO FOR SINGLE SCATTERING = 0.90

A0 = 1.0000 A1 = 1.7321 A2 = 1.0000 F = 0.9330 P = 0.4821

NUMBER OF ITERATIONS FOR M = 0, 1, AND 2 WERE 7, 11, AND 3

μ	ψ_0^0	ψ_1^0	ψ_2^0	ψ_1^1	ψ_2^1	ψ_2^2	ϕ_0^0	ϕ_1^0
0.04691	1.0972	0.0212	-0.5262	1.0537	0.1050	3.0528	0.0171	0.0100
0.23077	1.2882	0.1331	-0.4855	1.1115	0.5606	2.9792	0.1231	0.0702
0.50000	1.3974	0.3589	-0.1968	1.0214	1.1674	2.3981	0.4058	0.2253
0.76923	1.3398	0.6369	0.3314	0.7513	1.3850	1.3159	0.7702	0.4588
0.95309	1.2661	0.8436	0.8278	0.3516	0.8302	0.2963	0.9945	0.6437

μ	ϕ_2^0	ϕ_1^1	ϕ_2^1	ϕ_2^2	R	T_D	T
0.04691	0.0020	0.0027	0.0043	0.0007	0.551	0.215	0.215
0.23077	0.0118	0.0234	0.0341	0.0118	0.426	0.303	0.305
0.50000	0.2515	0.1108	0.1606	0.1347	0.286	0.397	0.447
0.76923	0.1036	0.1991	0.3554	0.2022	0.176	0.446	0.588
0.95309	0.2542	0.1496	0.3044	0.0653	0.118	0.456	0.663

PROBLEM 29 TAU = INFINITY ALBEDO FOR SINGLE SCATTERING = 0.90

A0 = 1.0000 A1 = 1.7321 A2 = 1.0000 F = 0.9330 P = 0.4821

NUMBER OF ITERATIONS FOR M = 0, 1, AND 2 WERE 21, 6, AND 4

μ	ψ_0^0	ψ_1^0	ψ_2^0	ψ_1^1	ψ_2^1	ψ_2^2	ϕ_0^0	ϕ_1^0
0.04691	1.1010	0.0190	-0.5257	1.0538	0.1049	3.0528	0.0000	0.0000
2.23077	1.3142	0.1180	-0.4825	1.1116	0.5604	2.9792	0.0000	0.0000
0.50000	1.4608	0.3115	-0.1869	1.0222	1.1659	2.3983	0.0000	0.0000
0.76923	1.4964	0.5451	0.3511	0.7530	1.3823	1.3163	0.0000	0.0000
0.95309	1.4740	0.7213	0.8547	0.3527	0.8283	0.2965	0.0000	0.0000

μ	ϕ_2^0	ϕ_1^1	ϕ_2^1	ϕ_2^2	R	T_D	T
0.04691	0.0000	0.0000	0.0000	0.0000	0.595	0.000	0.000
0.23077	0.0000	0.0000	0.0000	0.0000	0.489	0.000	0.000
0.50000	0.0000	0.0000	0.0000	0.0000	0.377	0.000	0.000
0.76923	0.0000	0.0000	0.0000	0.0000	0.292	0.000	0.000
0.95309	0.0000	0.0000	0.0000	0.0000	0.244	0.000	0.000

PROBLEM 30 TAU = 0.050 ALBEDO FOR SINGLE SCATTERING = 0.60

A0 = 1.0000 A1 = 1.7321 A2 = 1.0000 F = 0.9330 P = 0.4821

NUMBER OF ITERATIONS FOR M = 0, 1, AND 2 WERE 2, 3, AND 3

μ	ψ_0^0	ψ_1^0	ψ_2^0	ψ_1^1	ψ_2^1	ψ_2^2	ϕ_0^0	ϕ_1^0
0.04691	1.0402	0.0383	-0.5109	1.0247	0.1261	3.0240	0.3734	0.0249
0.23077	1.0492	0.2212	-0.4387	1.0051	0.6568	2.8803	0.8846	0.2046
0.50000	1.0421	0.4927	-0.1419	0.8937	1.2862	2.2836	0.9689	0.4716
0.76923	1.0287	0.7649	0.3755	0.6575	1.4676	1.2134	0.9992	0.7431
0.95309	1.0172	0.9509	0.8552	0.3108	0.8630	0.2791	1.0083	0.9290

μ	ϕ_2^0	ϕ_1^1	ϕ_2^1	ϕ_2^2	R	T_D	T
0.04691	-0.1790	0.3618	0.0626	1.0501	0.184	0.188	0.531
0.23077	-0.3659	0.8338	0.5738	2.3420	0.042	0.080	0.887
0.50000	-0.1291	0.8214	1.2013	2.0703	0.015	0.038	0.943
0.76923	0.3528	0.6286	1.4027	1.1661	0.006	0.029	0.966
0.95309	0.8127	0.3025	0.8327	0.2649	0.003	0.026	0.974

PROBLEM 31 TAU = 0.500 ALBEDO FOR SINGLE SCATTERING = 0.60

A0 = 1.0000 A1 = 1.7321 A2 = 1.0000 F = 0.9330 P = 0.4821

NUMBER OF ITERATIONS FOR M = 0, 1, AND 2 WERE 4, 4, AND 3

μ	ψ_0^0	ψ_1^0	ψ_2^0	ψ_1^1	ψ_2^1	ψ_2^2	ϕ_0^0	ϕ_1^0
0.04691	1.0558	0.0332	-0.5145	1.0331	0.1189	3.0322	0.0166	0.0077
0.23077	1.1385	0.1894	-0.4584	1.0523	0.6123	2.9283	0.2128	0.0737
0.50000	1.1435	0.4574	-0.1660	0.9453	1.2398	2.3389	0.5810	0.2827
0.76923	1.1049	0.7390	0.3567	0.6924	1.4399	1.2776	0.7966	0.5394
0.95309	1.0662	0.9338	0.8444	0.3252	0.8533	0.2871	0.8951	0.7262

μ	ϕ_2^0	ϕ_1^1	ϕ_2^1	ϕ_2^2	R	T_D	T
0.04691	-0.0010	0.0077	0.0091	0.0063	0.293	0.165	0.165
0.23077	-0.0526	0.1566	0.1321	0.3578	0.180	0.204	0.319
0.50000	-0.0593	0.4118	0.5804	0.8883	0.085	0.197	0.565
0.76923	0.2020	0.4266	0.8720	0.6804	0.040	0.178	0.701
0.95309	0.5251	0.2327	0.5743	0.1726	0.021	0.169	0.761

PROBLEM 32 TAU = INFINITY ALBEDO FOR SINGLE SCATTERING = 0.60

A0 = 1.0000 A1 = 1.7321 A2 = 1.0000 F = 0.9330 P = 0.4821

NUMBER OF ITERATIONS FOR M = 0, 1, AND 2 WERE 8, 5, AND 3

μ	ψ_0^0	ψ_1^0	ψ_2^0	ψ_1^1	ψ_2^1	ψ_2^2	ϕ_0^0	ϕ_1^0
0.04691	1.0568	0.0327	-0.5148	1.0332	0.1187	3.0322	0.0000	0.0000
0.23077	1.1499	0.1833	-0.4583	1.0551	0.6084	2.9300	0.0000	0.0000
0.50000	1.1795	0.4381	-0.1652	0.9540	1.2279	2.3452	0.0000	0.0000
0.76923	1.1517	0.7134	0.3589	0.7003	1.4298	1.2833	0.0000	0.0000
0.95309	1.1128	0.9074	0.8483	0.3286	0.8493	2.8867	0.0000	0.0000

μ	ϕ_2^0	ϕ_1^1	ϕ_2^1	ϕ_2^2	R	T_D	T
0.04691	0.0000	0.0000	0.0000	0.0000	0.303	0.000	0.000
0.23077	0.0000	0.0000	0.0000	0.0000	0.206	0.000	0.000
0.50000	0.0000	0.0000	0.0000	0.0000	0.124	0.000	0.000
0.76923	0.0000	0.0000	0.0000	0.0000	0.073	0.000	0.000
0.95309	0.0000	0.0000	0.0000	0.0000	0.048	0.000	0.000

PROBLEM 33 TAU = 0.050 ALBEDO FOR SINGLE SCATTERING = 0.30

A0 = 1.0000 A1 = 1.7321 A2 = 1.0000 F = 0.9330 P = 0.4821

NUMBER OF ITERATIONS FOR M = 0, 1, AND 2 WERE 2, 2, AND 2

μ	ψ_0^0	ψ_1^0	ψ_2^0	ψ_1^1	ψ_2^1	ψ_2^2	ϕ_0^0	ϕ_1^0
0.04691	1.0195	0.0427	-0.5036	1.0115	0.1335	3.0086	0.3581	0.0204
0.23077	1.0243	0.2260	-0.4293	0.9889	0.6652	2.8602	0.8435	0.1952
0.50000	1.0206	0.4964	-0.1333	0.8796	1.2928	2.2667	0.9371	0.4622
0.76923	1.0146	0.7671	0.3817	0.6480	1.4712	1.2341	0.9681	0.7321
0.95309	1.0086	0.9520	0.8590	0.3066	0.8643	0.2769	0.9785	0.9167

μ	ϕ_2^0	ϕ_1^1	ϕ_2^1	ϕ_2^2	R	T_D	T
0.04691	-0.1748	0.3526	0.0553	1.0404	0.090	0.089	0.435
0.23077	-0.3513	0.8080	0.5581	2.3141	0.021	0.040	0.845
0.50000	-0.1216	0.8028	1.1889	2.0531	0.008	0.019	0.924
0.76923	0.3581	0.6137	1.3923	1.1569	0.003	0.015	0.952
0.95309	0.8157	0.2948	0.8268	0.2629	0.001	0.013	0.961

PROBLEM 34 TAU = 0.500 ALBEDO FOR SINGLE SCATTERING = 0.30

A0 = 1.0000 A1 = 1.7321 A2 = 1.0000 F = 0.9330 P = 0.4821

NUMBER OF ITERATIONS FOR M = 0, 1, AND 2 WERE 4, 3, AND 3

μ	ψ_0^0	ψ_1^0	ψ_2^0	ψ_1^1	ψ_2^1	ψ_2^2	ϕ_0^0	ϕ_1^0
0.04691	1.0257	0.0408	-0.5052	1.0152	0.1304	3.0125	0.0064	0.0030
0.23077	1.0598	0.2135	-0.4374	1.0093	0.6460	2.8831	0.1539	0.0456
0.50000	1.0600	0.4831	-0.1432	0.9015	1.2733	2.2931	0.4567	0.2259
0.76923	1.0421	0.7579	0.3743	0.6624	1.4604	1.2504	0.6399	0.4624
0.95309	1.0249	0.9464	0.8553	0.3124	0.8609	0.2808	0.7238	0.6369

μ	ϕ_2^0	ϕ_1^1	ϕ_2^1	ϕ_2^2	R	T_D	T
0.04691	-0.0004	0.0033	0.0040	0.0030	0.131	0.064	0.063
0.23077	-0.0496	0.1312	0.1015	0.3409	0.075	0.083	0.198
0.50000	-0.0502	0.3602	0.5241	0.8568	0.034	0.084	0.452
0.76923	0.2046	0.3757	0.8164	0.6591	0.015	0.079	0.601
0.95309	0.5202	0.2030	0.5400	0.1674	0.008	0.076	0.668

PROBLEM 35 TAU = INFINITY ALBEDO FOR SINGLE SCATTERING = 0.30

A0 = 1.0000 A1 = 1.7321 A2 = 1.0000 F = 0.9330 P = 0.4821

NUMBER OF ITERATIONS FOR M = 0, 1, AND 2 WERE 4, 3, AND 3

μ	ψ_0^0	ψ_1^0	ψ_2^0	ψ_1^1	ψ_2^1	ψ_2^2	ϕ_0^0	ϕ_1^0
0.04691	1.0258	0.0407	-0.5052	1.0152	0.1304	3.0125	0.0000	0.0000
0.23077	1.0620	0.2124	-0.4374	1.0102	0.6448	2.8838	0.0000	0.0000
0.50000	1.0670	0.4793	-0.1433	0.9042	1.2696	2.2959	0.0000	0.0000
0.76923	1.0499	0.7538	0.3744	0.6646	1.4578	1.2530	0.0000	0.0000
0.95309	1.0313	0.9427	0.8559	0.3132	0.8602	2.8149	0.0000	0.0000

μ	ϕ_2^0	ϕ_1^1	ϕ_2^1	ϕ_2^2	R	T_D	T
0.04691	0.0000	0.0000	0.0000	0.0000	0.133	0.000	0.000
0.23077	0.0000	0.0000	0.0000	0.0000	0.080	0.000	0.000
0.50000	0.0000	0.0000	0.0000	0.0000	0.042	0.000	0.000
0.76923	0.0000	0.0000	0.0000	0.0000	0.020	0.000	0.000
0.95309	0.0000	0.0000	0.0000	0.0000	0.011	0.000	0.000

PROBLEM 36 TAU = INFINITY ALBEDO FOR SINGLE SCATTERING = 0.90

A0 = 1.0000 A1 = 1.7346 A2 = 2.2415 F = 0.9336 P = 0.5821

NUMBER OF ITERATIONS FOR M = 0, 1, AND 2 WERE 2, 5, AND 5
(NOTE--FOR M = 0 INITIAL FUNCTIONS WERE THOSE OF REFERENCE 4)

μ	ψ_0^0	ψ_1^0	ψ_2^0	ψ_1^1	ψ_2^1	ψ_2^2	ϕ_0^0	ϕ_1^0
0.04691	1.1185	0.0185	-0.5370	1.0516	0.1080	3.1376	0.0000	0.0000
0.23077	1.3550	0.1179	-0.5124	1.0896	0.5911	3.1955	0.0000	0.0000
0.50000	1.4868	0.3121	-0.2079	0.9561	1.2596	2.6406	0.0000	0.0000
0.76923	1.4699	0.5412	0.3771	0.6583	1.5179	1.4695	0.0000	0.0000
0.95309	1.3948	0.7094	0.9349	0.2909	0.9173	0.3331	0.0000	0.0000

μ	ϕ_2^0	ϕ_1^1	ϕ_2^1	ϕ_2^2	R	T_D	T
0.04691	0.0000	0.0000	0.0000	0.0000	0.604	0.000	0.000
0.23077	0.0000	0.0000	0.0000	0.0000	0.486	0.000	0.000
0.50000	0.0000	0.0000	0.0000	0.0000	0.376	0.000	0.000
0.76923	0.0000	0.0000	0.0000	0.0000	0.296	0.000	0.000
0.95309	0.0000	0.0000	0.0000	0.0000	0.256	0.000	0.000

Table B-1 Comparison of Exact and Approximate Integrated Reflectances

Isotropic and Rayleigh Scattering

	μ_o	Exact. iso*	Six-flux,iso.	Six-flux,Ray.	Two-flux,iso. and Ray.	Richards,iso.
$\omega_o = 0.90, \tau_1 = \infty$:						
	0.00000	0.677	0.569	0.569	0.519	0.500
	0.04691	0.654	0.560	0.560	0.519	0.489
	0.23077	0.581	0.527	0.527	0.519	0.448
	0.50000	0.507	0.484	0.485	0.519	0.400
	0.76923	0.452	0.440	0.441	0.519	0.361
	0.95309	0.422	0.389	0.391	0.519	0.338
	1.00000	0.416	0.333	0.337	0.519	0.333
$\omega_o = 0.90, \tau_1 = 1.0$:						
	0.00000	0.624	0.533	0.534	0.519	0.457
	0.04691	0.598	0.520	0.521	0.519	0.446
	0.23077	0.504	0.466	0.467	0.496	0.393
	0.50000	0.394	0.375	0.376	0.421	0.316
	0.76923	0.316	0.303	0.304	0.330	0.257
	0.95309	0.279	0.259	0.260	0.293	0.227
	1.00000	0.271	0.137	0.138	0.284	0.220
$\omega_o = 0.90, \tau_1 = 0.50$:						
	0.00000	0.584	0.507	0.509	0.519	0.428
	0.04691	0.548	0.492	0.494	0.519	0.413
	0.23077	0.417	0.390	0.391	0.420	0.333
	0.50000	0.279	0.267	0.267	0.285	0.231
	0.76923	0.209	0.199	0.199	0.211	0.171
	0.95309	0.179	0.166	0.167	0.182	0.147
	1.00000	0.170	0.079	0.079	0.175	0.141
$\omega_o = 0.90, \tau_1 = 0.25$:						
	0.00000	0.571	0.488	0.491	0.519	0.405
	0.04691	0.498	0.463	0.467	0.505	0.388
	0.23077	0.300	0.283	0.283	0.302	0.243
	0.50000	0.173	0.166	0.166	0.175	0.146
	0.76923	0.125	0.117	0.117	0.124	0.103
	0.95309	0.105	0.096	0.096	0.103	0.084
	1.00000	0.101	0.043	0.043	0.099	0.081
$\omega_o = 0.60, \tau_1 = 0.50$:						
	0.00000	0.345	0.235	0.234	0.225	0.176
	0.04691	0.318	0.228	0.227	0.225	0.168
	0.23077	0.232	0.190	0.193	0.211	0.135
	0.50000	0.154	0.137	0.138	0.164	0.092
	0.76923	0.114	0.103	0.104	0.129	0.069
	0.95309	0.097	0.082	0.084	0.112	0.059
	1.00000	0.093	0.034	0.035	0.108	0.057
$\omega_o = 0.30, \tau_1 = 0.50$:						
	0.00000	0.156	0.091	0.091	0.089	0.064
	0.04691	0.142	0.088	0.088	0.089	0.061
	0.23077	0.100	0.075	0.076	0.089	0.048
	0.50000	0.066	0.056	0.057	0.086	0.033
	0.76923	0.048	0.042	0.043	0.079	0.025
	0.95309	0.041	0.032	0.034	0.074	0.021
	1.00000	0.040	0.012	0.014	0.072	0.020

* Extrapolated by Lagrange's formula for μ_o = o and μ_o = 1.0.

Table B-2 Comparison of Exact and Approximate Total Transmissions

Isotropic and Rayleigh Scattering

	μ_o	Exact. iso.*	Six-flux,iso.	Six-flux,Ray.	Two-flux,iso.	Richards,iso.
ω_o=0.90,τ_1=1.0:						
	0.00000	0.183	0.181	0.173	0.000	0.211
	0.04691	0.197	0.194	0.185	0.001	0.221
	0.23077	0.271	0.265	0.256	0.189	0.269
	0.50000	0.398	0.413	0.412	0.420	0.385
	0.76923	0.498	0.523	0.523	0.549	0.492
	0.95309	0.536	0.567	0.568	0.609	0.549
	1.00000	0.541	0.562	0.564	0.621	0.562
ω_o=0.90,τ_1=0.50:						
	0.00000	0.257	0.240	0.228	0.000	0.278
	0.04691	0.286	0.256	0.244	0.025	0.292
	0.23077	0.426	0.415	0.412	0.395	0.403
	0.50000	0.607	0.611	0.610	0.621	0.582
	0.76923	0.693	0.710	0.710	0.724	0.686
	0.95309	0.728	0.749	0.749	0.767	0.733
	1.00000	0.742	0.742	0.743	0.776	0.742
ω_o=0.90,τ_1=0.25:						
	0.00000	0.277	0.277	0.264	0.000	0.322
	0.04691	0.365	0.308	0.294	0.137	0.340
	0.23077	0.607	0.597	0.594	0.600	0.574
	0.50000	0.768	0.768	0.768	0.776	0.748
	0.76923	0.827	0.837	0.837	0.844	0.823
	0.95309	0.854	0.864	0.864	0.871	0.853
	1.00000	0.861	0.859	0.859	0.877	0.859
ω_o=0.60,τ_1=0.50:						
	0.00000	0.139	0.102	0.095	0.000	0.104
	0.04691	0.154	0.109	0.101	0.000	0.108
	0.23077	0.283	0.255	0.252	0.061	0.217
	0.50000	0.502	0.491	0.491	0.269	0.454
	0.76923	0.623	0.618	0.620	0.421	0.588
	0.95309	0.675	0.668	0.670	0.496	0.648
	1.00000	0.694	0.661	0.664	0.512	0.661
ω_o=0.30,τ_1=0.50:						
	0.00000	0.057	0.038	0.035	0.000	0.039
	0.04691	0.064	0.040	0.037	0.000	0.041'
	0.23077	0.184	0.168	0.167	0.026	0.156
	0.50000	0.425	0.417	0.418	0.186	0.399
	0.76923	0.566	0.561	0.562	0.335	0.546
	0.95309	0.628	0.622	0.622	0.411	0.612
	1.00000	0.651	0.626	0.629	0.430	0.626

* Extrapolated by Lagrange's formula for μ_o = o and μ_o = 1.0.

Table B-3 Comparison of Exact and Approximate Integrated Reflectances

Anisotropic Scattering, F = 0.9330, P = 0.4821

	μ_o	Exact*	Six-flux	Two-flux		Exact*	Six-flux	Two-flux
$\omega_o = 0.90$, $\tau_1 = \infty$:	0.00000	0.628	0.467	0.195	$\omega_o = 0.90$, $\tau_1 = 1.0$:	0.567	0.433	0.195
	0.04691	0.595	0.457	0.195		0.530	0.421	0.195
	0.23077	0.489	0.425	0.195		0.396	0.362	0.143
	0.50000	0.377	0.383	0.195		0.239	0.259	0.089
	0.76923	0.292	0.338	0.195		0.131	0.196	0.064
	0.95309	0.244	0.286	0.195		0.079	0.161	0.054
	1.00000	0.233	0.228	0.195		0.067	0.074	0.052
$\omega_o = 0.90$, $\tau_1 = 0.50$:	0.00000	0.547	0.417	0.195	$\omega_o = 0.90$, $\tau_1 = 0.25$:	0.552	0.405	0.195
	0.04691	0.500	0.400	0.186		0.466	0.371	0.155
	0.23077	0.333	0.277	0.098		0.238	0.183	0.056
	0.50000	0.163	0.170	0.052		0.097	0.099	0.028
	0.76923	0.078	0.120	0.035		0.043	0.068	0.019
	0.95309	0.041	0.098	0.029		0.018	0.055	0.015
	1.00000	0.030	0.041	0.028		0.013	0.022	0.014
$\omega_o = 0.60$, $\tau_1 = \infty$:	0.00000	0.336	0.165	0.046	$\omega_o = 0.60$, $\tau_1 = 0.50$:	0.330	0.162	0.046
	0.04691	0.303	0.160	0.046		0.293	0.157	0.046
	0.23077	0.206	0.142	0.046		0.180	0.119	0.039
	0.50000	0.124	0.120	0.046		0.085	0.077	0.027
	0.76923	0.073	0.097	0.046		0.040	0.055	0.020
	0.95309	0.048	0.071	0.046		0.021	0.041	0.017
	1.00000	0.043	0.041	0.046		0.016	0.012	0.016
$\omega_o = 0.30$, $\tau_1 = \infty$:	0.00000	0.153	0.057	0.014	$\omega_o = 0.30$, $\tau_1 = 0.50$:	0.151	0.056	0.014
	0.04691	0.133	0.054	0.014		0.131	0.054	0.014
	0.23077	0.080	0.047	0.014		0.075	0.043	0.013
	0.50000	0.042	0.039	0.014		0.034	0.029	0.011
	0.76923	0.020	0.030	0.014		0.015	0.020	0.008
	0.95309	0.011	0.020	0.014		0.008	0.014	0.007
	1.00000	0.010	0.008	0.014		0.007	0.003	0.007

* Extrapolated by Lagrange's formula for μ_o = o and 1.0.

Table B-4 Comparison of Exact and Approximate Total Transmissions

Anisotropic Scattering, F = 0.9330, P = 0.4821

	μ_o	Exact*	Six-flux	Two-flux		Exact*	Six-flux	Two-flux
ω_o = 0.90 τ_1 = 1.0:					ω_o =0.90 τ_1=0.50:			
	0.00000	0.236	0.220	0.000		0.291	0.269	0.000
	0.04691	0.259	0.233	0.041		0.331	0.288	0.198
	0.23077	0.372	0.339	0.511		0.507	0.515	0.711
	0.50000	0.553	0.523	0.730		0.723	0.707	0.853
	0.76923	0.692	0.632	0.814		0.833	0.790	0.902
	0.95309	0.750	0.673	0.847		0.885	0.821	0.920
	1.00000	0.761	0.662	0.853		0.904	0.811	0.923
ω_o = 0.90 τ_1 = 0.25:								
	0.00000	0.297	0.299	0.000				
	0.04691	0.396	0.354	0.439				
	0.23077	0.669	0.698	0.842				
	0.50000	0.853	0.835	0.923				
	0.76923	0.933	0.889	0.949				
	0.95309	0.979	0.905	0.959				
	1.00000	0.991	0.900	0.961				
ω_o = 0.60 τ_1 = 0.50:					ω_o =0.30 τ_1 =0.50:			
	0.00000	0.143	0.100	0.000		0.041	0.033	0.000
	0.04691	0.165	0.106	0.009		0.063	0.035	0.000
	0.23077	0.319	0.303	0.386		0.198	0.175	0.210
	0.50000	0.565	0.550	0.644		0.452	0.445	0.487
	0.76923	0.701	0.671	0.751		0.601	0.587	0.626
	0.95309	0.761	0.716	0.794		0.668	0.645	0.686
	1.00000	0.783	0.710	0.803		0.692	0.652	0.698

* Extrapolated by Lagrange's formula for μ_o = o and 1.0

Table B-5 Comparison of Exact and Approximate Integrated Reflectances and Total Transmissions

Anisotropic Scattering, $F_1 = 0.750$, $\omega_o = 0.90$, $\tau_1 = 1.0$

	Integrated Reflectances				Total Transmissions		
	μ_o	Exact*	Six-flux	Two-flux	Exact*	Six-flux	Two-flux
P = 0.5821:	0.00000	0.600	0.484	0.402	0.184	0.171	0.000
	0.04691	0.563	0.471	0.402	0.203	0.183	0.006
	0.23077	0.438	0.411	0.357	0.303	0.289	0.310
	0.50000	0.303	0.315	0.261	0.472	0.467	0.560
	0.76923	0.213	0.248	0.201	0.605	0.583	0.678
	0.95309	0.173	0.209	0.173	0.657	0.629	0.727
	1.00000	0.165	0.105	0.168	0.665	0.621	0.738
P = 0.4821:							
	0.00000	0.598	0.483	0.402	0.203	0.191	0.000
	0.04691	0.564	0.470	0.402	0.222	0.203	0.006
	0.23077	0.445	0.410	0.357	0.318	0.295	0.310
	0.50000	0.310	0.314	0.261	0.476	0.469	0.560
	0.76923	0.218	0.247	0.201	0.598	0.582	0.678
	0.95309	0.175	0.207	0.173	0.645	0.626	0.727
	1.00000	0.166	0.103	0.168	0.652	0.618	0.738
P = 0.3889:							
	0.00000	0.589	0.480	0.402	0.221	0.213	0.000
	0.04691	0.559	0.467	0.402	0.238	0.225	0.006
	0.23077	0.448	0.409	0.357	0.329	0.308	0.310
	0.50000	0.315	0.314	0.261	0.480	0.471	0.560
	0.76923	0.220	0.246	0.201	0.596	0.580	0.678
	0.95309	0.174	0.205	0.173	0.646	0.622	0.727
	1.00000	0.164	0.101	0.168	0.656	0.614	0.738

* Extrapolated by Lagrange's formula for μ_o = o and 1.0

APPENDIX C

Computer Programs

All computer programs were written in MAD (Michigan Algorithm Decoder) compiler language and were compiled and executed on the IBM-704. A description of the language is given in reference (10). Programs written in this compiler language may be compiled and run on either the IBM-704 or 709 and in the near future provision will be made for processing MAD programs on the IBM-7090.

The programs are presented in the following pages with brief explanatory notes.

1. Solution of the Integral Equations

a) Input Data

One general program was written to solve any set of integral equations for an arbitrary value of N and m. In order to specify the problem and its method of solution the following information is required. (The MAD symbols are placed in parenthesis for use in reading the listing of the programs).

1)	Run Number	A five-character number for identification of output (RUN)
	N	Number of terms in expansion of phase function (NTERM)
	p	Number of points to be used in numerical integration (NPOINT)

τ_1	Optical thickness of dispersion (TAU)
TOL	Iteration will be stopped whenever the maximum absolute value of the change in any function is less than TOL (TOL)
ω_0	Albedo for single scattering (ALBEDO)
2) $a_0 \ldots a_N$	Coefficients of an angular distribution function, A(0)...A(NTERMS)
3) $\mu_0, \ldots, \mu_N$	Set of values of the independent variable for use in numerical integration (See equation I-35), U(0)...U(NPOINT)
4) $W_0, \ldots, W_N$	Weight coefficients for numerical integration (See equation I-35), W(0)...W(NPOINT)
5) $\psi_l^m(\mu_i)^{(0)}$ $i = 0, \cdots, p$ $l = m, \cdots, N$	Initial value of Psi Functions, PSI(0,L,I)
6) $\phi_l^m(\mu_i)^{(0)}$ $i = 0, \cdots, p$ $l = m, \cdots, N$	Initial values of PHi Functions, PHI(0,L,I)

The preceeding six groups of information are referred to as the "basic data package." They completely define a solution to the integral equations or any trial solution. Thus, the computer starts with a basic data package as an initial approximation; it then performs a certain number of iterations; and, finally, produces as output a new basic data package. The basic data package is preceeded by a header card giving the maximum number of iterations allowable and information as to which of the special options (described next) are desired.

b) Special Options

The following options are available in the routine:

1) Items 5 and 6 (i.e. the initial values of the functions) of the data above may be deleted in which case the following expressions will be used for the initial functions.

$$\psi_l^m(\mu_i)^{(o)} = P_l^m(\mu_i)$$

$$\phi_l^m(\mu_i)^{(o)} = e^{-\tau_1/\mu_i} P_l^m(\mu_i)$$

2) The results of a computation may be punched on cards in the form of a basic data package for use in performing more iterations later or for use in another program if desired.

3) A reversal of direction of the change in ψ_l^o or ϕ_l^o may be interpreted as convergence and the iteration stopped when this occurs if desired.

In addition the following features are available for speeding up the solution under certain special circumstances.

a) if $\tau_1 \geq 80$, calculation of the $\phi_l^m(\mu_i)$ will be deleted and $\phi_l^m(\mu_i)$ will be set to zero

b) if any angular distribution coefficient is zero (for example if $a_k = 0$) then calculation of the corresponding functions will not be calculated but set to zero

$$\psi_k^m(\mu) \equiv 0$$

$$\phi_k^m(\mu) \equiv 0$$

The number of equations in a set for use in estimating the computer time or cost in I-56 or I-58 is equal to the number of non-zero terms in the phase function.

c) Restrictions

At the time of compilation of the MAD program an upper limit must be placed on the following so that the computer may properly allocate storage to the various arrays.

1) Maximum value of N, i.e. the maximum number of terms in the phase function (MAXL)

2) Maximum number of consecutive approximations which may be stored in the computer at one time (NMAX). Iteration will proceed computing f*(0), f*(1),....,f*(NMAX) after which f*(0) will be set to f*(NMAX) and the process repeated.

3) Maximum number of points to be used in numerical integration (MAXPT)

Large values of NMAX result in a slightly more efficient program, however NMAX may be any value greater than zero. The total storage required including the approximately 5000 locations required by the program and its subroutines is given by

$$\text{STORAGE} = 2(\text{NMAX} + 1)(\text{MAXL} + 1)(\text{MAXPT} + 1) + 4(\text{MAXL} +1)(\text{MAXPT} +1) + 4(\text{MAXPT}) + 5000$$

In practice, the large computer time required for large values of N and/or large values of p is a more serious restriction than the limitation on storage.

An additional restriction is that all numbers expressed in floating point form must be of the following range.

$$10^{37} > |X| > 10^{-37}$$

Thus, if μ_{min} is the smallest non-zero value of μ_i, then

$$e^{-\tau_1/\mu_{min}} \text{ must be greater than } 10^{-18}$$

in order to evaluate $F_k^m(\tau_1, \mu_{min}, \mu_{min})$ which is needed in performing some of the integrations. (Note: This does not apply if $\tau_1 \geqslant 80$ as the ϕ_1^m are not computed in this case.)

The only restriction on the integration procedure is that

$$\sum_{i=0}^{p} W_i = 1.0 \pm 10^{-7}, \qquad \sum_{i=0}^{p} W_i \mu_i = 0.5 \pm 10^{-7},$$

or in other words, the integrals

$$\int_0^1 dx = 1, \qquad \int_0^1 x\,dx = 0.5$$

must be done correctly.

d) General Outline of the Programs

The general outline of the programs is as follows:

1) A few initializing calculations are performed to give the computer information as to the maximum size of various arrays.

2) The header card is read followed by the basic data package.

3) Iterations are performed until convergence is indicated or until the allowable number of iterations is exceeded.

4) The results are punched and/or printed.

A printed copy of the MAD Program follows.

```
$COMPILE MAD, PUNCH OBJECT, PRINT OBJECT                      5THI0001
          R
          R   GENERALIZED PROGRAM FOR SOLUTION OF LIGHT SCATTERING
          R   INTEGRAL EQUATIONS
          R
           BOOLEAN  C1, C2, C4, C5, PCH, BL,BJ,BK,CONVG,THICK,OMIT,SIGN,
          1   CONVG1
           INTEGER I,J,K,L,M,N,NPOINT,NTERMS,MAXIT,ITCNT,INT1,INT2,
          1   FIXARG,RUN,NMAX1,TYPE,TIME.,I0,I1,I2
           DIMENSION BLOOP(100)
           VECTOR VALUES NMAX = 3
           VECTOR VALUES MAXL = 3
           VECTOR VALUES MAXPT = 20
          R  FOLLOWING DIMENSIONED (NMAX+1)*(MAXL+1)*(MAXPT+1) - 1
           DIMENSION PSI(335,PSIDIM),PHI(335,PHIDIM)
          R  FOLLOWING DIMENSIONED (MAXL+1)*(MAXPT+1) + 1
           DIMENSION S(83,SDIM),POL(83,POLDIM),CHPSI(83,CHPSID),
          1   CHPHI(83,CHPHID)
          R  FOLLOWING DIMENSIONED MAXPT
           DIMENSION SS(20),W(20),U(20),BJ(20)
          R  FOLLOWING DIMENSIONED MAXL
           DIMENSION PSIDEV(3),PHIDEV(3),OMEGA(3),     A(3),BL(3),BK(3),
          1   OMIT(3)
           DIMENSION PSIDIM(3),PHIDIM(3),SDIM(2),POLDIM(2),CHPSID(2),
          1   CHPHID(2)
           VECTOR VALUES PSIDIM = 3
           VECTOR VALUES PHIDIM = 3
           VECTOR VALUES SDIM = 2
           VECTOR VALUES POLDIM = 2
           VECTOR VALUES CHPSID = 2
           VECTOR VALUES CHPHID = 2
           PSIDIM(3) = MAXPT + 1
           PSIDIM(2) = MAXL + 1
           PSIDIM(1) = PSIDIM(2)*PSIDIM(3) + PSIDIM(3) + 1
           PHIDIM(3) = MAXPT + 1
           PHIDIM(2) = MAXL + 1
           PHIDIM(1) = PHIDIM(2)*PHIDIM(3) + PHIDIM(3) + 1
           SDIM(2) = MAXPT + 1
           SDIM(1) = SDIM(2) + 1
           POLDIM(2) = MAXPT + 1
           POLDIM(1) = POLDIM(2) + 1
           CHPSID(2) = MAXPT + 1
           CHPSID(1) = CHPSID(2) + 1
           CHPHID(2) = MAXPT + 1
           CHPHID(1) = CHPHID(2) + 1
START      EXECUTE ZERO.(BL...BL(MAXL),BJ...BJ(MAXPT),BK...BK(MAXL))
           READ FORMAT INBOOL, I,BL...BL(I)
           READ FORMAT INBOOL, I,BJ...BJ(I)
           READ FORMAT INBOOL, I,BK...BK(I)
           READ FORMAT IN1, MAXIT, TYPE,PCH,SIGN
           READ FORMAT INA, RUN,M,NTERMS,NPOINT,TAU,TOL,ALBEDO
           PRINT FORMAT FORM1, RUN,M,NTERMS,NPOINT,MAXIT,TOL,TAU,ALBEDO
           THICK = 0B
           WHENEVER TAU .GE. 80., THICK = 1B
           WHENEVER TYPE .E. 0
              PRINT FORMAT FORM9, $READ IN$
           OR WHENEVER TYPE .E. 1
              PRINT FORMAT FORM9, $CALCULATED FROM LEGENDRE POLYNOMIALS$
           END OF CONDITIONAL
```

```
            READ FORMAT IN2, A...A(NTERMS)
            PRINT FORMAT FORM2
            PRINT FORMAT FF1, A...A(NTERMS)
            EXECUTE ZERO.(OMEGA...OMEGA(NTERMS),OMIT...OMIT(NTERMS))
            THROUGH ST6, FOR L = M,1,L.G.NTERMS
            OMEGA(L) =      A(L)*FACT.(L-M)/FACT.(L+M)*ALBEDO
ST6         WHENEVER OMEGA(L) .E. 0., OMIT(L) = 1B
            PRINT FORMAT FORM3
            PRINT FORMAT FF1, OMEGA...OMEGA(NTERMS)
            READ FORMAT IN2, U...U(NPOINT)
            PRINT FORMAT FORM4
            PRINT FORMAT FF1, U...U(NPOINT)
            READ FORMAT IN2, W...W(NPOINT)
            PRINT FORMAT FORM5
            TEST = -0.5
            WEIGHT = -1.0
            THROUGH SL6, FOR I = 0,1,I.G.NPOINT
            TEST = TEST + W(I)*U(I)
SL6         WEIGHT = WEIGHT + W(I)
            PRINT FORMAT FF1, W...W(NPOINT),WEIGHT,TEST
            WHENEVER .ABS.WEIGHT.G.0.0000001
               PRINT FORMAT BCD, $1SUM OF WEIGHTS NOT 1.0$
            OR WHENEVER .ABS.TEST.G.0.0000001
               PRINT FORMAT BCD, $1SUM OF W(I)*U(I) NOT 0.5$
            OR WHENEVER M.G.NTERMS
               PRINT FORMAT BCD, $1M IS GREATER THAN NTERMS$
            OTHERWISE
               TRANSFER TO AWAY
            END OF CONDITIONAL
            EXECUTE ERROR.
AWAY        CONTINUE
            THROUGH S1, FOR L = M,1,L.G.NTERMS
            THROUGH S1, FOR J = 0,1,J.G.NPOINT
S1          POL(L,J) = LEG.(U(J),M,L)
            PRINT FORMAT FORM6
            THROUGH ST3, FOR L =M,1,L.G.NTERMS
ST3         PRINT FORMAT FF4,L,M,POL(L,0)...POL(L,NPOINT)
            WHENEVER TYPE .E. 0
            THROUGH SL1, FOR L = M,1,L.G.NTERMS
SL1         READ FORMAT IN2, PSI(0,L,0)...PSI(0,L,NPOINT),PSIDEV(L)
            THROUGH SL2, FOR L = M,1,L.G.NTERMS
SL2         READ FORMAT IN2, PHI(0,L,0)...PHI(0,L,NPOINT),PHIDEV(L)
            OR WHENEVER TYPE .E. 1
            THROUGH SL7A, FOR L = M,1,L.G.NTERMS
            PSIDEV(L) = 0.
            PHIDEV(L) = 0.
            WHENEVER OMIT(L)
               EXECUTE ZERO.(PSI(0,L,0)...PSI(0,L,NPOINT))
               EXECUTE ZERO.(PHI(0,L,0)...PHI(0,L,NPOINT))
               TRANSFER TO SL7A
            END OF CONDITIONAL
            THROUGH SL7, FOR J = 0,1,J.G.NPOINT
            PSI(0,L,J) = POL(L,J)
            WHENEVER U(J) .G. 0.  .AND.  TAU/U(J).L.80.
               PHI(0,L,J) = EXP.(-TAU/U(J))*POL(L,J)
            OTHERWISE
               PHI(0,L,J) = 0.
SL7         END OF CONDITIONAL
```

```
SL7A         CONTINUE
             OTHERWISE
             EXECUTE ERROR.
             END OF CONDITIONAL
             PRINT FORMAT FORM7
             THROUGH ST4, FOR L = M,1,L.G.NTERMS
ST4          PRINT FORMAT FF2,L,M,PSI(0,L,0)...PSI(0,L,NPOINT)
             PRINT FORMAT FORM8
             THROUGH ST5, FOR L = M,1,L.G.NTERMS
ST5          PRINT FORMAT FF3,L,M,PHI(0,L,0)...PHI(0,L,NPOINT)
             CONVG = 0B
             ITCNT = 1
             PRINT FORMAT BCD,$1BEGINNING OF ITERATIVE SOLUTION$
             TIME1 = TIME.(0)
            R...........................................................
LOOP         THROUGH S6, FOR N = 1,1,N.G.NMAX.OR.CONVG.OR.ITCNT.G.MAXIT
             EXECUTE ZERO.(PSIDEV(M)...PSIDEV(NTERMS))
             THROUGH A1, FOR L = M,1,L.G.NTERMS
             WHENEVER OMIT(L)
                EXECUTE ZERO.(PSI(N,L,0)...PSI(N,L,NPOINT))
                TRANSFER TO A1
             END OF CONDITIONAL
             THROUGH A2, FOR J = 0,1,J.G.NPOINT
             WHENEVER U(J) .E. 0.
                PSI(N,L,J) = POL(L,J)
                TRANSFER TO A5
             END OF CONDITIONAL
             EXECUTE SETCHK.
             SUMX = 0.
             THROUGH A3, FOR I = 0,1,I.G.NPOINT
             XXX = 0.
             THROUGH A4, FOR K = M,1,K.G.NTERMS
             WHENEVER OMIT(K), TRANSFER TO A4
             TS1 = (-1.0).P.(K+L)*F.(0)
             WHENEVER C1,S(K,I) = TS1
             XXX = XXX + TS1
A4           CONTINUE
             WHENEVER C2,SS(I) = XXX
A3           SUMX = SUMX + W(I)*XXX*POL(L,I)
             PSI(N,L,J) = POL(L,J) + 0.5*U(J)*SUMX
A5           TS1 = PSI(N,L,J) - PSI(N-1,L,J)
             EXECUTE CHECK.
A2           WHENEVER .ABS.TS1.G.PSIDEV(L), PSIDEV(L) = .ABS. TS1
A1           CONTINUE
            R...........................................................
             EXECUTE ZERO.(PHIDEV(M)...PHIDEV(NTERMS))
             WHENEVER THICK
             THROUGH SL10, FOR L = M,1,L.G.NTERMS
SL10         EXECUTE ZERO.(PHI(N,L,0)...PHI(N,L,NPOINT))
             TRANSFER TO ON
             END OF CONDITIONAL
             THROUGH B1, FOR L = M,1,L.G.NTERMS
             WHENEVER OMIT(L)
                EXECUTE ZERO.(PHI(N,L,0)...PHI(N,L,NPOINT))
                TRANSFER TO B1
             END OF CONDITIONAL
             THROUGH B2, FOR J = 0,1,J.G.NPOINT
             WHENEVER U(J) .E. 0.
```

```
           PHI(N,L,J) = 0.
           TRANSFER TO B5
        OR WHENEVER TAU/U(J) .G. 80.
           EXPON = 0.
        OTHERWISE
           EXPON = EXP.(-TAU/U(J))
        END OF CONDITIONAL
        EXECUTE SETCHK.
        SUMY = 0.
        THROUGH B3, FOR I = 0,1,I.G.NPOINT
        YYY = 0.
        THROUGH B4, FOR K = M,1,K.G.NTERMS
        WHENEVER OMIT(K), TRANSFER TO B4
        TS1 = G.(0)
        WHENEVER C4, S(K,I) = TS1
        YYY = YYY + TS1
B4      CONTINUE
        WHENEVER C5, SS(I) = YYY
B3      SUMY = SUMY + W(I)*YYY*POL(L,I)
        PHI(N,L,J) = EXPON*POL(L,J) + 0.5*U(J)*SUMY
B5      TS1 = PHI(N,L,J) - PHI(N-1,L,J)
        EXECUTE CHECK.
B2      WHENEVER .ABS.TS1 .G. PHIDEV(L), PHIDEV(L) = .ABS. TS1
B1      CONTINUE
       R.................................................................
ON      PRINT FORMAT HEAD1,ITCNT
        PRINT FORMAT FF1, U...U(NPOINT)
        CONVG = 1B
        CONVG1 = 0B
        THROUGH S7A, FOR L = M,1,L.G.NTERMS
        WHENEVER OMIT(L),TRANSFER TO S7A
        THROUGH S11, FOR J = 0,1,J.G.NPOINT
        TS1  = PSI(N,L,J) - PSI(N-1,L,J)
        WHENEVER L.NE.0, TRANSFER TO S11
        WHENEVER TS1*CHPSI(L,J).L.0. .AND. ITCNT.G.1 .AND. SIGN,
       1   CONVG1 = 1B
S11     CHPSI(L,J) = TS1
        PRINT FORMAT FF2,L,M,CHPSI(L,0)...CHPSI(L,NPOINT),PSIDEV(L)
S7A     WHENEVER PSIDEV(L).G.TOL, CONVG = 0B
        THROUGH S7B, FOR L = M,1,L.G.NTERMS
        WHENEVER OMIT(L), TRANSFER TO S7B
        THROUGH S12, FOR J = 0,1,J.G.NPOINT
        TS1  = PHI(N,L,J)-PHI(N-1,L,J)
        WHENEVER L.NE.0, TRANSFER TO S12
        WHENEVER TS1*CHPHI(L,J).L.0. .AND. ITCNT.G.1 .AND. SIGN,
       1   CONVG1 = 1B
S12     CHPHI(L,J) = TS1
        PRINT FORMAT FF3,L,M,CHPHI(L,0)...CHPHI(L,NPOINT),PHIDEV(L)
S7B     WHENEVER PHIDEV(L).G.TOL, CONVG = 0B
        CONVG = CONVG .OR. CONVG1
S6      ITCNT = ITCNT + 1
        NMAX1 = N-1
        WHENEVER CONVG
           PRINT FORMAT BCD,$4SUCCESSFUL SOLUTION$
        OR WHENEVER ITCNT .G. MAXIT
           PRINT FORMAT BCD,$4DESIRED ACCURACY COULD NOT BE OBTAINED
       1 IN SPECIFIED NUMBER OF ITERATIONS$
        OTHERWISE
```

```
                   THROUGH ALPHA5, FOR L = M,1,L.G.NTERMS
                   THROUGH ALPHA5, FOR I = 0,1,I.G.NPOINT
                   PSI(0,L,I) = PSI(NMAX1,L,I)
ALPHA5             PHI(0,L,I) = PHI(NMAX1,L,I)
                   TRANSFER TO LOOP
                END OF CONDITIONAL
                TIME2 = TIME.(0)
                PRINT FORMAT FTIME,(TIME2-TIME1)*3.6
                PRINT FORMAT BCD, $0VALUES OF FUNCTIONS AFTER LAST ITERATION$
                N = NMAX1
                THROUGH S13, FOR L = M,1,L.G.NTERMS
S13             PRINT FORMAT FF2,L,M,PSI(N,L,0)...PSI(N,L,NPOINT),PSIDEV(L)
                THROUGH S14, FOR L = M,1,L.G.NTERMS
S14             PRINT FORMAT FF3,L,M,PHI(N,L,0)...PHI(N,L,NPOINT),PHIDEV(L)
                WHENEVER PCH
                PRINT FORMAT BCD, $0RESULTS PUNCHED$
                INA(4) = RUN
                IN2(3) = RUN
                PUNCH FORMAT INA,RUN,M,NTERMS,NPOINT,TAU,TOL,ALBEDO
                PUNCH FORMAT IN2, A...A(NTERMS)
                PUNCH FORMAT IN2, U...U(NPOINT)
                PUNCH FORMAT IN2, W...W(NPOINT)
                THROUGH SL3, FOR L = M,1,L.G.NTERMS
SL3             PUNCH FORMAT IN2, PSI(NMAX1,L,0)...PSI(NMAX1,L,NPOINT),
               1    PSIDEV(L)
                THROUGH SL4, FOR L = M,1,L.G.NTERMS
SL4             PUNCH FORMAT IN2, PHI(NMAX1,L,0)...PHI(NMAX1,L,NPOINT),
               1    PHIDEV(L)
                END OF CONDITIONAL
                TRANSFER TO START
               R
               R    INTERNAL FUNCTIONS
               R
                INTERNAL FUNCTION
                ENTRY TO F.
                FUNCTION RETURN OMEGA(K)*(PSI(N-1,K,J)*PSI(N-1,K,I)
               1       - PHI(N-1,K,J)*PHI(N-1,K,I))/(U(I)+U(J))
               R
                ENTRY TO G.
                WHENEVER I.NE.J
                FUNCTION RETURN OMEGA(K)*(PHI(N-1,K,J)*PSI(N-1,K,I)
               1        - PSI(N-1,K,J)*PHI(N-1,K,I))/(U(J)-U(I))
                OTHERWISE
                   WHENEVER I.E.0
                      I0 = 0
                      I1 = 1
                      I2 = 2
                   OR WHENEVER I.E.NPOINT
                      I0 = NPOINT - 2
                      I1 = NPOINT - 1
                      I2 = NPOINT
                   OTHERWISE
                      I0 = I-1
                      I1 = I
                      I2 = I+1
                   END OF CONDITIONAL
                X = U(I)
                X0 = U(I0)
```

```
             X1 = U(I1)
             X2 = U(I2)
             F0 = PHI(N-1,K,I0)
             F1 = PHI(N-1,K,I1)
             F2 = PHI(N-1,K,I2)
             PHPR = (F2-F0)/(X2-X0) + (2.*X-X0-X2)*
            1   ((X2-X1)*F0 - (X2-X0)*F1 + (X1-X0)*F2)/
            2   ((X1-X0)*(X2-X0)*(X2-X1))
             F0 = PSI(N-1,K,I0)
             F1 = PSI(N-1,K,I1)
             F2 = PSI(N-1,K,I2)
             PSPR = (F2-F0)/(X2-X0) + (2.*X-X0-X2)*
            1   ((X2-X1)*F0 - (X2-X0)*F1 + (X1-X0)*F2)/
            2   ((X1-X0)*(X2-X0)*(X2-X1))
             FUNCTION RETURN OMEGA(K)*(PSI(N-1,K,J)*PHPR
            1   - PHI(N-1,K,J)*PSPR)
             END OF CONDITIONAL
             END OF FUNCTION
            R
             INTERNAL FUNCTION (FIXARG)
             INTEGER FIXARG
             ENTRY TO FACT.
             TS1 = 1.
             THROUGH ALPHA2, FOR INT1 = 1,1,INT1.G.FIXARG
ALPHA2       TS1 = TS1*INT1
             FUNCTION RETURN TS1
             END OF FUNCTION
            R
             INTERNAL FUNCTION
             ENTRY TO CHECK.
             WHENEVER BJ(J).AND.BL(L)
                PRINT FORMAT NGFO,$J$,J,U(J)
                THROUGH SL5, FOR K = M,1,K.G.NTERMS
SL5             WHENEVER BK(K), PRINT FORMAT NGEO, $K$,K,
            1      S(K,0)...S(K,NPOINT)
                PRINT FORMAT  GEO,     J,SS...SS(NPOINT),TS1,
            1      PSI(N,L,J),PHI(N,L,J)
             END OF CONDITIONAL
             FUNCTION RETURN
            R
             ENTRY TO SETCHK.
             WHENEVER BL(L).AND.BJ(J)
                C1 = 1B
                C2 = 1B
                C4 = 1B
                C5 = 1B
             OTHERWISE
                C1 = 0B
                C2 = 0B
                C4 = 0B
                C5 = 0B
             END OF CONDITIONAL
             FUNCTION RETURN
             END OF FUNCTION
            R
            R   FORMAT SPECIFICATIONS AND PRE-SET CONSTANTS
            R
             VECTOR VALUES INA = $C5,3I5$,$,3F10.$,$5,S22,$,$    5H$,$ $,
```

```
1  $*$
 VECTOR VALUES IN1 = $4I5,5F10.5*$
 VECTOR VALUES IN2 = $1P5E14$,$.7,S2,$,$    5H$,$ $,$*$
 VECTOR VALUES INBOOL = $I5,S5,60I1/(70I1)*$
 VECTOR VALUES HEAD1 = $37H0CHANGE IN FUNCTION DURING ITERATIO
1N     I3*$
 VECTOR VALUES GIO = $1H0I9,11I10/(S10,11I10)*$
 VECTOR VALUES GFO = $1H0,I5,S4,10F11.6/(S10,10F11.6)*$
 VECTOR VALUES GEO = $1H0,I5,S4,1P10E11.3/(S10,1P10E11.3)*$
 VECTOR VALUES BCD = $20C6*$
 VECTOR VALUES NGFO = $1H0,C1,2H =,I3,S3,10F11.6/
1   (S10,10F11.6)*$
 VECTOR VALUES NGEO = $1H0,C1,2H =,I3,S3,1P10E11.3/
1   (S10,1P10E11.3)*$
 VECTOR VALUES FORM1 = $40H1SOLUTION OF INTEGRAL EQUATIONS FOR
1 RUN   C5//6H M =    I5,S5,18HNUMBER OF TERMS =    I5,S5,
219HNUMBER OF POINTS =    I5/32H MAXIMUM NUMBER OF ITERATIONS
3= I5,S5,33HDESIRED DEGREE OF CONVERGENCE =   1PE10.1/
4  7H TAU =   1PE11.3,S5,32H ALBEDO FOR SINGLE SCATTERING =
5  1PE11.3*$
 VECTOR VALUES FTIME = $19H0TIME REQUIRED WAS     F6.1,
1  8H SECONDS  *$
 VECTOR VALUES FORM2 = $30H0PHASE FUNCTION--A...A(NTERMS)   *$
 VECTOR VALUES FORM3 =
1 $47H0MODIFIED PHASE FUNCTION--OMEGA...OMEGA(NTERMS)   *$
 VECTOR VALUES FORM4 = $28H0VALUES OF MU--U...U(NPOINT)    *$
 VECTOR VALUES FORM5 = $30H0WEIGHT FACTORS--W...W(NPOINT)   *$
 VECTOR VALUES FORM6 =
1 $47H0LEGENDRE POLYNOMIALS--P(L,M) FOR U...U(NPOINT)   *$
 VECTOR VALUES FORM7 =
1$47H0INITIAL FUNCTIONS--PSI(L,M) FOR U...U(NPOINT)   *$
 VECTOR VALUES FORM8 =
1$47H0INITIAL FUNCTIONS--PHI(L,M) FOR U...U(NPOINT)   *$
 VECTOR VALUES FORM9 = $19H0INITIAL FUNCTIONS  8C6*$
 VECTOR VALUES FF1 = $1H0,S9,1P9E12.4/(S10,1P9E12.4)*$
 VECTOR VALUES FF2 = $5H0PSI( I1, 1H, I1, 2H)  1P 9E12.4/
1  (S10,1P 9E12.4)*$
 VECTOR VALUES FF3 = $5H0PHI( I1, 1H, I1, 2H)  1P 9E12.4/
1  (S10,1P 9E12.4)*$
 VECTOR VALUES FF4 = $5H0  P( I1, 1H, I1, 2H)  1P 9E12.4/
1  (S10,1P 9E12.4)*$
 END OF PROGRAM
```

2. Subroutine to Calculate Associated Legendre Polynomials

A subroutine was written to calculate $P_l^m(x)$.

The program should be self explanatory.

```
*COMPILE MAD,PUNCH OBJECT                                             LEG  001
            EXTERNAL FUNCTION (X,M,N)
            INTEGER N,M,K,FIXARG,INT1,TWOLOG.
            ENTRY TO LEG.
            WHENEVER X.G.1. .OR. X .L.0.  .OR. M.G. N
               PRINT FORMAT BCD,$4BAD ARGUMENT FOR LEGENDRE POLYNOMIAL SU
           1BROUTINE$
               PRINT FORMAT FORMA,X,M,N
               ERROR RETURN
            END OF CONDITIONAL
            U = (1.-X)/2.
            T = FACT.(N+M)/(FACT.(M)*FACT.(N-M))
            SUM = T
            THROUGH S1, FOR K = 1,1,K.G.N-M
            T = -T*(N+M+K)*(N-M-K+1)  /(K*(M+K))
            WHENEVER TWOLOG.(T) + TWOLOG.(U) .L. -115,
           1   TRANSFER TO OUT
            T = T*U
S1          SUM = SUM + T
OUT         WHENEVER M.E.0, FUNCTION RETURN SUM
            WHENEVER X.E.1., FUNCTION RETURN 0.
            V = SQRT.(1.-X*X)/2.
            FUNCTION RETURN SUM*V.P.M
            INTERNAL FUNCTION (FIXARG)
            ENTRY TO FACT.
            TS1 = 1.
            THROUGH BETA2, FOR INT1 = 1,1,INT1.G.FIXARG
BETA2       TS1 = TS1*INT1
            FUNCTION RETURN TS1
            END OF FUNCTION
            VECTOR VALUES BCD = $20C6*$
            VECTOR VALUES FORMA = $1H0 1PE15.6,2I15*$
            END OF FUNCTION
*ASSEMBLE,PUNCH OBJECT                                                2LOG 001
        ORG 0
        PGM
        PZE SIZE
        PZE
        BCD 1TWOLOG
        PZE START
        REL
        ORG 0
START   CAL 1,4
        STA *+1
        CLA **
        SSP
        SUB TWO
        ARS 27
        TRA 2,4
TWO     OCT 200000000000
SIZE    SYN *
        END
```

3. Program to Calculate Integrated Reflectance and Transmission

The program to calculate the integrated reflectance and transmission follows the same general outline as the program to solve the integral equations. It accepts as input a basic data package. The results for $R(\mu_o)$ and $T(\mu_o)$ are calculated using A-4.2 and A-4.4.

```
*COMPILE MAD, PUNCH OBJECT ,PRINT OBJECT                        REF  001
          R
          R   INTEGRATED REFLECTANCE PROGRAM
          R
           INTEGER I,J,K,L,M,N,NPOINT,NTERMS,INT1,INT2,FIXARG,RUN
           BOOLEAN PCH
           VECTOR VALUES NMAX = 0
           VECTOR VALUES MAXL = 6
           VECTOR VALUES MAXPT = 50
          R  FOLLOWING DIMENSIONED (NMAX+1)*(MAXL+1)*(MAXPT+1) - 1
           DIMENSION PSI(356,PSIDIM), PHI(356,PHIDIM)
          R  FOLLOWING DIMENSIONED MAXPT
           DIMENSION W(50),U(50),T(50),R(50)
          R  FOLLOWING DIMENSIONED MAXL
           DIMENSION OMEGA(6),A(6),PSIDEV(6),PHIDEV(6)
           DIMENSION PSIDIM(3),PHIDIM(3)
           VECTOR VALUES PSIDIM = 3
           VECTOR VALUES PHIDIM = 3
           PSIDIM(3) = MAXPT + 1
           PSIDIM(2) = MAXL + 1
           PSIDIM(1) = PSIDIM(2)*PSIDIM(3) + PSIDIM(3) + 1
           PHIDIM(3) = MAXPT + 1
           PHIDIM(2) = MAXL + 1
           PHIDIM(1) = PHIDIM(2)*PHIDIM(3) + PHIDIM(3) + 1
START      READ FORMAT IN1, PCH
           READ FORMAT IN1, RUN,M,NTERMS,NPOINT,TAU,TOL,ALBEDO
           WHENEVER M.NE. 0
              PRINT FORMAT BCD, $1M NOT ZERO$
              EXECUTE ERROR.
           END OF CONDITIONAL
           READ FORMAT IN2, A...A(NTERMS)
           EXECUTE ZERO.(OMEGA...OMEGA(NTERMS))
           THROUGH ST6, FOR L = M,1,L.G.NTERMS
ST6        OMEGA(L) = A(L)*FACT.(L-M)/FACT.(L+M)*ALBEDO
           READ FORMAT IN2, U...U(NPOINT)
           READ FORMAT IN2, W...W(NPOINT)
           THROUGH SL1, FOR L = M,1,L.G.NTERMS
SL1        READ FORMAT IN2, PSI(0,L,0)...PSI(0,L,NPOINT),PSIDEV(L)
           THROUGH SL2, FOR L = M,1,L.G.NTERMS
SL2        READ FORMAT IN2, PHI(0,L,0)...PHI(0,L,NPOINT),PHIDEV(L)
           PRINT FORMAT BCD, $1INPUT DATA$
           PRINT FORMAT GIO, NPOINT, M, NTERMS, RUN
           PRINT FORMAT GFO, NTERMS, OMEGA...OMEGA(NTERMS)
           PRINT FORMAT GFO, 0, U...U(NPOINT)
           PRINT FORMAT GFO,0,W...W(NPOINT)
           PRINT FORMAT GEO,0,TAU,TOL
           THROUGH ST3, FOR L = M,1,L.G.NTERMS
ST3        PRINT FORMAT GEO,L,PSI(0,L,0)...PSI(0,L,NPOINT),PSIDEV(L)
           THROUGH ST4, FOR L = M,1,L.G.NTERMS
ST4        PRINT FORMAT GEO,L,PHI(0,L,0)...PHI(0,L,NPOINT),PHIDEV(L)
           THROUGH A2, FOR J = 0,1,J.G.NPOINT
           SUMX = 0.
           SUMY = 0.
           THROUGH A3, FOR I = 0,1,I.G.NPOINT
           XXX = 0.
           YYY = 0.
           THROUGH A4, FOR K = M,1,K.G.NTERMS
           XXX = XXX + (-1.0).P.K * F.(0)
A4         YYY = YYY + G.(0)
```

```
             SUMX  =  SUMX  +  W(I)*U(I)*XXX
A3           SUMY  =  SUMY  +  W(I)*U(I)*YYY
             R(J)  =  0.5*SUMX
A2           T(J)  =  0.5*SUMY
             PRINT FORMAT BCD, $0REFLECTION AND TRANSMISSION COEFFICIENTS$
             PRINT FORMAT GEO,1,R...R(NPOINT)
             PRINT FORMAT GEO,2,T...T(NPOINT)
             WHENEVER PCH
                PUNCH FORMAT IN1, RUN, M, NTERMS, NPOINT, TAU, TOL, ALBEDO
                PUNCH FORMAT IN2, R...R(NPOINT)
                PUNCH FORMAT IN2, T...T(NPOINT)
             END OF CONDITIONAL
             TRANSFER TO START
            R
            R   INTERNAL FUNCTIONS
            R
             INTERNAL FUNCTION
             INTEGER I0,I1,I2
             ENTRY TO F.
             FUNCTION RETURN OMEGA(K)*(PSI(N-1,K,J)*PSI(N-1,K,I)
            1     - PHI(N-1,K,J)*PHI(N-1,K,I))/(U(I)+U(J))
            R
             ENTRY TO G.
             WHENEVER I.NE.J
             FUNCTION RETURN OMEGA(K)*(PHI(N-1,K,J)*PSI(N-1,K,I)
            1      - PSI(N-1,K,J)*PHI(N-1,K,I))/(U(J)-U(I))
             OTHERWISE
                WHENEVER I.E.0
                   I0 = 0
                   I1 = 1
                   I2 = 2
                OR WHENEVER I.E.NPOINT
                   I0 = NPOINT - 2
                   I1 = NPOINT - 1
                   I2 = NPOINT
                OTHERWISE
                   I0 = I-1
                   I1 = I
                   I2 = I+1
                END OF CONDITIONAL
             X = U(I)
             X0 = U(I0)
             X1 = U(I1)
             X2 = U(I2)
             F0 = PHI(N-1,K,I0)
             F1 = PHI(N-1,K,I1)
             F2 = PHI(N-1,K,I2)
             PHPR = (F2-F0)/(X2-X0) + (2.*X-X0-X2)*
            1   ((X2-X1)*F0 - (X2-X0)*F1 + (X1-X0)*F2)/
            2   ((X1-X0)*(X2-X0)*(X2-X1))
             F0 = PSI(N-1,K,I0)
             F1 = PSI(N-1,K,I1)
             F2 = PSI(N-1,K,I2)
             PSPR = (F2-F0)/(X2-X0) + (2.*X-X0-X2)*
            1   ((X2-X1)*F0 - (X2-X0)*F1 + (X1-X0)*F2)/
            2   ((X1-X0)*(X2-X0)*(X2-X1))
             FUNCTION RETURN OMEGA(K)*(PSI(N-1,K,J)*PHPR
            1   - PHI(N-1,K,J)*PSPR)
```

```
            END OF CONDITIONAL
            END OF FUNCTION
           R
            INTERNAL FUNCTION (FIXARG)
            INTEGER FIXARG
            ENTRY TO FACT.
            TS1 = 1.
            THROUGH ALPHA2, FOR INT1 = 1,1,INT1.G.FIXARG
ALPHA2      TS1 = TS1*INT1
            FUNCTION RETURN TS1
            END OF FUNCTION
           R
           R   FORMAT SPECIFICATIONS AND PRE-SET CONSTANTS
           R
            VECTOR VALUES IN1 = $4I5,5F10.5*$
            VECTOR VALUES IN2 = $1P5E14.7,S2*$
            VECTOR VALUES GIO = $1H0I9,11I10/(S10,11I10)*$
            VECTOR VALUES GFO = $1H0,I5,S4,10F11.6/(S10,10F11.6)*$
            VECTOR VALUES GEO = $1H0,I5,S4,1P10E11.3/(S10,1P10E11.3)*$
            VECTOR VALUES BCD = $20C6*$
            VECTOR VALUES N = 1
            END OF PROGRAM
```

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