Decay Rates and Probability Estimates for Massive Dirac Particles in the Kerr–Newman Black Hole Geometry

F. Finster[1,*], **N. Kamran**[2,**], **J. Smoller**[3,***], **S.-T. Yau**[4,†]

[1] Max Planck Institute for Mathematics in the Sciences, Inselstr. 22–26, 04103 Leipzig, Germany.
   E-mail: Felix.Finster@mis.mpg.de
[2] Department of Math. and Statistics, McGill University, Montréal, Québec, Canada, H3A 2K6.
   E-mail: nkamran@math.McGill.CA
[3] Mathematics Department, The University of Michigan, Ann Arbor, MI 48109, USA.
   E-mail: smoller@umich.edu
[4] Mathematics Department, Harvard University, Cambridge, MA 02138, USA.
   E-mail: yau@math.harvard.edu

Abstract: The Cauchy problem is considered for the massive Dirac equation in the non-extreme Kerr–Newman geometry, for smooth initial data with compact support outside the event horizon and bounded angular momentum. We prove that the Dirac wave function decays in L^∞_{loc} at least at the rate $t^{-5/6}$. For generic initial data, this rate of decay is sharp. We derive a formula for the probability p that the Dirac particle escapes to infinity. For various conditions on the initial data, we show that $p = 0, 1$ or $0 < p < 1$. The proofs are based on a refined analysis of the Dirac propagator constructed in [4].

Contents

1. Introduction

The Cauchy problem for the massive Dirac equation in the non-extreme Kerr–Newman black hole geometry outside the event horizon was recently studied [4], and it was proved that for initial data in L^∞_{loc} with L^2 decay at infinity, the probability for the Dirac particle to be located in any compact region of space tends to zero as $t \to \infty$. This result shows that the Dirac particle must eventually either disappear into the event horizon or escape to infinity. The questions of the likelihood of each of these possibilities and the rates of decay of the Dirac wave function in a compact region of space were left open. In the present paper, we shall study these questions by means of a detailed analysis of the integral representation of the Dirac propagator constructed in [4]. This analysis will also give us some insight into the physical mechanism which leads to the decay.

Recall that in Boyer–Lindquist coordinates $(t, r, \vartheta, \varphi)$ with $r > 0$, $0 \le \vartheta \le \pi$, $0 \le \varphi < 2\pi$, the Kerr–Newman metric is given by [2]

$$
\begin{aligned}
ds^2 &= g_{jk} dx^j x^k \\
&= \frac{\Delta}{U}(dt - a\sin^2\vartheta\, d\varphi)^2 - U\left(\frac{dr^2}{\Delta} + d\vartheta^2\right) \\
&\quad - \frac{\sin^2\vartheta}{U}(a\,dt - (r^2 + a^2)d\varphi)^2
\end{aligned}
\tag{1.1}
$$

with

$$
U(r, \vartheta) = r^2 + a^2\cos^2\vartheta, \qquad \Delta(r) = r^2 - 2Mr + a^2 + Q^2,
$$

and the electromagnetic potential is

$$
A_j dx^j = -\frac{Qr}{U}(dt - a\sin^2\vartheta\, d\varphi),
$$

where M, aM and Q denote the mass, the angular momentum and the charge of the black hole, respectively. Here a and/or Q are allowed to be zero, so that our results apply also to the Kerr, Reissner–Nordström, and Schwarzschild solutions. We shall restrict attention to the *non-extreme case* $M^2 > a^2 + Q^2$, in which case the function Δ has two distinct zeros,

$$
r_0 = M - \sqrt{M^2 - a^2 - Q^2} \qquad \text{and} \qquad r_1 = M + \sqrt{M^2 - a^2 - Q^2},
$$

corresponding to the Cauchy horizon and the event horizon, respectively. We will here consider only the region $r > r_1$ outside of the event horizon, and thus $\Delta > 0$.

Our starting point is the representation of the Dirac propagator for a Dirac particle of mass m and charge e established in [4, Thm. 3.6]

$$
\Psi(t, x) = \frac{1}{\pi} \sum_{k,n \in \mathbf{Z}} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} \sum_{a,b=1}^{2} t_{ab}^{k\omega n}\, \Psi_a^{k\omega n}(x) \langle \Psi_b^{k\omega n} | \Psi_0 \rangle.
\tag{1.2}
$$

Here Ψ_0 is the initial data, ω is the energy, and $\langle . | . \rangle$ is a positive scalar product (see [4] for details). The quantum number k arises from the usual separation $\sim \exp(-i(k + \frac{1}{2}))$ of the angular dependence around the axis of symmetry, whereas n labels the eigenvalues of generalized total angular momentum in Chandrasekhar's separation of the Dirac equation into ODEs [3]. The $\Psi_a^{k\omega n}$ are solutions of the Dirac equation and $t_{ab}^{k\omega n}$ are

complex coefficients; they can all be expressed in terms of the fundamental solutions to these ODEs. We postpone the detailed formulas for $\Psi_a^{k\omega n}$ and $t_{ab}^{k\omega n}$ to later sections, and here merely describe those qualitative properties of the wave functions $\Psi_a^{k\omega n}$ which are needed for understanding our results. Near the event horizon, the $\Psi_a^{k\omega n}$ go over asymptotically to spherical waves. In the region $|\omega| > m$, the solutions for $a = 1$ are the *incoming waves*, i.e. asymptotically near the event horizon they are waves moving towards the black hole. Conversely, the solutions for $a = 2$ are the *outgoing waves*, which near the event horizon move outwards, away from the black hole. Asymptotically near infinity, the $\Psi_a^{k\omega n}$, $|\omega| > m$, go over to spherical waves. In the region $|\omega| < m$, however, the fundamental solutions for $a = 1, 2$ near the event horizon are both linear combinations of incoming and outgoing waves, taken in such a way that $\Psi_1^{k\omega n}$ and $\Psi_2^{k\omega n}$ at infinity have exponential decay and growth, respectively.

For technical simplicity, we make the assumption that Ψ_0 is *smooth* and *compactly supported* outside the event horizon. We point out that, while the assumption of compact support is physically reasonable at infinity, it is indeed restrictive with respect to the behavior near the event horizon. Furthermore, we shall assume that the *angular momentum is bounded* in the strict sense that there exist constants k_0 and n_0 such that

$$\Psi(t, x) = \frac{1}{\pi} \sum_{|k| \leq k_0} \sum_{|n| \leq n_0} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \left[\sum_{a,b=1}^{2} t_{ab}^{k\omega n} \Psi_a^{k\omega n}(x) \langle \Psi_b^{k\omega n} | \Psi_0 \rangle \right]. \quad (1.3)$$

We expect that the rate of decay is the same if an infinite number of angular modes are present. Namely, away from the event horizon, modes with large angular momentum feel strong centrifugal forces and should therefore be quickly driven out to infinity, whereas the behavior near the event horizon is independent of the angular momentum. However, it seems a very delicate problem to rigorously establish decay rates without the assumption (1.3), because this would make it necessary to control the dependence of our estimates on k and n. Finally, we assume that the charge of the black hole is so small that the gravitational attraction is the dominant force at a large distance from the black hole. More precisely, we shall assume throughout this paper that

$$mM > |eQ|. \quad (1.4)$$

We now state our main results and discuss them afterwards.

Theorem 1.1 (Decay Rates). *Consider the Cauchy problem*

$$(i\gamma^j D_j - m)\Psi(t, x) = 0, \qquad \Psi(0, x) = \Psi_0(x)$$

for the Dirac equation in the non-extreme Kerr–Newman black hole geometry with small charge (1.4). Assume that the Cauchy data Ψ_0 is smooth with compact support outside the event horizon and has bounded angular momentum (1.3).

(i) *If for any k and n,*

$$\limsup_{\omega \searrow m} |\langle \Psi_2^{k\omega n} | \Psi_0 \rangle| \neq 0 \qquad or \qquad \limsup_{\omega \nearrow -m} |\langle \Psi_2^{k\omega n} | \Psi_0 \rangle| \neq 0, \quad (1.5)$$

then for large t,

$$|\Psi(t, x)| = ct^{-\frac{5}{6}} + \mathcal{O}(t^{-\frac{5}{6} - \varepsilon}), \quad (1.6)$$

with $c = c(x) \neq 0$ and any $\varepsilon < \frac{1}{30}$.

(ii) *If for all k, n and $a = 1, 2$,*

$$\langle \Psi_a^{k\omega n} | \Psi_0 \rangle = 0$$

for all ω in a neighborhood of $\pm m$, then $|\Psi(t, x)|$ has rapid decay in t (i.e. for any fixed x, $\Psi(t, x)$ decays in t faster than polynomially).

Theorem 1.2 (Probability Estimates). *Consider the Cauchy problem as in Theorem 1.1, with initial data Ψ_0 normalized by $\langle \Psi_0 | \Psi_0 \rangle = 1$. Let p be the probability for the Dirac particle to escape to infinity, defined for any $R > r_1$ by*

$$p = \lim_{t \to \infty} \int_{\{R < r < \infty\}} (\overline{\Psi} \gamma^j \Psi)(t, x) v_j d\mu, \tag{1.7}$$

where v denotes the future directed normal to the hypersurface $t = const$ and $d\mu$ denotes the induced invariant measure on that hypersurface. Then p is given by

$$p = \frac{1}{\pi} \sum_{|k| \leq k_0} \sum_{|n| \leq n_0} \int_{\mathbb{R} \setminus [-m, m]} d\omega \left(\frac{1}{2} - 2|t_{12}^{k\omega n}|^2 \right) \left| \langle \Psi_2^{k\omega n} | \Psi_0 \rangle \right|^2. \tag{1.8}$$

Accordingly, $1 - p$ gives the probability that the Dirac particles disappears into the event horizon. Furthermore,

(i)   *Suppose that the outgoing energy distribution for $|\omega| > m$ is non-zero, i.e.*

$$\langle \Psi_2^{k\omega n} | \Psi_0 \rangle \neq 0$$

      for some ω with $|\omega| > m$. Then $p > 0$.
(ii) *If the energy distribution of the Cauchy data has a non-zero contribution in the interval $[-m, m]$, then $p < 1$.*
(iii) *If the energy distribution of the Cauchy data is supported in $[-m, m]$, then $p = 0$.*
(iv) *If (1.5) holds, then $0 < p < 1$.*

The decay rate of $t^{-\frac{5}{6}}$ obtained in Theorem 1.1 quantifies the effect of the black hole's gravitational attraction on the long-time behavior of massive Dirac particles. Before discussing this effect in detail, it is instructive to recall the derivation of the decay rates in Minkowski space. We denote the plane-wave solutions of the Dirac equation by $\Psi_{ks\epsilon}$, where k is momentum, $\epsilon = \pm 1$ is the sign of energy, and $s = \pm$ refers to the two spin orientations. The plane-wave solutions are normalized according to

$$(\Psi_{ks\epsilon} | \Psi_{k's'\epsilon'}) = \delta(k - k') \delta_{ss'} \delta_{\epsilon\epsilon'},$$

where $(.|.)$ is the usual spatial scalar product

$$(\Psi | \Phi)(t) = \int \overline{\Psi(t, x)} \gamma^0 \Phi(t, x) dx.$$

The Dirac propagator is obtained by decomposing the initial data into the plane-wave solutions,

$$\Psi(t, x) = \sum_{s, \epsilon} \int dk \, \Psi_{ks\epsilon}(t, x)(\Psi_{ks\epsilon}(t = 0) | \Psi_0),$$

and a straightforward calculation using the explicit form of the plane-wave solutions yields that

$$\Psi(t, \boldsymbol{x}) = 2\pi \int \frac{d^4k}{(2\pi)^4} (\not{k} + m)\delta(k^2 - m^2)\Gamma(\omega)e^{-ikx}\gamma^0\hat{\Psi}_0(\boldsymbol{k}), \qquad (1.9)$$

where $\hat{\Psi}_0(\boldsymbol{k})$ is the Fourier transform of $\Psi_0(\boldsymbol{x})$ (and as usual $x = (t, \boldsymbol{x})$, $k = (\omega, \boldsymbol{k})$, $\not{k} = k_j\gamma^j$, and Γ is the step function $\Gamma(x) = \text{sgn}(x)$). Let us assume for simplicity that the initial data is a Schwartz function. We write (1.9) as a Fourier integral in ω,

$$\Psi(t, \boldsymbol{x}) = \int_{-\infty}^{\infty} \tilde{\Psi}(\omega, \boldsymbol{x})e^{-i\omega t}\, d\omega, \qquad (1.10)$$

where

$$\tilde{\Psi}(\omega, \boldsymbol{x}) = \int \frac{d\boldsymbol{k}}{(2\pi)^3} (\not{k} + m)\delta(\omega^2 - |\boldsymbol{k}|^2 - m^2)\Gamma(\omega)e^{i\boldsymbol{kx}}\gamma^0\hat{\Psi}_0(\boldsymbol{k}).$$

We consider the ω-dependence of $\tilde{\Psi}$ for fixed $\boldsymbol{x}$. The δ-distribution gives a contribution to the momentum integral only for $\boldsymbol{k}$ on the sphere $|\boldsymbol{k}|^2 = \omega^2 - m^2$. Thus $\tilde{\Psi}(\omega, \boldsymbol{x})$ vanishes for $|\omega| < m$ and has rapid decay at infinity. Furthermore, $\tilde{\Psi}$ is clearly smooth in the region $|\omega| > m$. For $|\omega|$ near m, $\tilde{\Psi}$ has the expansion

$$\tilde{\Psi}(\omega, \boldsymbol{x}) = \int_0^{\infty} \frac{k^2 dk}{4\pi^2} (\omega\gamma^0 + m)\delta(\omega^2 - k^2 - m^2)\Gamma(\omega)\gamma^0\hat{\Psi}_0(0)(1 + \mathcal{O}(k))$$

$$= \frac{\Gamma(\omega)}{8\pi^2}(\omega + m\gamma^0)\hat{\Psi}_0(0)\sqrt{\omega^2 - m^2} + \mathcal{O}(\omega^2 - m^2).$$

A typical plot of $|\tilde{\Psi}(\omega, \boldsymbol{x})|$ is shown in Fig. 1(a). If $\hat{\Psi}_0$ vanishes in a neighborhood of $\boldsymbol{k} = 0$, then $\tilde{\Psi}(., \boldsymbol{x})$ is a Schwartz function, and thus its Fourier transform (1.10) has rapid decay. This is the analogue of Case (ii) of Theorem 1.1. However, if $\hat{\Psi}_0(0) \neq 0$, the decay rate is determined by the square root behavior of $\tilde{\Psi}$ for $|\omega|$ near m. A change of variables gives that for any test function η which is supported in a neighborhood of the origin,

$$\int_m^{\infty} \sqrt{\omega - m}\, \eta(\omega - m)e^{-i\omega t}\, d\omega = e^{-imt}t^{-\frac{3}{2}}\int_0^{\infty} \sqrt{u}\, \eta\left(\frac{u}{t}\right)e^{-iu}\, du.$$

An integration-by-parts argument shows that the last integral is bounded uniformly in t, and is non-zero for large t if $\eta(0) \neq 0$. From this we conclude that in Minkowski space, $|\Psi(t, \boldsymbol{x})|$ decays polynomially at the rate $t^{-\frac{3}{2}}$.

We now proceed with a more detailed discussion of our results, beginning with the rates of decay obtained in Theorem 1.1. Naively speaking, a massive Dirac particle behaves near the event horizon similar to a massless particle, i.e. like a solution of the wave equation. In Minkowski space, solutions of the wave equation decay rapidly in time according to the Huygens principle. On the other hand, at large distance from the black hole the solutions should behave like those of the massive Dirac equation in Minkowski space, which decay at the rate $t^{-\frac{3}{2}}$. It is thus tempting to expect that the solutions of the massive Dirac equation in the Kerr–Newman black hole geometry should decay at a rate which "interpolates" between the behavior of a massive particle in Minkowski
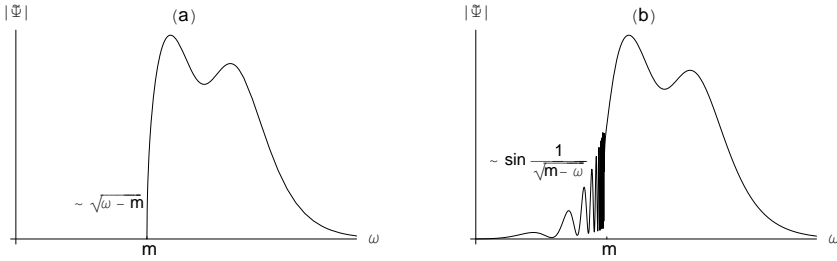
Fig. 1. Typical plot for $\tilde{\Psi}$ in Minkowski space (**a**) and in the Kerr–Newman black hole geometry (**b**)

space and that of a massless particle, and should thus decay at a rate no slower than $t^{-\frac{3}{2}}$. However, Theorem 1.1 shows that this naive picture is incorrect, since the rate of decay we have established for a massive Dirac particle in the Kerr–Newman black hole geometry is actually slower than that of a massive particle in Minkowski space. Thus the gravitational field of the black hole affects the behavior of massive Dirac particles in a more subtle way. One can understand this fact by comparing the plots in Fig. 1, which give typical examples for the energy distribution of the Dirac wave function in Minkowski space and in the Kerr–Newman geometry. One sees that in the Kerr–Newman geometry, there is a contribution to the energy distribution for $|\omega| < m$, which oscillates infinitely fast as ω approaches m. When taking the Fourier transform, these oscillations lead to the decay rate $t^{-\frac{5}{6}}$ given Theorem 1.1 (see the rigorous saddle point argument in Lemma 3.3).

The oscillations in the energy distribution in Fig. 1(b) are a consequence of the field behavior near spatial infinity. On a qualitative level, they can already be understood in Newtonian gravity and the semi-classical approximation. Namely, in the Newtonian limit of General Relativity, the momentum $\boldsymbol{k}$ of a relativistic particle is related to its energy ω by

$$|\boldsymbol{k}|^2 = \left(\omega + \frac{mM}{r}\right)^2 - m^2.$$

Thus the particle has positive momentum even if $\omega < m$, provided that the Newtonian potential is large enough, $\frac{mM}{r} > m - \omega$. This means in the semi-classical approximation that the wave function $\Psi(r)$ has an oscillatory behavior near the black hole,

$$\Psi(r) \sim \exp\left(\pm i \int^r \boldsymbol{k}\,d\boldsymbol{s}\right) \qquad \text{for} \qquad r < R \equiv \frac{mM}{m - \omega},$$

and will fall off exponentially for $r > R$. As a consequence, the fundamental solutions $\Psi_a^{k\omega n}$ for $|\omega| < m$ involve phase factors $\sim \exp(\pm i \int^R \boldsymbol{k}\,d\boldsymbol{s})$. In the limit $\omega \nearrow m$, $R \to \infty$, leading to infinitely fast oscillations in our integral representation. This simple argument even gives the correct quantitative behavior of the phases $\sim (m - \omega)^{-\frac{1}{2}}$.

The fact that the decay rate in the presence of a black hole is slower than in Minkowski space has the following direct physical interpretation. One can view the gravitational attraction of the black hole and the tendency of quantum mechanical wave functions to spread out in space as competing with each other over time. The component of the wave function for ω near m and $\omega < m$, which is responsible for the decay rate $t^{-\frac{5}{6}}$, has not

enough energy to propagate out to infinity. But since it is an outgoing wave near the event horizon (note that in (1.5) the fundamental solutions $\Psi_a^{k\omega n}$ enter only for $a = 2$), it is driven outwards and resists the gravitational attraction for a long time before it will eventually be drawn into the black hole. As a result, the Dirac particle stays in any compact region of space longer than it would in Minkowski space, and thus the rate of decay of the wave function is slower. According to this interpretation, our decay rates are a consequence of the far-field behavior of the black hole. Similar to the "power law tails" in the massless case (see [8]), our effect can be understood as a "backscattering" of the outgoing wave from the long-range potential, but clearly the rest mass drastically changes the behavior of the wave near infinity. We expect that result for the decay rates should be valid even in a more general setting, independent of the details of the local geometry near the event horizon. Furthermore, the decay rates should be independent of the spin. This view is supported by [6, 7], who obtained the rate $t^{-\frac{5}{6}}$ for massive scalar fields in a spherically symmetric geometry using asymptotic expansions of the Green's functions.

Theorem 1.2 gives a precise formula for the probability that the Dirac particle either disappears into the black hole or escapes to infinity. In cases **(i)**–**(iv)** we give sufficient conditions for these probabilities to occur. These results are consistent with the general behavior of quantum mechanical particles in the presence of a potential barrier and can be thought of as a tunnelling effect. In case **(iii)**, the particle does not have enough energy to escape to infinity. Thinking again in terms of a tunnelling effect, the Dirac particle cannot tunnel to infinity because the potential barrier (which has finite height m) has infinite width. Finally, one might ask whether $p = 1$ can occur; i.e., that the particles escape to infinity with probability one. This is indeed the case for very special initial data, whose energy distribution is supported outside the interval $[-m, m]$ (see Corollary 9.3 below).

We conclude by remarking that a number of significant results are known for the long-term behavior of *massless* fields in black hole geometries. These results do not capture our effect, which is intimately related to the presence of a mass gap in the energy spectrum. Price [8] discussed the rates of decay of massless fields in the Schwarzschild background for special choices of initial data. His decay rates depend on the angular momentum and are faster than the ones we have derived. A rigorous proof of the boundedness of the solutions of the wave equation in the Schwarzschild geometry has been given by Kay and Wald [5]. Beyer pursues an approach using C^0-semigroup theory, which also applies to the Kerr metric and the massive case [1]. An important contribution to the long-time behavior of gravitational perturbations of the Kerr metric has been given by Whiting [9].

2. The Long-Time Dynamics Under a Spectral Condition

We begin the analysis with the case when the energy distribution of the Cauchy data is zero in a neighborhood of $\omega = \pm m$. The following theorem is an equivalent formulation of Theorem 1.1(ii).

Theorem 2.1. *Consider the Cauchy problem*

$$(i\gamma^j D_j - m)\Psi(t, x) = 0, \qquad \Psi(0, x) = \Psi_0(x)$$

for smooth initial data with compact support outside the event horizon. Assume that angular momentum is bounded and that the energy is supported away from $\omega = \pm m$,

i.e.

$$\Psi_0 = \frac{1}{\pi} \sum_{|k| \leq k_0} \sum_{|n| \leq n_0} \left(\int_{-\infty}^{-m-\varepsilon} + \int_{-m+\varepsilon}^{m-\varepsilon} + \int_{m+\varepsilon}^{\infty} \right) d\omega \sum_{a,b=1}^{2} t_{ab}^{k\omega n} \Psi_a^{k\omega n} \langle \Psi_b^{k\omega n} | \Psi_0 \rangle \quad (2.1)$$

for suitable $\varepsilon > 0$. Then for all x, $\Psi(t, x)$ has rapid decay in t.

Before giving the proof, we recall a few basic formulas from [4]. The separation ansatz for the fundamental solutions $\Psi_a^{k\omega n}$ is

$$\Psi_a^{k\omega n}(t, r, \vartheta, \varphi) = e^{-i\omega t} e^{-i(k+\frac{1}{2})\varphi} \begin{pmatrix} X_-^{k\omega n}(r) Y_-^{k\omega n}(\vartheta) \\ X_+^{k\omega n}(r) Y_+^{k\omega n}(\vartheta) \\ X_+^{k\omega n}(r) Y_-^{k\omega n}(\vartheta) \\ X_-^{k\omega n}(r) Y_+^{k\omega n}(\vartheta) \end{pmatrix}, \quad (2.2)$$

where $X = (X_+, X_-)$ and $Y = (Y_+, Y_-)$ are the radial and angular components, respectively. The radial part $X(u)$ is a solution of the radial Dirac equation [4, Eq. (3.7)]

$$\left[\frac{d}{du} + i\Omega(u) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] X = \frac{\sqrt{\Delta}}{r^2 + a^2} \begin{pmatrix} 0 & imr - \lambda \\ -imr - \lambda & 0 \end{pmatrix} X, \quad (2.3)$$

where

$$\Omega(u) = \omega + \frac{(k + \frac{1}{2})a + eQr}{r^2 + a^2}, \qquad \Delta = r^2 - 2Mr + a^2 + Q^2,$$

λ is the angular eigenvalue (which depends smoothly on ω), and $u \in (-\infty, \infty)$ is related to the radius by

$$\frac{du}{dr} = \frac{r^2 + a^2}{\Delta}. \quad (2.4)$$

To analyze X in the asymptotic region $u \to -\infty$, one employs for X the ansatz

$$X(u) = \begin{pmatrix} e^{-i\Omega_0 u} f^+(u) \\ e^{i\Omega_0 u} f^-(u) \end{pmatrix} \quad (2.5)$$

and obtains for f the equation

$$\frac{d}{du} f = \left[i(\Omega_0 - \Omega(u)) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right.$$

$$\left. + \frac{\sqrt{\Delta}}{r^2 + a^2} \begin{pmatrix} 0 & e^{-2i\Omega u}(imr - \lambda) \\ e^{2i\Omega u}(-imr - \lambda) & 0 \end{pmatrix} \right] f. \quad (2.6)$$

Standard Gronwall estimates yield that the fundamental solutions of (2.3) have the asymptotic form [4, Lemma 3.1]

$$X_a(u) = \begin{pmatrix} e^{-i\Omega_0 u} f_{0a}^+ \\ e^{i\Omega_0 u} f_{0a}^- \end{pmatrix} + R_0(u), \quad (2.7)$$

where $|R_0(u)| \le c \exp(du)$ for suitable constants $c, d > 0$ and

$$f_{01} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad f_{02} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.8}$$

In the asymptotic region $u \to \infty$, one transforms the spinor basis with a matrix $B(u)$ such that the matrix potential in (2.3) becomes equal to the diagonal matrix $-i\Omega(u)\sigma^3$ (σ^j are the Pauli matrices). One must distinguish between the two cases $|\omega| < m$ and $|\omega| > m$. In the first case, $\Omega(u)$ is imaginary for large u, and thus there are two fundamental solutions X_1 and X_2 with exponential decay and growth, respectively, and we normalize them such that

$$\lim_{u \to -\infty} |X(u)| = 1. \tag{2.9}$$

In the case $|\omega| > m$, $\Omega(u)$ is real for all u. The ansatz

$$X = B \begin{pmatrix} e^{-i\Phi} f^+(u) \\ e^{i\Phi} f^-(u) \end{pmatrix} \qquad \text{with} \qquad \Phi'(u) = \Omega(u) \tag{2.10}$$

gives the differential equation

$$\frac{d}{du} f = M(u) f \qquad \text{with} \qquad |M(u)| \le \frac{c}{u^2}, \tag{2.11}$$

which can again be controlled by Gronwall estimates. Thus one obtains the asymptotic formula [4, Lemma 3.5]

$$X_a(u) = \begin{pmatrix} \cosh\Theta & \sinh\Theta \\ \sinh\Theta & \cosh\Theta \end{pmatrix} \begin{pmatrix} e^{-i\Phi(u)} f^+_{\infty a} \\ e^{i\Phi(u)} f^-_{\infty a} \end{pmatrix} + R_\infty(u), \tag{2.12}$$

where $|R_\infty| \le c/u$ for suitable $c > 0$ and

$$\Theta = \frac{1}{4} \log\left(\frac{\omega+m}{\omega-m}\right), \qquad \Phi = \Gamma(\omega)\left(\sqrt{\omega^2-m^2}u + \frac{\omega e Q + M m^2}{\sqrt{\omega^2-m^2}} \log u\right). \tag{2.13}$$

The complex factors $f^\pm_{\infty a}$ in (2.12) are the so-called *transmission coefficients*. Furthermore, we introduced the functions $t_a(\alpha)$, $0 \le \alpha \le 2\pi$, in terms of the transmission coefficients by [4, Eq. (3.47)]

$$t_1(\alpha) = f^+_{\infty 2} e^{-i\alpha} - f^-_{\infty 2} e^{i\alpha}, \qquad t_2(\alpha) = -f^+_{\infty 1} e^{-i\alpha} + f^-_{\infty 1} e^{i\alpha}. \tag{2.14}$$

Finally, the coefficients $(t_{ab})_{a,b=1,2}$ are given by

$$t_{ab} = \begin{cases} \delta_{a1}\delta_{b1} & \text{if } |\omega| \le m \\ \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \dfrac{t_a \overline{t_b}}{|t_1|^2 + |t_2|^2} d\alpha & \text{if } |\omega| > m. \end{cases} \tag{2.15}$$

Proof of Theorem 2.1. Since (2.1) contains only finite sums, we can fix k, n and consider one summand. The coefficients in the differential equation (2.6) are smooth in ω, and their ω-derivatives are integrable on the half-lines $u \in (-\infty, u_0]$ for u_0 sufficiently small. Apart from the singularities at $\omega = \pm m$, the same is true for the differential equation (2.11) for u on the half line $[u_1, \infty)$ and u_1 sufficiently large. Since furthermore the ansatz (2.5) is smooth in $\omega \in \mathbb{R} \setminus ((-m - \varepsilon, -m + \varepsilon) \cup (m - \varepsilon, m + \varepsilon))$

and (2.10) is smooth in $\omega \in (-\infty, -m - \varepsilon] \cup [m + \varepsilon, \infty)$, and also the coefficients of the differential equation (2.3) depend smoothly on ω in the bounded interval $u \in [u_0, u_1]$, we conclude that the fundamental solutions $\Psi_a^{k_0 \omega n_0}(x)$ and the transmission coefficients $f_a^{k_0 \omega n_0}$ depend smoothly on $\omega \in \mathbb{R} \setminus ((-m - \varepsilon, -m + \varepsilon) \cup (m - \varepsilon, m + \varepsilon))$. Hence the integrand in (2.1) is a smooth function in ω (which vanishes for $|\omega| - m < \varepsilon$). Since Ψ_0 has compact support and the fundamental solutions for $\omega \to \infty$ go over to plane waves, it is clear that the ω-derivatives of the integrand in (2.1) are all integrable. It follows that the Fourier integral (1.2) has rapid decay. $\quad \square$

According to this theorem and using linearity, it remains to analyze the energy distribution in a neighborhood of $\omega = \pm m$. Since all constructions and estimates are similar for positive and negative ω, we can in what follows restrict attention to a neighborhood of $\omega = +m$.

3. Decay Rates of Fourier Transforms – Basic Considerations

In this section, we derive estimates of some elementary Fourier integrals. Our decay rate $t^{-\frac{5}{6}}$ ultimately comes from Lemma 3.3. We always denote by ε a parameter in the range $0 < \varepsilon < \frac{1}{30}$.

Lemma 3.1. *Let $g \in L^\infty(\mathbb{R}) \cap C^1((0, \infty))$ with compact support and assume that for a suitable constant C,*

$$|g'(\alpha)| \leq \frac{C}{\alpha} \qquad \text{for all } \alpha > 0. \tag{3.1}$$

Then there is a constant $c = c(g)$ such that for all $t > 0$,

$$\left| \int_0^\infty e^{i\alpha t} g(\alpha) d\alpha \right| \leq c t^{-\frac{5}{6} - \varepsilon}.$$

Proof. Assume that $\text{supp} g \subset [-L, L]$. For given $\delta > 0$, we split up the integral as

$$\int_0^\infty e^{i\alpha t} g(\alpha) d\alpha = \int_0^\delta e^{i\alpha t} g(\alpha) d\alpha + \int_\delta^\infty e^{i\alpha t} g(\alpha) d\alpha.$$

The first term can be estimated by

$$\left| \int_0^\delta e^{i\alpha t} g(\alpha) d\alpha \right| \leq c_1 \delta$$

with $c_1 = \sup |g|$. In the second term, we integrate by parts,

$$\int_\delta^\infty e^{i\alpha t} g(\alpha) d\alpha = \frac{1}{it} \int_\delta^\infty \left(\frac{d}{d\alpha} e^{i\alpha t} \right) g(\alpha) d\alpha$$

$$= -\frac{1}{it} e^{i\delta t} g(\delta) - \frac{1}{it} \int_\delta^\infty e^{i\alpha t} g'(\alpha) d\alpha,$$

and estimate using (3.1),

$$\left| \int_\delta^\infty e^{i\alpha t} g(\alpha) d\alpha \right| \leq \frac{c_1}{t} + \frac{C}{t} (\log L - \log \delta).$$

We choose $\delta = t^{-\frac{5}{6}-\varepsilon}$ to conclude that

$$\left| \int_0^\infty e^{i\alpha t} g(\alpha) d\alpha \right| \leq c_1 t^{-\frac{5}{6}-\varepsilon} + \frac{c_1}{t} + \frac{C}{t}(\log L - \log t^{-\frac{5}{6}-\varepsilon}),$$

and this has the required decay properties in t. $\square$

In the next lemma we insert into the Fourier integral a phase factor which oscillates infinitely fast as $\alpha \searrow 0$.

Lemma 3.2. *Let g be as in Lemma 3.1. Then there is a constant $c = c(g)$ such that for all $t > 0$,*

$$\left| \int_0^\infty \exp\left(i\alpha t - \frac{i}{\sqrt{\alpha}} \right) g(\alpha) d\alpha \right| \leq ct^{-\frac{5}{6}-\varepsilon}. \tag{3.2}$$

Proof. We set

$$\phi(\alpha) = \alpha t - \frac{1}{\sqrt{\alpha}}.$$

Then

$$\phi'(\alpha) = t + \frac{1}{2}\alpha^{-\frac{3}{2}}, \qquad \phi''(\alpha) = -\frac{3}{4}\alpha^{-\frac{5}{2}}. \tag{3.3}$$

We integrate the Fourier integrals by parts,

$$\int_0^\infty e^{i\phi(\alpha)} g(\alpha) d\alpha = -i \int_0^\infty \left(\frac{d}{d\alpha} e^{i\phi(\alpha)} \right) \frac{g}{\phi'} d\alpha = i \int_0^\infty e^{i\phi(\alpha)} \left(\frac{g'}{\phi'} - \frac{g\phi''}{\phi'^2} \right) d\alpha,$$

and obtain the bound

$$\left| \int_0^\infty e^{i\phi(\alpha)} g(\alpha) d\alpha \right| \leq \int_0^\infty \left| \frac{g'}{\phi'} - \frac{g\phi''}{\phi'^2} \right| d\alpha$$

$$= \int_0^\infty |\phi'|^{-\frac{5}{6}-\varepsilon} \left\{ |g'||\phi'|^{-\frac{1}{6}+\varepsilon} + |g||\phi''||\phi'|^{-\frac{7}{6}+\varepsilon} \right\} d\alpha.$$

According to (3.3), we can estimate the factor $|\phi'|^{-\frac{5}{6}-\varepsilon}$ from above by $t^{-\frac{5}{6}-\varepsilon}$, whereas for the factors $|\phi'|$ in the curly brackets we use the bound $|\phi'| \geq \frac{1}{2}\alpha^{-\frac{3}{2}}$. Furthermore, we substitute in the formula for ϕ'' in (3.3) and obtain

$$\left| \int_0^\infty e^{i\phi(\alpha)} g(\alpha) d\alpha \right| \leq t^{-\frac{5}{6}-\varepsilon} \int_0^\infty \left(c_1 |g'|\alpha^{\frac{1}{4}-\frac{3}{2}\varepsilon} + c_2 |g|\alpha^{-\frac{3}{4}-\frac{3}{2}\varepsilon} \right) d\alpha$$

with two constants c_1 and c_2. Using that g is in L^∞ and g' satisfies the bound (3.1), one sees that the pole in the last integrand is integrable. $\square$

The following lemma deals with the Fourier integral when we replace the minus sign in the integrand of (3.2) by a plus sign. Reversing this sign completely changes the long-time asymptotics. We estimate the Fourier integral using a rigorous version of the "saddle point method".

Lemma 3.3. *Let g be as in Lemma 3.1. Then there are constants $c = c(g)$ and $c_1 = 2^{-\frac{4}{3}}3^{-\frac{1}{2}}\sqrt{\pi}$ such that for all sufficiently large t,*

$$\left| \int_0^\infty \exp\left(i\alpha t + \frac{i}{\sqrt{\alpha}} \right) g(\alpha)d\alpha - c_1 e^{i\phi_0} g(\alpha_0) t^{-\frac{5}{6}} \right| \le ct^{-\frac{5}{6}-\varepsilon}, \tag{3.4}$$

where α_0 and ϕ_0 are given by

$$\alpha_0 = (2t)^{-\frac{2}{3}}, \qquad \phi_0 = \left(\frac{9}{4}t \right)^{\frac{1}{3}}. \tag{3.5}$$

Proof. We introduce the function ϕ by

$$\phi(\alpha) = \alpha t + \frac{1}{\sqrt{\alpha}}.$$

Then

$$\phi'(\alpha) = t - \frac{1}{2}\alpha^{-\frac{3}{2}}, \qquad \phi''(\alpha) = \frac{3}{4}\alpha^{-\frac{5}{2}}. \tag{3.6}$$

One sees that $\phi(\alpha)$ has a minimum at α_0 with $\phi(\alpha_0) = \phi_0$ and

$$\phi''(\alpha_0) = 2^{-\frac{1}{3}}3t^{\frac{5}{3}}. \tag{3.7}$$

We set

$$\delta = t^{-\frac{5}{6}+\varepsilon}.$$

For large t, $\delta < \alpha_0$. We split the integration range into two regions D_1 and D_2 with

$$D_1 = (0, \alpha_0 - \delta) \cup (\alpha_0 + \delta, \infty), \qquad D_2 = [\alpha_0 - \delta, \alpha_0 + \delta].$$

Let us first estimate the integral over D_1. An integration-by-parts argument similar to that in the proof of Lemma 3.2 gives

$$\left| \int_{D_1} e^{i\phi} g\, d\alpha \right| \le \left| \frac{g(\alpha_0 + \delta)}{\phi'(\alpha_0 + \delta)} \right| + \left| \frac{g(\alpha_0 - \delta)}{\phi'(\alpha_0 - \delta)} \right| + \int_{D_1} \left| \frac{g'}{\phi'} - \frac{g\phi''}{\phi'^2} \right| d\alpha.$$

Putting in the formulas for ϕ' and ϕ'' given in (3.6), and using that $g \in L^\infty$ together with (3.1), one sees that for suitable c,

$$\left| \int_{D_1} e^{i\phi} g\, d\alpha \right| \le ct^{-\frac{5}{6}-\varepsilon}.$$

Next we show that the leading contribution to the integral over D_2 is given by the saddle point approximation. To this end, we introduce the quadratic polynomial

$$\phi_S(\alpha) = \phi_0 + \frac{1}{2}\phi''(\alpha_0)(\alpha - \alpha_0)^2.$$

Then the mean value theorem gives for sufficiently large t,

$$\left| \int_{D_2} \left(e^{i\phi} g - e^{i\phi_S} g(\alpha_0) \right) d\alpha \right| \leq \sup_{D_2} (|g'| + |(\phi - \phi_S)'g|) \, \delta^2$$

$$\leq \left(\frac{2C}{\alpha_0} + \frac{1}{2} \sup_{D_2} |\phi'''| \delta^2 \|g\|_\infty \right) \delta^2 \leq \left(c_2 t^{-1+2\varepsilon} + c_3 t^{-1+4\varepsilon} \right) \leq c t^{-\frac{5}{6} - \varepsilon},$$

where in the last step we used that $5\varepsilon < \frac{1}{6}$. Finally, we compute the contribution of the saddle point approximation,

$$\int_{D_2} e^{i\phi_S} g(\alpha_0) d\alpha = e^{i\phi_0} g(\alpha_0) \int_{-\delta}^{\delta} e^{\frac{i}{2}\phi''(\alpha_0)\alpha^2} d\alpha.$$

Introducing the new variable $s = \frac{1}{2}\phi''(\alpha_0)\alpha^2$ gives

$$\int_{D_2} e^{i\phi_S} g(\alpha_0) d\alpha = e^{i\phi_0} g(\alpha_0) \sqrt{\frac{2}{\phi''(\alpha_0)}} \int_0^L \frac{e^{is}}{\sqrt{s}} ds$$

with

$$L = \frac{1}{2}\phi''(\alpha_0)\delta^2 = 2^{-\frac{4}{3}} 3 t^{2\varepsilon}. \tag{3.8}$$

Using (3.7), we conclude that

$$\int_{D_2} e^{i\phi_S} g(\alpha_0) d\alpha = e^{i\phi_0} g(\alpha_0) t^{-\frac{5}{6}} (c_1 + R(t))$$

with

$$c_1 = 2^{\frac{1}{3}} 3^{-\frac{1}{2}} \int_0^\infty \frac{e^{is}}{\sqrt{s}} ds = 2^{\frac{4}{3}} 3^{-\frac{1}{2}} \sqrt{\pi} \neq 0$$

$$R(t) = -2^{\frac{1}{3}} 3^{-\frac{1}{2}} \int_L^\infty \frac{e^{is}}{\sqrt{s}} ds.$$

The error term $R(t)$ can be integrated by parts,

$$\left| \int_L^\infty \frac{e^{is}}{\sqrt{s}} ds \right| \leq \frac{1}{\sqrt{L}} + \frac{1}{2} \int_L^\infty s^{-\frac{3}{2}} ds = \frac{5}{4} \frac{1}{\sqrt{L}},$$

and this shows according to (3.8) that $R(t)$ decays in t at the desired rate, $|R(t)| \leq c t^{-\varepsilon}$.
□

4. The Planar Equation

Let us transform the radial Dirac equation (2.3) into an equation for a real 2-spinor as follows. We first unitarily transform the spinor X according to

$$X \to \tilde{X} = UX \qquad \text{with} \quad U = \exp\left(i\frac{\beta}{2}\sigma^3\right), \qquad \beta = \arctan\frac{\lambda}{mr}. \qquad (4.1)$$

Then $\tilde{X}$ satisfies the equation

$$\frac{d}{du}\tilde{X} = i\begin{pmatrix} -a & b \\ -b & a \end{pmatrix}\tilde{X} \qquad (4.2)$$

with

$$a(u) = \Omega(u) + \frac{\lambda m}{m^2 r^2 + \lambda^2}\frac{\Delta}{2(r^2 + a^2)}, \qquad (4.3)$$

$$b(u) = \frac{\sqrt{\Delta}}{r^2 + a^2}\sqrt{m^2 r^2 + \lambda^2}. \qquad (4.4)$$

Notice that the transformation U is regular for all $u \in \mathbb{R}$, and that the second summand in (4.3) has nice decay properties for $u \to \pm\infty$. Next we employ the ansatz

$$\tilde{X} = \begin{pmatrix} \Gamma(a)\psi^+ - i\psi^- \\ -\psi^+ - i\Gamma(a)\psi^- \end{pmatrix}, \qquad \psi = \frac{1}{2}\begin{pmatrix} \Gamma(a)\tilde{X}^+ - \tilde{X}^- \\ i\tilde{X}^+ + i\Gamma(a)\tilde{X}^- \end{pmatrix} \qquad (4.5)$$

with a complex 2-spinor ψ. Then ψ satisfies the equation

$$\frac{d}{du}\psi = \begin{pmatrix} 0 & -g \\ f & 0 \end{pmatrix}\psi, \qquad (4.6)$$

with

$$f = |a| + b, \qquad g = |a| - b. \qquad (4.7)$$

The coefficients in (4.6) are all real, and so we can study the real and imaginary parts of ψ separately. Thus we assume in what follows that ψ is real and then call (4.6) the *planar equation*.

We bring the planar equation into a form more appropriate for our estimates. For given u_0 we introduce the new variable

$$x(u) = 2\int_{u_0}^u \sqrt{|fg|}(\tau)d\tau \qquad (4.8)$$

and set

$$h = \frac{1}{2}\log\left|\frac{g}{f}\right|. \qquad (4.9)$$

In the case $g > 0$, (4.6) becomes

$$\psi' \equiv \frac{d}{dx}\psi = \frac{1}{2}\begin{pmatrix} 0 & -e^h \\ e^{-h} & 0 \end{pmatrix}\psi.$$

Employing the ansatz

$$\psi = e^{-\frac{L}{2}} \begin{pmatrix} e^{\frac{h}{2}} \cos \frac{x+\vartheta}{2} \\ e^{-\frac{h}{2}} \sin \frac{x+\vartheta}{2} \end{pmatrix} \tag{4.10}$$

with real functions $L(x)$ and $\phi(x)$ gives the equation

$$-L'a_1 + h'b + (1 + \vartheta')a_2 = a_2 \tag{4.11}$$

with

$$a_1 = \begin{pmatrix} \cos \frac{x+\vartheta}{2} \\ \sin \frac{x+\vartheta}{2} \end{pmatrix}, \qquad a_2 = \begin{pmatrix} -\sin \frac{x+\vartheta}{2} \\ \cos \frac{x+\vartheta}{2} \end{pmatrix}, \qquad b = \begin{pmatrix} \cos \frac{x+\vartheta}{2} \\ -\sin \frac{x+\vartheta}{2} \end{pmatrix}.$$

Elementary trigonometry shows that

$$b = \cos(x + \vartheta)a_1 - \sin(x + \vartheta)a_2.$$

Hence the planar equation takes the form

$$\vartheta' = h' \sin(x + \vartheta), \qquad L' = h' \cos(x + \vartheta). \tag{4.12}$$

In the case $g < 0$, the ansatz

$$\psi = e^{-\frac{L}{2}} \begin{pmatrix} e^{\frac{h}{2}} \cosh \frac{x+\vartheta}{2} \\ e^{-\frac{h}{2}} \sinh \frac{x+\vartheta}{2} \end{pmatrix} \tag{4.13}$$

gives similarly the equations

$$\vartheta' = h' \sinh(x + \vartheta), \qquad L' = h' \cosh(x + \vartheta). \tag{4.14}$$

We can now give the strategy for the proof of Theorem 1.1(**i**). First, in the next section, we will obtain estimates which will enable us to control the function h' which appears in the planar equations (4.12) and (4.14). Then we will carefully analyze the solutions (ϑ, L) of these planar equations, and this will allow us to study the time-dependence of the propagator (1.3). For the analysis of the planar equations, it is necessary to consider both cases $\omega > m$ and $\omega < m$ separately; this will be done in Sects. 6 and 7, respectively.

5. Uniform Bounds for the Potentials

In this section, we shall derive estimates for the function $h(x)$ (as introduced in (4.9) with x according to (4.8)) as well as for its partial derivatives with respect to x and ω. The usefulness of our estimates lies in the fact that they are uniform in ω for ω in a small neighborhood of m, $\omega \in (m - \delta, m + \delta)$. The main technical difficulty is that x is defined via an integral transformation (4.8), and thus $h(x)$ depends on ω in a nonlocal way. On the other hand, our estimates also show the advantage of working with the variable x. Namely, by introducing x, the ω-dependence of h becomes small in the critical regions near infinity and near the poles of h, in the sense that $(\omega - m)\partial_\omega h(x)$ has bounded total variation in x, uniformly in ω. This will be essential for getting control of the ω-dependence of the solutions to the planar equation (see Lemmas 6.3 and 7.5). Since the

technical details of the proofs of Lemmas 5.1, 5.2, and 5.3 will not be needed later on, the reader may consider skipping these proofs in a first reading.

In what follows, we often denote derivatives by a lower index, e.g. $h_\omega \equiv \partial_\omega h$. Furthermore, we denote constants which are independent of ω by c; the value of c may change throughout our calculations. For clarity, we sometimes add a subscript to c to mean a fixed constant. According to their definition (4.7) and (4.3),(4.4), the functions f and g have for large u the expansion

$$f = (\omega + m) - \frac{mM - eQ}{u} + \mathcal{O}\left(\frac{1}{u^2}\right),$$

$$g = (\omega - m) + \frac{mM + eQ}{u} + \mathcal{O}\left(\frac{1}{u^2}\right).$$

Our notation $\mathcal{O}(u^{-n})$ implies that the error terms depend smoothly on ω, and that their u-derivatives have the natural scaling behavior, i.e.

$$\partial_\omega \mathcal{O}(u^{-n}) = \mathcal{O}(u^{-n}) \qquad \text{and} \qquad \partial_u \mathcal{O}(u^{-n}) = \mathcal{O}(u^{-n-1}).$$

Our assumption (1.4) ensures that for large u, g is monotone decreasing, whereas f is increasing.

We begin with the case $\omega > m$. In this parameter range, we fix u_0 independent of ω. By choosing u_0 sufficiently large, we can arrange that the following estimates hold.

Lemma 5.1. *There are constants $c, \delta > 0$ such that for all $\omega \in (m, m + \delta)$ and $x > 0$,*

$$0 < -h'(x) \le \frac{c}{1 + x}, \tag{5.1}$$

$$|h''(x)| \le \frac{c}{(1 + x)^2}, \tag{5.2}$$

$$\int_0^\infty |h'_\omega(x)| dx \le \frac{c}{\omega - m}. \tag{5.3}$$

Proof. We set $\varepsilon = \omega^2 - m^2$ and introduce the function

$$\rho = 2\sqrt{fg}.$$

Then h and ρ have the asymptotic expansions

$$h(u) = \frac{1}{2} \log\left(\frac{\varepsilon}{(\omega + m)^2} + \frac{\alpha}{u} + \mathcal{O}\left(\frac{1}{u^2}\right)\right), \tag{5.4}$$

$$\rho(u) = 2\sqrt{\varepsilon + \frac{\beta}{u} + \mathcal{O}\left(\frac{1}{u^2}\right)} \tag{5.5}$$

with positive constants α and β, which depend smoothly on ω and are bounded away from zero as $\varepsilon \to 0$. Our first step is to bound the function $x(u)$, (4.8), as well as its inverse $u(x)$. According to (5.5), there are (possibly after increasing u_0), constants $a_1, a_2 > 0$ such that

$$2\sqrt{\varepsilon + \frac{a_1}{u}} \le \rho(u) \le 2\sqrt{\varepsilon + \frac{a_2}{u}} \tag{5.6}$$

for all $\omega \in (m, m + \delta)$ and $u > u_0$. We introduce the functions $\underline{x}$ and $\overline{x}$ by

$$\underline{x} = 2\sqrt{u(a_1 + \varepsilon u)} - b_1, \qquad \overline{x} = 4\sqrt{u(a_2 + \varepsilon u)} - b_2,$$

where the constants b_1 and b_2 are chosen such that $\underline{x}(u_0) = 0 = \overline{x}(u_0)$,

$$b_1 = 2\sqrt{u_0(a_1 + \varepsilon u_0)}, \qquad b_2 = 4\sqrt{u_0(a_2 + \varepsilon u_0)}. \tag{5.7}$$

Then

$$\underline{x}'(u) = \frac{a_1 + 2\varepsilon u}{\sqrt{u(a_1 + \varepsilon u)}} \leq 2\sqrt{\frac{a_1 + \varepsilon u}{u}} \leq \rho(u),$$

$$\overline{x}'(u) = \frac{2(a_2 + 2\varepsilon u)}{\sqrt{u(a_2 + \varepsilon u)}} \geq 2\sqrt{\frac{a_2 + \varepsilon u}{u}} \geq \rho(u),$$

and integration yields that $\underline{x}$ and $\overline{x}$ are bounds for x,

$$\underline{x}(u) \leq x(u) \leq \overline{x}(u) \qquad \text{for all } u \geq u_0. \tag{5.8}$$

The functions $\underline{x}$ and $\overline{x}$ are strictly monotone and thus invertible. Their inverses are computed as follows,

$$\underline{x}^{-1}(x) = \frac{1}{2\varepsilon}\left(\sqrt{a_1^2 + \varepsilon(x + b_1)^2} - a_1\right)$$

$$= \frac{1}{2\varepsilon}\frac{\varepsilon(x + b_1)^2}{\sqrt{a_1^2 + \varepsilon(x + b_1)^2} + a_1} \leq \frac{1}{2}\frac{(x + b_1)^2}{\sqrt{\varepsilon}(x + b_1) + a_1},$$

$$\overline{x}^{-1}(x) = \frac{1}{4\varepsilon}\left(\sqrt{4a_2^2 + \varepsilon(x + b_2)^2} - 2a_2\right)$$

$$= \frac{1}{4\varepsilon}\frac{\varepsilon(x + b_2)^2}{\sqrt{4a_2^2 + \varepsilon(x + b_2)^2} + 2a_2} \geq \frac{1}{4}\frac{(x + b_2)^2}{\sqrt{\varepsilon}(x + b_2) + 4a_2},$$

where in the last step we applied the inequality $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$ $(a, b > 0)$. The inequalities (5.8) yield for the inverses that $\underline{x}^{-1}(x) \geq u(x) \geq \overline{x}^{-1}(x)$. Thus the functions $\underline{u}$ and $\overline{u}$ defined by

$$\underline{u}(x) = \frac{1}{4}\frac{(x + b_2)^2}{\sqrt{\varepsilon}(x + b_2) + 4a_2}, \qquad \overline{u}(x) = \frac{1}{2}\frac{(x + b_1)^2}{\sqrt{\varepsilon}(x + b_1) + a_1} \tag{5.9}$$

are bounds for $u(x)$,

$$\underline{u}(x) \leq u(x) \leq \overline{u}(x) \qquad \text{for all } x \geq 0. \tag{5.10}$$

Let us derive (5.1). Since f and g are monotone increasing and decreasing, respectively, $h(u)$ is decreasing and thus $h'(x) < 0$. Furthermore,

$$|h'(x)| = \frac{1}{\rho}|h'(u)| = \frac{1}{2\rho}\left|\frac{g'(u)}{g} - \frac{f'(u)}{f}\right|$$

$$= \frac{2}{\rho^3}|g'(u)f - f'(u)g| \leq \frac{c_1}{u^2\rho^3}. \tag{5.11}$$

We employ (5.6), (5.10), and (5.9) to obtain

$$|h'(x)| \leq \frac{c_1}{8} u^{-\frac{1}{2}} (a_1 + \varepsilon u)^{-\frac{3}{2}} \leq \frac{c_1}{8} \underline{u}^{-\frac{1}{2}} (a_1 + \varepsilon \underline{u})^{-\frac{3}{2}}$$

$$= 2c_1 \left(\frac{\sqrt{\varepsilon}(x + b_2) + 4a_2}{(x + b_2)^2} \right)^{\frac{1}{2}} \left(\frac{\sqrt{\varepsilon}(x + b_2) + 4a_2}{\varepsilon(x + b_2)^2 + 4a_1\sqrt{\varepsilon}(x + b_2) + 16a_1a_2} \right)^{\frac{3}{2}}$$

$$\leq \frac{2c_1}{x + b_2} \left[\frac{(\sqrt{\varepsilon}(x + b_2) + 4a_2)^2}{(\varepsilon(x + b_2)^2 + 16a_1a_2)^{\frac{3}{2}}} \right].$$

The square bracket is bounded uniformly in ε and x, proving (5.1). The second derivative of h is computed to be

$$h''(x) = \frac{1}{\rho} \frac{d}{du} \left(\frac{2}{\rho^3} (g'(u)f - f'(u)g) \right)$$

$$= -\frac{6}{\rho^5} \left(\rho'(u)(g'(u)f - f'(u)g) \right) + \frac{2}{\rho^4} \frac{d}{du} (g'(u)f - f'(u)g).$$

Thus

$$|h''(x)| \leq \frac{c_2}{u^4\rho^6} + \frac{c_2}{u^3\rho^4}, \tag{5.12}$$

and (5.6) shows that

$$|h''(x)| \leq 8c_2 u^{-1} (a_1 + \varepsilon u)^{-3} + c_2 u^{-1} (a_1 + \varepsilon u)^{-2} \leq c_3 \underline{u}^{-1} (a_1 + \varepsilon \underline{u})^{-2}.$$

We substitute in (5.9),

$$|h''(x)| \leq \frac{16c_3}{(x_1 + b_1)^2} \left[\frac{(\sqrt{\varepsilon}(x + b_2) + 4a_2)^3}{(\varepsilon(x + b_2)^2 + 16a_1a_2)^2} \right].$$

The square bracket is again bounded uniformly in ε and x, and this gives (5.2). We finally estimate $h'_\omega(x)$. Since the relation between ω and ε is one-to-one and smooth, we can just as well consider the ε-derivative $h'_\varepsilon(x)$. Since $h(x)$ is not given in closed form, we need to compute $h_\varepsilon(x)$ via the formula

$$\varepsilon h_\varepsilon(x) = \varepsilon h_\varepsilon(s) + \varepsilon h'(s)s_\varepsilon(x), \tag{5.13}$$

where $s = s(u)$ is a suitable variable. Clearly, $h_\varepsilon(x)$ is independent of how s is chosen. However, if we take for s too simple a function (e.g. $s = u$), then it turns out that $h_\varepsilon(s)$ will develop singularities in the limit $\varepsilon \to 0$, which are compensated in (5.13) by corresponding singular contributions to the second summand, making the analysis very delicate. To bypass these difficulties, it is convenient to choose for $s(u)$ a function with a similar qualitative behavior as $x(u)$; this will make it possible to estimate the two summands in (5.13) separately. We set $s = [\bar{u}\text{-inverse} + b_1]$, so that $u = \bar{u}(s(u) - b_1)$; i.e.

$$h(s) = h(u = \bar{u}(s - b_1))$$

with $\bar{u}$ according to (5.9) and $s \in [b_1, \infty)$. Then the expansion (5.4) becomes

$$h(s) = \frac{1}{2} \log \left(\varepsilon\lambda_1 + \frac{2\lambda_2\sqrt{\varepsilon}}{s} + \frac{\lambda_3}{s^2} + \sqrt{\varepsilon}\mathcal{O}(s^{-2}) + \mathcal{O}(s^{-3}) \right)$$

with positive constants λ_i which depend smoothly on $\sqrt{\varepsilon}$ and are uniformly bounded away from zero. Differentiating with respect to ε gives

$$\varepsilon h_\varepsilon(s) = \frac{1}{2} \frac{\lambda_1 \varepsilon s^2 + \lambda_2 \sqrt{\varepsilon} s + \sqrt{\varepsilon} \mathcal{O}(s^0)}{\lambda_1 \varepsilon s^2 + 2\lambda_2 \sqrt{\varepsilon} s + \lambda_3 + \sqrt{\varepsilon} \mathcal{O}(s^0) + \mathcal{O}(s^{-1})}.$$

We want to show that this function has bounded total variation. To this end, we differentiate with respect to s and obtain

$$\varepsilon h'_\varepsilon(s) = \frac{c_1 \varepsilon^{\frac{3}{2}} s^2 + c_2 \varepsilon s + c_3 \sqrt{\varepsilon} + \varepsilon^{\frac{3}{2}} \mathcal{O}(s) + \varepsilon \mathcal{O}(s^0) + \sqrt{\varepsilon} \mathcal{O}(s^{-1})}{2(\lambda_1 \varepsilon s^2 + 2\lambda_2 \sqrt{\varepsilon} s + \lambda_3 + \sqrt{\varepsilon} \mathcal{O}(s^0) + \mathcal{O}(s^{-1}))^2}.$$

Hence by choosing δ small enough and u_0 (and thus b_2) large enough, we can arrange that

$$\varepsilon |h'_\varepsilon| \leq c \frac{\varepsilon^{\frac{3}{2}} s^2 + \varepsilon s + \sqrt{\varepsilon}}{(\lambda_1 \varepsilon s^2 + \lambda_3)^2}$$

for all ε and all $s \geq b_1$. The L^1 norm of the rhs is bounded uniformly in ω. Namely, setting $t = \sqrt{\lambda_1 \varepsilon} s$ shows that for $n = 0, 1, 2$,

$$\int_{b_2}^\infty \frac{(\sqrt{\varepsilon} s)^n}{(\lambda_1 \varepsilon s^2 + \lambda_3)^2} \sqrt{\varepsilon} ds \leq \lambda_1^{-\frac{n+1}{2}} \int_0^\infty \frac{t^n}{(t^2 + \lambda_3)^2} dt,$$

and the last integral is finite, independent of ε. It remains to estimate the total variation of the second summand in (5.13). More precisely, in order to finish the proof of (5.3), we shall show that

$$\int_0^\infty \left| \frac{d}{dx}(h'(s)\varepsilon s_\varepsilon(x)) \right| dx < c. \tag{5.14}$$

We first derive sufficient conditions for (5.14). The relations

$$0 = \partial_\varepsilon s(x(s)) = s_\varepsilon(x) + s'(x)x_\varepsilon(s) \qquad h'(x) = h'(s)s'(x)$$

yield that

$$h'(s)s_\varepsilon(x) = -h'(x)x_\varepsilon(s).$$

Differentiating with respect to x, one sees that it suffices to bound the L^1 norms of the expressions

$$h'(x)\frac{\varepsilon x'_\varepsilon(s)}{x'(s)} \qquad \text{and} \qquad h''(x)\varepsilon x_\varepsilon(s)$$

uniformly in ε. Substituting in the bounds (5.1) and (5.2), we conclude that the following inequalities imply that (5.14) holds,

$$\frac{1}{c} \leq x'(s) \leq c, \tag{5.15}$$

$$\int_0^\infty \frac{\varepsilon |x'_\varepsilon(s)|}{1+x} dx \leq c, \tag{5.16}$$

$$\int_0^\infty \frac{\varepsilon |x_\varepsilon(s)|}{(1+x)^2} dx \leq c. \tag{5.17}$$

We begin the proof of (5.15)–(5.17) by computing $x'(s)$,

$$x'(s) = \frac{d}{ds} \int_{u_0}^{\bar{u}(s-b_1)} \rho(v) dv = \rho(\bar{u}(s-b_1)) \bar{u}'(s-b_1). \tag{5.18}$$

A short calculation using (5.9) and (5.5) gives

$$\bar{u}'(s-b_1) = \frac{s(\sqrt{\varepsilon}s + 2a_1)}{2(\sqrt{\varepsilon}s + a_1)^2} > 0,$$

$$\rho^2(\bar{u}(s-b_1)) = \frac{1}{s^2}(4\varepsilon s^2 + 8\beta\sqrt{\varepsilon}s + 8\beta a_1) + \sqrt{\varepsilon}\mathcal{O}(s^{-2}) + \mathcal{O}(s^{-3}),$$

and thus

$$x'(s)^2 = \frac{(\sqrt{\varepsilon}s)^4 + a_3(\sqrt{\varepsilon}s)^3 + \cdots + a_0 + \varepsilon^{\frac{3}{2}}\mathcal{O}(s^2) + \varepsilon\mathcal{O}(s) + \sqrt{\varepsilon}\mathcal{O}(s^0) + \mathcal{O}(s^{-1})}{(\sqrt{\varepsilon}s)^4 + b_3(\sqrt{\varepsilon}s)^3 + \cdots + b_0} \tag{5.19}$$

with coefficients $a_j, b_j \geq 0$ and $a_0, b_0 > 0$. Possibly after increasing u_0 and decreasing δ, we can neglect the error terms. The fraction in (5.19) is clearly uniformly bounded from above and below. This proves (5.15). Integrating (5.15), we obtain that the ratio $(1+x)/s$ is uniformly bounded from above and below, and thus in (5.16) and (5.17) we may replace the factors $(1+x)$ by s. Using (5.15) we may furthermore replace the integral over $x \in (0, \infty)$ by the integral over $s \in (b_1, \infty)$. Next we differentiate (5.19) with respect to ε. A short computation shows that

$$\varepsilon x'_\varepsilon(s) x'(s) = \frac{a_7(\sqrt{\varepsilon}s)^7 + \cdots + a_1(\sqrt{\varepsilon}s)}{(\sqrt{\varepsilon}s)^8 + b_7(\sqrt{\varepsilon}s)^7 + \cdots + b_0}\left(1 + \mathcal{O}(s^{-1})\right),$$

where the coefficients b_j are non-negative and $b_0 > 0$ (but the a_j might be zero or negative). Using the bounds (5.15), one sees that $\varepsilon x'_\varepsilon(s)$ can be estimated by

$$\varepsilon|x'_\varepsilon(s)| \leq c\frac{\sqrt{\varepsilon}s((\sqrt{\varepsilon}s)^6 + 1)}{(\sqrt{\varepsilon}s)^8 + 1}. \tag{5.20}$$

A scaling argument shows that

$$\int_{b_1}^\infty \frac{\varepsilon|x'_\varepsilon(s)|}{s} ds \leq c\int_{b_1}^\infty \frac{(\sqrt{\varepsilon}s)^6 + 1}{(\sqrt{\varepsilon}s)^8 + 1}\sqrt{\varepsilon} ds \leq c\int_0^\infty \frac{t^6 + 1}{t^8 + 1} < \infty, \tag{5.21}$$

proving (5.16). To derive (5.17), we use that

$$\varepsilon|x_\varepsilon(s)| \leq \int_{b_1}^s \varepsilon|x'_\varepsilon(t)| dt + \varepsilon|x_\varepsilon(b_1)|$$

and obtain

$$\int_{b_1}^\infty \frac{\varepsilon|x_\varepsilon(s)|}{s^2} ds \leq \int_{b_1}^\infty \frac{1}{s^2}\left(\int_{b_1}^s \varepsilon|x'_\varepsilon(t)| dt + \varepsilon|x_\varepsilon(b_1)|\right) ds$$

$$= -\int_{b_1}^\infty \frac{d}{ds}\left(\frac{1}{s}\right)\left(\int_{b_1}^s \varepsilon|x'_\varepsilon(t)| dt + \varepsilon|x_\varepsilon(b_1)|\right) ds.$$
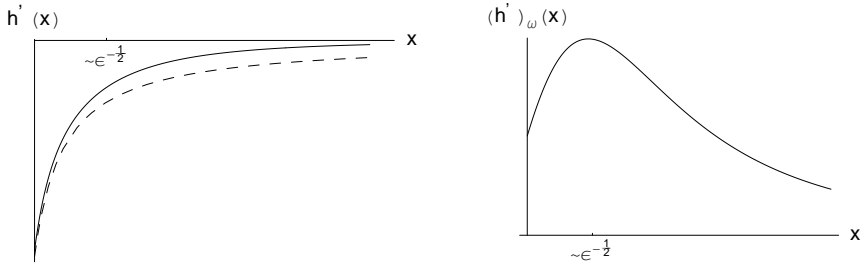
Fig. 2. Typical plots for $h'(x)$ and $h'_\omega(x)$ in the case $\omega > m$

According to (5.20), the inner integral diverges at most logarithmically as $s \to \infty$. Therefore, integrating by parts gives no boundary terms at infinity,

$$\int_{b_1}^\infty \frac{\varepsilon |x_\varepsilon(s)|}{s^2} ds \le \int_{b_1}^\infty \frac{\varepsilon |x'_\varepsilon(s)|}{s} ds + \frac{\varepsilon |x_\varepsilon(b_1)|}{b_1}.$$

The integral on the right was estimated in (5.21). The last summand is computed to be

$$\varepsilon \frac{|x_\varepsilon(b_1)|}{b_1} = \varepsilon \frac{|x'(b_1)|}{b_1} \frac{\partial b_1}{\partial \varepsilon},$$

and this is bounded uniformly in ε in view of (5.18) and the fact that b_1 is smooth in $\sqrt{\varepsilon}$ and bounded away from zero, (5.7). This completes the proof of Lemma 5.1. □

The above estimates are illustrated in Fig. 2, where h' and h'_ω are plotted in a typical example. The dashed curve describes the asymptotics near $x = 0$; it is the graph of $h'(x)$, where for $x(u)$ one uses the approximate formula $x \approx 4\sqrt{\beta}(\sqrt{u} - \sqrt{u_0})$, obtained by setting ε in (5.5) equal to zero, dropping the error term and integrating, (4.8).

In the case $\omega < m$, we fix u_{min} independent of $\omega \in (m - \delta, m)$. By choosing u_{min} large and δ sufficiently small, we can arrange that the function g has exactly one zero on the half line (u_{min}, ∞). We set u_0 equal to this zero,

$$g(u_0) = 0.$$

Clearly, u_0 depends on ω. The variable $x(u)$, (4.8), is positive for $u > u_0$ and negative on the interval (u_{min}, u_0). We set $x_{min} = x(u_{min})$. The following lemma is the analogue of Lemma 5.1 for $\omega < m$. The method of proof is also similar, but the pole of h at $x = 0$ makes the situation a bit more complicated.

Lemma 5.2. *There are constants $c, \delta > 0$ such that for all $\omega \in (m - \delta, m)$ and $x > x_{min}$,*

$$0 \le \Gamma(x) h'(x) \le \frac{c}{x - x_{min} + 1} + \frac{c}{|x|}, \tag{5.22}$$

$$|h''(x)| \le \frac{c}{(x - x_{min} + 1)^2} + \frac{c}{|x|^2}, \tag{5.23}$$

$$\int_{x_{min}}^\infty |h'_\omega(x)| dx \le \frac{c}{m - \omega}. \tag{5.24}$$

Furthermore, for every $x_1 > 0$ the constants $c, \delta > 0$ can be chosen such that for all $\omega \in (m - \delta, m)$ and $x \in [-x_1, x_1]$,

$$\left| h(x) - \frac{1}{3} \log |\varepsilon^2 x| \right| \leq c. \tag{5.25}$$

Proof. We now set $\varepsilon = m^2 - \omega^2$ and $\rho = 2\sqrt{|fg|}$. Then

$$\rho^2(u) = 4 \left| \varepsilon - \frac{\beta}{u} + \mathcal{O}(u^{-2}) \right|$$

with $\beta > 0$. Since $\rho(u_0) = 0$,

$$u_0 = \frac{\beta}{\varepsilon}(1 + \mathcal{O}(\varepsilon)), \tag{5.26}$$

and furthermore, the functions h and ρ have the expansions

$$h(u) = \frac{1}{2} \log \left(\frac{\varepsilon}{(\omega + m)^2} \frac{|u - u_0|}{u} \left(1 + \mathcal{O}(u^{-1})\right) \right), \tag{5.27}$$

$$\rho(u) = 2\sqrt{\varepsilon \frac{|u - u_0|}{u}} \left(1 + \mathcal{O}(u^{-1})\right). \tag{5.28}$$

Since global bounds for $x(u)$ and $u(x)$ would be more difficult to obtain than those in Lemma 5.1, we here construct the bounds piecewise. We set $\Delta u = u_0/2$. By decreasing δ, we can arrange that $u_0/4 > u_{\min}$, and furthermore we can also make the error terms in (5.27) and (5.28) as small as we like. Thus we may assume that

$$\frac{3}{2}\sqrt{\frac{\varepsilon}{u_0}}\sqrt{|u - u_0|} \leq \rho(u) \leq 3\sqrt{\frac{\varepsilon}{u_0}}\sqrt{|u - u_0|} \qquad \text{for } |u - u_0| \leq \Delta u. \tag{5.29}$$

Integrating from u_0 to u gives

$$\sqrt{\frac{\varepsilon}{u_0}}|u - u_0|^{\frac{3}{2}} \leq |x(u)| \leq 2\sqrt{\frac{\varepsilon}{u_0}}|u - u_0|^{\frac{3}{2}} \qquad \text{for } |u - u_0| \leq \Delta u. \tag{5.30}$$

Taking the inverses, we obtain for $|u - u_0|$ the bounds

$$\underline{v}(x) \leq |u(x) - u_0| \leq \overline{v}(x) \qquad \text{for } |x| \leq \Delta x, \tag{5.31}$$

where we set

$$\underline{v}(x) = \left(\frac{x^2 u_0}{4\varepsilon} \right)^{\frac{1}{3}}, \qquad \overline{v}(x) = \left(\frac{x^2 u_0}{\varepsilon} \right)^{\frac{1}{3}}, \tag{5.32}$$

$$\Delta x = \sqrt{\frac{\varepsilon}{u_0}}(\Delta u)^{\frac{3}{2}} = 2^{-\frac{3}{2}}\sqrt{\varepsilon}u_0 = 2^{-\frac{3}{2}}\beta\varepsilon^{-\frac{1}{2}}(1 + \mathcal{O}(\varepsilon)). \tag{5.33}$$

If $x > \Delta x$, the second inequality in (5.30) shows that $u - u_0 \geq 2^{-\frac{5}{3}}u_0$, and thus in this region there are constants $a_1, a_2 > 0$ such that

$$a_1\sqrt{\varepsilon} \leq \rho(u) \leq a_2\sqrt{\varepsilon}. \tag{5.34}$$

Hence for $x > \Delta x$,

$$\frac{1}{a_2\sqrt{\varepsilon}} \leq \frac{du}{dx} \leq \frac{1}{a_1\sqrt{\varepsilon}}.$$

Integration shows that, possibly after decreasing a_1 and increasing a_2,

$$\underline{u}(x) \leq u(x) \leq \overline{u}(x) \qquad \text{for } x > \Delta x \tag{5.35}$$

with

$$\underline{u}(x) = \frac{x}{a_2\sqrt{\varepsilon}}, \qquad \overline{u}(x) = \frac{x}{a_1\sqrt{\varepsilon}}. \tag{5.36}$$

If on the other hand $x < -\Delta x$, we see from (5.30) that $u - u_0 \leq -2^{-\frac{5}{3}}u_0$, and thus ρ can be estimated by

$$\frac{b_1}{2}u^{-\frac{1}{2}} \leq \rho(u) \leq \frac{b_2}{2}u^{-\frac{1}{2}} \tag{5.37}$$

with $b_1, b_2 > 0$. We integrate from $u_{\min}$ to u,

$$b_1\sqrt{u} - b_1\sqrt{u_{\min}} \leq x(u) - x_{\min} \leq b_2\sqrt{u} - b_2\sqrt{u_{\min}},$$

and solve for u. This gives

$$\underline{u}(x) \leq u(x) \leq \overline{u}(x) \qquad \text{for } x_{\min} \leq x < -\Delta x \tag{5.38}$$

with

$$\underline{u}(x) = \left(\frac{x - x_{\min}}{b_2} + \sqrt{u_{\min}}\right)^2, \qquad \overline{u}(x) = \left(\frac{x - x_{\min}}{b_1} + \sqrt{u_{\min}}\right)^2. \tag{5.39}$$

For any $x_1 > 0$, we can, by choosing δ small enough, arrange that Δx, (5.33), is greater than x_1. Thus $u(x)$ is on the interval $[-x_1, x_1]$ bounded by (5.31). Substituting these bounds into (5.27) and using (5.26) gives (5.25).

To show that $\Gamma(x)h'(x) \geq 0$, note that

$$h'(x) = h'(u)\frac{du}{dx},$$

where $\frac{du}{dx} = \rho^{-1}$ is positive, and the sign of $h'(u)$ is obtained from (5.27).

For the derivation of the inequalities (5.22)–(5.24), we consider the three regions $x < -\Delta x$, $|x| \leq \Delta x$, and $x > \Delta x$ separately. We begin with the case $|x| \leq \Delta x$. For $h'(x)$ and $h''(x)$, we have again the bounds (5.11) and (5.12), respectively. Using that $u \geq u_0/2$ as well as (5.29), (5.31), and (5.32), we obtain

$$|h'(x)| \leq \frac{c}{u^2\rho^3} \leq \frac{c}{\sqrt{u_0}\varepsilon^{\frac{3}{2}}}|u - u_0|^{-\frac{3}{2}} \leq \frac{c}{u_0\varepsilon x},$$

$$|h''(x)| \leq \frac{c}{u^4\rho^6} + \frac{c}{u^3\rho^4} \leq \frac{c}{u_0^2\varepsilon^2 x^2} + \frac{c}{u_0^{\frac{5}{3}}\varepsilon^{\frac{4}{3}}x^{\frac{4}{3}}},$$

and in view of (5.26) and (5.33), this implies (5.22) and (5.23). To compute $h_\varepsilon(x)$, we again use (5.13), but now with

$$s(u) = \sqrt{\frac{\varepsilon}{u_0}} |u - u_0|^{\frac{3}{2}} \Gamma(u - u_0) \qquad \text{for } |u - u_0| \le \Delta u,$$

where as before Γ is the step function $\Gamma(\tau) = \text{sgn}(\tau)$. The first summand in (5.13) is computed as follows,

$$u(s) = u_0 + u_0^{\frac{1}{3}} \varepsilon^{-\frac{1}{3}} s^{\frac{2}{3}} \Gamma(s) \qquad \text{for } |s| \le \Delta x, \tag{5.40}$$

$$\overset{(5.26)}{=} \frac{1}{\varepsilon} (\beta + (\beta \varepsilon s^2)^{\frac{1}{3}} \Gamma(s))(1 + \mathcal{O}(\varepsilon)), \tag{5.41}$$

$$h(s) \overset{(5.27)}{=} \frac{1}{2} \log \left(\frac{\beta^{\frac{1}{3}} \varepsilon^{\frac{4}{3}} s^{\frac{2}{3}}}{\beta + (\beta \varepsilon s^2)^{\frac{1}{3}} \Gamma(s)} \left(1 + \varepsilon \mathcal{O}(s^0) + \varepsilon^{\frac{4}{3}} \mathcal{O}(s^{\frac{2}{3}})\right) \right) - \log(\omega + m)$$

$$\varepsilon h_\varepsilon(s) = \left(\frac{2}{3} - \frac{1}{6} \frac{(\beta \varepsilon s^2)^{\frac{1}{3}} \Gamma(s)}{\beta + (\beta \varepsilon s^2)^{\frac{1}{3}} \Gamma(s)} \right) \left(1 + \varepsilon \mathcal{O}(s^0) + \varepsilon^{\frac{4}{3}} \mathcal{O}(s^{\frac{2}{3}})\right).$$

Differentiating with respect to s gives the bound

$$\varepsilon |h'_\varepsilon(s)| \le c \varepsilon^{\frac{1}{3}} s^{-\frac{1}{3}} + c \varepsilon^{\frac{2}{3}} s^{\frac{1}{3}},$$

and thus

$$\int_{-\Delta x}^{\Delta x} \varepsilon |h'_\varepsilon(s)| ds \le c \varepsilon^{\frac{1}{3}} (\Delta x)^{\frac{2}{3}} + c \varepsilon^{\frac{2}{3}} (\Delta x)^{\frac{4}{3}}.$$

Using (5.33), we conclude that the total variation of $\varepsilon h_\varepsilon(s)$ is bounded uniformly in ε. In order to estimate the total variation of the second summand in (5.13), we first compute $x'(s)$,

$$x'(s)^2 = (\rho(s) u'(s))^2 = \frac{16}{9} \frac{u_0}{u(s)} \left(1 + \varepsilon \mathcal{O}(s^0) + \varepsilon^{\frac{4}{3}} \mathcal{O}(s^{\frac{2}{3}})\right).$$

This is uniformly bounded from above and below, proving (5.15). Differentiating with respect to ε using (5.41) gives the estimate

$$\varepsilon |x'_\varepsilon(s)| \le c \varepsilon^{\frac{1}{3}} s^{\frac{2}{3}}. \tag{5.42}$$

Since $x(s = 0) = 0$ for all ε (from (5.40)), integration yields that

$$\varepsilon |x_\varepsilon(\pm \Delta x)| \le \left| \int_0^{\pm \Delta x} \varepsilon |x'_\varepsilon(s)| ds \right| \le c \varepsilon^{\frac{1}{3}} (\Delta x)^{\frac{5}{3}} \overset{(5.33)}{\le} c \varepsilon^{-\frac{1}{2}} \tag{5.43}$$

$$\int_{-\Delta x}^{\Delta x} \frac{\varepsilon |x'_\varepsilon(s)|}{s} ds \le c \varepsilon^{\frac{1}{3}} (\Delta x)^{\frac{2}{3}} \le c. \tag{5.44}$$

Furthermore,

$$\int_{-\Delta x}^{\Delta x} \frac{\varepsilon |x_\varepsilon(s)|}{s^2} ds \le \int_{-\Delta x}^{\Delta x} \frac{1}{s^2} \left(\int_0^s \varepsilon |x'_\varepsilon(t)| dt \right) ds. \tag{5.45}$$

Using (5.42), the inner integral is for small s bounded by a constant times $t^{\frac{5}{3}}$. Thus when integrating by parts, we get no boundary terms at $s = 0$ and obtain

$$\int_{-\Delta x}^{\Delta x} \frac{\varepsilon |x_\varepsilon(s)|}{s^2} ds \leq \frac{1}{\Delta x} \int_{-\Delta x}^{\Delta x} \varepsilon |x_\varepsilon'(t)| dt + \int_{-\Delta x}^{\Delta x} \frac{\varepsilon |x_\varepsilon'(t)|}{s} ds \leq c, \qquad (5.46)$$

where in the last step we used (5.42) and (5.44). Combining (5.44) and (5.46) with the estimates (5.22) and (5.23), we conclude that the total variation of the second summand in (5.13) is bounded uniformly in ε. This shows that (5.24) holds if the integration domain is restricted to $x \in (-\Delta x, \Delta x)$.

In the case $x > \Delta x$, (5.34), (5.35), and (5.36) yield, again using (5.11) and (5.12),

$$|h'(x)| \leq \frac{c}{u^2 \rho^3} \leq \frac{c}{x^2 \sqrt{\varepsilon}} \leq \frac{c}{x} \frac{1}{\sqrt{\varepsilon} \Delta x} \overset{(5.33)}{\leq} \frac{c}{x},$$

$$|h''(x)| \leq \frac{c}{u^4 \rho^6} + \frac{c}{u^3 \rho^4} \leq \frac{c}{x^4 \varepsilon} + \frac{c}{x^3 \sqrt{\varepsilon}} \leq \frac{c}{x^2},$$

proving (5.22) and (5.23). To compute the total variation of $h_\varepsilon(x)$, we apply (5.13) with

$$s(u) = \frac{\sqrt{\varepsilon}}{3\sqrt{2}} u \qquad \text{for } u > u_0 + \Delta u.$$

Using that

$$s(u_0 + \Delta u) = s\left(\frac{3u_0}{2}\right) \overset{(5.33)}{=} \Delta x,$$

we see that

$$u(s) = \frac{3\sqrt{2}}{\sqrt{\varepsilon}} s \qquad \text{for } s > \Delta x.$$

Moreover, from (5.27) and (5.28),

$$h(s) = \frac{1}{2} \log\left(\frac{\varepsilon}{(\omega+m)^2} \frac{3\sqrt{2}s - \beta\varepsilon^{-\frac{1}{2}}}{3\sqrt{2}s} \left(1 + \varepsilon\mathcal{O}(s^0) + \sqrt{\varepsilon}\mathcal{O}(s^{-1})\right)\right),$$

$$\varepsilon h_\varepsilon(s) = \frac{1}{2} \left(1 + \frac{\beta\varepsilon^{-\frac{1}{2}}}{6\sqrt{2}s - 2\beta\varepsilon^{-\frac{1}{2}}}\right) \left(1 + \varepsilon\mathcal{O}(s^0) + \sqrt{\varepsilon}\mathcal{O}(s^{-1})\right),$$

$$\varepsilon |h_\varepsilon'(s)| \leq \frac{c}{\sqrt{\varepsilon} s^2},$$

$$\int_{\Delta x}^{\infty} \varepsilon |h_\varepsilon'(s)| ds \leq \frac{c}{\sqrt{\varepsilon} \Delta x} \leq c,$$

$$x'(s)^2 = (\rho(s)u'(s))^2 = 72 \left(1 - \frac{\beta}{3\sqrt{2}s\sqrt{\varepsilon}}\right) \left(1 + \varepsilon\mathcal{O}(s^0) + \sqrt{\varepsilon}\mathcal{O}(s^{-1})\right).$$

Since $s > \Delta x$, we conclude from (5.33) that (5.15) holds. Differentiating the last relation with respect to ε and integrating gives

$$\varepsilon |x'_\varepsilon(s)| \le \frac{c}{s\sqrt{\varepsilon}},$$

$$\int_{\Delta x}^\infty \frac{\varepsilon |x'_\varepsilon(s)|}{s} ds \le \frac{c}{\sqrt{\varepsilon}\Delta x} \le c, \tag{5.47}$$

$$\int_{\Delta x}^\infty \frac{\varepsilon |x_\varepsilon(s)|}{s^2} ds \le \int_{\Delta x}^\infty \frac{1}{s^2} \left(\int_{\Delta x}^s \varepsilon |x'_\varepsilon(t)| dt + \varepsilon x_\varepsilon(\Delta x) \right) ds$$

$$\le \frac{\varepsilon |x_\varepsilon(\Delta x)|}{\Delta x} + \int_{\Delta x}^\infty \frac{\varepsilon |x_\varepsilon(s)|}{s} ds \le c,$$

where in the last step we used (5.43) and (5.47). This proves (5.24) if $x > \Delta x$.

Finally, if $x < -\Delta x$, the bounds (5.38) and (5.39) give

$$|h'(x)| \overset{(5.11)}{\le} \frac{c}{u^2 \rho^3} \le \frac{8c}{b_1^3} u^{-\frac{1}{2}} = \frac{c}{b_2\sqrt{u}}$$

$$\le \frac{c}{x - x_{\min} + b_2\sqrt{u_{\min}}} \le \frac{c}{x - x_{\min} + 1},$$

$$|h''(x)| \overset{(5.12)}{\le} \frac{c}{u^4 \rho^6} + \frac{c}{u^3 \rho^4} \le \frac{c}{u} \le \frac{c}{(x - x_{\min} + 1)^2}.$$

This concludes the proof of (5.22) and (5.23). In order to prove (5.24), we apply (5.13) with

$$s(u) = \frac{1}{2}\sqrt{\beta u} - 2\Delta x \qquad \text{for } u_{\min} \le u \le u_0 - \Delta u.$$

Similar as in the case $x > \Delta x$,

$$u(s) = \frac{4}{\beta}(s + 2\Delta x)^2 \qquad \text{for } s_{\min} \equiv \tfrac{1}{2}\sqrt{\beta u_{\min}} - 2\Delta x \le s \le -\Delta x, \tag{5.48}$$

$$h(s) = \frac{1}{2}\log \left(\frac{1}{(\omega + m)^2} \left(\frac{\beta^2(1 + \mathcal{O}(\varepsilon))}{4(s + 2\Delta x)^2} - \varepsilon \right) \left(1 + \mathcal{O}((s + 2\Delta x)^{-2}) \right) \right),$$

$$\varepsilon h_\varepsilon(s) = -\frac{\varepsilon}{2} \frac{4(s + 2\Delta x)^2 + \beta^2 \mathcal{O}(\varepsilon)}{\beta^2(1 + \mathcal{O}(\varepsilon)) - 4\varepsilon(s + 2\Delta x)^2} + \varepsilon\mathcal{O}(s^0) + \frac{\varepsilon\mathcal{O}((s + 2\Delta x)^{-2})}{1 + \mathcal{O}((s + 2\Delta x)^{-2})}$$

$$= -\frac{\varepsilon}{2} \frac{4(s + 2\Delta x)^2 + \epsilon\mathcal{O}(s^0)}{\beta^2 - 4\varepsilon(s + 2\Delta x)^2} + \varepsilon\mathcal{O}(s^0) + \varepsilon\mathcal{O}((s + 2\Delta x)^{-2}),$$

$$\varepsilon |h'_\varepsilon(s)| \le c\varepsilon(s + 2\Delta x).$$

Integration yields that

$$\int_{s_{\min}}^{-\Delta x} \varepsilon |h'_\varepsilon(s)| ds \overset{(5.48)}{\le} c\varepsilon \left(\Delta x^2 - \frac{1}{4}\beta u_{\min} \right) \overset{(5.33),(5.26)}{\le} ; c\varepsilon(u_0 - \Delta u - u_{\min}) \le c.$$

Moreover,

$$x'(s)^2 = (\rho(s)u'(s))^2$$

$$= 64 \left(1 - \frac{4}{\beta^2}\varepsilon(s + 2\Delta x)^2 \right) \left(1 + \varepsilon\mathcal{O}(s^0) + \mathcal{O}((s + 2\Delta x)^{-2}) \right).$$
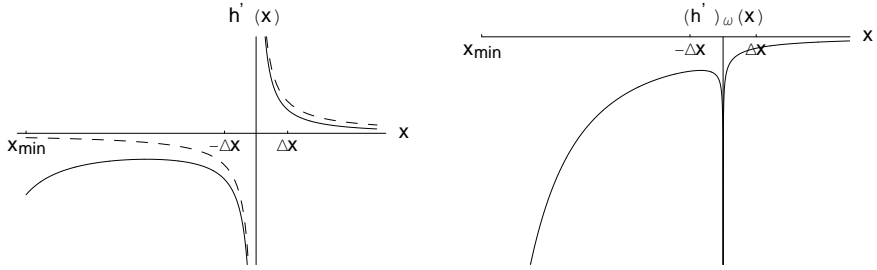
Fig. 3. Typical plots for $h'(x)$ and $h'_\omega(x)$ in the case $\omega < m$

Using that $(s + 2\Delta x)^2 \le (\Delta x)^2$, we conclude that $x'(s)$ is uniformly bounded from above and below. We differentiate with respect to ε and integrate to finally obtain similar to (5.15)–(5.17),

$$\varepsilon|x'_\varepsilon| \le c\varepsilon(s + 2\Delta x)^2, \tag{5.49}$$

$$\int_{s_{\min}}^{-\Delta x} \frac{\varepsilon|x'_\varepsilon(s)|}{s + 2\Delta x} ds \le c\varepsilon(u_0 - \Delta u - u_{\min}) \le c, \tag{5.50}$$

$$\int_{s_{\min}}^{-\Delta x} \frac{\varepsilon|x_\varepsilon(s)|}{(s + 2\Delta x)^2} ds \le \int_{s_{\min}}^{-\Delta x} \frac{1}{(s + 2\Delta x)^2} \left(\int_{-\Delta x}^{s} \varepsilon|x'_\varepsilon(t)|dt + \varepsilon|x_\varepsilon(-\Delta x)| \right) ds$$

$$\le \frac{\varepsilon|x_\varepsilon(-\Delta x)|}{\Delta x} + \frac{1}{s_{\min} + 2\Delta x} \int_{s_{\min}}^{-\Delta x} \varepsilon|x'_\varepsilon(t)|dt + \int_{s_{\min}}^{-\Delta x} \frac{\varepsilon|x'_\varepsilon(s)|}{s + 2\Delta x} ds \le c,$$

where in the last line we integrated by parts and used (5.43) and (5.49),(5.50). This yields (5.24) and completes the proof of Lemma 5.2. $\quad\square$

The above estimates are illustrated in Fig. 3. The dashed curve is the graph of $(3x)^{-1}$; it is the x-derivative of the asymptotic function $\frac{1}{3} \log(\varepsilon^2 x)$ which appears in (5.25).

The next lemma controls the behavior of $x_{\min}$.

Lemma 5.3. *There are constants $c, \delta > 0$ such that for all $\omega \in (m - \delta, m)$,*

$$\left| x_{min} + 4\beta(m^2 - \omega^2)^{-\frac{1}{2}} \right| \le c, \tag{5.51}$$

$$\left| \partial_\omega \left(x_{min} + 4\beta(m^2 - \omega^2)^{-\frac{1}{2}} \right) \right| \le \frac{c}{m - \omega}. \tag{5.52}$$

Proof. According to the definition of x and (5.28),

$$x_{\min} = -2 \int_{u_{\min}}^{u_0} \rho(u)du = -2 \int_{u_{\min}}^{u_0} \sqrt{\varepsilon \frac{|u - u_0|}{u}} \left(1 + \mathcal{O}(u^{-1})\right)du, \tag{5.53}$$

and a calculation using (5.26) shows that the leading contribution in $\sqrt{\varepsilon}$ to this integral is $4\beta\varepsilon^{-\frac{1}{2}}$ (this can be readily verified using Mathematica). This proves (5.51). Differentiating (5.53) with respect to ω and estimating the resulting integral gives (5.52). $\square$

6. The Region $\omega > m$

We turn now to the planar equation (4.12). Consider the two solutions $(\vartheta^{(b)}, L^{(b)})$, $b = 1, 2$, with

$$\lim_{x \to -\infty} (\vartheta^{(1)}, L^{(1)}) = (0, 0), \qquad \lim_{x \to -\infty} (\vartheta^{(2)}, L^{(2)}) = (\pi, 0) \tag{6.1}$$

and define in analogy to the transmission coefficients, the quantities $(\vartheta_\infty^{(b)}, L_\infty^{(b)})$ by

$$(\vartheta_\infty^{(b)}, L_\infty^{(b)}) = \lim_{x \to \infty} (\vartheta^{(b)}(x), L^{(b)}(x)). \tag{6.2}$$

The next lemma expresses the coefficients t_{ab} in the integral representation (1.2) in terms of these "transmission coefficients."

Lemma 6.1. *The coefficients t_{ab}, (2.15), are for $\omega > m$ given by*

$$t_{11} = t_{22} = \frac{1}{2}, \tag{6.3}$$

$$t_{12} = \overline{t_{21}} = \frac{1}{2} e^{-i\beta_0} \tanh z, \tag{6.4}$$

where

$$\beta_0 = \arctan \frac{\lambda}{m r_1}, \tag{6.5}$$

$$z = \frac{L_\infty^{(1)} - L_\infty^{(2)}}{4} + i \frac{\vartheta_\infty^{(1)} - \vartheta_\infty^{(2)} + \pi}{4}. \tag{6.6}$$

Proof. According to (4.10), (4.5), and (4.1), the functions (ϑ, L) correspond to the 2-spinor

$$X = e^{-\frac{L}{2}} \begin{pmatrix} e^{-\frac{i\beta}{2}} & 0 \\ 0 & e^{\frac{i\beta}{2}} \end{pmatrix} \begin{pmatrix} e^{\frac{h}{2}} \cos \frac{x+\vartheta}{2} - i e^{-\frac{h}{2}} \sin \frac{x+\vartheta}{2} \\ -e^{\frac{h}{2}} \cos \frac{x+\vartheta}{2} - i e^{-\frac{h}{2}} \sin \frac{x+\vartheta}{2} \end{pmatrix}. \tag{6.7}$$

In the limit $u \to -\infty$, the function $h \to 0$ (cf. (4.9) and observe that $\lim_{r \searrow r_1} f(r) = \omega = \lim_{r \searrow r_1} g(r)$ according to (4.7) and (4.3),(4.4)), and x coincides asymptotically with $\Omega_0 u$, up to an (irrelevant) additive constant. Thus comparing (6.7) with (2.7) and using (6.1) gives

$$f_0^{(1)} = \begin{pmatrix} e^{-\frac{i\beta_0}{2}} \\ -e^{\frac{i\beta_0}{2}} \end{pmatrix}, \qquad f_0^{(2)} = \begin{pmatrix} -i e^{-\frac{i\beta_0}{2}} \\ -i e^{\frac{i\beta_0}{2}} \end{pmatrix}.$$

Hence the fundamental solutions X_0 and X_1, which are characterized by (2.8), are the linear combinations

$$X_1 = \frac{1}{2} e^{\frac{i\beta_0}{2}} (X^{(1)} + i X^{(2)}), \qquad X_2 = \frac{1}{2} e^{-\frac{i\beta_0}{2}} (-X^{(1)} + i X^{(2)}).$$

We next consider (6.7) in the limit $u \to +\infty$. According to (4.9), (4.7), (4.3)–(4.4), and (2.13),

$$\lim_{x \to \infty} h(x) = \frac{1}{2} \log \frac{\omega - m}{\omega + m} = -2\Theta.$$

Also, β goes to zero in this limit. Hence using (2.12) and (6.2), one sees that

$$f_{\infty 1/2} = \frac{1}{2}e^{\pm \frac{i\beta_0}{2}} \begin{pmatrix} \pm \exp\left(-\frac{L_\infty^{(1)} + i\vartheta_\infty^{(1)}}{2}\right) + i\exp\left(-\frac{L_\infty^{(2)} + i\vartheta_\infty^{(2)}}{2}\right) \\ \pm \exp\left(-\frac{L_\infty^{(1)} - i\vartheta_\infty^{(1)}}{2}\right) + i\exp\left(-\frac{L_\infty^{(2)} - i\vartheta_\infty^{(2)}}{2}\right) \end{pmatrix}.$$

Substituting this last formula into (2.14) yields

$$t_{1/2}(\alpha) = e^{\mp \frac{i\beta_0}{2}} \left[ie^{-\frac{L_\infty^{(1)}}{2}} \sin\left(\alpha + \frac{\vartheta_\infty^{(1)}}{2}\right) \pm e^{-\frac{L_\infty^{(2)}}{2}} \sin\left(\alpha + \frac{\vartheta_\infty^{(2)}}{2}\right) \right].$$

A short calculation shows that

$$|t_1|^2 = |t_2|^2. \tag{6.8}$$

Together with (2.15), this immediately yields (6.3). Furthermore, it is obvious from (2.15) that $\overline{t_{12}} = t_{21}$. Thus it remains to compute t_{12}. According to (2.15) and (6.8), we have

$$\begin{aligned} t_{12} &= \frac{1}{2\pi}\int_0^{2\pi} \frac{t_1\overline{t_2}}{2|t_2|^2} = \frac{1}{4\pi}\int_0^{2\pi}\frac{t_1}{t_2} \\ &= \frac{e^{-i\beta_0}}{4\pi}\int_0^{2\pi}\frac{i\rho_1\sin(\alpha + \varphi_1) + \rho_2\sin(\alpha + \varphi_2)}{i\rho_1\sin(\alpha + \varphi_1) - \rho_2\sin(\alpha + \varphi_2)}d\alpha, \end{aligned}$$

where we set $\rho_i = L_\infty^{(i)}/2$ and $\varphi_i = \vartheta_\infty^{(i)}/2$. It is convenient to shift the integration variable by $\alpha \to \alpha - \varphi_2$ and to divide the numerator and denominator by ρ_2. This gives

$$t_{12} = \frac{e^{-i\beta}}{4\pi}\int_0^{2\pi}\frac{i\rho\sin(\alpha + \varphi) + \sin\alpha}{i\rho\sin(\alpha + \varphi) - \sin\alpha}d\alpha$$

with $\rho \equiv \rho_1/\rho_2$ and $\varphi \equiv \varphi_1 - \varphi_2$. We express the trigonometric functions as exponentials and set $\mu = \rho e^{-i\varphi}$,

$$t_{12} = \frac{e^{-i\beta_0}}{4\pi}\int_0^{2\pi}\frac{(\overline{\mu} - i)e^{2i\alpha} - (\mu - i)}{(\overline{\mu} + i)e^{2i\alpha} - (\mu + i)}d\alpha.$$

Setting $z = e^{2i\alpha}$, the α-integral can be regarded as an integral along the unit circle in the complex plane; more precisely,

$$t_{12} = \frac{e^{-i\beta_0}}{4\pi i}\oint_{|z|=1}\frac{(\overline{\mu} - i)z - (\mu - i)}{(\overline{\mu} + i)z - (\mu + i)}\frac{dz}{z}. \tag{6.9}$$

This contour integral can be computed with residues as follows. According to (6.1), $\lim_{x\to -\infty}\vartheta_\infty^{(1)} - \vartheta_\infty^{(1)} = -\pi$. A comparison argument using the differential equation for ϑ, (4.12), shows that $\vartheta^{(1)} - \vartheta^{(2)}$ takes values in the interval $(-2\pi, 0)$ for all x. Hence $-\pi < \varphi < 0$, or equivalently,

$$\mathrm{Im}\,\mu > 0.$$

As a consequence, the integrand in (6.9) has only one pole in the unit circle, at $z = 0$. We conclude that

$$t_{12} = \frac{1}{2} e^{-i\beta_0} \frac{\mu - i}{\mu + i} = \frac{1}{2} e^{-i\beta_0} \frac{\sqrt{\mu/i} - \sqrt{i/\mu}}{\sqrt{\mu/i} + \sqrt{i/\mu}},$$

and this coincides with (6.4). $\quad\square$

The following two lemmas control the behavior of (ϑ, L) for large x.

Lemma 6.2. *There is $c > 0$ such that for all $\omega \in (m, m + \delta)$ and $x \in (0, \infty]$,*

$$|\vartheta(x) - \vartheta(0)| \leq c, \qquad |L(x) - L(0)| \leq c. \tag{6.10}$$

Proof. According to (5.1), there is $x_0 > 0$ such that

$$1 + h' \sin(x + \vartheta) > \frac{1}{2} \qquad \text{for all } x > x_0. \tag{6.11}$$

On the interval $[0, x_0]$, we can control ϑ by integrating the ϑ-equation in (4.12),

$$|\vartheta(x_0) - \vartheta(0)| = \left| \int_0^{x_0} h' \sin(x + \vartheta) dx \right| \overset{(5.1)}{=} cx_0. \tag{6.12}$$

In the region $x > x_0$, we again integrate the equation,

$$\vartheta(x) - \vartheta(x_0) = \int_{x_0}^x h' \sin(x + \vartheta) dx = -\int_{x_0}^x \frac{h'}{1 + h' \sin(\tau + \vartheta)} \frac{d}{d\tau} (\cos(\tau + \vartheta)) d\tau.$$

We integrate by parts and, using (6.11) and Lemma 5.1, we find

$$|\vartheta(x) - \vartheta(x_0)| \leq 2(|h'(x)| + |h'(x_0)|) + 4 \int_{x_0}^x \left(|h''| + h'^2 \right) d\tau \leq c. \tag{6.13}$$

The second statement in (6.10) follows similarly by integrating the L-equation in (4.12). $\quad\square$

Lemma 6.3. *There is $c > 0$ such that for all $\omega \in (m, m + \delta)$ and $x \in (0, \infty]$,*

$$|\vartheta_\omega(x) - \vartheta_\omega(0)| \leq \frac{c}{\omega - m}, \qquad |L_\omega(x) - L_\omega(0)| \leq \frac{c}{\omega - m}. \tag{6.14}$$

Proof. Differentiating through the ODEs in (4.12) with respect to ω gives

$$\vartheta'_\omega = h' \cos(x + \vartheta)\vartheta_\omega + h'_\omega \sin(x + \vartheta)$$
$$= L'\vartheta_\omega + h'_\omega \sin(x + \vartheta), \tag{6.15}$$
$$L'_\omega = -h' \sin(x + \vartheta)\vartheta_\omega + h'_\omega \cos(x + \vartheta). \tag{6.16}$$

The differential equation (6.15) can be solved using the method of variation of constants. The solution is

$$\vartheta_\omega(x) - \vartheta_\omega(0) = e^{L(x)} \int_0^x e^{-L(\tau)} h'_\omega(\tau) \sin(\tau + \vartheta) d\tau. \tag{6.17}$$

Lemma 6.2 yields that

$$|\vartheta_\omega(x) - \vartheta_\omega(0)| \leq \int_0^x |h'_\omega(\tau)|d\tau,$$

and the estimate (5.3) in Lemma 5.1 gives the first part of (6.14). To derive the second part, we integrate (6.16) and apply again the integration-by-parts technique of Lemma 6.2,

$$|L_\omega(x) - L_\omega(0)|$$
$$\leq \left| \int_0^x \frac{h'\vartheta_\omega}{1 + h'\sin(\tau + \vartheta)} \left(\frac{d}{d\tau} \cos(\tau + \vartheta) \right) d\tau + \int_0^x h'_\omega \cos(\tau + \vartheta)d\tau \right|$$
$$\leq 2|h'\vartheta_\omega|(x) + 2|h'\vartheta_\omega|(0) + 2 \int_0^x \left(|h''\vartheta_\omega| + h'^2|\vartheta_\omega| + |h'\vartheta'_\omega| \right) + \int_0^x |h'_\omega|.$$
$$(6.18)$$

Using the estimates of Lemma 5.1 and Lemma 6.2, the only problematic term is the integral of $|h'\vartheta'_\omega|$. But from (6.15) and (4.12) we have

$$|h'\vartheta'_\omega| = |h'^2 \sin(x + \vartheta)\vartheta_\omega + h'h'_\omega \sin(x + \vartheta)|$$
$$\leq |h'^2\vartheta_\omega| + |h'h'_\omega| \leq \frac{1}{\omega - m} \frac{c}{(1 + x)^2},$$

where in the last step we used (5.1) and the first part of (6.14).   □

We remark that by combining (6.16) and (4.12), we can write the L_ω-equation as

$$L'_\omega = -\vartheta'\vartheta_\omega + h'_\omega \cos(x + \vartheta).$$

Although this looks very similar to (6.15), it seems difficult to deduce the second part of (6.14) by integration (note that the total variation of ϑ need not be bounded uniformly in ω). This is the reason why we instead used an integration-by-parts argument.

We are now in the position to prove that in the integral representation (1.3), the contributions for $\omega > m$ decay in t at least at the rate $t^{-\frac{5}{6}-\varepsilon}$. Consider the two fundamental solutions $(\vartheta^{(b)}, L^{(b)})$, (6.1). For negative x, the function $h'(x)$ is smooth in ω. Furthermore, $h'(x)$ is computed to be

$$h'(x) = h'(r)\frac{dr}{du}\frac{du}{dx} \overset{(2.4)}{=} h'(r)\frac{\Delta}{r^2 + a^2}\frac{du}{dx}. \qquad (6.19)$$

Using that Δ decays exponentially as $u \to -\infty$, and that for large negative x, $u'(x)$ is bounded away from zero, we see that $h'(x)$ decays rapidly as $x \to -\infty$, locally uniformly in ω. Thus standard Gronwall estimates applied to the differential equations (4.12) yield that $(\vartheta^{(b)}(0), L^{(b)}(0))$ depends smoothly on ω. Hence Lemma 6.2 and Lemma 6.3 give us information on the transmission coefficients, namely

$$|\vartheta_\infty^{(b)}|, |L_\infty^{(b)}| \leq c \qquad \text{and} \qquad |\partial_\omega\vartheta_\infty^{(b)}|, |\partial_\omega L_\infty^{(b)}| \leq \frac{c}{\omega - m}. \qquad (6.20)$$

Next we consider the propagator (1.3) for x in a compact set K and Ψ_0 with compact support. Again, standard Gronwall estimates starting from the event horizon yield that the fundamental solutions $\Psi_a^{k\omega n}(x)$ depend smoothly on ω, uniformly for $x \in K$. Hence

the only non-smooth terms are the coefficients $t_{ab}^{k\omega n}$. According to Lemma 3.2, these coefficients have the same regularity as the transmission coefficients, (6.20). Furthermore, Theorem 2.1 allows us to restrict attention to a neighborhood of $\omega = m$, and thus we may assume that the square bracket in (1.3) has compact support. We conclude that this square bracket satisfies the assumption of Lemma 3.1 (with $\alpha = \omega - m$), and thus its Fourier transform decays like $t^{-\frac{5}{6}-\varepsilon}$.

7. The Region $\omega < m$

For $\omega < m$, the coefficients t_{ab} in the integral representation (1.3) have the simple explicit form (2.15), and thus our task is to analyze the ω-dependence of $\Psi_1^{k\omega n}(x)$. We again work in the variables (ϑ, L) and set

$$\phi(x) = x + \vartheta(x).$$

Recall that $\Psi_1^{k\omega n}$ is the fundamental solution with exponential decay at infinity. The following lemma shows that this implies that $\lim_{x\to\infty} \phi(x) = -\infty$.

Lemma 7.1. *There is a constant C independent of ω such that for all $x > 0$,*

$$\phi(x) < C - \log x. \tag{7.1}$$

Proof. Using the bounds (5.22) and that h' is positive, we have

$$\phi' \geq 1 - \frac{c}{2x} e^{-\phi} \tag{7.2}$$

with c independent of ω. Suppose that (7.1) were false for some $x = x_0$ and $C = \log c$. Then (7.2) implies that

$$\phi'(x) \geq \frac{1}{2}. \tag{7.3}$$

Hence at x, ϕ is monotone increasing, whereas the right side of (7.1) is monotone decreasing. As a consequence (7.1) is violated on an open interval $(x, x+\epsilon)$. Furthermore, by continuity (7.1) is violated on a closed set. We conclude that (7.1) is violated for all $x \in [x_0, \infty)$. This means that (7.3) holds for all $x \geq x_0$, and integration yields that

$$\lim_{x\to\infty} \phi(x) = \infty. \tag{7.4}$$

To finish the proof, we shall show that (7.4) implies that the corresponding two-spinor ψ, (4.13), grows exponentially at infinity, giving the desired contradiction (note that since $\Psi_1^{k\omega n}$ decays at infinity, X, $\tilde{X}$, and ψ also vanish at infinity, see (2.2), (4.1) and (7.17)). According to (7.4), ψ behaves for large x asymptotically as

$$\psi = e^{\frac{\phi-L}{2}} \begin{pmatrix} e^{\frac{h}{2}} \\ e^{-\frac{h}{2}} \end{pmatrix} (1 + \mathcal{O}(e^{-2\phi})).$$

Furthermore, using (4.12),

$$(\phi - L)' = 1 + h'(\sinh\phi - \cosh\phi) = 1 - h'e^{-\phi} \stackrel{(5.1)}{=} 1 + \mathcal{O}(e^{-\phi}).$$

Hence for large x, $\phi - L \sim x$, and so ψ grows asymptotically like $\psi \sim e^x$. $\square$

The inequality (7.1) shows in particular that

$$\phi(x) < -\frac{1}{2} \qquad \text{for all } x \geq x_1 \tag{7.5}$$

with $x_1 = \exp(C + \frac{1}{2})$. We next show that ϕ leaves the region $\{\phi < -\frac{1}{2}\}$ for positive x.

Lemma 7.2. *There is $x_0 \geq v > 0$ with v independent of ω such that*

$$\phi(x_0) \geq -\frac{1}{2}. \tag{7.6}$$

Proof. We introduce for $x > 0$ the function

$$\underline{\vartheta}(x) = \log \frac{1}{4} \int_x^\infty h'(\tau) e^{-\tau} d\tau.$$

Since by (5.22) the integrand is positive, $\underline{\vartheta}$ is monotone decreasing. According to (5.22) and (5.25),

$$\lim_{x \to \infty} \underline{\vartheta}(x) = -\infty, \qquad \lim_{x \to 0} \underline{\vartheta}(x) = \infty,$$

and so there is a unique x_0 with

$$\underline{\vartheta}(x_0) = -\frac{1}{2}.$$

Now, choosing $0 < y < z$, we have

$$\int_y^z h'(\tau) e^{-\tau} d\tau \geq e^{-z}(h(z) - h(y)).$$

Using (5.25), we see that for small y,

$$\int_y^\infty h'(\tau) e^{-\tau} d\tau > 4,$$

implying that x_0 is bounded away from zero, uniformly in ω.

We shall now prove that $\underline{\vartheta}(x)$ is a lower bound for ϑ, i.e.

$$\vartheta(x) > \underline{\vartheta}(x) \qquad \text{for all } x \geq x_0. \tag{7.7}$$

Thus in the region $x \geq x_1$, we apply (7.5) to get the estimate

$$\vartheta' = h' \sinh \phi < -\frac{1}{4} h' e^{-\phi} = -\frac{1}{4} h' e^{-(x+\vartheta)}.$$

We separate variables,

$$(e^\vartheta)'(x) < -\frac{1}{4} h'(x) e^{-x},$$

and integrating (using that $e^{\vartheta(\infty)} = 0$), we find

$$e^{\vartheta(x)} > \frac{1}{4} \int_x^\infty h'(\tau) e^{-\tau} d\tau.$$

Thus $\underline{\vartheta}$ is indeed a lower bound in the region $x \geq x_1$.

It remains to show that $\vartheta > \underline{\vartheta}$ on the interval $[x_0, x_1]$. Let us assume the contrary. Then ϑ and $\underline{\vartheta}$ meet somewhere on this interval. Let

$$y = \sup_{[x_0, x_1]} \{x | \vartheta(x) = \underline{\vartheta}(x)\}.$$

Then $\vartheta(y) = \underline{\vartheta}(y) < -\frac{1}{2}$, and thus

$$\vartheta'(y) < -\frac{1}{4} h' e^{-(x+\vartheta)} = -\frac{1}{4} h' e^{-(x+\underline{\vartheta})} = \underline{\vartheta}'(y).$$

This contradicts the fact that $\vartheta(x) > \underline{\vartheta}(x)$ for all $x > y$. $\quad\square$

The next lemma controls the behavior of ϕ near the origin and "matches" the solution across the singularity at $x = 0$.

Lemma 7.3. *Suppose that for given $\kappa_1 \leq 0$ and $\kappa_2 > 0$,*

$$-\frac{1}{2} \leq \phi(x) \leq 0 \qquad \text{for all } x \in (\kappa_1, \kappa_2). \tag{7.8}$$

Then there is $\tilde{\kappa}_1$ with $\tilde{\kappa}_1 < 0$, $\tilde{\kappa}_1 \leq \kappa_1$ and a parameter $\lambda \geq 0$ such that

$$\begin{cases} -\lambda e^{h(x)} + r(x) < \phi(x) < -\lambda e^{h(x)} & \text{for } \tilde{\kappa}_1 < x < 0 \\ -\lambda e^{h(x)} < \phi(x) < -\lambda e^{h(x)} + r(x) & \text{for } 0 < x < \kappa_2 \end{cases} \tag{7.9}$$

with

$$r(x) = e^{h(x)} \int_0^x e^{-h(\tau)} d\tau. \tag{7.10}$$

Note that the function $r(x)$, (7.10) is finite according to (5.25).

Proof of Lemma 7.3. Let us first assume that $\kappa_1 < 0$. We set $\tilde{\kappa}_1 = \kappa_1$. We choose η with $0 < \eta < \min(-\kappa_1, \kappa_2)$. For negative x, ϕ satisfies according to (4.12), the equation $\phi' = 1 + h' \sin \phi$. Using the bounds (7.8) as well as the fact that h' is negative, we obtain that

$$h'(\phi + \phi^2) < \phi' < 1 + h'\phi \tag{7.11}$$

for all $x \in (\kappa_1, 0)$. We choose x in the interval $(\kappa_1, -\eta)$ and consider the inequality (7.11) on the interval $(x, -\eta)$. The inequality on the lhs can be solved by separation of variables and the rhs by variation of constants. This gives the explicit bounds

$$e^{h(x)-h(-\eta)}\phi(-\eta) + e^{h(x)} \int_x^{-\eta} e^{-h(\tau)} d\tau < \phi(x) < \frac{\alpha}{1-\alpha} \qquad \text{for } -\kappa < x < -\eta \tag{7.12}$$

with

$$\alpha = e^{h(x)-h(-\eta)} \frac{\phi(-\eta)}{1 + \phi(-\eta)}. \tag{7.13}$$

If x is positive, then according to (4.14), ϕ satisfies the equation $\phi' = 1 + h' \sinh \phi$. Using (7.8) and that h' is now positive, we get the bounds

$$-1 - h'\phi < -\phi' < -h'(\phi - \phi^2) \qquad \text{on } (0, x). \tag{7.14}$$

We choose x in the interval (η, κ) and integrate these bounds from η to x. This gives the bounds

$$\frac{\beta}{1 + \beta} < \phi(x) < e^{h(x) - h(\eta)} \phi(\eta) + e^{h(x)} \int_\eta^x e^{-h(\tau)} d\tau \qquad \text{for } \eta < x < \kappa \tag{7.15}$$

with

$$\beta = e^{h(x) - h(\eta)} \frac{\phi(\eta)}{1 - \phi(\eta)}.$$

We now show that

$$\lim_{\eta \searrow 0} \phi(-\eta) = 0 = \lim_{\eta \searrow 0} \phi(\eta). \tag{7.16}$$

Consider $\phi(-\eta)$. From (7.12) and (7.8) we have for fixed x in the interval $-\kappa < x < -\eta$,

$$-\frac{1}{2} \leq \phi(x) < \frac{\alpha}{1 - \alpha} < 0, \tag{7.17}$$

and thus there is some $\alpha_0 < 0$ for which $\alpha > \alpha_0$ if η is sufficiently small. According to (5.25), the factor $e^{h(x) - h(-\eta)}$ in (7.13) tends to $+\infty$ as $\eta \searrow 0$. We conclude from (7.13) that

$$\lim_{\eta \searrow 0} \frac{\phi(-\eta)}{1 + \phi(-\eta)} = 0,$$

implying the lhs of (7.16). A similar argument using the rhs of (7.15) gives the rhs of (7.16).

Since the planar equation (4.6) has smooth coefficients, it is obvious that $\psi(u)$ is smooth and non-zero. Using the ansatz' (4.10) and (4.13) as well as (7.16), we see that the following limits exist,

$$\lim_{\eta \searrow 0} e^{-\frac{L(-\eta) - h(-\eta)}{2}} \begin{pmatrix} 1 \\ e^{-h(-\eta)} \frac{\phi(-\eta)}{2} \end{pmatrix} = \psi|_{x=0} = \lim_{\eta \searrow 0} e^{-\frac{L(\eta) - h(\eta)}{2}} \begin{pmatrix} 1 \\ e^{-h(\eta)} \frac{\phi(\eta)}{2} \end{pmatrix}.$$

We consider the two cases $\lim_{\eta \searrow 0}(L(\eta) - h(\eta)) = 0$ and $\neq 0$ separately. In the first case, the second components must have a non-zero limit (because $\psi(0) \neq 0$), and thus $\lim_{\eta \searrow 0} e^{-h(\pm\eta)} \phi(\pm\eta) = -\infty$. In the second case, the limits $\lim_{\eta \searrow 0} e^{-h(\pm\eta)} \phi(\pm\eta)$ must exist and be equal. We conclude that

$$\lim_{\eta \to 0} e^{-h(-\eta)} \phi(-\eta) = -\lambda = \lim_{\eta \to 0} e^{-h(\eta)} \phi(\eta) \tag{7.18}$$

for some $\lambda \in [0, \infty]$. In the case $\kappa_1 = 0$, this matching of the two ansatz' shows that $\phi \leq 0$ for negative x, and thus we can make κ_1 slightly negative and repeat the above construction. Again using (7.17), one deduces that λ must in fact be finite. We finally take the limit $\eta \to 0$ in (7.12) and (7.15) to obtain (7.9). $\quad\square$

Our next goal is to bound $\vartheta(x_{\min})$ uniformly in ω. To this end, we combine the a-priori estimates for large x (Lemmas 7.1 and Lemma 7.2) with the estimates in a neighborhood of $x = 0$ (Lemma 7.3). For negative x outside of this neighborhood we can use similar methods as in Lemma 6.3.

Lemma 7.4. *There is $c > 0$ such that for all $\omega \in (m - \delta, m)$,*

$$|\vartheta(x_{min})| \leq c.$$

Proof. First let us verify that the assumptions of Lemma 7.3 are satisfied for a particular choice of κ_1 and κ_2. To this end, observe that $\phi(x)$ has no zero for $x > 0$, because otherwise

$$\phi'(x) = 1 + h'(x) \sinh \phi = 1,$$

violating the fact that $\phi'(x) \leq 0$ at the largest zero (recall that Lemma 7.1 implies that ϕ is negative for large x). Thus

$$\phi(x) < 0 \qquad \text{for all } x > 0. \tag{7.19}$$

As a consequence, $\sinh \phi < \phi$, and thus using (5.22),

$$\phi'(x) < 1 + h'(x)\phi \qquad \text{for all } x > 0.$$

Integrating this inequality from a given positive $x < x_0$ to x_0 and using (7.6), we obtain the lower bound

$$\phi(x) > \underline{\phi}(x) = -\frac{1}{2} e^{h(x)-h(x_0)} - e^{h(x)} \int_x^{x_0} e^{-h(\tau)} d\tau \qquad \text{for } 0 < x < x_0 \tag{7.20}$$

(this is indeed quite similar to the second part of (7.15), but now we have solved for ϕ at the lower limit of the integration range). According to (5.25), $\lim_{x \to 0} \underline{\phi}(x) = 0$. We conclude that the assumptions (7.8) are satisfied for $\kappa_1 = 0$ and $\kappa_2 > 0$ sufficiently small. We can further decrease κ_1 and increase κ_2, provided that the bounds in (7.9) all take values in the strip $(-\frac{1}{2}, 0)$.

The parameter λ in (7.9) can be bounded a-priori. Namely, were λ sufficiently large, we would get a contradiction to (7.20), whereas a very small value of λ would be inconsistent with (7.5). Thus we can find parameters $0 < \lambda_{min} < \lambda_{max}$ such that

$$\lambda_{min} < \lambda < \lambda_{max}.$$

As a consequence, in (7.9) the lower bound for $\lambda = \lambda_{max}$ and the upper bound for $\lambda = \lambda_{min}$ are a-priori bounds for ϕ. We choose x_2 such that these bounds take values in the strip $(-\frac{1}{2}, 0)$ on the interval $[x_2, 0]$. Then we have a-priori bounds for $\phi(x_2)$, and thus also for $\vartheta(x_2) = \phi(x_2) - x_2$,

$$\vartheta_{min} < \vartheta(x_2) < \vartheta_{max}. \tag{7.21}$$

The bounds (7.21) are uniform in ω. This is not surprising since the differential equation for ϑ involves only h', which according to Lemma 5.2 is bounded uniformly in ω. To see this rigorously, one must be careful because λ_{min} and λ_{max} *do* depend on ω. Namely, according to (5.25), h involves the additive constant $\frac{1}{3} \log \varepsilon^2$, which diverges as $\omega \nearrow m$. This implies, according to (7.18), that

$$\varepsilon^{\frac{2}{3}} \vartheta_{min}, \qquad \varepsilon^{\frac{2}{3}} \vartheta_{max}$$

can be chosen uniformly in ε. Using these scalings in (7.9), one sees that the estimates for ϑ and x_2 are indeed uniform.

It remains to control ϑ on the interval $[x_{\min}, x_2]$. According to (5.22), there is $R > 0$ independent of ω such that

$$1 + h'(x) \sin(x + \vartheta) > \frac{1}{2} \qquad \text{for } x \in [x_{\min} + R, x_2 - R]$$

(note that this last interval is non-empty in view of Lemma 5.3). On the bounded intervals $[x_{\min}, x_{\min} + R)$ and $(x_2 - R, x_2]$ we can control ϑ directly by integrating the equations in a method similar to (6.12). In the intermediate region, we integrate by parts and obtain similar to (6.13),

$$|\vartheta(x_{\min} + R) - \vartheta(x_2 - R)|$$
$$\le 2(|h'(x_{\min} + R)| + |h'(x_2 - R)|) + 4 \int_{x_{\min}+R}^{x_2-R} (|h''| + h'^2) d\tau,$$

and the terms on the right are all uniformly bounded according to Lemma 5.2. □

The next lemma controls the ω-dependence of ϑ.

Lemma 7.5. *There is $c > 0$ such that for all $\omega \in (m - \delta, m)$,*

$$|\vartheta_\omega(x_{min})| \le \frac{c}{m - \omega}.$$

Proof. In the proof of Lemma 7.4, we have verified that the hypothesis of Lemma 7.3 are satisfied, and thus $\vartheta(0) = 0$ for all ω. Hence $\vartheta_\omega(x_{\min})$ is obtained by integrating the differential equation (6.15) from $x_{\min}$ to zero. This gives in analogy to (6.17),

$$\vartheta_\omega(x_{\min}) = e^{-L(x_{\min})} \int_{x_{\min}}^0 e^{-L(\tau)} h'_\omega(\tau) \sin \phi(\tau) d\tau.$$

By definition of $\Psi_1^{k\omega n}$, $\lim_{u \to -\infty} L(u) = 1$ (see [4, eqn (3.31)] and (4.10)). Standard Gronwall estimates on the interval $(-\infty, u_{\min})$ show that $L(x_{\min})$ is bounded uniformly in ω. Furthermore, it was shown in Lemma 5.2 that $(m - \omega)h_\omega$ has bounded total variation. Thus to finish the proof, it suffices to show that there is c independent of ω such that

$$\left| e^{-L(\tau)} \sin \phi(\tau) \right| \le c \qquad \text{for all } \tau \in [x_{\min}, 0). \tag{7.22}$$

The integration-by-parts technique of Lemma 6.3 yields that L is uniformly bounded in the region $[x_{\min}, x_2]$ with x_2 as in the proof of Lemma 7.4 (for more details see the last paragraph of Lemma 7.4, where this method is used to estimate ϑ). On the interval $(x_2, 0)$, the a-priori bounds for ϕ, (7.9), show that

$$|\sin \phi(\tau)| \le |\phi(\tau)| \le c e^{h(\tau)-h(x_2)} \tag{7.23}$$

(with c independent of ω). Furthermore,

$$(h - L)' = h'(1 - \cos \phi) \le |h'|\phi^2 \le c x^{-\frac{1}{3}},$$

where in the last step we used (7.23), (5.25), and (5.22). Since $x^{-\frac{1}{3}}$ is integrable,

$$(h - L)|_{x_2}^\tau \le c.$$

We exponentiate and use that $L(x_2)$ is bounded to obtain

$$e^{-L(\tau)} e^{h(\tau)-h(x_2)} \le c. \tag{7.24}$$

The inequality (7.22) follows by combining (7.23) and (7.24). □

8. Proof of the Decay Rates

We are now ready to finish the proof of Theorem 1.1. In view of Theorem 2.1 and the consideration in the last paragraph of Sect. 6, it remains to show that the contribution to the propagator (1.2) for $\omega \in (m - \delta, m)$ has the decay (1.6). Since the coefficients t_{ab} are trivial for $\omega < m$, (2.15), the contribution to the propagator (1.3) simplifies to

$$\Psi(t, x) = \frac{1}{\pi} \sum_{|k| \leq k_0} \sum_{|n| \leq n_0} \int_{m-\delta}^{m} d\omega\, e^{-i\omega t} \left[\sum_{a,b=1}^{2} \Psi_1^{k\omega n}(x) \langle \Psi_1^{k\omega n} | \Psi_0 \rangle \right]. \qquad (8.1)$$

Since Ψ_0 has compact support, it suffices to analyze the ω-dependence of $\Psi_1^{k\omega n}(u)$ for u in a compact set.

According to the separation ansatz (2.2), we must only analyze the radial function X (the angular part Y is clearly smooth in ω). To see the ω-dependence of X in detail, we substitute (4.10) into (4.5) and (4.1). This gives, exactly as in the case $|\omega| > m$, the formula (6.7). For fixed u, the function h in (6.7) depends smoothly on ω. Using that h vanishes at the event horizon (because $\lim_{r \searrow r_1} f(r) = \omega = \lim_{r \searrow r_1} g(r)$ according to (4.7) and (4.3),(4.4)), our normalization condition for $\Psi_1^{k\omega n}$ near the event horizon (2.9) yields that

$$1 = \lim_{u \to -\infty} |X(u)|^2 = 2 \lim_{u \to -\infty} e^{-L(u)}$$

and thus $\lim_{u \to -\infty} L(u) = \log 2$, independent of ω. Furthermore, an argument similar to (6.19) shows that $h'(u)$ has exponential decay as $u \to -\infty$. Hence standard Gronwall estimates yield that $L(u)$ is bounded and depends smoothly on ω. Furthermore, Gronwall estimates in the finite region between $u_{\min}$ and u show that the difference $\vartheta(u) - \vartheta(u_{\min})$ is uniformly bounded and smooth in ω. Writing

$$(x + \vartheta)(u) = (x + \vartheta)(u_{\min}) + ((x(u) - x(u_{\min})) + (\vartheta(u) - \vartheta(u_{\min}))),$$

we conclude that the only possible non-smooth terms in (6.7) are the factors $\cos(\phi_{\min}/2)$ and $\sin(\phi_{\min}/2)$ with $\phi_{\min} \equiv x_{\min} + \vartheta(x_{\min})$.

We next consider the factors $\langle \Psi_1^{k\omega n} | \Psi_0 \rangle$ in (8.1). Again from Gronwall estimates, one sees that for $\omega > m$, the expectation values $\langle \Psi_2^{k\omega n} | \Psi_0 \rangle$ depend smoothly on ω, and thus our assumption (1.5) implies that

$$r_2 \equiv \lim_{\omega \searrow m} \langle \Psi_2^{k\omega n} | \Psi_0 \rangle \neq 0.$$

Except for the additional phase factors, the expectation values are smooth even for $\omega < m$. To compute the phases, we consider (6.7) in the asymptotic regime $u \to -\infty$, and compare with (2.7) and (2.8). This shows that for $\omega \in (m - \delta, m)$,

$$\langle \Psi_1^{k\omega n} | \Psi_0 \rangle = r_1 \alpha_1 \exp\left(-i \frac{\phi_{\min}}{2}\right) - r_2 \alpha_2 \exp\left(i \frac{\phi_{\min}}{2}\right) \qquad (8.2)$$

with coefficients α_a which depend smoothly on ω and are non-zero (indeed $\lim_{\omega \nearrow m} \alpha_a = 1$). Since the factor r_2 is non-zero, we conclude that $\langle \Psi_1^{k\omega n} | \Psi_0 \rangle$ has a non-vanishing

contribution which oscillates like $\exp(i\phi_{\min}/2)$. Using (8.2) and (6.7) in (8.1), we can write the propagator in the region $\omega \in (m - \delta, m)$ as the Fourier integral

$$\int_{m-\delta}^{m} e^{-i\omega t} \left(s_1 e^{-i(x_{\min} + \vartheta(x_{\min}))} + s_2 + s_3 e^{i(x_{\min} + \vartheta(x_{\min}))} \right) d\omega$$

with coefficients s_j which are smooth in ω and $s_3 \neq 0$. According to Lemma 5.3, Lemma 7.4, and Lemma 7.5, the three contributions to this Fourier integral satisfy the hypotheses of Lemma 3.2, Lemma 3.1, and Lemma 3.3, respectively (with $\alpha = m - \omega$). Hence the first two terms decay like $t^{-\frac{5}{6}-\varepsilon}$, whereas the last term gives the desired decay rate $\sim t^{-\frac{5}{6}}$. This concludes the proof of Theorem 1.1.

9. Probability Estimates

We now proceed with the proof of Theorem 1.2. We want to compute the probability p, (1.7). We begin with the following lemma.

Lemma 9.1. *For any Schwartz function $f \in \mathcal{S}(\mathbb{R} \times \mathbb{R})$, let $A_{\pm}$ be defined by*

$$A_{\pm} = \lim_{t \to \infty} \int_{-\infty}^{u_0} du \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' e^{-i(\omega - \omega')(t \pm u)} f(\omega, \omega').$$

Then

$$A_{+} = 2\pi \int_{-\infty}^{\infty} f(\omega, \omega) d\omega \qquad and \qquad A_{-} = 0. \tag{9.1}$$

Proof. We integrate by parts to obtain

$$\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' e^{-i(\omega - \omega')(t \pm u)} f(\omega, \omega')$$

$$= \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \frac{1}{(t \pm u)^2 + 1} \left((\partial_{\omega} + 1)(\partial_{\omega'} + 1) e^{-i(\omega - \omega')(t \pm u)} \right) f(\omega, \omega')$$

$$= \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \frac{1}{(t \pm u)^2 + 1} e^{-i(\omega - \omega')(t \pm u)} g(\omega, \omega'),$$

where g is the Schwartz function

$$g(\omega, \omega') = (-\partial_{\omega} + 1)(-\partial_{\omega'} + 1) f(\omega, \omega'). \tag{9.2}$$

Since the factor $((t \pm u)^2 + 1)^{-1}$ is integrable in u, we can integrate over u, apply Fubini, and use Lebesgue's dominated convergence theorem to take the limit $t \to \infty$ inside the integrand,

$$A_{\pm} = \lim_{t \to \infty} \int_{-\infty}^{u_0} du \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \frac{1}{(t \pm u)^2 + 1} e^{-i(\omega - \omega')(t \pm u)} g(\omega, \omega')$$

$$= \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' g(\omega, \omega') \lim_{t \to \infty} \int_{-\infty}^{u_0} \frac{1}{(t \pm u)^2 + 1} e^{-i(\omega - \omega')(t \pm u)} du. \tag{9.3}$$

In the case "$t - u$", we introduce a new integration variable $\alpha = t - u$ and get for the inner integral

$$\lim_{t \to \infty} \int_{-\infty}^{u_0} du \frac{1}{(t \pm u)^2 + 1} e^{-i(\omega - \omega')(t-u)} = \lim_{t \to \infty} \int_{t-u_0}^{\infty} \frac{1}{\alpha^2 + 1} e^{-i(\omega - \omega')\alpha} d\alpha = 0.$$

This proves that $A_- = 0$.

In the case "$t + u$", we obtain similarly an integral over the real line, which can be computed with residues,

$$\lim_{t \to \infty} \int_{-\infty}^{u_0} du \frac{1}{(t \pm u)^2 + 1} e^{-i(\omega - \omega')(t+u)} = \int_{-\infty}^{\infty} \frac{1}{\alpha^2 + 1} e^{-i(\omega - \omega')\alpha} d\alpha = \pi e^{-|\omega - \omega'|}.$$

We substitute this formula as well as (9.2) into (9.3) and integrate by parts "backwards",

$$
\begin{aligned}
A_+ &= \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' g(\omega, \omega') \pi e^{-|\omega - \omega'|} \\
&= \pi \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' f(\omega, \omega') \left((\partial_\omega + 1)(\partial_{\omega'} + 1) e^{-|\omega - \omega'|} \right).
\end{aligned}
\tag{9.4}
$$

A short explicit calculation shows that the derivatives can be computed in the distributional sense to be

$$(\partial_\omega + 1)(\partial_{\omega'} + 1) e^{-|\omega - \omega'|} = 2\delta(\omega - \omega').$$

Substitution into (9.4) gives the desired formula for A_+. $\quad\square$

We remark that the above lemma cannot be obtained by naively interchanging the orders of integration.

Theorem 9.2. *The probability q for the Dirac particle to disappear into the event horizon, defined for any $\varepsilon > 0$ by*

$$q = \lim_{t \to \infty} \int_{\{r_1 < r < r_1 + \varepsilon\}} (\overline{\Psi} \gamma^j \Psi)(t, x) \nu_j d\mu, \tag{9.5}$$

is given by

$$q = \frac{1}{\pi} \sum_{|k| \le k_0} \sum_{|n| \le n_0} \int_{-\infty}^{\infty} d\omega \sum_{a,b=1}^{2} s_{ab}^{k\omega n} \langle \Psi_b^{k\omega n} | \Psi_0 \rangle \langle \Psi_0 | \Psi_a^{k\omega n} \rangle \tag{9.6}$$

with

$$s_{ab}^{k\omega n} = \begin{cases} \delta_{a1}\delta_{b1} & \text{if } |\omega| \le m \\ 2t_{2a}^{k\omega n} t_{2b}^{k\omega n} & \text{if } |\omega| > m. \end{cases} \tag{9.7}$$

We remark that $p + q = 1$, since we know from [4] that the probability for the Dirac particle to be in any compact set tends to zero as $t \to \infty$.

Proof of Theorem 9.2. In the variable u, we need to compute the probability for the Dirac particle to be in the region $u < u_0$, where u_0 may be chosen as small as we like. Thus we can work with the asymptotic formulas near the event horizon, with error terms which decay exponentially fast as $u_0 \to -\infty$. More precisely, a straightforward calculation shows that the probability integral in (9.5) coincides asymptotically with the integral of the scalar product $\langle . | . \rangle$ on the transformed spinors (see [4, Eq. (2.15)]). Thus it suffices to consider the probability

$$q(t) = \int_{-\infty}^{u_0} du \int_{-1}^{1} d\cos\vartheta \int_{0}^{2\pi} d\varphi \langle \Psi | \Psi \rangle_{|(t,u,\vartheta,\varphi)}, \tag{9.8}$$

and let $t \to \infty$. Due to the additivity of the probabilities corresponding to the angular momentum modes (which are orthogonal with respect to the scalar product $\langle . | . \rangle$), it suffices to consider a solution of the Dirac equation with fixed angular momentum quantum numbers k and n, i.e.

$$\Psi(t, x) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \sum_{a,b=1}^{2} t_{ab}^{\omega} \Psi_a^{k\omega n}(x) \langle \Psi_b^{k\omega n} | \Psi_0 \rangle.$$

We substitute this formula for the propagator into (9.8) and carry out the angular integrals to obtain

$$q(t) = \frac{1}{\pi^2} \int_{-\infty}^{u_0} du \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' e^{-i(\omega-\omega')t}$$
$$\times \sum_{a,b,c,d=1}^{2} t_{ab}^{\omega} \langle \Psi^{\omega} | \Psi_0 \rangle \overline{t_{cd}^{\omega'} \langle \Psi^{\omega'} | \Psi_0 \rangle} \langle X_d^{\omega'} | X_a^{\omega} \rangle(u).$$

Substituting for X^{ω} the asymptotic formulas (2.5), valid near the event horizon, we obtain with an exponentially small error term

$$q(t) = \frac{1}{\pi^2} \int_{-\infty}^{u_0} du \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' e^{-i(\omega-\omega')t} \sum_{a,b,c,d=1}^{2}$$
$$\times t_{ab}^{\omega} \langle \Psi_b^{\omega} | \Psi_0 \rangle \overline{t_{cd}^{\omega'} \langle \Psi_d^{\omega'} | \Psi_0 \rangle} \left(\overline{f_{c+}^{\omega'}} f_{a+}^{w} e^{-i(\omega-\omega')u} + \overline{f_{c-}^{\omega'}} f_{a-}^{w} e^{i(\omega-\omega')u} \right). \tag{9.9}$$

Since we cannot expect the integrand to be smooth when ω or ω' is equal to $\pm m$, we must use an approximation argument. Namely, the integrand is bounded and has rapid decay in ω and ω'. Thus we can approximate the integrand in L^1 by a Schwartz function, and applying Lemma 9.1 we obtain

$$q = \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega \sum_{a,b,c,d=1}^{2} 2t_{ab}^{\omega} \overline{t_{cd}^{\omega} f_{c+}^{\omega}} f_{a+}^{\omega} \langle \Psi_b^{\omega} | \Psi_0 \rangle \overline{\langle \Psi_d^{\omega} | \Psi_0 \rangle}. \tag{9.10}$$

It remains to compute the factors $\overline{f_{c+}^{\omega}} f_{a+}^{\omega}$. In the case $|\omega| > m$, we conclude from (2.8) that

$$\overline{f_{c+}^{\omega}} f_{a+}^{\omega} = \delta_{c1} \delta_{a1} \qquad \text{(for } |\omega| \geq m). \tag{9.11}$$

On the other hand if $|\omega| \le m$, using (2.15) in (9.10) we must only compute $|f_{1+}^\omega|^2$. To this end, we again consider (6.7). Using that h vanishes asymptotically near the event horizon, one sees that

$$\lim_{r \searrow r_1} (|X^+|^2 - |X_-|^2) = 0,$$

and thus $|f_{1+}^\omega|^2 = |f_{1-}^\omega|^2$. Furthermore, our normalization of the fundamental solutions near the event horizon (2.9) yields that $|f_{1+}^\omega|^2 + |f_{1-}^\omega|^2 = 1$, and thus we conclude that

$$|f_{1+}^\omega|^2 = \frac{1}{2} \qquad \text{(for } |\omega| < m). \tag{9.12}$$

Substituting (9.11) and (9.12) into (9.10) and using (2.15) completes the proof. □

Proof of Theorem 1.2. Since the initial data is normalized by $\langle \Psi_0 | \Psi_0 \rangle = 1$, by taking the inner product of Ψ_0 with (1.3), evaluated at $t = 0$, we obtain that

$$1 = \frac{1}{\pi} \sum_{|k| \le k_0} \sum_{|n| \le n_0} \int_{-\infty}^\infty d\omega \sum_{a,b=1}^2 t_{ab}^{k\omega n} \langle \Psi_b^{k\omega n} | \Psi_0 \rangle \langle \Psi_0 | \Psi_a^{k\omega n} \rangle. \tag{9.13}$$

As remarked after the statement of Theorem 9.2, $p = 1 - q$. Thus p is obtained by taking the difference of (9.13) and (9.6). Using (2.15), we get (1.8).

For the proofs of **(i)**–**(iv)**, it again suffices to consider a fixed angular momentum mode. Since the energy distribution in the interval $[-m, m]$ is absent from (1.8), it is obvious that **(iii)** holds.

To prove **(ii)**, we introduce a vector $v^\omega \in \mathbb{C}^2$ by

$$v_a^\omega = \langle \Psi_a^\omega | \Psi_0 \rangle, \qquad a = 1, 2$$

and remark that in the region $|\omega| > m$ we can write the integrands in (9.13) and (1.8) as

$$\langle T^\omega v^\omega | v^\omega \rangle \qquad \text{and} \qquad \langle A^\omega v^\omega | v^\omega \rangle,$$

respectively, where, using Lemma 6.1,

$$T^\omega = \begin{pmatrix} 1/2 & t_{12}^\omega \\ t_{12}^\omega & 1/2 \end{pmatrix} \qquad \text{and} \qquad A^\omega = \begin{pmatrix} 0 & 0 \\ 0 & 1/2 - 2|t_{12}^\omega|^2 \end{pmatrix}.$$

An easy calculation shows that $T \ge A$. Thus from (9.13) and (2.15),

$$1 = \frac{1}{\pi} \int_{-m}^m d\omega |\langle \Psi_1^\omega | \Psi_1^\omega \rangle|^2 + \frac{1}{\pi} \int_{\mathbb{R} \setminus [-m,m]} d\omega \langle T^\omega v^\omega | v^\omega \rangle \tag{9.14}$$

$$\ge \frac{1}{\pi} \int_{-m}^m d\omega |\langle \Psi_1^\omega | \Psi_1^\omega \rangle|^2 + \frac{1}{\pi} \int_{\mathbb{R} \setminus [-m,m]} d\omega \langle A^\omega v^\omega | v^\omega \rangle \tag{9.15}$$

$$= \frac{1}{\pi} \int_{-m}^m d\omega |\langle \Psi_1^\omega | \Psi_1^\omega \rangle|^2 + p, \tag{9.16}$$

and this is strictly larger than p because in case **(ii)** the first summand is positive.

To prove **(i)**, we note that the factor $|\langle \Psi_2^\omega | \Psi_0 \rangle|^2$ is positive on a set of positive measure (by continuity in ω). Thus it suffices to show that

$$\frac{1}{2} - 2|t_{12}^\omega|^2 > 0 \qquad \text{for all } \omega \in \mathbb{R} \setminus [-m, m].$$

Using the explicit formula (6.4) in Lemma 6.1, this holds iff

$$|\tanh z| < 1 \tag{9.17}$$

with z as in (6.6). Using (4.12) together with (6.1), we see that $-2\pi < \vartheta_\infty^{(1)} - \vartheta_\infty^{(2)} < 2\pi$ (by the uniqueness theorem for ODEs). Then from (6.6),

$$-\frac{\pi}{4} < \arg z < \frac{\pi}{4}.$$

It follows that $|e^{2z} - 1| < |e^{2z} + 1|$, giving (9.17). This proves **(i)**.

Finally, if (1.5) holds, then we saw in (8.2) that $\langle \Psi_1^\omega | \Psi_0 \rangle$ is non-zero for $\omega \in (m - \delta, m)$. Thus **(iv)** follows from **(i)** and **(ii)**. $\square$

Given the fact that the Fourier transform of a C^∞ function with compact support is analytic, one might think that $\langle \Psi_b^{k\omega n} | \Psi_0 \rangle$ should be analytic in ω, implying that the cases **(ii)** and **(iii)** cannot occur. However, it is not at all obvious that the solutions of our ODEs should depend analytically on ω. Should this be the case, one could still make sense of **(ii)** and **(iii)** by slightly weakening the assumptions on the initial data.

We conclude by describing the class of initial data for which the Dirac particle must escape to infinity, with probability one.

Corollary 9.3. *The probability p is equal to one if and only if the initial data satisfies for all k, ω, and n the following conditions,*

$$\begin{cases} \langle \Psi_1^{k\omega n} | \Psi_0 \rangle = 0 & \text{if } |\omega| \leq m \\ \langle \Psi_1^{k\omega n} | \Psi_0 \rangle = -2t_{12}^{k\omega n} \langle \Psi_2^{k\omega n} | \Psi_0 \rangle & \text{if } |\omega| > m. \end{cases}$$

Proof. It again suffices to consider a fixed angular momentum mode. In view of (9.16), $p = 1$ only if

$$\int_{-m}^m d\omega |\langle \Psi_1^\omega | \Psi_1^\omega \rangle|^2 = 0, \tag{9.18}$$

and thus the energy distribution of the initial data must be supported in the outside the interval $(-m, m)$. Furthermore, the inequality in (9.15) must be replaced by equality, and thus

$$\langle S^\omega v^\omega | v^\omega \rangle = 0 \qquad \text{for all } \omega \in \mathbb{R} \setminus [-m, m], \tag{9.19}$$

where the matrix S^ω is defined by

$$S^\omega = T^\omega - A^\omega = \begin{pmatrix} 1/2 & t_{12}^\omega \\ t_{12}^\omega & 2|t_{12}^\omega|^2 \end{pmatrix}.$$

The eigenvalues of S^ω are zero and $\frac{1}{2} + 2|t_{12}^\omega|^2 > 0$. Hence (9.19) implies that v^ω must be in the kernel of S^ω, i.e.

$$v_1^\omega = -2t_{12}^\omega v_2^\omega. \tag{9.20}$$

Conversely, if (9.18) and (9.20) hold, it is obvious from (9.14)–(9.16) that $p = 1$. $\square$

One can also understand directly why $p = 1$ for special choices of the initial data. Indeed, to obtain such initial data, one can consider the physical situation where a Dirac particle at time $t = -\infty$ comes in from spatial infinity. If we take as our initial data the corresponding $\Psi(t, x)$ at $t = 0$ and reverse the direction of time, the solution to this Cauchy problem will clearly escape to infinity with probability one.

References

1. Beyer, H.: On the stability of the Kerr metric. Commun. Math. Phys. **221**, 659–676 (2001)
2. Carter, B.: Black hole equilibrium states. In: *Black holes/Les astres occlus*, Ecole d' été Phys. Théor., Les Houches, 1972
3. Chandrasekhar, S.: The solution of Dirac's equation in Kerr geometry. Proc. Roy. Soc. Lond. A **349**, 571–575 (1976)
4. Finster, F., Kamran, N., Smoller, J., Yau, S.-T.: The long-time dynamics of Dirac particles in the Kerr–Newman black hole geometry. gr-qc/0005088
5. Kay, B., Wald, R.: Linear stability of Schwarzschild under perturbations which are nonvanishing on the bifurcation 2-sphere. Class. Qu. Grav. **4**, no. 4, 893–898 (1987)
6. Koyama, H., Tomimatsu, A.: Asymptotic tails of massive scalar fields in a Reissner-Nordström background. Phys. Rev. D **63**, 064032 (2001)
7. Koyama, H., Tomimatsu, A.: Asymptotic tails of massive scalar fields in a Schwarzschild background. Phys. Rev. D **64**, 044014 (2001)
8. Price, R.: Nonspherical perturbations of relativistic gravitational collapse. I. Scalar and gravitational perturbations. Phys. Rev. D (3)**5**, 2419–2438 (1972)
   Price, R.: Nonspherical perturbations of relativistic gravitational collapse. II. Integer-spin, zero-rest-mass fields. Phys. Rev. D (3)**5**, 2439–2454 (1972)
9. Whiting, B.: Mode stability of the Kerr black hole. J. Math. Phys. **30**, no. 6, 1301–1305 (1989)

Communicated by H. Nicolai