

On Existence of Mini-Boson Stars

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Abstract: We prove the existence of a countable family of globally regular solutions of spherically symmetric Einstein–Klein–Gordon equations. These solutions, known as mini-boson stars, were discovered numerically many years ago.

1. Introduction

Boson stars are compact gravitationally bound soliton-like equilibrium configurations of bosonic fields. The simplest kind of boson star, which is made up of a self-gravitating free complex massive scalar field, was conceived over thirty years ago by Kaup [1] and Ruffini and Bonazzola [2] who found numerically the ground state solution to the spherically symmetric Einstein–Klein–Gordon (EKG) equations. A decade later the systematic numerical analysis of these equations was performed by Friedberg, Lee, and Pang [3] who rediscovered and extended the results of [1, 2], in particular they found a countable sequence of excited states.

The aim of this paper is to give a rigorous proof of existence of solutions found in [1–3]. In the physics literature these solutions are usually referred to as mini-boson stars (“mini” because they are tiny objects with mass $\sim \frac{1}{Gm}$, where m is the boson mass). What are the motivations for studying such objects? Let us mention three possible reasons varying from physical to purely mathematical. First, most theories of elementary particles predict the existence of massive bosons which interact weakly with baryonic matter. To the extent one believes in these models, one should accept their consequences, like boson stars. From this standpoint, the recent surge of interest in boson stars is largely due to the suggestion that the dark matter could be bosonic since then some fraction of the missing mass of the universe would float around in the form of boson stars. Second, even if massive scalar fields do not exist in nature, they provide one of the simplest fundamental matter sources for the Einstein equations and, as such, are ideal theoretical “laboratories” for studying the dynamics of gravitational collapse. Mathematically, these studies amount to the analysis of the Cauchy problem for the EKG equations. Boson stars

play an important role in this context as candidates for intermediate or final attractors of dynamical evolution. Finally, and admittedly most interestingly for us, mini-boson stars are *non-perturbative* solutions of the EKG equations in the sense that they have no regular flat-spacetime limit (one manifestation of this property is the fact mentioned above that the total mass of a mini-boson star is inversely proportional to the gravitational constant G). In this respect mini-boson stars are similar to the Bartnik–McKinnon solutions of the Einstein–Yang–Mills equations [4]. However, in contrast to the Bartnik–McKinnon solutions, the mini-boson stars are not static: although the metric and the stress-energy tensor of the scalar field are time-independent, the scalar field itself has the form of a standing wave $\phi(r, t) = e^{i\omega t} \tilde{\phi}(r)$. This fact has an important consequence at the ode level, namely the lapse function does not decouple from the Klein–Gordon equation and the hamiltonian constraint which means that we have to deal with a 4-dimensional (nonautonomous) dynamical system¹. Below we analyze this system using a shooting method which is similar in spirit (but quite different in implementation) to the proof of existence of the Bartnik–McKinnon solutions [5].

The paper is organized as follows. In Sect. 2 we derive the field equations together with the boundary conditions and discuss some basic properties of solutions. We also formulate the main theorem and sketch the heuristic idea of its proof. In Sect. 3 we prove the local existence of solutions near the origin. In Sects. 4 and 5 we discuss the limiting behavior of solutions for small and large values of the shooting parameter, respectively. In Sect. 6 we derive the asymptotics of globally regular solutions. Sect. 7 contains some technical results concerning the behavior of singular solutions. Finally, in Sect. 8, using the results of Sects. 4–7, we complete the proof of the main theorem by a shooting argument.

2. Preliminaries

The action for the EKG system is given by

$$S = \int d^4x \sqrt{-g} \left(\frac{R}{16\pi G} - \frac{1}{2} \partial_a \phi^* \partial^a \phi - \frac{1}{2} m^2 \phi^* \phi \right), \tag{2.1}$$

where R is the scalar curvature of the spacetime metric g_{ab} , ϕ is the complex scalar field, and m is a real constant called the boson mass. The associated field equations are the Einstein equations

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G T_{ab} \tag{2.2}$$

with the stress-energy tensor of the scalar field

$$T_{ab} = \frac{1}{2} (\partial_a \phi^* \partial_b \phi + \partial_a \phi \partial_b \phi^*) - \frac{1}{2} g_{ab} (g^{cd} \partial_c \phi^* \partial_d \phi + m^2 \phi^* \phi), \tag{2.3}$$

and the Klein–Gordon equation

$$(\square - m^2)\phi = 0, \tag{2.4}$$

¹ For comparison, the static spherically symmetric Einstein–Yang–Mills equations reduce (within the purely magnetic ansatz) to a 3-dimensional (nonautonomous) dynamical system.

where \square is the d'Alembertian operator associated with the metric g_{ab} . Now, we assume that the fields are spherically symmetric. We write the metric using areal radial coordinate and polar slicing

$$ds^2 = -e^{-2\delta} Adt^2 + A^{-1} dr^2 + r^2 d\Omega^2, \tag{2.5}$$

where $d\Omega^2$ is the standard metric on the unit two-sphere, and A and δ are functions of (t, r) . In this parametrization the (relevant components of) Einstein equations have the particularly simple form

$$\partial_r A = \frac{1 - A}{r} - 8\pi Gr T_{00}, \tag{2.6}$$

$$\partial_r \delta = -4\pi Gr A^{-1}(T_{00} + T_{11}), \tag{2.7}$$

$$\partial_t A = -8\pi Gr e^{-\delta} AT_{01}, \tag{2.8}$$

where the components of stress-energy tensor T_{ab} are expressed in the orthonormal frame determined by the metric (2.5) ($e_0 = e^\delta A^{-1/2} \partial_t$, $e_1 = A^{1/2} \partial_r$). From (2.3) we obtain

$$T_{00} = \frac{1}{2}(A|\partial_r \phi|^2 + A^{-1} e^{2\delta} |\partial_t \phi|^2 + m^2 |\phi|^2), \tag{2.9}$$

$$T_{11} = \frac{1}{2}(A|\partial_r \phi|^2 + A^{-1} e^{2\delta} |\partial_t \phi|^2 - m^2 |\phi|^2), \tag{2.10}$$

$$T_{01} = \frac{1}{2} e^\delta (\partial_r \phi^* \partial_t \phi + \partial_r \phi \partial_t \phi^*). \tag{2.11}$$

The remaining components of Einstein's equations are equivalent to the Klein-Gordon equation.

For the scalar field ϕ we assume the standing wave ansatz $\phi(r, t) = \exp(i\omega t) \tilde{\phi}(r)$, where ω is a real constant. Then, due to the $U(1)$ symmetry of the action, the stress-energy tensor and the metric are time-independent. Moreover, $T_{01} = 0$ so Eq. (2.8) is trivially satisfied. In terms of the dimensionless variables

$$x = mr, \quad f(x) = \sqrt{4\pi G} \tilde{\phi}(r) \tag{2.12}$$

and the auxiliary variable

$$C(x) = \frac{\omega}{m} A^{-1} e^\delta, \tag{2.13}$$

Eqs. (2.4),(2.6) and (2.7) reduce to the following system of ordinary differential equations (hereafter prime denotes $\frac{d}{dx}$):

$$\left(\frac{x^2 f'}{C}\right)' = \frac{x^2}{AC} (1 - AC^2) f, \tag{2.14a}$$

$$A' = \frac{1 - A}{x} - x(Af'^2 + AC^2 f^2 + f^2), \tag{2.14b}$$

$$C' = \frac{C}{xA} (A - 1 + x^2 f^2). \tag{2.14c}$$

Instead of A , it is sometimes convenient to use the “mass” function $M(x)$ defined by $A(x) = 1 - 2M(x)/x$. From (2.14b) we have

$$M(x) = \frac{1}{2} \int_0^x s^2 (A f'^2 + AC^2 f^2 + f^2) ds. \tag{2.15}$$

A spacetime is said to be asymptotically flat if $\delta(\infty)$ is finite and

$$\lim_{x \rightarrow \infty} M(x) = M_\infty < \infty. \tag{2.16}$$

The limiting value M_∞ is interpreted as the total mass of a solution (in our case it is measured in units $\frac{1}{Gm}$). In Sect. 6 we will show that the finiteness of mass implies that C has a finite limit and f decays exponentially as $x \rightarrow \infty$.

Besides the singularity at infinity, the field equations (2.14) have the fixed singular point at $x = 0$ and a moving singularity at \bar{x} , where $A(\bar{x}) = 0$. Regularity of solutions at $x = 0$ requires the following behavior:

$$f(x) = a + O(x^2), \quad A(x) = 1 + O(x^2), \quad C(x) = \alpha + O(x^2), \tag{2.17}$$

where $a = f(0)$ and $\alpha = C(0)$ are arbitrary parameters (assumed positive without loss of generality). In Sect. 3 we will show that these parameters determine uniquely a smooth local solution to Eqs. (2.14).

Definition 2.1. *The solution of Eqs. (2.14) starting at $x = 0$ with the behavior (2.17) is called the **a-orbit**.*

In the following whenever we write “a solution” we always mean the a -orbit. Also when we write that some property holds for all x we always mean for all $x \geq 0$. We will frequently refer to the behavior of a -orbits in the (f, f') -plane; when we write, say, that the a -orbit enters the first quadrant (Q1 for brevity), we mean that the projection of the a -orbit in the (f, f') -plane does so.

Definition 2.2. *The a -orbits which exist for all x and are asymptotically flat are called **globally regular**.*

Now, we are ready to formulate our main result:

Theorem 2.1. *For each $\alpha > 1$, there is a decreasing sequence of parameters a_n ($n = 0, 1, 2, \dots$) such that the corresponding a_n -orbits are globally regular. The index n labels the number of nodes of the function $f(x)$.*

This theorem makes rigorous the numerical results obtained in [1–3]. Notice that although the a -orbits are determined by two parameters, only the parameter a has to be fine-tuned so the shooting is essentially one-dimensional.

In order to prepare the ground for the proof of Theorem 2.1 we discuss now some elementary global properties of a -orbits.

Lemma 2.2. *$A(x) < 1$ for all $x > 0$ unless $f(x) \equiv 0$.*

Proof. From (2.14b), $A'(x_0) < 0$ if $A(x_0) = 1$ so A cannot cross 1 from below. Since $A(x) < 1$ for small x , the lemma follows. \square

Lemma 2.3. *An a -orbit exists as long as $A(x) > 0$.*

Proof. If $A(x) > 0$ for $x < \bar{x} < \infty$, then $\lim_{x \rightarrow \bar{x}} M(x)$ exists (because $M' > 0$ and $M(x) < x < \bar{x}$) so $\lim_{x \rightarrow \bar{x}} A$ exists as well. We will show that the orbit can be continued beyond \bar{x} provided that $\lim_{x \rightarrow \bar{x}} A(x) > 0$. Since $0 < A < 1$, the only obstruction to extending the solution is the possibility that C, f , or f' might be unbounded. To see that f is bounded we note that $(xA)' < 1 - x^2 A f'^2$. Choose $\epsilon > 0$ such that $A(x) > A(\bar{x})/2$ for $\bar{x} - \epsilon < x < \bar{x}$; then $(xA)' < 1 - (\bar{x} - \epsilon)^2 A(\bar{x}) f'^2/2$ and integrating from $\bar{x} - \epsilon < x$ gives that $\int_{\bar{x}-\epsilon}^x f'(x)^2 < \infty$ and hence, by the Cauchy-Schwarz inequality, $\int_{\bar{x}-\epsilon}^x |f'(x)| < \infty$. Thus, f is bounded. This implies by Eq. (2.14c) that $(\ln C)'$ is also bounded so both C and $1/C$ are bounded. Now, (2.14a) says that $x^2 f'/C$ is bounded so f' is bounded. \square

Remark. It follows from Lemma 2.3 that the only possible obstruction to extendability of a -orbits to arbitrarily large x is $\lim_{x \rightarrow \bar{x}} A = 0$ for some \bar{x} . If that happens we will say that the solution *crashes* at \bar{x} .

Let us define the function $g = 1 - AC^2$. The following two properties of this function will play an important role in our discussion.

Lemma 2.4. *We have*

- (a) *If $g(x) \leq 0$, then $g'(x) > 0$;*
- (b) *If $g(x_0) \geq 0$, then $g(x) > 0$ for all $x > x_0$.*

Proof. A simple calculation yields

$$g' = C^2 \left(\frac{1-A}{x} + xA f'^2 - xg f^2 \right). \tag{2.18}$$

Part (a) follows immediately from (2.18). To prove Part (b) note that $g'(x_1) > 0$ if $g(x_1) = 0$, so g cannot cross zero from above. \square

The restriction $\alpha > 1$ in Theorem 2.1 can be easily seen as follows. Suppose that there is a globally regular solution with $\alpha \leq 1$. Since $g(0) = 1 - \alpha^2$, it follows from Lemma 2.4 that $g(x)$ is positive for all x . Multiplying Eq. (2.14a) by f and integrating by parts we get that $ff' > 0$ for all x , hence f^2 is monotone increasing which is obviously impossible for globally regular solutions (in fact such solutions crash at finite x as follows easily from Eq. (2.14b)). Thus we have

Lemma 2.5. *There are no (nontrivial) globally regular solutions for $\alpha \leq 1$.*

Note that Lemma 2.5 implies in particular that there are no static ($\alpha = 0$) globally regular solutions. In view of Lemma 2.5 from now on we always assume that $\alpha > 1$.

Definition 2.3. *The rotation function $\theta(x, a)$ of an a -orbit is defined by $\theta(0, a) = 0$, $\tan \theta(x, a) = -f'(x)/f(x)$ and $\theta(x, a)$ is continuous in x . We will drop the second argument of θ if there is no danger of confusion.*

Now we list the basic properties of the rotation function of a -orbits which we will need below.

Lemma 2.6. *For any nonnegative integer n we have:*

- (a) *If $\theta(x_1) > (n + 1/2)\pi$ for some x_1 , then $\theta(x) > (n + 1/2)\pi$ for all $x > x_1$.*
- (b) *If $\theta(x_1) < n\pi$ for some x_1 and $g(x_1) \geq 0$, then $\theta(x) < n\pi$ for all $x > x_1$.*
- (c) *There are at most two values of x with $\theta(x) = n\pi$.*

Proof. (a) We note that $\theta'(x) = (f'^2 - ff'')/(f^2 + f'^2)$, so $\theta'(x) = 1$ if $\theta(x) = (n + 1/2)\pi$.

(b) If $x > x_1$ and $g(x_1) \geq 0$, then $g(x) > 0$ by Lemma 2.4. Next, we note that $\theta'(x) = -g(x)/A(x) < 0$ if $\theta(x) = n\pi$ and $g(x) > 0$. If $g(x) = 0$ then $\theta'(x) = 0$ but $\theta''(x) = -g'(x)/A(x) < 0$ since $g'(x) > 0$ when $g(x) = 0$ by Eq. (2.18).

(c) The function $\theta(x) - n\pi$ changes sign at each zero for which $g(x) \neq 0$. From Lemma 2.4, g changes sign at most once. Thus, for $n > 0$, $\theta(0) - n\pi < 0$ and at x_1 , the first zero of $\theta(x) - n\pi$, if $g(x_1) \geq 0$ then by part b) $\theta(x) - n\pi < 0$ for all $x > x_1$. If $g(x_1) < 0$ then $\theta(x) - n\pi$ changes sign at x_1 , and hence, at x_2 , the next zero of $\theta(x) - n\pi$, $g(x_2) \geq 0$ and hence $\theta(x) - n\pi < 0$ for all $x > x_2$. For $n = 0$, $\theta(0) - n\pi = 0$, $\theta(x) > 0$ near $x = 0$ and if $\theta(x_1) = 0$ then $g(x_1) \geq 0$, hence, $\theta(x) < 0$ for all $x > x_1$. \square

Before going into details of Sects. 3–8, let us outline the main idea of the proof of Theorem 2.1. According to this theorem there exists a countable family of globally regular solutions distinguished by nodal class. We first show (Sect. 3) that there is a continuous one-parameter family of local solutions depending on $a = f(0)$; we all these solutions a -orbits. In Sect. 6 we show that an a -orbit that has bounded rotation and that is defined for all x is a globally regular solution, that is, it has the correct asymptotic behavior as $x \rightarrow \infty$. The existence of a -orbits with bounded rotation that are defined for all x is proven in each nodal class by an inductive application of a shooting argument. The zeroth solution we construct has $\theta(x, a_0) < \pi/2$ for all x ; the first solution has $\theta(x, a_1) < 3\pi/2$ (and greater than $\pi/2$ for large x), etc. This is shown in Fig. 1. The crucial step of our argument is the control of behavior of a -orbits for large and small values of the parameter a . In Sect. 4 we show that for sufficiently small a the a -orbit has arbitrarily large rotation; more precisely, there is a number b_n such that $\theta(x, a) > n\pi$ for some x if $a < b_n$. In contrast, we show in Sect. 5 that for $a \gg 1$ the a -orbit exits Q4 directly to Q1 (see Fig. 1).

Now, to prove the existence of a globally regular solution in the zeroth nodal class we let $a_0 = \inf\{a \mid \theta(x, a) < \pi/2 \text{ for all } x \text{ for which the } a\text{-orbit is defined}\}$. Note that $a_0 \geq b_1 > 0$. We then prove that the a_0 -orbit is the globally regular solution in the zeroth nodal class. It is clear that the a_0 -orbit has rotation $\theta(x, a_0) \leq \pi/2$ for otherwise all nearby orbits would have rotation $> \pi/2$ which contradicts the definition of a_0 . It is also easy to see that the a_0 -orbit cannot exit Q4 to Q1 because again, nearby orbits would also do so which contradicts the definition of a_0 . Hence, the a_0 -orbit must stay in Q4; it either crashes or is defined for all x and is a globally regular solution in the zeroth nodal class. Thus, it remains to show that the a_0 -orbit does not crash. The (technical) crash lemma of Sect. 7 shows that if an orbit crashes in Q4 then nearby orbits either crash in Q4 or exit Q4 to Q1. Thus the a_0 -orbit cannot crash because nearby orbits would all be in $\{a \mid \theta(x, a) < \pi/2 \text{ for all } x \text{ for which the } a\text{-orbit is defined}\}$ and a_0 would not be the infimum of that set.

To show the existence of globally regular solutions in higher nodal classes we proceed as above. We let $a_n = \inf\{a \mid \theta(x, a) < (n + 1/2)\pi \text{ for all } x \text{ for which the } a\text{-orbit is defined}\}$. We then show that $\theta(x, a_n) < (n + 1/2)\pi$. We again use the crash lemma as

we did in the $n = 0$ case to show the a_n -orbit is defined for all x . The only difference is that we must show that $\theta(x, a_n) > n\pi$. That fact follows easily from Lemmas 2.6b and 6.3.

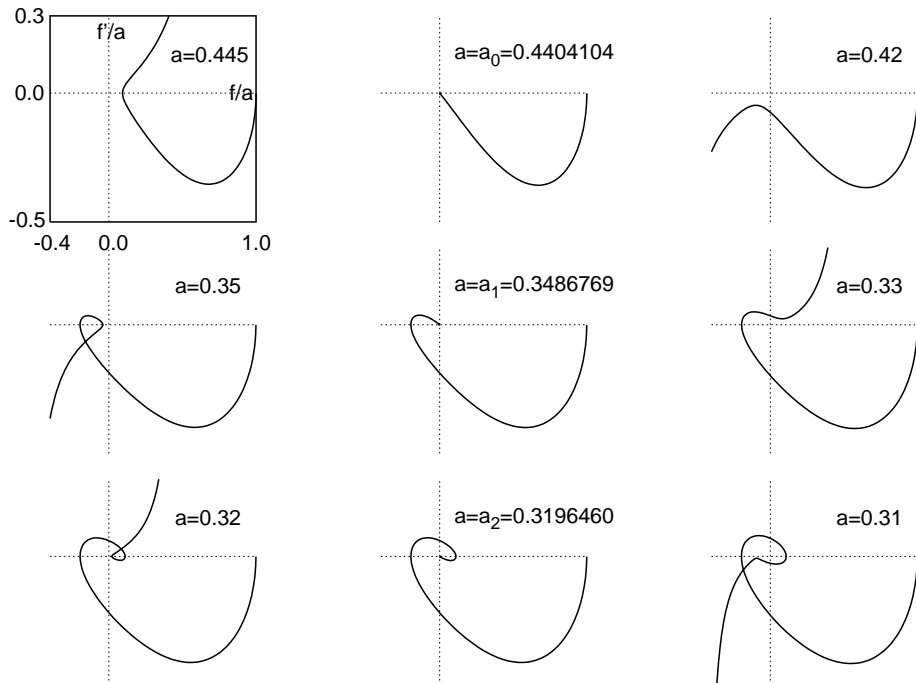


Fig. 1. The projection of a -orbits on the (f, f') phase plane for several selected values of the shooting parameter a

3. Local Existence

Proposition 3.1. *There exists a two-parameter family of local solutions of Eqs. (2.14) near $x = 0$ satisfying the initial conditions (2.17).*

Proof. The proof is standard so we just outline it. We introduce new variables $w = f'$, $z = \ln(C)$, $B = (1 - A)/x$, and rewrite Eqs. (2.14) as the first order system

$$f' = w, \tag{3.1a}$$

$$(x^2w)' = \frac{x^2}{A} \left((xf^2 - B)w + f(1 - AC^2) \right), \tag{3.1b}$$

$$(x^2B)' = x^2(Aw^2 + f^2AC^2 + f^2), \tag{3.1c}$$

$$z' = \frac{xf^2 - B}{A}. \tag{3.1d}$$

We will use the sup norm throughout this discussion: $\|h\|$ means the $\sup\{|h(x)| : 0 \leq x \leq r\}$.

Consider the space X of quadruples of functions (f, y, B, z) , where $\|f - a\| \leq 1, \|w\| \leq 1, \|B\| \leq M$, and $\|z - \ln(\alpha)\| \leq 1$ and each of the four functions is in $C^0([0, r])$, the space of continuous functions defined on the interval $0 \leq x \leq r$ with the sup norm. X is a complete metric space if we take as metric the maximum of the four components. We define a map $T : X \rightarrow X$ by $T(f, w, B, z) = (T_1, T_2, T_3, T_4)$ where

$$T_1 = a + \int_0^x w \, ds, \tag{3.2a}$$

$$T_2 = \frac{1}{x^2} \int_0^x \frac{s^2}{A} \left((sf^2 - B)w + f(1 - AC^2) \right) ds, \tag{3.2b}$$

$$T_3 = \frac{1}{x^2} \int_0^x s^2 \left(Aw^2 + f^2 AC^2 + f^2 \right) ds, \tag{3.2c}$$

$$T_4 = \ln \alpha + \int_0^x \frac{1}{A} (sf^2 - B) ds. \tag{3.2d}$$

One verifies easily that T does in fact take X to X and that T is a contracting map if r is sufficiently small, and that a fixed point of T is a solution to our equations. The proof that the solution depends continuously on a is also routine. \square

4. Behavior of Solutions for Small a

In this section we show that the rotation $\theta(x, a)$ of the a -orbit is arbitrarily large if a is sufficiently small and x is sufficiently large.

Proposition 4.1. *For any $n > 0$, there exists a b_n such that for $a < b_n$ there is an x with $\theta(x, a) > n\pi$.*

Proof. Let $\tilde{f} = f/a$. Then, Eqs. (2.14) become

$$\left(\frac{x^2 \tilde{f}'}{C} \right)' = \frac{x^2}{AC} (1 - AC^2) \tilde{f}, \tag{4.1a}$$

$$A' = \frac{1 - A}{x} - a^2 x (A \tilde{f}'^2 + AC^2 \tilde{f}^2 + \tilde{f}^2), \tag{4.1b}$$

$$C' = \frac{C}{xA} (A - 1 + a^2 x^2 \tilde{f}^2) \tag{4.1c}$$

with the behavior at the origin

$$\tilde{f}(0) = 1, \quad A(0) = 1, \quad C(0) = \alpha. \tag{4.2}$$

For $a = 0$ (decoupling of gravity) Eqs. (4.1bc) with conditions (4.2) have constant flat-spacetime solutions $A \equiv 1, C \equiv \alpha$. Inserting these solutions into Eq. (4.1a) gives the Bessel equation

$$(x^2 \tilde{f}')' + x^2 (\alpha^2 - 1) \tilde{f} = 0, \tag{4.3}$$

whose unique solution satisfying (4.2) is

$$\tilde{f}(x) = \frac{\sin \sqrt{\alpha^2 - 1} x}{\sqrt{\alpha^2 - 1} x}. \tag{4.4}$$

This solution has infinite rotation as $x \rightarrow \infty$. If $x > \frac{n\pi}{\sqrt{\alpha^2 - 1}}$ then $\theta(x, 0) > n\pi$ so for a close to 0, say $a < b_n$, we have $\theta(x, a) > n\pi$ because solutions of Eqs. (4.1) are continuous in a and x . This concludes the proof of Proposition 4.1. \square

5. Behavior of Solutions for Large a

Proposition 5.1. *The a -orbits with sufficiently large a exit Q4 directly to Q1.*

We define new variables

$$y = ax, \quad \tilde{v}(y) = a(f(x) - a), \quad \tilde{A}(y) = A(x), \quad \tilde{C}(y) = C(x). \tag{5.1}$$

Then, Eqs. (2.14) become (where now the prime denotes the derivative with respect to y)

$$\left(\frac{y^2 v'}{\tilde{C}}\right)' = \frac{y^2}{\tilde{A}\tilde{C}}(1 - \tilde{A}\tilde{C}^2)\left(1 + \frac{\tilde{v}}{a^2}\right), \tag{5.2a}$$

$$\tilde{A}' = \frac{1 - \tilde{A}}{y} - y \left(\frac{1}{a^2} A \tilde{v}'^2 + (1 + \tilde{A}\tilde{C}^2) \left(1 + \frac{\tilde{v}}{a^2}\right)^2\right), \tag{5.2b}$$

$$\tilde{C}' = \frac{\tilde{C}}{y\tilde{A}} \left(\tilde{A} - 1 + y^2\left(1 + \frac{\tilde{v}}{a^2}\right)^2\right). \tag{5.2c}$$

The initial conditions at $y = 0$ are

$$\tilde{A}(0) = 1, \quad \tilde{C}(0) = \alpha > 1, \quad \tilde{v}(0) = 0, \quad \tilde{v}'(0) = 0. \tag{5.3}$$

As $a \rightarrow \infty$, the solutions of Eqs.(5.2) tend uniformly on compact intervals to the solutions of the following limiting system:

$$\left(\frac{y^2 v'}{C}\right)' = \frac{y^2}{AC}(1 - AC^2), \tag{5.4a}$$

$$A' = \frac{1 - A}{y} - y(1 + AC^2), \tag{5.4b}$$

$$C' = \frac{C}{yA}(A - 1 + y^2), \tag{5.4c}$$

satisfying the same initial conditions as in (5.3). The rest of this section is devoted to the analysis of Eqs. (5.4). Our goal is to show that $v'(y)$ becomes positive at a point $y_1 < \bar{y}$. This would imply that $v(y)$ is bounded below, i.e., there is $d > 0$ such that $v(y) > -d$ for $y < y_1$, and therefore $\tilde{v}(y) > -d - 1$ if a is sufficiently large. Then $f(x) > a - (d + 1)/a$ and $f'(x) > 0$ for $x = y_1/a$, hence, if $a > \sqrt{d + 1}$, the a -orbit exits Q4 to Q1 directly without entering Q3.

Note that the function v decouples from Eqs. (5.4bc) for the metric coefficients – this fact considerably simplifies the analysis.

Lemma 5.2. *The solution of Eqs. (5.4) crashes at some \bar{y} , that is, $A(\bar{y}) = \lim_{y \rightarrow \infty} A(y) = 0$. Moreover, $1 < \bar{y} < \sqrt{3}$.*

Proof. Note that $(yA)' < 1 - y^2$, so integrating gives $A < 1 - y^2/3$. Therefore, $A(\bar{y}) = 0$ for $\bar{y} < \sqrt{3}$. To show that $\bar{y} > 1$ assume $\bar{y} \leq 1$. Then $(\frac{y}{C})' = \frac{1-y^2}{AC} \geq 0$, so if $0 < \tau < y < \bar{y}$ we have $y/C(y) > \tau/C(\tau)$ or $C(y) < C(\tau)/\tau$, so C is bounded. Since $(AC)' = -yAC^3$, $(\ln(AC))' = -yC^2$ is bounded below. Thus, by integrating one concludes that $\lim_{y \rightarrow \bar{y}} AC(y) > 0$. But $(AC)^2 = A(AC^2)$; $AC^2 \leq \alpha^2$ and $\lim_{y \rightarrow \bar{y}} A(y) = 0$, so $\lim_{y \rightarrow \bar{y}} (AC(y))^2 = 0$. This contradicts $\lim_{y \rightarrow \bar{y}} AC(y) > 0$, so we must have $\bar{y} > 1$. \square

Proof of Proposition 5.1. In order to prove that $v'(y)$ becomes positive at some point $y_1 < \bar{y}$, we will show that $v'(\bar{y}) > 0$. By Eq. (5.4a) we have $v'(y) = \frac{C}{y^2} \int_0^y \frac{z^2(1-AC^2)}{AC} dz$, so we must show that $\int_0^{\bar{y}} \frac{y^2(1-AC^2)}{AC} dy > 0$.

The proof of this fact is divided into two cases: (i) $\bar{y}^2 \geq 3/2$, and (ii) $\bar{y}^2 < 3/2$. Before considering these cases we list some useful properties of the function $g = 1 - AC^2$.

Lemma 5.3. *We have:*

- (a) $g' = (1 - A - y^2g)C^2/y$;
- (b) if $g(y_0) \geq 0$, then $g(y) > 0$ for all $y > y_0$;
- (c) $g' > 0$ if $g \leq 1/3$.

Proof. Part (a) is a calculation. For (b) note that $g'(g = 0) > 0$ so g cannot cross zero from above. For (c) we have $(yA)' < 1 - y^2$, so integrating gives $1 - A > y^2/3$ and hence, $g' > y(1/3 - g)C^2$. \square

We return now to the proof that $\int_0^{\bar{y}} \frac{y^2g}{AC} dy > 0$. We first consider the case (i) $\bar{y}^2 > 3/2$.

A calculation shows that $y^2g = (2y^3/3 + yA - y)'$, hence $\int_0^{\bar{y}} y^2g dy = 2\bar{y}^3/3 - \bar{y} > 0$ if $\bar{y}^2 > 3/2$. Since $g(0) = 1 - \alpha^2 < 0$, this implies that $g(\sigma) = 0$ for some $\sigma < \bar{y}$ and therefore $g(y) > 0$ for $y > \sigma$. Note that AC is monotone decreasing because $(AC)' = -yAC^3 < 0$. Thus

$$\frac{y^2g(y)}{A(y)C(y)} \geq \frac{y^2g(y)}{A(\sigma)C(\sigma)} \quad \text{for } 0 \leq y \leq \bar{y}, \tag{5.5}$$

and therefore

$$\int_0^{\bar{y}} \frac{y^2g}{AC} dy \geq \frac{1}{A(\sigma)C(\sigma)} \int_0^{\bar{y}} y^2g dy > 0. \tag{5.6}$$

Now we consider the case (ii) $\bar{y}^2 \leq 3/2$.

Lemma 5.4. *Define the function $p = 1 + y^2g - y^2$. If $y^2 \leq 3/2$, then $p(y) > 0$.*

Proof. Note that $p(0) = 1$. Let y_1 be the first zero of p , that is, $p(y_1) = 0$ and $p'(y_1) \leq 0$. If $g(y_1) > 1/3$ then $p = y^2g + 1 - y^2 > y^2/3 + 1 - y^2 = 1 - 2y^2/3$. Thus p can have a zero for $y_1^2 \leq 3/2$ only if $g(y_1) \leq 1/3$. Then, from Lemma (5.3), $g'(y) > 0$ for all $y \leq y_1$. Define a function $k(y) = 2 - 3A - y^2$. A calculation gives $y^3g' = (y(k + p))'$, so by integrating we get $k(y_1) > 0$. On the other hand we have $p' = yC^2(k - p)$, so $k(y_1) \leq 0$; contradiction. \square

To show that $\int_0^{\bar{y}} \frac{y^2 g}{AC} dy > 0$, we rewrite it as

$$\int_0^{\bar{y}} \frac{y^2 g}{AC} dy = \int_0^{\bar{y}} \frac{p - 1 + y^2}{AC} dy = \int_0^{\bar{y}} \frac{p}{AC} dy - \int_0^{\bar{y}} \frac{1 - y^2}{AC} dy. \tag{5.7}$$

The first term on the right hand side of (5.7) is positive because p is positive. To compute the second term, note that

$$\left(\frac{y}{C}\right)' = \frac{1 - y^2}{AC}, \tag{5.8}$$

hence $L = \lim_{y \rightarrow \bar{y}} (y/C)$ exists and is finite since $\bar{y} > 1$ by Lemma 5.2. If $L > 0$ then $\lim_{y \rightarrow \bar{y}} C(y) = \bar{y}/L < \infty$, so C is bounded. Since $\lim_{y \rightarrow \bar{y}} A(y) = 0$ we conclude that $\lim_{y \rightarrow \bar{y}} AC(y) = 0$. But $(\ln AC)' = yC^2$ is bounded, so $\ln AC$ is bounded below and hence $\lim AC \neq 0$. This contradiction shows that $L = 0$. Thus, the second term on the right-hand side of (5.7) is zero. This concludes the proof of Proposition 5.1. \square

6. Asymptotics of Globally Regular Solutions

In this section we derive the leading asymptotic behavior of globally regular solutions. We use \lim to denote $\lim_{x \rightarrow \infty}$.

Proposition 6.1. *An a -orbit which exists for all x and has bounded rotation is asymptotically flat. The leading asymptotic behavior for $x \rightarrow \infty$ is*

$$A(x) \sim 1 - \frac{2M_\infty}{x}, \quad C(x) \sim C_\infty e^{\frac{2M_\infty}{x}}, \quad f(x) \sim f_\infty e^{-bx}, \tag{6.1}$$

where $0 < M_\infty < \infty$, $0 < C_\infty < 1$, and $b = \sqrt{1 - C_\infty^2}$.

To prove this proposition we need several partial results.

Lemma 6.2. *An a -orbit which exists for all x and has bounded rotation is ultimately in the second (Q2) or fourth (Q4) quadrant.*

Proof. If $\theta(x)$ is bounded above then there is an integer $n \geq 0$ such that $\theta(x) < (n + 1/2)\pi$ for all x but $\theta(x_1) > (n - 1/2)\pi$ for some x_1 and hence, by Lemma 2.6a for all $x > x_1$, $(n - 1/2)\pi < \theta(x) < (n + 1/2)\pi$. We next show that there is an x_2 such that for all $x > x_2$, $n\pi < \theta(x) < (n + 1/2)\pi$ (that is, the orbit is ultimately in Q2 or Q4). Note that, by Lemma 2.6c the orbit must satisfy either $n\pi < \theta(x) < (n + 1/2)\pi$ or $(n - 1/2)\pi < \theta(x) < n\pi$, that is the orbit must lie in Q3 or Q2 if n is odd and in Q1 or Q4 if n is even. We must rule out the possibility that the orbit is in Q1 or Q3. Assume that the orbit lies in Q1 or Q3 for all $x > x_1$. Then $f(x)f'(x) > 0$ for all $x > x_1$, so $f^2(x) \geq f^2(x_1)$ for all $x > x_1$. From Eq. (2.15b) we have $(xA)' = 1 - x^2 A f'^2 - x^2 f^2 AC^2 - x^2 f^2$, so $(xA)' < 1 - x^2 f^2 < 1 - x^2 f^2(x_1)$, and hence A goes to zero in finite x . This contradiction concludes the proof. \square

Lemma 6.3. *Under the assumptions of Proposition 6.1 the function $g = 1 - AC^2$ is eventually positive.*

Proof. Suppose that $g(x) \leq 0$ for all x . We claim that this implies $\lim A = 1$. To see this, suppose that $\liminf A = 1 - 4\epsilon$ for some $\epsilon > 0$. Let $-\beta = \lim g \leq 0$ which exists because $g' > 0$. Note that $g(x) < -\beta$ for all x . Choose an x_1 such that $g(x_1) > -\beta - \epsilon$. If $A(x_2) < 1 - 3\epsilon$ for some $x_2 > x_1$, then by (2.18) $g'(x) > C^2(1 - A)/x > (1 - A)/x = x(1 - A)/x^2 > x_2(1 - A(x_2))/x^2 > 3\epsilon x_2/x^2$ for $x > x_2$, where the last but one inequality follows from the fact that $x(1 - A(x))$ is monotone increasing. Integrating this inequality from x_2 to $2x_2$ say, we get $g(2x_2) > g(x_2) + 3\epsilon/2 > -\beta - \epsilon + 3\epsilon/2 > -\beta$; contradiction. Thus, $\liminf A = 1$ and hence $\lim A = 1$. Since $\lim g = \lim(1 - AC^2)$ exists, $\lim C$ also exists and is finite. Next, from Lemma 6.2 we know that the a -orbit is ultimately in Q2 or in Q4. For concreteness we consider the case of Q4 (the proof of the Q2-case is identical), that is $f(x) > 0$ and $f'(x) < 0$ for sufficiently large x . Then, from (2.14a), $\lim(x^2 f'/C)$ exists, so $\lim(x^2 f') = -\tau < 0$ exists as well (where τ might be infinite; the point is that $\tau \neq 0$). Now, by L'Hôpital's rule, $\lim xf = -\lim(x^2 f') = \tau$. But (2.14c) says $(\ln C)' > \tau^2/4x$ which implies $\lim C = \infty$, a contradiction. \square

Proof of Proposition 6.2. From the previous lemma we know that there exists an x_1 such that $g(x) > 0$ for $x > x_1$. Let $u = ACf/g$ for $x > x_1$. A calculation shows that $u' = -AC(fC^2(1 - A)/x - f'g + xf f'^2)/g^2$ so $u' < 0$ if $g > 0$. Multiplying Eq. (2.14a) by u we obtain

$$(x^2 A f f' / g)' = x^2 f^2 + x^2 f' u' / C. \tag{6.2}$$

The right-hand side is positive for $x > x_1$, so $x^2 A f f' / g$ is negative and increasing, hence it has a finite non-positive limit. This implies that $x^2 f^2$ is integrable. Similarly, multiplying Eq. (2.14a) by f we obtain

$$x^2 f f' / C = (x^2 f^2 g + A x^2 f'^2) / (AC). \tag{6.3}$$

The right-hand side is positive for $x > x_1$, so $x^2 f f' / C$ is negative and increasing, hence it has a finite non-positive limit. This implies that $A x^2 f'^2$ is integrable (recall that AC is monotone decreasing). The integrability of $x^2 f^2$ and $A x^2 f'^2$ implies via Eq. (2.15) that $\lim M = M_\infty < \infty$ exists. This concludes the proof that $A(x) \sim 1 - 2M_\infty/x$.

Having $\lim A = 1$ we can strengthen Lemma 6.3 by showing that $\lim g = g_\infty > 0$ exists. To see this choose an x_1 such that $g(x_1) > 0$. Then $AC^2(x_1) < 1$, hence $AC(x_1) < 1$. Since AC is monotone decreasing, we have $AC(x) < AC(x_1)$ for $x > x_1$ and thus $\lim AC < 1$. Hence, $\lim AC^2 = (\lim AC)^2 / \lim A < 1$. Since $g = 1 - AC^2$, $\lim g$ exists and $\lim g > 0$.

Now we have all we need to derive the asymptotics of f . Let $r = f'/f$. Then $r' = f''/f - r^2 = -r(1 + A - x^2 f^2)/(xA) + g/A = g_\infty - r^2 + \epsilon(x)$, where $\lim \epsilon = 0$. Let $\sigma(x_2) = \max(|\epsilon(x)|)$ for $x > x_2$ and assume that x_2 is sufficiently large so that $g_\infty > \sigma(x_2)$. If $r(x_2) > -\sqrt{g_\infty - \sigma(x_2)}$, then clearly r becomes eventually positive which contradicts that the orbit is eventually in Q2 or Q4. If $r(x_2) < -\sqrt{g_\infty + \sigma(x_2)}$, then $\lim r = -\infty$; this is impossible because then by L'Hôpital's rule $\lim r = \lim f''/f' = \lim g/r = 0$. Therefore $r(x_2)$ must be sandwiched in the interval $-\sqrt{g_\infty + \sigma(x_2)} < r(x_2) < -\sqrt{g_\infty - \sigma(x_2)}$. Since x_2 is arbitrarily large and $\lim \sigma = 0$, we conclude that $\lim r = -\sqrt{g_\infty}$. The asymptotics of f given in (6.1) follows immediately from this.

Finally, inserting the derived leading asymptotic behavior of A and f into Eq. (2.14c), we obtain $C'/C \sim -2M_\infty/x$, from which the asymptotics of C follows trivially. \square

7. Solutions that Crash

Proposition 7.1. *If the b -orbit crashes at some \bar{x} then $g(x) > 0$ for x near \bar{x} .*

Proof. Suppose that $g(x) < 0$ for all $x < \bar{x}$, so $AC^2(x) > 1$ for all $x < \bar{x}$. We have from (2.18) that $g' > AC^2 x f'^2 > x f'^2$. Integrating this inequality from some $x_1 > 0$ to some $x_2 < \bar{x}$, we obtain

$$x_1 \int_{x_1}^{x_2} f'^2 dx < \int_{x_1}^{x_2} x f'^2 dx < g(x_2) - g(x_1) < \alpha^2 - 1, \tag{7.1}$$

which implies (by the Cauchy–Schwartz inequality) that f is bounded.

Next, $A(\bar{x}) = 0$, $AC^2 > 1$, implies that $\lim_{x \rightarrow \bar{x}^-} C = \infty$; moreover, by (2.14c) $(\ln C)' < x f^2/A$, hence $x f^2/A$ is not integrable near \bar{x} . Since f is bounded, this shows that $1/A$ is not integrable near \bar{x} . But from (2.18), $g' > C^2(1 - A)/x = AC^2(1 - A)/(xA) > 1/(2xA)$, so g' is not integrable near \bar{x} , which contradicts the fact that g is a bounded function. \square

The importance of Proposition 7.1 derives from Lemma 2.6b which says that if $g > 0$ then rotation stops. The main result of this section is the crash theorem which states that if an orbit has bounded rotation and crashes, then nearby orbits also have similarly bounded rotation. The precise statement is given in Proposition 7.2. Since we consider more than one orbit in this section, we use the notation $A(x, a)$ to denote the value of A at x for the a -orbit, etc.

Proposition 7.2 (Crash Theorem). *If the b -orbit crashes at $x = \bar{x}$ and*

- (a) *if $(k - 1/2)\pi < \theta(x, b) < k\pi$, $k \geq 1$, for x near \bar{x} , then nearby orbits have rotation $< k\pi$ for $x \geq \bar{x}$;*
- (b) *if $k\pi < \theta(x, b) < (k + 1/2)\pi$, then nearby orbits have rotation $< (k + 1/2)\pi$.*

Proof. Part (a): Suppose the b -orbit crashes in Q3 or Q1. By Proposition 7.1, $g(x_1, b) > 0$ for some $x_1 < \bar{x}$ with $(k - 1/2)\pi < \theta(x_1, b) < k\pi$; hence, for a sufficiently near b we have $g(x_1, a) > 0$ with $(k - 1/2)\pi < \theta(x, a) < k\pi$. By Lemma 2.6b, $\theta(x, a) < k\pi$ for all $x > x_1$. Part (b): This case is much more difficult and will require several auxiliary results. It follows from part (a) that nearby orbits have rotation $< (k + 1)\pi$; we must prove a much more difficult result, namely that nearby orbits have rotation $< (k + 1/2)\pi$. \square

Remark. It is clear from numerical observations that no a -orbit crashes in Q2 or Q4; however, that appears to be quite difficult to prove. Moreover, one can easily construct orbit segments that start, for example, at $x = 1$ with $f = 5$, $f' = 0$, $A = 0.2$, $C = 3$, say, that crash in Q4. Such orbit segments have $\lim_{x \rightarrow \bar{x}^-} f'(x) = -\infty$. Nevertheless, the next lemma shows that $A f'^2$ remains bounded at crash.

Lemma 7.3. *If an a -orbit is defined for $x < x_2$, $f f'(x) < 0$ for $x_1 < x < x_2$, $f^2(x_1) < B$, and $f'(x_1) = 0$, then $A f'^2(x) \leq \max(B, \alpha^2/3)$. In particular, if an orbit crashes in Q2 or Q4, $\lim_{x \rightarrow \bar{x}^-} A(x) f'(x) = 0$.*

Proof. We set $q = A f'^2$ and then compute that

$$xq' = -(3 + x^2 f'^2 + x^2 C^2 f^2)q - f'^2 + 2x f f' + x^2 f^2 f'^2 - 2AC^2 x f f'. \tag{7.2}$$

Note that $q \geq 0$ and all terms on the right side of (7.3) are negative except for the last two. If $q > B$, we combine the term $-qx^2 f'^2$ with $x^2 f^2 f'^2$; clearly, $x^2 f^2 f'^2 - qx^2 f'^2 = (f^2 - q)x^2 f'^2 \leq 0$. Next, we combine the term $-qx^2 f'^2 C^2$ with $-2x f f' AC^2$ to get $-AC^2(y^2 - 2y)$, where $y = -x f f'$; the maximum value of this expression occurs when $y = 1$ and that value is $AC^2 \leq \alpha^2$ by Lemma 2.4. Hence, if $q \geq \alpha^2/3$, then $-q(x^2 f'^2 C^2) - 3q - 2x f f' AC^2 \leq 0$. Thus, $q \geq \max(B, \alpha^2/3)$ implies that $q' < 0$; consequently, $A f'^2(x) \leq \max(B, \alpha^2/3)$. Since $AA f'^2 = (A f')^2$, and $A f'^2$ is bounded and $\lim x \rightarrow \bar{x}^- A(x) = 0$, $\lim_{x \rightarrow \bar{x}^-} (A(x) f'(x))^2 = 0$, hence $\lim_{x \rightarrow \bar{x}^-} A(x) f'(x) = 0$. \square

We can now discuss the strategy of the proof of part (b) of Proposition 7.2. We want to show that if an orbit is sufficiently close to an orbit that crashes in Q4 then it must either crash or exit Q4 to Q1 (the case in which the orbit crashes in Q2 is completely symmetric). To that end, let $v(x) = A(x) f'(x)$. We will prove that $v(x, a)$ goes to 0 if a is sufficiently close to b and $f(x, a) > 0$. This means either $f' = 0$ and hence the orbit is exiting Q4 to Q1, or $A = 0$, that is, the orbit is crashing in Q4. Note that $v'(x) = -(2A f' - x f + x AC^2 f + x^2 A f'^2 + x^2 f^2 f' AC^2)/x = -v(2 + x^2 f'^2 + x^2 f^2 C^2)/x + f g > f g$. We know that $v(x, b)$ goes to 0 at crash so nearby orbits will also have v small for x near \bar{x} . We will show that f and g are both uniformly bounded away from 0 in an interval about \bar{x} . That is, the size of the interval and the bounds work for all a near b . That is enough to force v positive. The most technical part of the proof involves showing that nearby orbits stay in Q4 long enough to have v go positive. Since f' goes to $-\infty$ at crash, nearby orbits have f' large also. Now, (2.14a) can be written as $x A f'' + (1 + A - x^2 f^2) f' - x g f = 0$; moreover, to get to Q3 orbits must pass through $x f(x) < 1$ which means that the coefficient of f' , $(1 + A - x^2 f^2)$, is positive. That is enough to bound f' .

The details of the proof, especially Lemma 7.5, are tedious. We will restrict ourselves to an interval $0.99 \bar{x} < x < 1.01 \bar{x}$ and replace x by \bar{x} (whenever justified) in making estimates.

We show next that if the b -orbit crashes at $x = \bar{x}$ with rotation $k\pi < \theta(x, b) < (k + 1/2)\pi$, then $|\bar{x} f(\bar{x})| \geq 1$.

Lemma 7.4. *If the b -orbit crashes at $x = \bar{x}$ with $\theta(x, b) < (k + 1/2)\pi$ for all $x < \bar{x}$ and $\theta(x, b) > k\pi$ for x near \bar{x} , then $|\bar{x} f(\bar{x})| \geq 1$, in particular $f(\bar{x}) \neq 0$.*

Proof. The assumption on $\theta(x, b)$ tells us that the orbit lies in Q2 or Q4 for x near \bar{x} . For simplicity of exposition we only discuss the case of Q4, i.e., $f(x) \geq 0, f'(x) \leq 0$. In particular, f is a monotone function and hence has a limit at \bar{x} . Thus, $h(x) = x f(x)$ is continuous; in particular, if we suppose that $\bar{x} f(\bar{x}) < 1$, then $h(x) < 1$ for x near \bar{x} . Since $A(\bar{x}) = 0$, we get from (2.14c) that $x AC' = C(A - 1 + x^2 f^2) < 0$ for x near \bar{x} . We conclude that C is bounded above, hence $\lim_{x \rightarrow \bar{x}^-} AC^2 = 0$ and $\lim_{x \rightarrow \bar{x}^-} g = 1$. Since $g > 0$, the right hand side of Eq. (2.14a) is positive and hence $x^2 f'/C$ is bounded and since C is bounded we conclude that f' is bounded; thus $\lim_{x \rightarrow \bar{x}^-} A f'^2 = 0$. Then, from (2.14b), $x A' = 1 - A - x^2 f^2 - x^2 (A f'^2 + AC^2 f^2)$, we see that $A' > 0$ near \bar{x} so there is no crash. This is a contradiction so we conclude that $\bar{x} f(\bar{x}) \geq 1$ and hence $f(\bar{x}) > 0$. \square

Lemma 7.5. *There is a $\gamma > 0$ such that $h(x, a) = x f(x, a) > 1/4$ for all a sufficiently near b and $\bar{x} < x < \bar{x} + \gamma$.*

Proof. If the b -orbit crashes at $x = \bar{x}$ with rotation $\theta(x, b) > k\pi$, then there is a y such that $\theta(y, b) = k\pi$. Let $B = (f(y, b) + 1)^2$. By Proposition 7.3, if a is sufficiently close to b , Af'^2 (along the a -orbit) is bounded in Q4 by $D = \max(\alpha^2/3, B)$; D is a uniform bound on Af'^2 in Q4 for all a sufficiently near b . Next, choose x_1 such that $0.99\bar{x} < x_1 < \bar{x}$ and such that $A(x_1, b) < 0.01$, $g(x_1, b) = 2\tau > 0$, and $h(x_1, b) > 0.9$; this is possible by Lemma 7.4 and Proposition 7.1. Then, for a sufficiently near b we have $A(x_1, a) < 0.02$, $g(x_1, a) > \tau > 0$, $f(x_1, a) < f(x_1, b) + 0.01/\bar{x}$ and $h(x_1) > 3/4$. We shall find a $\gamma \in (0, 0.01\bar{x})$ that works for all a , that is, it satisfies $h(x, a) > 1/4$ for all a sufficiently near b and $\bar{x} < x < \bar{x} + \gamma$. So let a satisfy: i) Af'^2 (along the a -orbit) is bounded by D , ii) $A(x_1, a) < 0.02$, iii) $h(x_1, a) > 3/4$, and iv) $g(x_1, a) > \tau > 0$. If $h(x, a) > 1/4$ for all $x < 1.01\bar{x}$ and all a near b we are done – let $\gamma = 0.01\bar{x}$. Otherwise, we define $x_2 = x_2(a)$, etc. by $h(x_2) = 3/4$, $h(x_3) = 1/2$, $h(x_4) = 1/4$, where x_2, x_3 , and x_4 are the largest values of $x < 1.01\bar{x}$ with that property. For $x > x_2$ we have from (2.14a) $xAf'' = xgf - (1 + A - h^2)f' \geq -(1 + A - h^2)f' \geq -f'/4$ since $h \leq 3/4$ so $f'' \geq -f'^3/(4xAf'^2) \geq -f'^3/(4 * 1.01\bar{x}D)$ or $f''/f'^2 \geq -f'/(4.04\bar{x}D)$. We now integrate the above from x_2 to $x > x_3$ to get

$$\frac{-1}{f'(x)} \geq \frac{-1}{f'(x_2)} + \frac{1}{f'(x_2)} \geq \int_{x_2}^x \frac{f''}{f'^2} dx \geq \int_{x_2}^x \frac{-f'}{4.04\bar{x}D} dx = \frac{f(x) - f(x_2)}{4.04\bar{x}D}. \tag{7.3}$$

Now, $f(x) \geq f(x_3)$, so $-f'(x) \leq \frac{5\bar{x}D}{f(x_2)-f(x_3)} \approx \frac{5\bar{x}^2D}{h(x_2)-h(x_3)} = 20\bar{x}^2D$. Using the uniform bound on f' in the interval $x_3 \leq x \leq x_4$, we have $x_4 - x_3 = (f(x_4) - f(x_3))/f'(\xi)$ for some $\xi \in [x_3, x_4]$. But $(f(x_4) - f(x_3))/f'(\xi) \geq (h(x_4) - h(x_3))/(\bar{x}f'(\xi)) \geq 1/80\bar{x}^3D$ and hence we may take $\gamma = 1/80\bar{x}^3D$. \square

Lemma 7.6. *In the interval $x_1 < x < \bar{x} + \gamma$, $g(x, a) > \min(\tau, 0.9/h^2(\bar{x}, b))$.*

Proof. From (2.18) we have $xg' = C^2(1 - A + x^2Af'^2 - x^2gf^2) \geq C^2(1 - A - x^2gf^2)$. Moreover, since $A(x_1, a) < 0.02$ and $xA' < 1$, $A(x, a) = A(x_1, a) + A'(z)(x - x_1) < 0.02 + 1/z(0.02\bar{x}) < 0.04$, so if $g < 0.96/h^2(x)$ then $g' > 0$. Since $f(x_1, a) < f(x_1, b) + 0.01/\bar{x}$, $h(x, a) \leq 1.01\bar{x}f(x_1, a) < 1.01(\bar{x}f(x_1, b) + 0.01) < 1.02\bar{x}f(x_1, b)$, we have $g' > 0$ if $g(x_1, a) < 0.9/h^2(\bar{x}, b)$. Thus, if $\tau < g(x_1, a) < 0.9/h(\bar{x}, b)$, $g' > 0$, and $g(x, a) > \tau$ in the interval $x_1 < x < \bar{x} + \gamma$; if $g(x_1, a) > 0.9/h^2(\bar{x}, b)$, then $g(x, a) > 0.9/h^2(\bar{x}, b)$ for all x in the interval $x_1 < x < \bar{x} + \gamma$ because g cannot cross that value from above. \square

Note that the above lower bound on g is uniform – it applies to all a satisfying the conditions i) Af'^2 (along the a -orbit) is bounded by D , ii) $A(x_1, a) < 0.02$, iii) $h(x_1, a) > 3/4$, and iv) $g(x_1, a) > \tau > 0$.

Lemma 7.7. *For all a sufficiently near b , $v(x, a)$ goes to 0 for some $x < \bar{x} + \gamma$.*

Proof. To show that $v(x, a)$ goes to 0, we note that $h(x, a) \geq 1/4$ for all a near b and $\bar{x} < x < \bar{x} + \gamma$ by Lemma 7.5. Hence, $f(x, a) = h(x, a)/x > 1/4\bar{x}$. By Lemma 7.6, $g(x, a) > \min(\tau, 1/h(\bar{x}))$, hence $v' \geq 1/4\bar{x} \min(\tau, 1/h(\bar{x})) = \eta > 0$ for $\bar{x} < x < \bar{x} + \gamma$. Thus, $v(\bar{x} + \gamma) - v(\bar{x}) = \int_{\bar{x}}^{\bar{x}+\gamma} v' dx \geq \int_{\bar{x}}^{\bar{x}+\gamma} \eta dx \geq \eta\gamma$. Let x_1 be chosen so that $v(x_1, a) > -\eta\gamma/2$. Then, if a is sufficiently close to b , $|v(x_1, a) - v(x_1, b)| > \eta\gamma/2$ so $v(x_1, a) > -\eta\gamma$. For such a we then have $v(\bar{x} + \gamma, a) > v(\bar{x}, a) + \eta\gamma$ and $v(\bar{x}, a) > v(x_1, a) > -\eta\gamma$ because $v' > fg > 0$; thus, $v(\bar{x} + \gamma, a) > 0$. \square

We now complete the proof of Proposition 7.2.

Proof of Proposition 7.2 b). Suppose that the b -orbit crashes at $x = \bar{x}$ with $\theta(x, b) < (k + 1/2)\pi$ for all $x < \bar{x}$ and $\theta(x, b) > k\pi$ for x near \bar{x} . For a near b there is an $x < \bar{x} + \gamma$ with $v(x, a) = 0$ by Lemma 7.7. Since $x < \bar{x} + \gamma$, $h(x) > 1/4$, i.e., $f(x, a) > 0$, so the a -orbit crashes, $A(x, a) = 0$, or exits Q4 to Q1 (or Q2 to Q3), $f'(x, a) = 0$, never to return. In either case, the a -orbit has rotation $\theta(x, a) < (k + 1/2)\pi$. \square

8. Proof of the Main Theorem

Proof of Theorem 2.1. Let $X_n = \{a > 0 \mid \theta(x, a) < (n + 1/2)\pi \text{ for all } x \text{ for which the } a\text{-orbit is defined}\}$. Note that $X_{n-1} \subset X_n$ and $X_0 \neq \emptyset$ by Proposition 5.1 and hence, $X_n \neq \emptyset$. Also note that $b_{n+1} > 0$ is a lower bound for X_n by Proposition 4.1; hence, X_n has a greatest lower bound $a_n = \inf(X_n) \geq b_{n+1} > 0$. We will show that the a_n -orbit is a globally regular solution and $n\pi < \theta(x, a_n) < (n + 1/2)\pi$ for large x .

We first show that $a_n \in X_n$, i.e., a_n is the smallest element in X_n . If $\theta(x, a_n) > (n + 1/2)\pi$ for some x then $\theta(x, a) > (n + 1/2)\pi$ for all a near a_n so $a \notin X_n$ for these a 's and this contradicts the fact that a_n is the greatest lower bound of X_n . Thus, $a_n \in X_n$. In particular, the a_n -orbit has bounded rotation.

Next we show that the a_n -orbit does not crash. Recall from Proposition 7.1 that if the a_n -orbit crashes at $x = \bar{x}$ then $g(x, a_n) > 0$ for x near \bar{x} . If the a_n -orbit crashes in Q1 or Q3, that is, if $\theta(x, a_n) < n\pi$ for x near \bar{x} then $\theta(x, a) < n\pi$ and $g(x, a) > 0$ for all a near a_n which implies by Lemma 2.6b that the a -orbit must have $\theta(x, a) < n\pi$ for all x . Thus, $a \in X_n$ for all a near a_n and this contradicts the fact that a_n is the greatest lower bound of X_n .

Similarly, if the a_n -orbit crashes in Q2 or Q4, that is, at some \bar{x} with $(n + 1/2)\pi > \theta(\bar{x}, a_n) > n\pi$, then by the crash lemma $(n + 1/2)\pi > \theta(x, a)$ for all x in the domain of definition of the a -orbit for all a near a_n and this contradicts the fact that a_n is the greatest lower bound of X_n .

Thus, the a_n -orbit is defined for all x and hence is a globally regular solution by Propositions 6.1. Also, by Proposition 6.2, the a_n -orbit is in Q2 or Q4 for large x . It remains to prove that $\theta(x, a_n) > n\pi$ for large x . Suppose that $\theta(x, a_n) < n\pi$ for large x . By Lemma 6.3 we have that $g(x, a_n) > 0$ for large x and hence, $g(x, a) > 0$ for all a near a_n . Then, by Lemma 2.6b the a -orbit must have $\theta(x, a) < n\pi$ for all x and thus $a \in X_n$, and this contradicts the fact that a_n is the greatest lower bound for X_n . This completes the proof of Theorem 2.1. \square

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