# Non-Formation of Vacuum States for Compressible Navier–Stokes Equations

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**Abstract:** We prove that weak solutions of the Navier–Stokes equations for compressible fluid flow in one space dimension do not exhibit vacuum states, provided that no vacuum states are present initially. The solutions and external forces that we consider are quite general: the essential requirements are that the mass and energy densities of the fluid be locally integrable at each time, and that the $L^2_{\text{loc}}$-norm of the velocity gradient be locally integrable in time. Our analysis shows that, if a vacuum state were to occur, the viscous force would impose an impulse of infinite magnitude on the adjacent fluid, thus violating the hypothesis that the momentum remains locally finite.

## 1. Introduction

We prove that weak solutions of the Navier–Stokes equations for compressible fluid flow in one space dimension do not exhibit vacuum states, provided that no vacuum states are present initially. The solutions and external forces that we consider are quite general: the essential requirements are that the mass and energy densities of the fluid be locally integrable at each time, and that the $L^2_{\text{loc}}$-norm of the velocity gradient be locally integrable in time.

Our result is motivated by the existence theorem of Hoff [8], in which global solutions are constructed with large, discontinuous initial data, possibly having different limits at $x = \pm\infty$, and with large external forces. In particular, arbitrary Riemann initial data is allowed. These constructed solutions have strictly positive densities, so that vacuum states cannot form in finite time. Uniqueness of weak solutions is not known, however, in any class which includes solutions with vacuum states. Indeed, the uniqueness results of which we are aware are based upon analyses in Lagrangian coordinates, in which the reciprocal of the density is a fundamental variable; see Hoff [7] and Hoff and Zarnowski

[10], for example. This change of coordinates clearly fails when vacuum states are allowed. The question therefore arises whether there are *any* solutions in which vacuum states occur in positive time. In the present paper we give a definitive answer by defining a vacuum state to be an open set in physical space in which there is no mass, and proving that no such vacuum states can occur at positive times if none are present initially. We recall in this regard that the physical derivation of the Navier–Stokes system presupposes that the fluid in question is *nondilute*. Our result therefore establishes an important self–consistency for this model.

We now give a precise formulation of our results. The Navier–Stokes equations express the conservation of mass and the balance of momentum as follows:

$$\rho_t + (\rho u)_x = 0, \tag{1.1}$$

$$(\rho u)_t + (\rho u^2 + P)_x = \mu u_{xx} + \rho f, \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+, \tag{1.2}$$

where $\rho, u$, and $P$ denote respectively the density, velocity, and pressure, $f = f(x, t)$ is an external force, and $\mu$ is a positive viscosity coefficient. *We do not assume that P is a function only of $\rho$.* Rather, $P$ may depend upon other unknowns, and there may be appended to (1.1)–(1.2) other equations for these unknowns. For example, for the nonbarotropic flow of an ideal gas, $P = (\gamma - 1)\rho e$, where $e$ is the specific internal energy and $\gamma$ is the adiabatic constant, and a third equation, the energy–balance equation, is appended to close the system. We shall therefore assume only that $P = P(\rho, x, t)$, and that

$$P(0, x, t) = 0, x \in \mathbb{R}, 0 \leq t \leq T, \tag{1.3}$$

where $T$ is a positive time which will be fixed throughout. Concerning the external force $f$ we assume only that

$$f \in L^1\left([0; T]; L^\infty_{\text{loc}}(\mathbb{R})\right), \tag{1.4}$$

which is a somewhat weaker requirement than that made in [8].

Weak solutions are defined in the usual way: we say that $(\rho, u)$ is a weak solution of (1.1)–(1.2) on $\mathbb{R} \times [0, T]$ provided that

(A$_1$)   $\rho$ and $\rho u$ are in $C\left([0, T]; H^{-1}_{\text{loc}}(\mathbb{R})\right)$ with $\rho$ nonnegative; $\rho(\cdot, t)$ and $(\rho u)(\cdot, t)$ are in $L^1_{\text{loc}}(\mathbb{R})$ for each $t \in [0, T]$; $\rho u^2$, $P(\rho, \cdot, \cdot)$, and $u_x$ are in $L^1([-L, L] \times [0, T])$ for every $L$; and for all $C^1$ test functions $\phi$ supported in $\mathbb{R} \times \mathbb{R}$,

$$\int \rho\phi \Big|_{t_1}^{t_2} dx = \int_{t_1}^{t_2} \int [\rho\phi_t + (\rho u)\phi_x] \, dx \, dt \tag{1.5}$$

and

$$\int \rho u\phi \Big|_{t_1}^{t_2} dx = \int_{t_1}^{t_2} \int [\rho u(\phi_t + u\phi_x) + (P - \mu u_x)\phi_x + \rho f\phi] \, dx \, dt \tag{1.6}$$

for all $t_1, t_2 \in [0, T]$.

It follows as a consequence of ($A_1$) that, for any $L > 0$, there is a constant $C = C(L)$ such that

$$\int_{-L}^{L} \rho(x, t)dx \leq C(L) \tag{1.7}$$

for all $t \in [0, T]$.

Now, in the existence theory of [8] (which deals only with the barotropic case $P = P(\rho)$), smooth reference functions $\bar{\rho}(x)$ and $\bar{u}(x)$ are defined which are constant for $x \leq -1$, constant for $x \geq 1$, and monotone for $-1 \leq x \leq 1$. The constructed solutions are then shown to satisfy a number of regularity conditions and estimates, among which the following are particularly important:

$$\int_{\mathbb{R}} \left[ \rho u^2 - \bar{\rho}\bar{u}^2 + G(\rho, \bar{\rho}) \right](x, t)dx + \int_0^T \int_{\mathbb{R}} u_x^2 dxdt < \infty.$$

(Thus $u_x(\cdot, t) \in L^2(\mathbb{R})$ for almost all $t \in [0, T]$.) Here $G$ is the potential energy density relative to the reference state $\bar{\rho}$, defined by

$$G(\rho, x) = \rho \int_{\bar{\rho}(x)}^{\rho} \frac{P(s) - P(\bar{\rho}(x))}{s^2} ds.$$

Thus $G$ is a smooth, nonnegative function. It was also assumed in [8] that

$$\liminf_{\rho \to 0} G(\rho, x) \geq C^{-1}$$

for some constant $C$, independent of $x$. It is easily seen that this condition is satisfied in the representative case that $P = P(\rho) = K\rho^{\gamma}$, $\gamma \geq 1$.

In the present paper we shall deal with weak solutions which are assumed to satisfy analogous, but somewhat weaker conditions. These conditions are formulated to be the minimum required for the proof of our theorem, and are consequently slightly technical. It is easy to see, however, that they are indeed weaker than the conditions described above, which are known to be satisfied by the solutions constructed in [8].

We thus assume that

($A_2$) $u_x \in L^1\left([0, T]; L^2_{\text{loc}}(\mathbb{R})\right)$.

In particular, $u_x(\cdot, t) \in L^2_{\text{loc}}(\mathbb{R})$ for almost all $t \in [0, T]$. Next we assume that

($A_3$) there is a function $\gamma(t) \in L^1([0, T])$ such that, for all $L > 0$ and almost all $t \in [0, T]$,

$$\left[ \int_{-L}^{L} (\rho u^2)(x, t)dx \right]^{1/2} \leq \gamma(t)(1 + L), \tag{1.8}$$

and

$$\left[ \int_{-L}^{L} u_x(x, t)^2 dx \right]^{1/2} \leq \gamma(t)(1 + L). \tag{1.9}$$

We assume also that

(A$_4$)  for every $L > 0$ there is a constant $C = C(L)$ such that

$$\int_{-L}^{L} (\rho|u|)(x, t)dx \leq C(L) \tag{1.10}$$

for all $t \in [0, T]$.

(We note, however, that, if (A$_3$) were strengthened slightly by replacing the right side of (1.8) by $C(1 + L)$ for some constant $C$ and requiring (1.8) to hold for all $t$, then (A$_4$) would be a *consequence* of (1.7) and (1.8); that is, finite local mass and kinetic energy would imply finite local momentum.)

Finally we assume that

(A$_5$)  there is a "potential energy density" function $G(\rho, x, t)$, which is nonnegative and continuous on $\mathbb{R}_{\geq 0} \times \mathbb{R} \times [0, T]$, and for which:

a) there exist positive constants $C_0 > 0$ and $\underline{\rho} > 0$ such that, for all $x \in \mathbb{R}$, $t \in [0, T]$, and $\rho \in [0, \underline{\rho}]$,

$$G(\rho, x, t) \geq C_0^{-1}; \tag{1.11}$$

b) there exist constants $C_1 > 0$ and $\theta \in [0, 1)$ such that, for all $x_0, L \in \mathbb{R}$ and all $t \in [0, T]$,

$$\int_{x_0}^{x_0+L} G(\rho(x, t), x, t)dx \leq C_1 + \theta C_0^{-1} L. \tag{1.12}$$

We remark that, for solutions of the nonbarotropic system alluded to earlier, in which $P = P(\rho, e) = (\gamma - 1)\rho e$, the negative of the entropy density, that is,

$$S(\rho, e) \equiv \rho(e - 1 - \log e) + (\gamma - 1)(1 - \rho + \rho \log \rho),$$

has locally finite spatial integral at all times, at least in all known constructed solutions which could be regarded as physical (see [5], for example). The hypothesis (A$_5$) above may therefore be met by taking $G = (\gamma - 1)(1 - \rho + \rho \log \rho)$. The results of the present paper are thus seen to apply as well to the equations of *nonbarotropic* flow for an ideal fluid.

The following theorem is the main result of this paper.

**Theorem.** *Assume that $P$ and $f$ satisfy conditions (1.3)–(1.4) above, and let $(\rho, u)$ be a solution of (1.1)–(1.2) on $[0, T]$ satisfying assumptions (A$_1$)–(A$_5$). If*

$$\int_E \rho(x, 0)dx > 0 \tag{1.13}$$

*for every open set $E \subset \mathbb{R}$, then*

$$\int_E \rho(x, t)dx > 0 \tag{1.14}$$

*for every open subset $E \subset \mathbb{R}$ and for every $t \in [0, T]$.*

We now give a brief, heuristic overview of the proof and explain some of the underlying physical motivations. The rigorous proof is detailed in a sequence of lemmas in Sect. 2.

We first show that $u \in L^1([0, T]; L^\infty([-L, L]))$ for every $L$, and that the norm in the latter space grows at most linearly in $L$. These facts would be immediate from (1.8) and (1.9) if we knew that $\rho$ were bounded below away from 0. We instead apply the hypotheses (1.11) and (1.12), which imply the weaker fact that $\rho$ cannot be close to zero on too a large set. This turns out to be sufficient for the required estimate, which is given in Lemma 2.1 below. Observe that $\int_0^T \|u(\cdot, t)\|_{L^\infty([-L,L])} dt$ dominates the distance that a fluid particle travels between times 0 and $T$, provided that it remains within $[-L, L]$. The fact that this integral grows at most linearly in $L$ therefore shows, at least at the heuristic level, that *a fluid particle can travel at most a finite distance in finite time.*

Now suppose that $\rho(x, t_1) = 0$ a.e. on $(a, b)$, where $a$ is minimal and $b$ is maximal. Our observation above concerning finite average convection speeds then implies that there must be nearby vacuum states at nearby times. Specifically, we construct curves $y(t)$ and $z(t)$ starting from $a$ and $b$ respectively, such that $\rho(\cdot, t) = 0$ a.e. on $(y(t), z(t))$, and such that $y(t)$ is minimal and $z(t)$ is maximal. By comparing with the time-antiderivative of $\|u(\cdot, t)\|_{L^\infty}$, we are able to prove that these curves are in fact absolutely continuous, and can be extended backward to a minimal time $t_0 \geq 0$, and that $y(t_0) = z(t_0)$. Thus a vacuum exists in the wedge–shaped region $V$ given by

$$V = \{(x, t) : y(t) \leq x \leq z(t), \ t_0 \leq t \leq t_1\} .$$

Since $\rho = 0$ in $V$, $u$ is evidently linear in $V$, say $u(x, t) = \alpha(t)x + \beta(t)$, in a suitable sense. Now, in what is the most difficult part of the analysis, we show that *integral curves of $u$ which start in $V$ must remain in $V$* on $[t_0, t_1]$. This result depends in a crucial way on the linearity of $u$ in $V$ and on the absolute continuity of the boundary curves $y$ and $z$, and is given in Lemma 2.6 below. This invariance of $V$ for the fluid flow thus implies that any two integral curves of $u$ in $V$, proceeding backward in time, must come together at time $t_0$. It therefore follows that *$\alpha$ cannot be integrable on $[t_0, t_1]$.* We now apply this fact to derive a contradiction, motivated by the following physical intuition. First recall that, in the Navier–Stokes model, the term $\mu u_x$ represents the viscous force applied at the surface of a fluid particle by an adjacent fluid particle. (The second derivative $\mu u_{xx}$ in (1.2) results from an application of the divergence theorem.) Recall also from elementary mechanics that the time-integral of a given force, which is called the impulse, equals the corresponding change in momentum of the system. Now, in the situation described above, $\mu u_x = \mu\alpha$ is therefore the viscous force applied by the "massless fluid particles" in $V$ at the boundary of the fluid to the right of $V$. The nonintegrability of $\alpha$ on $[t_0, t_1]$ therefore implies that *the change in momentum from time $t$ to time $t_1$ becomes infinite as $t \to t_0$.* But this contradicts the fact (1.8) that the momentum is locally finite, thus completing the proof.

The initial-value problem for the Navier–Stokes equations (1.1)–(1.2) has been studied by many authors. See for example Kanel [12], Hoff [4, 5], and [8], Kazhikov and Shelukhin [13], and Serre [19] for existence of solutions with constant time-asymptotic states, as well as Liu [14], Hoff and Liu [9], Liu and Xin [15], Szepessy and Xin [21], and Matsumura and Nishihara [16] and [17] for cases in which the time-asymptotic state contains a viscous shock or rarefaction wave, usually of small strength. There are a number of results concerning solutions of (1.1)–(1.2) on a finite interval with suitable boundary conditions, among which we mention those of Amosov and Zlotnick [1], Chen,

Hoff and Trivisa [2], Fujita-Yashima et. al. [3], Hoff and Ziane [11], Matsumura and Yanagi [18], and Shelukhin [20]. See also Hoff and Zarnowski [10], Hoff [5] and [7], and Hoff and Ziane [11] for uniqueness and continuous dependence results for solutions with strictly positive densities.

Finally we call attention to the result of Hoff [6], in which solutions are obtained for the multidimensional, spherically symmetric version of (1.1)–(1.2) with large, possibly discontinuous data. The density is assumed to be strictly positive at $t = 0$, but the existence theory allows for the possibility that a vacuum state forms in a ball centered at the origin in positive time. It is not known whether there are in fact solutions with such vacuum states, or whether such solutions can be precluded. Indeed, the question of the spontaneous formation of vacuum states in solutions of the Navier–Stokes equations in several space variables remains an important open question.

## 2. Proof of the Theorem

In this section we give the details of the proof outlined above. The hypotheses (1.3), (1.4), and (A$_1$)–(A$_5$) will be in force throughout this section, the constants $C_0$, $C_1$ and $\theta$ defined in (1.11) and (1.12) will be fixed, and, unless otherwise stated, $C$ will denote a generic positive constant whose precise meaning will be clear from the context.

**Lemma 2.1.** $u \in L^1([0, T]; L^\infty_{\text{loc}}(\mathbb{R}))$; in fact, there is a constant $C > 0$ such that for any $L > 0$,

$$\|u(\cdot, t)\|_{L^\infty(-L, L)} \leq C\gamma(t)(1 + L)$$

for almost all $t \in [0, T]$, where $\gamma$ is as in (1.8) and (1.9).

*Proof.* From hypothesis (A$_3$), $u(\cdot, t) \in H^1_{\text{loc}}$, for almost all $t \in [0, T]$; pick such a $t$. If $\ell > 0$ is given, and $x_0 \in [-L, L]$, let

$$A_\ell = \{x \in [x_0, x_0 + \ell] : \rho(x, t) \leq \underline{\rho}\}.$$

Since (1.12) implies that $C_0 G(\rho, x, t) \geq 1$ if $0 \leq \rho \leq \underline{\rho}$, we have, using (1.12),

$$\text{meas}(A_\ell) \leq C_0 \int_{x_0}^{x_0+\ell} G(\rho(x, t), x, t)dx \leq C_0 C_1 + \theta\ell. \tag{2.1}$$

Now choose $\ell_0$ such that

$$C_0 C_1 + \theta\ell_0 \leq \frac{1+\theta}{2}\ell_0, \tag{2.2}$$

so that $\text{meas}(A_{\ell_0}) \leq \frac{1+\theta}{2}\ell_0$. Thus if $B_{\ell_0} = [x_0, x_0 + \ell_0] - A_{\ell_0}$, then

$$\text{meas}(B_{\ell_0}) \geq \ell_0 - \frac{1+\theta}{2}\ell_0 = \frac{1-\theta}{2}\ell_0; \tag{2.3}$$

that is, $\text{meas}(B_{\ell_0})$ is strictly positive. Now if $x_1 \in B_{\ell_0}$, then $\rho(x_1, t) \geq \underline{\rho}$, and therefore

$$|u(x_0, t)| \leq |u(x_1, t)| + \int_{x_0}^{x_1} |u_x|dx$$

$$\leq \underline{\rho}^{-\frac{1}{2}}\left(\rho^{1/2}|u|\right)(x_1, t) + \left(\int_{x_0}^{x_0+\ell_0} u_x^2 dx\right)^{1/2}\ell_0^{\frac{1}{2}}.$$

Integrating with respect to $x_1$ over the set $B_{\ell_0}$ gives

$$\text{meas}(B_{\ell_0})|u(x_0, t)| \leq \underline{\rho}^{-\frac{1}{2}} \left( \int_{B_{\ell_0}} (\rho u^2)(x_1, t) dx_1 \right)^{1/2} \left( \text{meas}(B_{\ell_0}) \right)^{1/2}$$
$$+ \ell_0^{1/2} \text{meas}(B_{\ell_0}) \left( \int_{x_0}^{x_0+\ell_0} u_x^2 dx \right)^{1/2},$$

so

$$|u(x_0, t)| \leq \left[ \underline{\rho} \, \text{meas}(B_{\ell_0}) \right]^{-\frac{1}{2}} \left[ \int_{x_0}^{x_0+\ell_0} \rho u^2 dx \right]^{1/2} + \ell_0^{\frac{1}{2}} \left[ \int_{x_0}^{x_0+\ell_0} u_x^2 dx \right]^{1/2}$$
$$\leq \left[ \underline{\rho} \, \text{meas}(B_{\ell_0}) \right]^{-\frac{1}{2}} (1 + L + \ell_0)\gamma(t) + \ell_0^{\frac{1}{2}} (1 + L + \ell_0)\gamma(t)$$
$$\leq C'(1 + L + \ell_0)\gamma(t) \leq C(1 + L)\gamma(t).$$

This proves the lemma since $(A_3)$ implies that $\gamma \in L^1[0, T]$.   □

We shall show that the hypothesis $\rho(\cdot, t) = 0$ a.e. on some open subset of $\mathbb{R}^1$ leads to a contradiction. In preparation for this, we first make a remark.

*Remark.* If $\rho(\cdot, t) = 0$ on some open interval $(a, b)$, then $b - a$ is bounded above by a constant depending only on the parameters $C_0$, $C_1$, and $\theta$ appearing in (1.11) and (1.12). Indeed, it follows from (1.11) and (1.12), that

$$C_1 + \theta C_0^{-1}(b - a) \geq \int_a^b G(\rho(x, t), x, t) dx \geq C_0^{-1}(b - a),$$

and as $0 \leq \theta < 1$, we see that $b - a$ is bounded, as required.

The following lemma shows that if $\rho(\cdot, t)$ is zero a.e. on some interval, then, if $t'$ is near $t$, $\rho(\cdot, t')$ is zero a.e. on a nearby, but possibly smaller interval.

**Lemma 2.2.** *Let $t_1 < T$ and suppose that $\rho(\cdot, t_1) = 0$ a.e. on an open interval $(a, b)$. Let*

$$t_0 = \inf \left\{ t \in [0, t_1] : \int_t^{t_1} \|u(\cdot, s)\|_{L^\infty(a,b)} < \frac{1}{2}(b - a) \right\}$$

*and*

$$t_2 = \sup \left\{ t \in [t_1, T] : \int_{t_1}^t \|u(\cdot, s)\|_{L^\infty(a,b)} < \frac{1}{2}(b - a) \right\}.$$

*Then $t_0 < t_1 < t_2$, and for any $t \in (t_0, t_2)$, $\rho(\cdot, t) = 0$ on the interval*

$$\left( a + \left| \int_{t_1}^t \|u(\cdot, s)\|_{L^\infty(a,b)} ds \right|, b - \left| \int_{t_1}^t \|u(\cdot, s)\|_{L^\infty(a,b)} ds \right| \right).$$
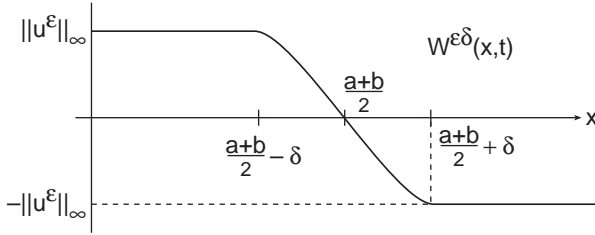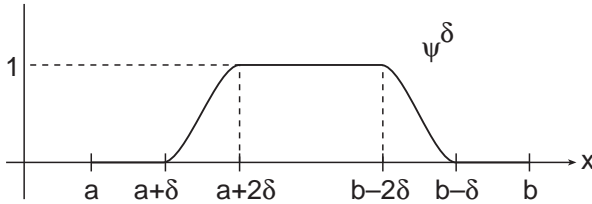
Fig. 1.



Fig. 2.

*Proof.* It is clear that $t_0 \leq t_1 \leq t_2$, and Lemma 2.1 shows that strict inequalities must hold because $\gamma$ is integrable.

Now suppose $t > t_1$; the proof for $t < t_1$ is similar, and will be omitted. Fix $\delta > 0$ satisfying $\delta < \frac{b-a}{6}$, and for small $\varepsilon > 0$, let $u^\varepsilon$ denote the usual spatial regularization of $u$. Then for almost all $t$, $T > t \geq t_1$,

$$\left\| u^\varepsilon(\cdot, t) \right\|_{L^\infty(a+\delta, b-\delta)} \leq \left\| u(\cdot, t) \right\|_{L^\infty(a,b)}.$$

For ease in notation, let

$$\left\| u^\varepsilon \right\|_\infty = \left\| u^\varepsilon(\cdot, t) \right\|_{L^\infty(a+\delta, b-\delta)} \quad \text{and} \quad \left\| u \right\|_\infty = \left\| u(\cdot, t) \right\|_{L^\infty(a,b)}.$$

Now define the smooth function $w^{\varepsilon\delta}(\cdot, t)$ by

$$w^{\varepsilon\delta}(x, t) = \begin{cases} \left\| u^\varepsilon(\cdot, t) \right\|_\infty, & \text{if } x < \frac{a+b}{2} - \delta \\ -\left\| u^\varepsilon(\cdot, t) \right\|_\infty, & \text{if } x > \frac{a+b}{2} + \delta, \end{cases}$$

and $w^{\varepsilon\delta}$ is decreasing on $\left(\frac{a+b}{2} - \delta, \frac{a+b}{2} + \delta\right)$; cf. Fig. 1 (where we take $a > 0$). Next, define the smooth function $\Psi^\delta(x)$ by

$$\Psi^\delta(x) = \begin{cases} 0, & \text{if } x < a + \delta \\ 1, & \text{if } a + 2\delta \leq x \leq b - 2\delta \\ 0, & \text{if } x > b - \delta \end{cases}$$

and $\Psi^\delta$ is increasing on the interval $(a + \delta, a + 2\delta)$, and decreasing on $(b - 2\delta, b - \delta)$; cf. Fig. 2.

Now let $\phi^{\varepsilon\delta}$ be the solution to the problem

$$\phi_t + w^{\varepsilon\delta}\phi_x = 0, \quad t > t_1, \tag{2.4}$$
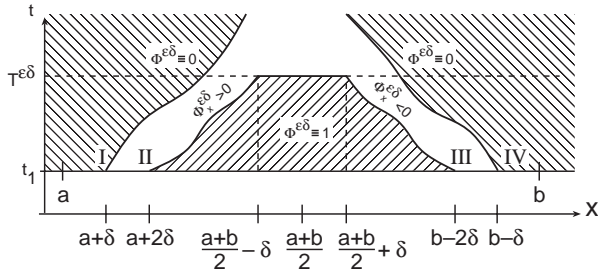$$\phi(\cdot, t_1) = \Psi^\delta.$$

**Fig. 3.**

It is easy to check that $\phi^{\varepsilon\delta}(x,t)$ is of the form depicted in Fig. 3, where the curves I–IV are characteristics. That is, $\phi^{\varepsilon\delta}$ is a smooth, compactly supported function, and can thus serve as a test function for the (weak) formulation of a solution of (1.1), (1.2).

In particular, from (1.5) we have

$$\int_{a+\delta}^{b-\delta} \rho\phi^{\varepsilon\delta}\Big|_{t_1}^{t} = \int_{a+\delta}^{b-\delta} \int_{t_1}^{t} \rho\left(\phi_t^{\varepsilon\delta} + u\phi_x^{\varepsilon\delta}\right)$$

$$= \int_{a+\delta}^{b-\delta} \int_{t_1}^{t} \rho\left(u - w^{\varepsilon\delta}\right)\phi_x^{\varepsilon\delta},$$

so that, since $\rho(x,t_1) = 0$ for $x \in [a,b]$, we have

$$\int_{a+\delta}^{b-\delta}(\rho\phi^{\varepsilon\delta})(x,t)dx$$

$$= \int_{a+\delta}^{b-\delta} \int_{t_1}^{t} \rho\left(u^{\varepsilon} - w^{\varepsilon\delta}\right)\phi_x^{\varepsilon\delta} + \int_{a+\delta}^{b-\delta} \int_{t_1}^{t} \rho(u - u^{\varepsilon})\phi_x^{\varepsilon\delta}. \quad (2.5)$$

Now in Fig. 3, $T^{\varepsilon\delta}$ is defined by

$$T^{\varepsilon\delta} = \sup\{t \in [t_1, T] : \text{II \& III}$$

$$\text{stay } \delta \text{ units away from} \frac{a+b}{2} \text{ on } [t_1, t]\}.$$

We now estimate $T^{\varepsilon\delta}$ from below. For this, we first notice that since the characteristics of (2.4) are given by $\dot{x} = w^{\varepsilon\delta}$, it follows that

$$\left(\frac{a+b}{2} - \delta\right) - (a + 2\delta) = \int_{t_1}^{T^{\varepsilon\delta}} w^{\varepsilon\delta}dt$$

$$\leq \int_{t_1}^{T^{\varepsilon\delta}} \|u^{\varepsilon}\|_{\infty}dt \leq \int_{t_1}^{T^{\varepsilon\delta}} \|u\|_{\infty}dt,$$

and thus

$$\int_{t_1}^{T^{\varepsilon\delta}} \|u\|_{\infty}dt \geq \frac{b-a}{2} - 3\delta. \quad (2.6)$$

Therefore if $T^\delta$ is defined by

$$T^\delta = \sup\left\{t \in [t_1, T] : \int_{t_1}^t \|u\|_\infty < \frac{b-a}{2} - 3\delta\right\},\tag{2.7}$$

then

$$T^{\varepsilon\delta} \geq T^\delta.\tag{2.8}$$

Thus if $t \in [t_1, T^\delta]$, then $t \in [t_1, T^{\varepsilon\delta}]$, so from Fig. 3, if $\phi_x^{\varepsilon\delta}(x, t) > 0$, then $x < \frac{a+b}{2} - \delta$, so from Fig. 1, $w^{\varepsilon\delta}(x, t) = \|u^\varepsilon\|_\infty$. If for such $t$, $\phi_x^{\varepsilon\delta}(x, t) < 0$, then $x > \frac{a+b}{2} + \delta$, and $w^{\varepsilon\delta}(x, t) = -\|u^\varepsilon\|_\infty$. It follows that (cf. (2.5)),

$$\int_{a+\delta}^{b-\delta}\int_{t_1}^t \rho(u^\varepsilon - w^{\varepsilon\delta})\phi_x^{\varepsilon\delta} \leq 0.\tag{2.9}$$

Next, we claim that

$$\lim_{\varepsilon\to 0}\int_{a+\delta}^{b-\delta}\int_{t_1}^t \rho(u - u^\varepsilon)\phi_x^{\varepsilon\delta} = 0.\tag{2.10}$$

Granting this for the moment, we complete the proof of Lemma 2.2 as follows. First, from (2.5), (2.9), and (2.10), we get

$$\overline{\lim_{\varepsilon\to 0}}\int_{a+\delta}^{b-\delta}(\rho\phi)^{\varepsilon\delta}(x, t)dx \leq 0, \quad t \in [t_1, T^\delta].\tag{2.11}$$

Then from Fig. 3, we see that the support of $\phi^{\varepsilon\delta}$ is the region bounded by the characteristics I and IV. As before, the $x$-distance traversed by these characteristics is bounded from above by

$$\int_{t_1}^t \|u^\varepsilon\|_\infty \leq \int_{t_1}^t \|u\|_\infty,$$

so that the interval

$$\left(a + \delta + \int_{t_1}^t \|u^\varepsilon\|_\infty, \ b - \delta - \int_{t_1}^t \|u^\varepsilon\|_\infty\right) \equiv I_\delta\tag{2.12}$$

is contained in the support of $\phi^{\varepsilon\delta}(\cdot, t)$. Hence (2.11) gives that for all $t \in [t_1, T^\delta]$,

$$\rho(\cdot, t) = 0 \quad \text{a.e. on } I_\delta.\tag{2.13}$$

If now $t < t_2$ (cf. the statement of Lemma 2.2), then

$$\int_{t_1}^t \|u\|_\infty < \frac{1}{2}(b - a),$$

and thus there is a $\delta_0 > 0$ such that if $\delta \leq \delta_0$, then

$$\int_{t_1}^t \|u\|_\infty < \frac{1}{2}(b - a) - 4\delta.$$

For such $\delta$, (2.7) implies that $t \leq T^\delta$. Thus for such $t$ and $\delta$, $\rho(\cdot, t) = 0$ a.e. on $I_\delta$. Taking a sequence $\delta_i \searrow 0$, we get that $\rho(\cdot, t) = 0$ on the interval

$$\left(a + \int_{t_1}^t \|u\|_\infty, \ b - \int_{t_1}^t \|u\|_\infty\right)$$

for all $t \in [t_1, t_2]$, and this completes the proof of the lemma.

It remains to prove (2.10). To this end, we first differentiate (2.4) with respect to $x$ to obtain

$$\phi_{xt}^{\varepsilon\delta} + w^{\varepsilon\delta}\phi_{xx}^{\varepsilon\delta} = -w_x^{\varepsilon\delta}\phi_x^{\varepsilon\delta},$$

so that along the characteristics $x = x(t)$,

$$\phi_x^{\varepsilon\delta}(x(t), t) = \Psi_x^\delta(x(t_1)) \exp\left(-\int_{t_1}^t w_x^{\varepsilon\delta}(x(s), s)ds\right). \tag{2.14}$$

But from Fig. 1, we see that

$$|w_x^{\varepsilon\delta}(\cdot, s)| \leq C(\delta)\|u^\varepsilon(\cdot, s)\|_\infty \leq C(\delta)\|u(\cdot, s)\|_\infty,$$

(where $|C(\delta)| \to \infty$ as $\delta \to 0$), and thus from (2.14),

$$\|\phi_x^{\varepsilon\delta}\|_\infty \leq C'(\delta),$$

where $C'(\delta)$ is a constant depending only on $\delta$. Hence

$$\left|\int_{a+\delta}^{b-\delta}\int_{t_1}^t \rho(u - u^\varepsilon)\phi_x^{\varepsilon\delta}\right| \leq C'(\delta) \int_{t_1}^t \|\rho u - \rho u^\varepsilon\|_{L^1(a+\delta, b-\delta)}dt$$

$$\leq C'(\delta) \int_{t_1}^T \|u(\cdot, t) - u^\varepsilon(\cdot, t)\|_{L^\infty(a+\delta, b-\delta)}\|\rho(\cdot, t)\|_{L^1(a+\delta, b-\delta)}dt. \tag{2.15}$$

But from hypotheses (A$_4$), we have that for almost all $t \in [t_1, T]$, $u(\cdot, t) \in H_{\text{loc}}^1$ and from (1.7) $\|\rho(\cdot, t)\|_{L^1(a+\delta, b-\delta)}$ is bounded; thus for each fixed $t$ the integrand on the right-hand side of (2.15) tends to zero as $\varepsilon \searrow 0$. Since

$$\|u(\cdot, t) - u^\varepsilon(\cdot, t)\|_{L^\infty(a+\delta, b-\delta)}\|\rho(\cdot, t)\|_{L^1(a+\delta, b-\delta)} \leq C(a, b)\|u(\cdot, t)\|_{L^\infty(a+\delta, b-\delta)}$$

and $\|u(\cdot, t)\|_{L^\infty(a+\delta, b-\delta)}$ is integrable (by Lemma 2.1), the Lebesgue dominated convergence theorem applies to the right-hand side of (2.15) and shows that (2.10) holds. $\square$

Now suppose that $\rho(\cdot, t_1) = 0$ a.e. on $(a, b)$, where, without loss of generality, $a$ is minimal and $b$ is maximal (cf. the remark following the proof of Lemma 2.1). The interval $(a, b)$ and the time $t_1$ will be fixed for the remainder of the argument.

Let $t_0$ be as in the statement of Lemma 2.2, and define for $t \in (t_0, t_1)$,

$$y(t) = \inf\left\{x : \rho(\cdot, t) = 0 \text{ a.e. on } \left(x, \frac{a+b}{2}\right)\right\}, \tag{2.16}$$

$$z(t) = \sup\left\{x : \rho(\cdot, t) = 0 \text{ a.e. on } \left(\frac{a+b}{2}, x\right)\right\}. \tag{2.17}$$

Clearly, $y(t_1) = a$ and $z(t_1) = b$.

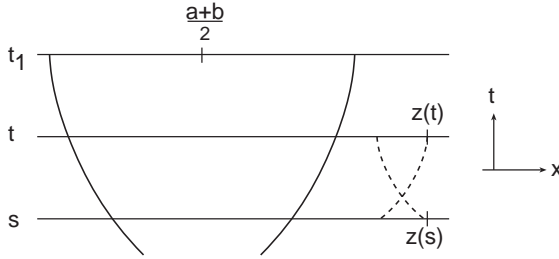In the following lemma we prove an important regularity property for the curves $y$ and $z$.

**Fig. 4.**

**Lemma 2.3.** *There exists a constant* $h = h(a, b) > 0$ *such that* $y$ *and* $z$ *are absolutely continuous functions on* $[t_1 - h, t_1]$.

*Proof.* First, it follows from the remark preceding Lemma 2.2 that there exists an $L > 0$ such that, for all $t \in (t_0, t_2)$,

$$-L \leq y(t), z(t) \leq L . \tag{2.18}$$

Next, choose $h > 0$ such that

$$\int_{t_1-h}^{t_1} \|u\|_{L^\infty(-L,L)} dt < \frac{b - a}{2}. \tag{2.19}$$

In order to prove that $z$ is AC, let $s$ and $t$ be such that

$$t_1 - h \leq s < t \leq t_1,$$

and compare $z(s)$ with $z(t)$; cf. Fig. 4, where all depicted curves have speeds $\pm\|u\|_{L^\infty(-L,L)}$, and thus comprise two families of horizontal translates.

Applying Lemma 2.2, we see that if $\rho(\cdot, t) = 0$ on $(y(t), z(t))$, then $\rho(\cdot, s) = 0$ a.e. on

$$\left( y(t) + \int_s^t \|u\|_{L^\infty(-L,L)}, \quad z(t) - \int_s^t \|u\|_{L^\infty(-L,L)} \right)$$

so that

$$z(s) \geq z(t) - \int_s^t \|u\|_{L^\infty(-L,L)}. \tag{2.20}$$

Similarly, if $\rho(\cdot, s) = 0$ a.e. on $(y(s), z(s))$ then

$$z(t) \geq z(s) - \int_s^t \|u\|_{L^\infty(-L,L)}. \tag{2.21}$$

Hence (2.20) and (2.21) give, for $t_1 - h \leq s < t \leq t_1$,

$$|z(t) - z(s)| \leq \int_s^t \|u\|_{L^\infty(-L,L)}. \tag{2.22}$$

Now let $\varepsilon > 0$ be given; then Lemma 2.1 implies that we can find $\delta > 0$ such that if $\mathrm{meas}(E) < \delta$, then

$$\int_E \|u\|_{L^\infty(-L,L)} dt \leq \varepsilon. \tag{2.23}$$

Thus given points $\{s_j\}_1^k$ and $\{\tau_j\}_1^k$ satisfying

$$t_1 - h \leq s_1 < \tau_1 < s_2 < \tau_2 < \cdots < s_k < \tau_k < t_1,$$

with $\sum_j (\tau_j - s_j) \leq \delta$, (2.22) and (2.23) give

$$\sum_j |z(\tau_j) - z(s_j)| \leq \sum_j \int_{s_j}^{\tau_j} \|u\|_{L^\infty(-L,L)}$$

$$= \int_{\cup [s_j,\,\tau_j]} \|u\|_{L^\infty(-L,L)} \leq \varepsilon.$$

This proves that $z$ is AC on $[t_1 - h, t_1]$; similarly, $y$ is AC on the same interval. □

In the next lemma, we obtain further results concerning the functions $y(t)$ and $z(t)$. To this end, let $S$ be defined as the set of all $t \geq 0$ such that there are extensions of $y$ and $z$ to $[t, t_1]$ such that the following three properties hold:

(i)   $y$ and $z$ are absolutely continuous on $[t, t_1]$,
(ii)  $y < z$ on $[t, t_1]$,
(iii) $\displaystyle\int_{y(s)-\varepsilon}^{z(s)} \rho(x, s)ds$ and $\displaystyle\int_{y(s)}^{z(s)+\varepsilon} \rho(x, s)dx$ are both positive for all $\varepsilon > 0$ and all $s \in [t, t_1]$, and $\displaystyle\int_{y(s)}^{z(s)} \rho(x, s)dx > 0$.

Notice that the last lemma implies that $S$ is nonempty; thus let

$$\tau = \inf S. \tag{2.24}$$

Concerning $\tau$ we have the following result.

**Lemma 2.4.** *$y$ and $z$ have AC extensions to time $\tau$, $y(\tau) = z(\tau)$, and there is an $L > 0$ such that for all $t \in [\tau, t_1]$, $-L \leq y(t) \leq z(t) \leq L$.*

*Proof.* We prove the last assertion first. Let

$$\tau < c < d < f < g < t_1,$$

and for $t \in (\tau, t_1)$, let

$$w(t) = \max\{z(t), -y(t)\} \geq 0. \tag{2.25}$$

Let $t \in [c, g]$; then by definition $\rho(\cdot, t) = 0$ a.e. on $(y(t), z(t))$, and since $y(t) < z(t)$, Lemma 2.2 shows that there is an $h = h(t) > 0$ such that if $|t - s| \leq h$, then $\rho(\cdot, s) = 0$ a.e. on the interval

$$\left( y(t) + \left| \int_t^s \|u\|_{L^\infty(-w(t),w(t))} \right|, z(t) - \left| \int_t^s \|u\|_{L^\infty(-w(t),w(t))} \right| \right),$$

and

$$C \int_{t-h}^{t+h} \gamma(s)ds \leq \tfrac{1}{2}, \tag{2.26}$$

where $C$ is as in Lemma 2.1. Thus

$$z(s) \geq z(t) - \left| \int_s^t \|u\|_{L^\infty(-w(t),w(t))} \right|,$$

$$y(s) \leq y(t) + \left| \int_s^t \|u\|_{L^\infty(-w(t),w(t))} \right|,$$

so that using Lemma 2.1, we get

$$w(s) \geq w(t) - \left| \int_s^t \|u\|_{L^\infty(-w(t),w(t))} \right|$$

$$\geq w(t) - C(1 + w(t)) \left| \int_s^t \gamma(\sigma)d\sigma \right|$$

$$= \left(1 - C \left| \int_s^t \gamma \right| \right) w(t) - C \left| \int_s^t \gamma \right|.$$

Thus for $|t - s| \leq h(t)$, (2.26) gives

$$w(t) \leq \left(1 - C \left| \int_s^t \gamma \right| \right)^{-1} \left[ w(s) + C \left| \int_s^t \gamma \right| \right]$$

$$\leq \left(1 + C \left| \int_s^t \gamma \right| \right) \left[ w(s) + C \left| \int_s^t \gamma \right| \right], \tag{2.27}$$

for some positive constant $C$.

Now choose constants $A < B$ (depending on $t$, which is fixed), such that $-w(t) < A < B < w(t)$. If $h(t)$ is further reduced, and if $|t - s| \leq h(t)$, then $y(s) < A < B < z(s)$, as follows from the continuity of $y$ and $z$ (Lemma 2.3). For such $s$, using Lemma 2.2, we find that there is a $\sigma$, depending on $\frac{B-A}{2}$, (so $\sigma = \sigma(t)$), such that if $s \leq \tilde{s} \leq s + \sigma$, then $\rho(\cdot, \tilde{s}) = 0$ on

$$\left( y(s) + \int_s^{\tilde{s}} \|u\|_{L^\infty(-w(t),w(t))}, \, z(s) - \int_s^{\tilde{s}} \|u\|_{L^\infty(-w(t),w(t))} \right).$$

It follows that

$$y(\tilde{s}) \leq y(s) + \int_s^{\tilde{s}} \|u\|_{L^\infty(-w(t),w(t))} \quad \text{and} \quad z(\tilde{s}) \geq z(s) - \int_s^{\tilde{s}} \|u\|_{L^\infty(-w(t),w(t))}.$$

We can further reduce $h(t)$ so that $h(t) \leq \sigma(t)$. Thus if $t - h(t) \leq s \leq t$, then $s \leq t \leq s + \sigma(t)$, and we may take $\tilde{s} = t$, to obtain

$$w(t) \geq w(s) - \int_s^t \|u\|_{L^\infty(-w(t),w(t))}$$

$$\geq w(s) - C(1 + w(t)) \int_s^t \gamma,$$

where we have used Lemma 2.1. Thus if $t - h(t) \leq s \leq t$, then

$$w(s) \leq \left(1 + C \int_s^t \gamma\right) \left[w(t) + C \int_s^t \gamma\right]. \tag{2.28}$$

We now cover the interval $[d, f]$ by a finite number of intervals $B_{h_j}(s_j)$, where $s_1 > s_2 > \cdots > s_p$ and $h_j = h(s_j)$. If $\tau_j \in B_{h_{j+1}}(s_{j+1}) \cap B_{h_j}(s_j)$, then by (2.27)

$$w(s_{j+1}) \leq \left(1 + C \int_{s_{j+1}}^{\tau_j} \gamma\right) \left[w(\tau_j) + C \int_{s_{j+1}}^{\tau_j} \gamma\right].$$

Also, from (2.28)

$$w(\tau_j) \leq \left(1 + C \int_{\tau_j}^{s_j} \gamma\right) \left[w(s_j) + C \int_{\tau_j}^{s_j} \gamma\right].$$

If we set $w_p = w(s_p)$, and $w_1 = w(s_1)$, then iterating these inequalities gives

$$w_p \leq \prod \left(1 + C \int_{s_{j+1}}^{\tau_j} \gamma\right) \left(1 + C \int_{\tau_j}^{s_j} \gamma\right) \left[w_1 + C \int_s^t \gamma\right]. \tag{2.29}$$

Now if $\varepsilon_1 + \cdots + \varepsilon_q = \varepsilon$, and each $\varepsilon_i > 0$, then

$$\prod (1 + \varepsilon_j) \leq \left(1 + \frac{\varepsilon}{q}\right)^q \leq e^\varepsilon.$$

Thus applying this to (2.29) gives

$$w_p \leq e^{C \int_0^T \gamma} \left[w_1 + C \int_0^T \gamma\right] \leq C'(w_1 + 1) \tag{2.30}$$

for some constant $C'$. As $w_1 = w(s_1)$, it follows that for $s_1$ near $t_1$, then as noted in (2.18), we can bound $w_1$ independent of $t$, and so (2.30) and (2.27) bound $w$ on $[d, t_1]$, for all $d > \tau$, independent of $t$. Thus we have proved that there is an $L > 0$ such that

$$-L \leq y(t) \leq z(t) < L, \quad t \in (\tau, t_1]. \tag{2.31}$$

We now show that $z$ and $y$ are uniformly continuous on the interval $(\tau, t_1]$. Once this is shown then the first and third assertions of Lemma 2.4 will be proved. Thus to prove the uniform continuity of $z$ on $(\tau, t_1]$, let $\varepsilon > 0$ be given. Choose $\delta > 0$ such that if $0 \leq s < t \leq T$, and $|s - t| \leq \delta$, then

$$\int_s^t \|u\|_{L^\infty(-L,L)} \leq \varepsilon.$$

Now just as earlier in this proof, if $t \in (\tau, t_1]$, we can find $h(t) > 0$ such that if $|t - s| \leq h(t)$, then

$$|z(s) - z(t)| \leq \left|\int_s^t \|u\|_{L^\infty(-L,L)}\right|. \tag{2.32}$$

Now fix $s < t$ with $|s - t| \leq \delta$ and $s, t \in (\tau, t_1]$; then the interval $[s, t]$ is covered by $\cup_1^q B_{\frac{h_k}{2}}(s_k)$, $s_1 < s_2 < \cdots < s_q$, where $s_j + \frac{h_j}{2} > s_{j+1} - \frac{h_{j+1}}{2}$, and $h_j < \delta$ for each $j$. Then $|s_{j+1} - s_j| \leq \frac{h_j + h_{j+1}}{2} \leq \max\{h_j, h_{j+1}\} < \delta$. Thus by (2.32),

$$|z(s_j) - z(s_{j+1})| \leq \int_{s_j}^{s_{j+1}} \|u\|_{L^\infty(-L, L)}.$$

Now for some $j$ and $k$, $s \in B_{\frac{h_k}{2}}(s_k)$, $t \in B_{\frac{h_j}{2}}(s_j)$, and we have

$$|z(t) - z(s)| \leq |z(s) - z(s_j)| + |z(s_j) - z(s_{j-1})| + \cdots + |z(s_k) - z(t)|$$
$$\leq \int_s^{s_j} + \cdots + \int_{s_k}^t = \int_s^t \|u\|_{L^\infty(-L, L)}$$
$$\leq \varepsilon.$$

To complete the proof, we have to show that $y(\tau) = z(\tau)$. But this is clear, since otherwise $y(\tau) < z(\tau)$, and if $\tau > 0$, then $\tau$ would not be minimal, whereas if $\tau = 0$, then the hypothesis that $\int_{y(\tau)}^{z(\tau)} \rho(x, 0)\, dx > 0$ would be violated.   $\square$

We next study the function $u$ in the vacuum region. To this end, we define the set $V$ by

$$V = \{(x, t) : y(t) < x < z(t), \quad \tau < t \leq t_1\}.$$

Note that for $\tau < t \leq t_1$, $\rho(\cdot, t) = 0$ a.e. on $(y(t), z(t))$.

**Lemma 2.5.** *There exist functions $\alpha, \beta \in L^1_{\text{loc}}((\tau, t_1])$ such that $u = \alpha(t)x + \beta(t)$ in $\mathcal{D}'(V)$ and $u(x, t) = \alpha(t)x + \beta(t)$ for all $x$ and almost all $t$ in $V$.*

*Proof.* From (1.2), we see that $u_{xx} = 0$ in $\mathcal{D}'(V)$, and thus $u^\varepsilon_{xx} = 0$ in $\mathcal{D}'(V)$, where $u^\varepsilon_{xx}$ is the standard regularization of $u_{xx}$. Thus $u^\varepsilon(x, t) = \alpha^\varepsilon(t)x + \beta^\varepsilon(t)$. Now from (1.10),

$$0 = \lim_{\varepsilon_1, \varepsilon_2 \to 0} \int_\tau^{t_1} \left( \int_{y(t)}^{z(t)} \left( u^{\varepsilon_1}_x - u^{\varepsilon_2}_x \right)^2 dx \right)^{1/2} dt$$
$$= \lim_{\varepsilon_1, \varepsilon_2 \to 0} \int_\tau^{t_1} |\alpha^{\varepsilon_1}(t) - \alpha^{\varepsilon_2}(t)| [z(t) - y(t)]^{1/2}\, dt,$$

and thus $\{\alpha^\varepsilon\}$ is a Cauchy sequence in $L^1([\tau + \delta, t_1])$ for every $\delta > 0$; that is, $\{\alpha^\varepsilon\}$ is a Cauchy sequence in $L^1_{\text{loc}}((\tau, t_1])$. Also, if $I$ is a compact set in $(\tau, t_1]$, and $t \in I$,

$$|\beta^{\varepsilon_1}(t) - \beta^{\varepsilon_2}(t)| \leq C \|u^{\varepsilon_1}(\cdot, t) - u^{\varepsilon_2}(\cdot, t)\|_{L^\infty(y(t), z(t))}$$

for some constant $C$. Since $u^\varepsilon \to u$ in $L^1(\{(x, t) : t \in I, \ y(t) \leq x \leq z(t)\})$, we see that $\{\beta^\varepsilon\}$ is a Cauchy sequence in $L^1(I)$ so $\beta^\varepsilon \to \beta$ in $L^1(I)$; thus $\beta^\varepsilon \to \beta$ in $L^1_{\text{loc}}((\tau, t_1])$. Since $u^\varepsilon \to u$ in $\mathcal{D}'(V)$, and $\alpha^\varepsilon x + \beta^\varepsilon \to \alpha x + \beta$ in $L^1_{\text{loc}}((\tau, t_1]; L^\infty)$ we obtain that $u = \alpha x + \beta$ in $V$.   $\square$

The last lemma which we need is

**Lemma 2.6.** *Fix $w_1 \in (a, b)$ and for $\tau < t \le t_1$ define $w(t)$ by*

$$w(t) = w_1 \exp\left(-\int_t^{t_1} \alpha(s)ds\right) - \int_t^{t_1} \exp\left(-\int_s^{t_1} \alpha\right) \beta(s)ds.$$

*Then $y(t) < w(t) < z(t)$ for $\tau < t \le t_1$.*

*Proof.* We claim that

$$\frac{dz}{dt} \le \alpha z + \beta \tag{2.33}$$

for almost all $t \in (\tau, t_1]$. If this holds, then since

$$\frac{dw}{dt} = \alpha w + \beta,$$
$$w(t_1) = w_1 < b = z(t_1),$$

we find

$$\frac{d}{dt}(z - w) \le \alpha(z - w), \quad a.e.$$

so that

$$\frac{d}{dt}\left[\exp\left(-\int_{t_1}^t \alpha\right)(z - w)\right] \le 0 \quad a.e.$$

Integrating from $t$ to $t_1$ and using Lemma 2.3 gives

$$\exp\left(-\int_{t_1}^t \alpha\right)[z(t) - w(t)] \ge z(t_1) - w(t_1) > 0,$$

so that $z(t) > w(t)$; similarly, $w(t) > y(t)$.

We now prove (2.33). For this, we define the following sets of zero measure:

$$A = \{t \in (\tau, t_1] : u_x(\cdot, t) \notin L^2(y(z), z(t))\},$$
$$D = \{(x, t) \in V : u(x, t) \ne \alpha(t)x + \beta(t)\},$$
$$E = \{t \in (\tau, t_1] : z \text{ is not differentiable at } t\}.$$

Let $\{r_k\}$ be the set of rational numbers, and let $B_{jk} = \{x : |x - r_k| < \frac{1}{j}\}, j, k = 1, 2, \ldots$. From Lemma 2.1, we have $\|u(\cdot, t)\|_{L^\infty(B_{jk})} \in L^1([0, T])$. Let

$$F_{jk} = \{t \in (\tau, t_1] : t \text{ is not a Lebesgue point of } \|u(\cdot, t)\|_{L^\infty(B_{jk})}\}$$

and set $F = \cup F_{jk}$; then $\text{meas}(F) = 0$, and if $\bar{t} \notin F$,

$$\lim_{t \searrow \bar{t}} \frac{1}{t - \bar{t}} \int_{\bar{t}}^t \|u(\cdot, t)\|_{L^\infty(B_{jk})} = \|u(\cdot, \bar{t})\|_{L^\infty(B_{jk})}, \tag{2.34}$$
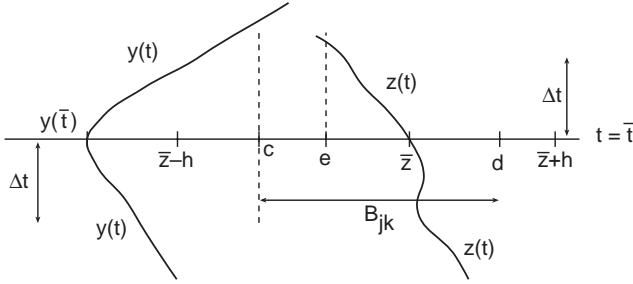
for every $j$ and $k$.

**Fig. 5.**

Let $\bar{t} \notin A \cup D \cup E \cup F$; we will prove that (2.33) holds at $\bar{t}$. Suppose not; then there is an $\varepsilon > 0$ such that for $t$ near $\bar{t}$ and $t > \bar{t}$,

$$\frac{z(t) - \bar{z}}{t - \bar{t}} \geq \alpha(\bar{t})\bar{z} + \beta(\bar{t}) + \varepsilon \equiv \bar{u} + \varepsilon,$$

where $\bar{z} = z(\bar{t})$; that is, for $t$ near $\bar{t}$,

$$z(t) \geq \bar{z} + (t - \bar{t})(\bar{u} + \varepsilon). \tag{2.35}$$

Because $u(\cdot, t)$ is in $H^1_{\text{loc}}$, we can find $h > 0$ such that if $|x - \bar{z}| \leq h$,

$$|u(x, \bar{t}) - \bar{u}| \leq \frac{\varepsilon}{2}, \tag{2.36}$$

and

$$y(\bar{t}) < \bar{z} - h. \tag{2.37}$$

Then choose $B_{jk}$ such that

$$\bar{z} \in B_{jk} \subset [\bar{z} - h, \bar{z} + h].$$

Let $B_{jk} = (c, d)$ and choose $e$ such that

$$\bar{z} - h < c < e < \bar{z} < d < \bar{z} + h.$$

We can thus can find $\Delta t > 0$ such that

$$|t - \bar{t}| < \Delta t \implies y(t) < c, \quad e \leq z(t) \leq d;$$

(this can be done since $y$ and $z$ are continuous functions); cf. Fig. 5. Then if $|t - \bar{t}| < \Delta t$,

$$\rho(\cdot, t) = 0 \text{ a.e. on } (y(t), z(t)) \supset (c, e),$$

so by Lemma 2.2, there is a $\sigma > 0$ such that $\rho(\cdot, s) = 0$ on

$$\left( c + \left| \int_t^s \|u\|_{L^\infty(c,e)} \right|, z(t) - \left| \int_t^s \|u\|_{L^\infty(c,e)} \right| \right)$$

if $|t - s| \leq \sigma$, $|t - \bar{t}| < \Delta t$. Thus for these $s$ and $t$,

$$z(s) \geq z(t) - \left| \int_t^s \|u\|_{L^\infty[c,e]} \right|$$

$$\geq z(t) - \left| \int_t^s \|u\|_{L^\infty(B_{jk})} \right|.$$

Let $s = \bar{t}$, and take $t$ within $\sigma$ of $\bar{t}$, $t > \bar{t}$, to get

$$z(\bar{t}) \geq z(t) - \int_{\bar{t}}^t \|u\|_{L^\infty(B_{jk})}.$$

Thus using (2.35), we have

$$\bar{z} + (t - \bar{t})(\bar{u} + \varepsilon) \leq z(t) \leq \bar{z} + \int_{\bar{t}}^t \|u\|_{L^\infty(B_{jk})} ,$$

so that

$$\bar{u} + \varepsilon \leq \frac{1}{t - \bar{t}} \int_{\bar{t}}^t \|u\|_{L^\infty(B_{jk})}.$$

If we let $t \searrow \bar{t}$ in this last inequality, we get

$$\bar{u} + \varepsilon \leq \|u\|_{L^\infty(B_{jk})}.$$

Since $B_{jk} \subset [\bar{z} - h, \ \bar{z} + h]$, this contradicts (2.36). This proves (2.33) and completes the proof of Lemma 2.6. $\quad \square$

**Corollary 2.1.** $\displaystyle\lim_{t \searrow \tau} \int_t^{t_1} \alpha(s)ds = \infty.$

*Proof.* With $w_1 < w_2$, $w_i \in (a, b)$, $i = 1, 2$, and $w_i(t)$ the corresponding functions $w$ as in the last lemma, we have

$$w_1(t) - w_2(t) = (w_1 - w_2) \exp\left( -\int_t^{t_1} \alpha(s)ds \right).$$

From Lemma 2.6

$$\lim_{t \searrow \tau} (w_1(t) - w_2(t)) = 0,$$

and the last equation gives the result. $\quad \square$

We now complete the proof of the theorem as follows. Let $c(t) \equiv w_1(t) < w_2(t) \equiv d(t)$ be two curves as in Lemma 2.6, corresponding to points $w_1$, $w_2$ respectively; then from Lemma 2.7,

$$0 \leq d(t) - c(t) \longrightarrow 0 \quad \text{as} \quad t \searrow \tau.$$

Define functions $\psi(x)$ and $\chi(x)$ as in Fig. 6, and define for $t \in (\tau, t_1]$,

$$w^\varepsilon(x, t) = \left( \alpha^\varepsilon(t)x + \beta^\varepsilon(t) \right)\chi(x),$$
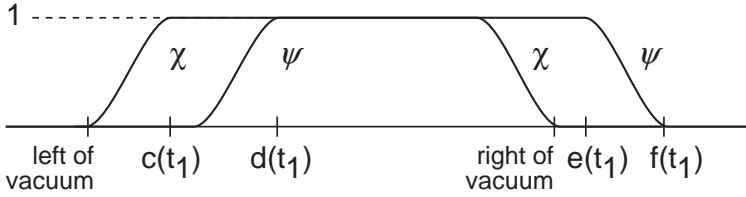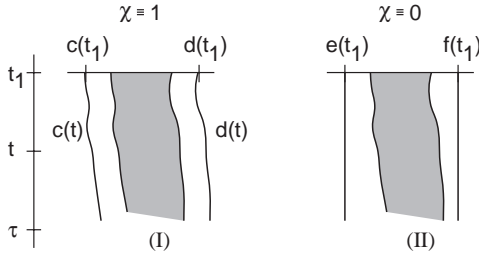
**Fig. 6.**



**Fig. 7.**

where $\alpha^\varepsilon$ and $\beta^\varepsilon$ are regularizations of $\alpha$ and $\beta$. Consider the initial-value problem

$$\phi_t^\varepsilon + w^\varepsilon \phi_x^\varepsilon = 0,$$
$$\phi^\varepsilon(x, t_1) = \psi(x). \tag{2.38}$$

Using Fig. 6, we see that $\phi^\varepsilon$ is a smooth compactly supported function. Thus from (1.2), we have, for $\tau < t < t_1$,

$$\int \rho u \phi^\varepsilon \Big|_t^{t_1} dx = \int \int \left[ \rho u (\phi_t^\varepsilon + u \phi_x^\varepsilon) + (P - \mu u_x) \phi_x^\varepsilon + \rho f \phi^\varepsilon \right]$$
$$= \int \int \left[ \rho u (u - w^\varepsilon) \phi_x^\varepsilon + (P - \mu u_x) \phi_x^\varepsilon + \rho f \phi^\varepsilon \right]. \tag{2.39}$$

Now $\phi^\varepsilon$ is constant along the characteristics of (2.38) so that the support of $\phi_x^\varepsilon$, in the region $[t, t_1]$, consists of two disjoint "strip-like" regions as depicted in Fig. 7. That is, the characteristics of (2.38) which start on (spt $\psi_x$) $\cap [c(t_1), d(t_1)]$ are given by $\dot{x} = \alpha^\varepsilon x + \beta^\varepsilon$, so for small $\varepsilon$ (depending on $t$) they stay between the curves $c(t)$ and $d(t)$; the corresponding support of $\phi_x^\varepsilon$ is the shaded region (I) in Fig. 7. Similarly the characteristics of (2.38) outside of the vacuum, which start on (spt $\psi_x$) $\cap [e(t_1), f(t_1)]$ are given by $\dot{x} = 0$; the corresponding support of $\phi_x^\varepsilon$ is depicted in Fig. 7 as the shaded region II.

We now consider (2.39). First, the left-hand side is bounded independent of $t$, for $\tau < t < t_1$ by virtue of (1.10). Similarly, the term

$$\int \int \rho f \phi^\varepsilon$$

is bounded because of (1.4). Also

$$\int \int_I \rho u (u - w^\varepsilon) \phi_x^\varepsilon = 0,$$

since $\rho = 0$ here. In II, $w^\varepsilon = 0$ and $\phi_x^\varepsilon = \psi_x$, so that in view of hypothesis (A$_3$),

$$\left| \int \int_{II} \rho u (u - w^\varepsilon) \phi_x^\varepsilon \right| = \left| \int \int_{II} \rho u^2 \psi_x \right|$$

$$\leq C \int \int \rho u^2 \leq C.$$

Next

$$\left| \int \int_{II} (P - \mu u_x) \phi_x^\varepsilon \right| \leq \left| \int \int_{II} (|P| + \mu |u_x|) |\psi_x| \right| \leq C$$

because of (A$_1$), (A$_2$), and (1.9). Since $P(0, x, t) = 0$ (by (1.3)) we have

$$\int \int_I (P - \mu u_x) \phi_x^\varepsilon = \int \int_I -\mu u_x \phi_x^\varepsilon$$

$$= -\int_t^{t_1} \int_{c(s)}^{d(s)} \mu u_x \phi_x^\varepsilon$$

$$= -\int_t^{t_1} \mu \alpha(s) \left[ \phi^\varepsilon(d(s), s) - \phi^\varepsilon(c(s), s) \right] ds$$

$$= -\int_t^{t_1} \mu \alpha(s) ds,$$

because $\phi^\varepsilon(d(s), s) = 1$ and $\phi^\varepsilon(c(s), s) = 0$. Thus from (2.41), we obtain that

$$\left| \int_t^{t_1} \alpha(s) ds \right|$$

is bounded, independent of $t$. Letting $t \searrow \tau$ contradicts Corollary 2.7. This completes the proof of the theorem.

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