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Ricci-Flat Metrics, Harmonic Forms and Brane Resolutions

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Abstract: We discuss the geometry and topology of the complete, non-compact, Ricci-flat Stenzel metric, on the tangent bundle of S^{n+1}. We obtain explicit results for all the metrics, and show how they can be obtained from first-order equations derivable from a superpotential. We then provide an explicit construction for the harmonic self-dual (p, q)-forms in the middle dimension $p + q = (n + 1)$ for the Stenzel metrics in $2(n+1)$ dimensions. Only the (p, p)-forms are L^2-normalisable, while for (p, q)-forms the degree of divergence grows with $|p-q|$. We also construct a set of Ricci-flat metrics whose level surfaces are $U(1)$ bundles over a product of N Einstein-Kähler manifolds, and we construct examples of harmonic forms there. As an application, we construct new examples of deformed supersymmetric non-singular M2-branes with such 8-dimensional transverse Ricci-flat spaces. We show explicitly that the fractional D3-branes on the 6-dimensional Stenzel metric found by Klebanov and Strassler is supported by a pure (2, 1)-form, and thus it is supersymmetric, while the example of Pando Zayas-Tseytlin is supported by a mixture of (1, 2) and (2, 1) forms. We comment on the implications for the corresponding dual field theories of our resolved brane solutions.

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1. Introduction

Fractional D3-branes have been extensively studied recently, since they can provide supergravity solutions that are dual to four-dimensional $N = 1$ super-Yang-Mills theories in the infra-red regime [1–8]. The idea is that by turning on fluxes for the R-R and NS-NS 3-form fields of the type IIB supergravity, in addition to the usual flux for the self-dual 5-form that supports the ordinary D3-brane, a deformed solution can be found that is free of the usual small-distance singular behaviour on the D3-brane horizon. This is achieved by first replacing the usual flat 6-metric transverse to the D3-brane by a non-compact Ricci-flat Kähler metric. It can then be shown that if there exists a suitable harmonic 3-form $G_{(3)}$ satisfying a complex self-duality condition, then the type IIB equations of motion are satisfied if the R-R and NS-NS fields are set equal to the real and imaginary parts of the harmonic 3-form, with the usual harmonic function H of the D3-brane solution now satisfying the modified equation $\Box H = -\frac{1}{12} m^2 |G_{(3)}|^2$ in the transverse space. A key feature of the type IIB equations that allows such a solution to arise is that there is a Chern-Simons or "transgression" modification in the Bianchi identity for the self-dual 5-form, bilinear in the R-R and NS-NS 3-forms.

The construction can be extended to encompass other examples of p-brane solutions, and in [6] a variety of such cases were analysed. These included heterotic 5-branes, dyonic strings, M2-branes, D2-branes, D4-branes and type IIA and type IIB strings. The case of M2-branes was also discussed in [9]. In all these cases, the ability to construct deformed solutions depends again upon the existence of certain Chern-Simons or transgression terms in Bianchi identities or equations of motion. The additional field strength contribution that modifies the standard p-brane configuration then comes from an appropriate harmonic form in the transverse space. One again replaces the usual flat transverse space by a more general complete non-compact Ricci-flat manifold. In order to get deformed solutions that are still supersymmetric, a necessary condition on this

manifold is that it must have an appropriate special holonomy that admits the existence of covariantly-constant spinors.

One can easily establish that if the harmonic form is L^2-normalisable, then it is possible to choose integration constants in such a way that the deformed solution is completely non-singular [6]. In particular, it can be arranged that the horizon is completely eliminated, with the metric instead smoothly approaching a regular "endpoint" at small radial distances. At large distances, the metric then has the same type of asymptotic structure as in the undeformed case, with a well-defined ADM mass per unit spatial world-volume. If, on the other hand, the harmonic form in the transverse manifold is not L^2-normalisable, then the deformed solution will suffer from some kind of pathology. Usually, one chooses a harmonic form that is at least square-integrable in the small-radius regime, and this can be sufficient to allow a solution which gives a useful infra-red description of the dual super-Yang-Mills theory.

If the harmonic form fails to be square-integrable at large radius, then this will lead to some degree of pathology in the asymptotic structure of the deformed solution in that region. For example, the deformed KS D3-brane solution [2] is based on a non-normalisable harmonic 3-form in the six-dimensional Ricci-flat Kähler transverse space, for which the integral of $|G_{(3)}|^2$ diverges as the logarithm of the proper distance at large radius. This leads to a deformed D3-brane metric that is complete and everywhere non-singular, and for which the harmonic function H has the asymptotic structure

$$H \sim c_0 + \frac{Q + m^2 \log \rho}{\rho^4} \tag{1.1}$$

at large proper distance ρ. Although the metric is still asymptotic to $dx^\mu\, dx_\mu + ds_c^2$, where ds_c^2 is the metric on the six-dimensional Ricci-flat conifold, the effect of the deformation involving the logarithm is that the associated ADM mass per unit 3-volume is no longer well-defined. This is because the effect of the $\log \rho$ term in H is to cause a slower fall-off at infinity than the normal ρ^{-4} dependence that picks up a finite and non-zero ADM contribution.[1] This change in the asymptotic structure implies that the solution may not admit an AdS_5 region, even when the constant c_0 in (1.1) goes to zero in a decoupling limit. Of course this feature is itself of great interest, since it is associated with a breaking of conformal symmetry in the dual field theory picture.

One might wonder whether there could be some other Ricci-flat Kähler 6-manifold for which an L^2-normalisable harmonic 3-form might exist. In fact rather general arguments establish that this is not possible, at least for the case where the 6-metric is asymptotically of the form of a cone, and the middle homology is one-dimensional.[2] On the other hand, L^2-normalisable harmonic forms can exist in non-compact Ricci-flat manifolds in other dimensions, and indeed some examples of fully resolved p-brane solutions based on such harmonic forms were obtained in [6]. We shall obtain further examples in this paper, using Ricci-flat Kähler 8-manifolds to obtain smooth deformed M2-branes. Since the ADM mass is then well-defined, the asymptotic structure correspondingly may still allow an approach to AdS, if the constant term in the metric function H goes to zero, implying that the dual field theory will still be a conformal one (three-dimensional in the case of M2-branes).

[1] For practical purposes, the ADM mass measured relative to the fiducial metric $dx^\mu\, dx_\mu + ds_c^2$ is a certain constant times the limit of $\rho^5\, \partial H/\partial\rho$ as ρ goes to infinity.

[2] We are grateful to Nigel Hitchin for extensive discussions on this point.

In this paper, we explore some of these questions in greater detail. To begin, in Sect. 2, we study the class of complete non-compact Ricci-flat Kähler manifolds whose metrics were constructed by Stenzel [10]. These are asymptotically conical, with level surfaces that are described by the coset space $SO(n+2)/SO(n)$, and they have real dimension $d = 2n+2$. The $n = 1$ example is the Eguchi-Hanson instanton [11], and the $n = 2$ example is the six-dimensional "deformed conifold" found by Candelas and de la Ossa [13]. It is this example that is used in the fractional D3-brane KS solution in [2]. In Sect. 2.1 we describe the geometry and topology of the general Stenzel manifolds, and then in Sect. 2.2 we carry out detailed calculations of the curvature, and show how Ricci-flat solutions can be obtained from a system of first-order equations derivable from a superpotential. In subsequent subsections we then obtain the explicit Ricci-flat Stenzel metrics and their Kähler forms, and then we derive integrability conditions for the covariantly-constant spinors.

In Sect. 3 we obtain explicit results for harmonic forms in the middle dimension, that is to say, for harmonic $(n+1)$-forms in the $2(n+1)$-dimensional Stenzel metrics.[3] More precisely, we construct harmonic (p, q)-forms for all integers p and q satisfying $p + q = n + 1$, where p and q count the number of holomorphic and antiholomorphic indices. We show that these are L^2-normalisable if and only if $p = q$, which can, of course, occur only in dimensions $d = 4p$.

In Sect. 4, we make use of some of these results in order to construct deformed p-brane solutions. Specifically, we first review the fractional D3-brane solution of [2]. Our results on harmonic forms allow us to give an explicit proof that their solution has a harmonic 3-form of type (2, 1), which therefore ensures supersymmetry. We then construct a smooth deformed M2-brane, using the L^2-normalisable (2, 2)-form in the 8-dimensional Stenzel metric. This is also supersymmetric.

In Sect. 5 we construct another class of complete non-compact Ricci-flat Kähler manifolds. These are again of the form of resolved cones, but in this case the level surfaces are themselves $U(1)$ bundles over the product of N Einstein-Kähler manifolds. Typical examples would be to take the base space to be $\mathcal{M} = \prod_{i=1}^{N} \mathbb{CP}^{m_i}$, for an arbitrary set of integers m_i. In fact the requirements of regularity of the metric mean that one of the factors in the base space $\mathcal{M}$ must be a complex projective space, but the others might be other Einstein-Kähler manifolds. Topologically, the total space is a $\mathbb{C}^k$ bundle over the remaining Einstein-Kähler factors.[4] Having obtained general results for Ricci-flat Kähler metrics in all the cases, we present some more detailed explicit formulae for three 8-dimensional examples, corresponding to taking the base space to be $S^2 \times \mathbb{CP}^2$, $\mathbb{CP}^2 \times S^2$ and $S^2 \times S^2 \times S^2$. We also discuss some well-known examples corresponding to complex line bundles over $\mathbb{CP}^m$.

In Sect. 6 we make use of our results for these Ricci-flat metrics, to obtain further examples of deformed p-brane solutions. We begin by considering the case where the base space is $\mathcal{M} = S^2 \times S^2$ (*i.e.* $m_1 = m_2 = 1$), meaning that the level surfaces are the 5-dimensional space known as $T^{1,1}$ or $Q(1, 1)$, which is a $U(1)$ bundle over $S^2 \times S^2$. Topologically, the 6-dimensional manifold is a $\mathbb{C}^2$ bundle over $\mathbb{CP}^1$. Its Ricci-flat metric is present in [13], and it was discussed recently in [5], where it was used to provide an alternative resolution of the D3-brane. We construct the self-dual harmonic 3-form that was used in [5] in a complex basis, and by this means demonstrate that it contains

[3] Nigel Hitchin has informed us that Daryl Noyce has independently constructed the unique harmonic form in the middle dimension in the $4N$-dimensional Stenzel manifolds.

[4] There are certain topological restrictions on the possible choices for the other Einstein-Kähler factors in the base space. For a detailed discussion, see [12].

both (2, 1) and (1, 2) pieces. This implies that the resolved D3-brane solution of [5] is not supersymmetric [6]. We also construct L^2-normalisable harmonic 4-forms of type (2, 2) in the 8-dimensional examples based on $S^2 \times \mathbb{CP}^2$ and $S^2 \times S^2 \times S^2$, and then use these in order to construct additional deformed M2-branes, which are supersymmetric. A further smooth deformed M2-brane example, which is non-supersymmetric, results from taking the 8-dimensional transverse space to be the complex line bundle over $\mathbb{CP}^3$. We also include a discussion of a fifth completely smooth deformed M2-brane, which was obtained previously in [6]. This solution uses an 8-manifold of exceptional Spin(7) holonomy rather than a Ricci-flat Kähler manifold. We give a simple proof of its supersymmetry.

The paper ends with conclusions and discussions in Sect. 7.

2. Stenzel Metrics

In this section we shall construct a sequence of complete non-singular Ricci-flat Kähler metrics, one for each even dimension, on the co-tangent bundle of the $(n + 1)$ sphere $T^\star S^{n+1}$. Restricted to the base space S^{n+1}, the metric coincides with the standard round sphere metric. The sequence, which begins with the Eguchi-Hanson metric for $n = 1$, was first constructed in generality by Stenzel [10] following a method discussed in [14]. The case $n = 2$ was originally given, in rather different guise, by Candelas and de la Ossa [13] as a "deformation" of the conifold. The isometry group of these metrics is $SO(n+2)$, acting in the obvious way on $T^\star S^{n+1}$. The principal (*i.e.* generic) orbits are of co-dimension one, corresponding to the coset $SO(n+2)/SO(n)$. There is a degenerate orbit (*i.e.* a generalized "bolt") corresponding to the zero section, *i.e.* to the base space $S^{n+1} \equiv SO(n+2)/SO(n+1)$. It is therefore possible to obtain the ordinary differential equations satisfied by the metric functions using coset techniques, and this we shall do shortly. Before doing so, however, we wish to make some comments about the geometry and topology of the metrics, which are intended to illuminate the subsequent calculations.

2.1. Geometrical and topological considerations. Any Kähler metric is necessarily symplectic, and in the present case the symplectic structure coincides with the standard symplectic structure on $T^\star S^{n+1}$. The sphere S^{n+1} is thus automatically a Lagrangian sub-manifold. In other words the Kähler form restricted to the $(n + 1)$-sphere vanishes. The complex structure on $T^\star S^{n+1}$ is however non-obvious, and arises from the fact that we may view $T^\star S^{n+1}$ as a complex quadric in $\mathbb{C}^{n+2}$,

$$z^a\, z^a = a^2, \tag{2.1}$$

where $a = 1, 2, \ldots, n + 2$. Setting

$$z^a = \cosh\left(\sqrt{p^b\, p^b}\right)\, x^a + \mathrm{i}\, \frac{\sinh\left(\sqrt{p^b\, p^b}\right)}{\sqrt{p_b\, p_b}}\, p_a, \tag{2.2}$$

one obtains $x^b\, x^b = a^2$ and $p_b\, x^b = 0$. These are the equations defining a point x^b lying on an $(n + 1)$-sphere of radius a in $\mathbb{E}^{n+1}$, and a cotangent vector p_b. Note that as the radius a is sent to zero we obtain the conifold, which makes contact with the work of Candelas and de la Ossa [13].

The strategy of Stenzel [10] is now to assume that the Kähler potential K depends only on the quantity

$$\tau = \bar{z}^a z^a = \cosh(2\sqrt{p_b\, p_b}). \tag{2.3}$$

From this it is clear that the principal orbits of the isometry group correspond to the surfaces of constant energy $H = \frac{1}{2} p_b\, p_b$ on the phase space $T^\star S^{n+1}$. The stabliser of each point on the orbit consists of rotations leaving fixed a point on S^{n+1} and a tangent vector p_b. The transitivity of the action is equally obvious. Thus $\sqrt{p_b\, p_b}$, or some function of it, it will serve as a radial variable.

In fact the level sets $H =$ constant can be viewed as circle bundles over the Grassmannian $SO(n+2)/(SO(n) \times SO(2))$. To see why, recall that the Hamiltonian H generates the geodesic flow on $T^\star S^{n+1}$. Each such geodesic is a great circle consisting of the intersection of a two-plane through the origin of $\mathbb{E}^{n+2}$ with the $(n+1)$-sphere. The circle factor in the denominator of the coset corresponds to the fact that geodesics or great circles are the orbits of a circle subgroup of the isometry group $SO(n+2)$ of the $(n+1)$-sphere.

Thus the circle fibre of the circle bundle is an orbit of the isometry group of the Ricci-flat Kähler metric. In terms of Kähler geometry, the quotient of $T^\star S^{n+1}$ by the circle action corresponds to the Marsden-Weinstein or symplectic quotient, and gives at each radius a homogeneous Kähler metric of two less dimensions.

At large distances the Stenzel metric tends to a Ricci-flat cone over the Einstein-Sasaski manifold $SO(n+2)/SO(n)$. At small radius the orbits collapse to the zero-section of $T^\star S^{n+1}$. Thus it is clear that the $(n+1)$-sphere $\Sigma \in H_{n+1}(T^\star S^{n+1})$ provides the only interesting homology cycle, and it is in the middle dimension. In the case that n is odd, its self-intersection number $\Sigma \cdot \Sigma \in \mathbb{Z}$ is, depending upon orientation convention, 2, while if n is even its self-intersection number vanishes. This is equivalent to the statement that the Euler characteristic of the even-dimensional spheres is 2, while for the odd-dimensional spheres it vanishes. To see this equivalence, recall that the topology of the co-tangent bundle is the same as that of the tangent bundle. Now the Euler characteristic of any closed orientable manifold is given by the number of intersections, suitably counted, of the zero section with any other section of its tangent bundle. In other words it is the number of zeros, suitably counted, of a vector field on the manifold.

We shall see that these facts have consequences for the cohomology. In the case of a closed $(2n+2)$-manifold $\mathcal{M}$ (*i.e.* compact, without boundary), one may use Poincaré duality to see that if α and β are closed middle-dimensional $(n+1)$-forms representing elements of $H^{n+1}(\mathcal{M})$, then the cup product $\alpha \cup \beta$ is an integer-valued bilinear form on $H^{n+1}(\mathcal{M})$ given by

$$\int_{\mathcal{M}} \alpha \wedge \beta\,. \tag{2.4}$$

The cup product is symmetric or skew-symmetric depending upon whether n is odd or even respectively. Thus if n is even,

$$\int_{\mathcal{M}} \alpha \wedge \alpha = 0\,. \tag{2.5}$$

Moreover, the Hodge duality operator $\star$ acts on $H^{n+1}(\mathcal{M})$, and

$$\star\star = (-1)^{n+1}\,. \tag{2.6}$$

Thus if n is odd, $H^{n+1}(\mathcal{M})$ decomposes into real self-dual or anti-self dual $(n+1)$ forms. Any such closed form must necessarily be harmonic, and its L^2 norm will be proportional to the self-intersection number. The total number of linearly-independent harmonic middle-dimensional forms will depend only on the topology of the closed manifold $\mathcal{M}$.

If n is even, we can find a complex basis of self-dual harmonic forms in L^2, but there is no relation between their normalisability and the integral in (2.4).

Our manifolds are non-compact, and the situation is therefore more complicated and we must proceed with caution. The usual one-to-one correspondence between harmonic forms and geometric cycles may break down. One generally expects at least as many L^2 harmonic forms as topology requires, but there may be more (cf. [15]). It is still true that L^2 harmonic forms must be closed and co-closed [16]. However, the notion of exactness must be modified since we are interested in whether closed forms in L^2 are the exterior derivatives of forms of one lower degree which are also in L^2. For example, the Taub-NUT metric admits an exact harmonic two-form in L^2, but it is the exterior derivative of a Killing 1-form which is not in L^2.

In the present case, if n is odd it seems reasonable to expect at least one harmonic form in the middle dimension, which is Poincaré dual to the $(n+1)$-sphere. Because the Stenzel metric behaves like a cone near infinity, all the Killing vectors are of linear growth. It follows [17] that any harmonic form must be invariant under the action of the isometry group. In the case of the Taub-NUT and Schwarzschild metrics, this observation permits the complete determination of the L^2 cohomology [17, 18]. We shall obtain an L^2 harmonic form in the middle dimension for all the Stenzel manifolds with odd n.

We obtain a general explicit construction of harmonic (p, q)-forms in all the Stenzel manifolds, where $p+q = n+1$. These middle-dimension harmonic forms include (p, p) forms when n is odd, and these are the L^2-normalisable examples mentioned above. All the others are non-normalisable, with a "degree of non-normalisability" that increases with $|p-q|$ at fixed $p+q$. In particular, this accords with the expectation that if n is even we should not find any harmonic form in L^2.[5]

2.2. Detailed calculations. Let L_{AB} be the left-invariant 1-forms on the group manifold $SO(n+2)$. These satisfy

$$dL_{AB} = L_{AC} \wedge L_{CB} \,. \tag{2.7}$$

We consider the $SO(n)$ subgroup, by splitting the index as $A = (1, 2, i)$. The L_{ij} are the left-invariant 1-forms for the $SO(n)$ subgroup. We make the following definitions:

$$\sigma_i \equiv L_{1i} \,, \qquad \tilde{\sigma}_i \equiv L_{2i} \,, \qquad \nu \equiv L_{12} \,. \tag{2.8}$$

These are the 1-forms in the coset $SO(n+2)/SO(n)$. We have

$$\begin{aligned} &d\sigma_i = \nu \wedge \tilde{\sigma}_i + L_{ij} \wedge \sigma_j \,, \quad d\tilde{\sigma}_i = -\nu \wedge \sigma_i + L_{ij} \wedge \tilde{\sigma}_j \,, \quad d\nu = -\sigma_i \wedge \tilde{\sigma}_i \,, \\ &dL_{ij} = L_{ik} \wedge L_{kj} - \sigma_i \wedge \sigma_j - \tilde{\sigma}_i \wedge \tilde{\sigma}_j \,. \end{aligned} \tag{2.9}$$

[5] Nigel Hitchin and Tamas Hausel have both pointed out to us that results of Atiyah, Patodi and Singer on asyptotically cylindrical manifolds [19] and some propeties of Kähler manifolds used in [17] can be extended to asymptotically conical metrics, and they imply that the L^2 cohomology is toplogical, *i.e.* isomorphic to the compactly-supported cohomolgy in ordinary cohomolgy. The results reported here are consistent with those theorems. We thank them for helpful communications.

Note that the 1-forms L_{ij} lie outside the coset, and so one finds that they do not appear eventually in the expressions for the curvature (see also [20]).

We now consider the metric

$$ds^2 = dt^2 + a^2\sigma_i^2 + b^2\,\tilde{\sigma}_i^2 + c^2\,\nu^2\,, \tag{2.10}$$

where a, b and c are functions of the radial coordinate t, and then we define the vielbeins

$$e^0 = dt\,, \qquad e^i = a\,\sigma_i\,, \qquad e^{\tilde{i}} = b\,\tilde{\sigma}_i\,, \qquad e^{\tilde{0}} = c\,\nu\,. \tag{2.11}$$

Calculating the spin connection, we find

$$\begin{aligned}
&\omega_{0i} = -\frac{\dot{a}}{a}\,e^i\,, \qquad \omega_{0\tilde{i}} = -\frac{\dot{b}}{b}\,e^{\tilde{i}}\,, \qquad \omega_{0\tilde{0}} = -\frac{\dot{c}}{c}\,e^{\tilde{0}}\,,\\
&\omega_{\tilde{0}i} = B\,e^{\tilde{i}}\,, \qquad \omega_{\tilde{0}\tilde{i}} = -A\,e^i\,, \qquad \omega_{i\tilde{j}} = C\,\delta_{ij}\,e^{\tilde{0}}\,,\\
&\omega_{ij} = -L_{ij}\,, \qquad \omega_{\tilde{i}\tilde{j}} = -L_{ij}\,,
\end{aligned} \tag{2.12}$$

where a dot means d/dt, and

$$A = \frac{(a^2-b^2-c^2)}{2a\,b\,c}\,, \quad B = \frac{(b^2-c^2-a^2)}{2a\,b\,c}\,, \quad C = \frac{(c^2-a^2-b^2)}{2a\,b\,c}\,. \tag{2.13}$$

From this, we obtain the curvature 2-forms

$$\begin{aligned}
\Theta_{0i} &= -\frac{\ddot{a}}{a}\,e^0\wedge e^i - \left(\frac{\dot{a}}{b\,c} + \frac{C\,\dot{b}}{b} + \frac{B\,\dot{c}}{c}\right) e^{\tilde{0}}\wedge e^{\tilde{i}}\,,\\
\Theta_{0\tilde{i}} &= -\frac{\ddot{b}}{b}\,e^0\wedge e^{\tilde{i}} + \left(\frac{\dot{b}}{a\,c} + \frac{C\,\dot{a}}{a} + \frac{A\,\dot{c}}{c}\right) e^{\tilde{0}}\wedge e^i\,,\\
\Theta_{0\tilde{0}} &= -\frac{\ddot{c}}{c}\,e^0\wedge e^{\tilde{0}} + \left(\frac{\dot{c}}{a\,b} + \frac{B\,\dot{a}}{a} + \frac{A\,\dot{b}}{b}\right) e^i\wedge e^{\tilde{i}}\,,\\
\Theta_{ij} &= \left(\frac{1}{a^2} - \frac{\dot{a}^2}{a^2}\right) e^i\wedge e^j + \left(\frac{1}{b^2} - B^2\right) e^{\tilde{i}}\wedge e^{\tilde{j}}\,,\\
\Theta_{\tilde{i}\tilde{j}} &= \left(\frac{1}{b^2} - \frac{\dot{b}^2}{b^2}\right) e^{\tilde{i}}\wedge e^{\tilde{j}} + \left(\frac{1}{a^2} - A^2\right) e^i\wedge e^j\,,\\
\Theta_{i\tilde{j}} &= A\,B\,e^{\tilde{i}}\wedge e^j - \frac{\dot{a}\,\dot{b}}{a\,b}\,e^i\wedge e^{\tilde{j}} - \frac{C\,c}{a\,b}\,\delta_{ij}\,e^k\wedge e^{\tilde{k}} + \left(\dot{C} + \frac{C\,\dot{c}}{c}\right)\delta_{ij}\,e^0\wedge e^{\tilde{0}}\,,\\
\Theta_{\tilde{0}i} &= -\left(\frac{\dot{a}\,\dot{c}}{a\,c} + A\,C + \frac{B\,b}{a\,c}\right) e^{\tilde{0}}\wedge e^i + \left(\dot{B} + \frac{B\,\dot{b}}{b}\right) e^0\wedge e^{\tilde{i}}\,,\\
\Theta_{\tilde{0}\tilde{i}} &= -\left(\frac{\dot{b}\,\dot{c}}{b\,c} + B\,C + \frac{A\,a}{b\,c}\right) e^{\tilde{0}}\wedge e^{\tilde{i}} - \left(\dot{A} + \frac{A\,\dot{a}}{a}\right) e^0\wedge e^i\,.
\end{aligned} \tag{2.14}$$

This implies that the Ricci tensor is diagonal, and that its vielbein components are given by

$$\begin{aligned}
R_{00} &= -\frac{n\,\ddot{a}}{a} - \frac{n\,\ddot{b}}{b} - \frac{\ddot{c}}{c}\,,\\
R_{\tilde{0}\tilde{0}} &= -\frac{\ddot{c}}{c} - n\left(\frac{\dot{a}\,\dot{c}}{a\,c} + \frac{\dot{b}\,\dot{c}}{b\,c} + \frac{(a^2-b^2)^2-c^4}{2a^2\,b^2\,c^2}\right),\\
R_{ij} &= \left[-\frac{\ddot{a}}{a} + (n-1)\left(\frac{1}{a^2} - \frac{\dot{a}^2}{a^2}\right) - \frac{n\,\dot{a}\,\dot{b}}{a\,b} - \frac{\dot{a}\,\dot{c}}{a\,c} + \frac{a^4-(b^2-c^2)^2}{2a^2\,b^2\,c^2}\right]\\
&\quad\times\delta_{ij}\,,\\
R_{\tilde{i}\tilde{j}} &= \left[-\frac{\ddot{b}}{b} + (n-1)\left(\frac{1}{b^2} - \frac{\dot{b}^2}{b^2}\right) - \frac{n\,\dot{a}\,\dot{b}}{a\,b} - \frac{\dot{b}\,\dot{c}}{b\,c} + \frac{b^4-(a^2-c^2)^2}{2a^2\,b^2\,c^2}\right]\delta_{ij}\,,
\end{aligned} \tag{2.15}$$

Defining $a = e^{\alpha}$, $b = e^{\beta}$, $c = e^{\gamma}$, and introducing the new coordinate η by $a^n\,b^n\,c\,d\eta = dt$, we find that the Ricci-flat equations can be derived from the Lagrangian $L = T - V$, where

$$\begin{aligned}
T &= \alpha'\,\gamma' + \beta'\,\gamma' + n\,\alpha'\,\beta' + \tfrac{1}{2}(n-1)\,\alpha'^2 + \tfrac{1}{2}(n-1)\,\beta'^2\,,\\
V &= \tfrac{1}{4}(a\,b)^{2n-2}\,(a^4+b^4+c^4-2a^2\,b^2-2n\,a^2\,c^2-2n\,b^2\,c^2)\,,
\end{aligned} \tag{2.16}$$

where a prime means $d/d\eta$, together with the constraint that the Hamiltonian vanishes, $T + V = 0$. (Note that the Hamiltonian comes from the G_{00} component of the Einstein tensor.)

Writing the Lagrangian as $L = \frac{1}{2}g_{ij}\,(d\alpha^i/d\eta)\,(d\alpha^j/d\eta) - V$, where $\alpha^i = (\alpha, \beta, \gamma)$, we find that the potential can be written in terms of a superpotential, as

$$V = -\tfrac{1}{2}g^{ij}\,\frac{\partial W}{\partial\alpha^i}\,\frac{\partial W}{\partial\alpha^j} \tag{2.17}$$

with

$$W = \tfrac{1}{2}(a\,b)^{n-1}\,(a^2+b^2+c^2)\,. \tag{2.18}$$

It follows that the Lagrangian can be written, after dropping a total derivative, as

$$L = \tfrac{1}{2}g_{ij}\left(\frac{d\alpha^i}{d\eta} \pm g^{ik}\,\partial_k W\right)\left(\frac{d\alpha^j}{d\eta} \pm g^{j\ell}\,\partial_\ell W\right), \tag{2.19}$$

where $\partial_i W \equiv \partial W/\partial\alpha^i$. This implies that the second-order equations for Ricci-flatness are satisfied if the first-order equations $d\alpha^i/d\eta = \mp g^{ij}\,\partial_j W$ are satisfied. Thus we arrive at the first-order equations

$$\begin{aligned}
\dot{\alpha} &= \tfrac{1}{2}e^{-\alpha-\beta-\gamma}\,(e^{2\beta}+e^{2\gamma}-e^{2\alpha})\,,\\
\dot{\beta} &= \tfrac{1}{2}e^{-\alpha-\beta-\gamma}\,(e^{2\alpha}+e^{2\gamma}-e^{2\beta})\,,\\
\dot{\gamma} &= \tfrac{1}{2}n\,e^{-\alpha-\beta-\gamma}\,(e^{2\alpha}+e^{2\beta}-e^{2\gamma})\,,
\end{aligned} \tag{2.20}$$

where the dot again denotes the radial derivative d/dt. Note that in terms of the quantities defined in (2.13), these equations take the simple form

$$\dot{\alpha} + A = 0\,, \qquad \dot{\beta} + B = 0\,, \qquad \dot{\gamma} + n\,C = 0\,. \tag{2.21}$$

If we now make use of the first-order Ricci-flat conditions (2.20) in the expressions (2.14) for the curvature 2-forms, we find that they can be simplified to

$$
\begin{aligned}
\Theta_{0i} &= -\frac{\ddot{a}}{a}\left(e^0\wedge e^i - e^{\tilde{0}}\wedge e^{\tilde{i}}\right), \qquad \Theta_{0\tilde{i}} = -\frac{\ddot{b}}{b}\left(e^0\wedge e^{\tilde{i}} + e^{\tilde{0}}\wedge e^i\right),\\
\Theta_{0\tilde{0}} &= -\frac{\ddot{c}}{c}\left(e^0\wedge e^{\tilde{0}} + \frac{1}{n}\,e^i\wedge e^{\tilde{i}}\right), \qquad \Theta_{ij} = \left(\frac{1}{a^2} - \frac{\dot{a}^2}{a^2}\right)\left(e^i\wedge e^j + e^{\tilde{i}}\wedge e^{\tilde{j}}\right),\\
\Theta_{\tilde{i}\tilde{j}} &= \left(\frac{1}{b^2} - \frac{\dot{b}^2}{b^2}\right)\left(e^i\wedge e^j + e^{\tilde{i}}\wedge e^{\tilde{j}}\right),\\
\Theta_{i\tilde{j}} &= A\,B\left(e^{\tilde{i}}\wedge e^j - e^i\wedge e^{\tilde{j}} + \frac{2}{n}\,\delta_{ij}\,e^k\wedge e^{\tilde{k}}\right)\\
&\quad -\left(2A\,B + \frac{n\,C\,c}{a\,b}\right)\left(e^0\wedge e^{\tilde{0}} + \frac{1}{n}\,e^k\wedge e^{\tilde{k}}\right)\delta_{ij},\\
\Theta_{\tilde{0}i} &= -\left(\frac{\dot{a}\,\dot{c}}{a\,c} + A\,C + \frac{B\,b}{a\,c}\right)\left(e^{\tilde{0}}\wedge e^i + e^0\wedge e^{\tilde{i}}\right),\\
\Theta_{\tilde{0}\tilde{i}} &= -\left(\frac{\dot{b}\,\dot{c}}{b\,c} + B\,C + \frac{A\,a}{b\,c}\right)\left(e^{\tilde{0}}\wedge e^{\tilde{i}} - e^0\wedge e^i\right).
\end{aligned}
\tag{2.22}
$$

2.3. Covariantly-constant spinors. Since the Stenzel metrics are Kähler, it follows that if they are Ricci flat then there should be two covariantly-constant spinors η. The integrability condition is

$$R_{abcd}\,\Gamma^{cd}\,\eta = 0\,. \tag{2.23}$$

From the expressions for the curvature that we obtained in (2.22), we can then read off that the covariantly-constant spinors must satisfy

$$(\Gamma_{0i} - \Gamma_{\tilde{0}\tilde{i}})\,\eta = 0\,, \tag{2.24}$$

and it is easy to check that all the integrability conditions are satisfied if (2.24) is satisfied. It is useful to note that one can directly read off from (2.22) other consequent results (which can also be derived from (2.24)), such as $\Gamma_{ij}\,\eta = -\Gamma_{\tilde{i}\tilde{j}}\,\eta$.

The covariant-constancy condition $D\,\eta \equiv d\,\eta + \frac{1}{4}\omega_{ab}\,\Gamma^{ab}\,\eta = 0$ itself is now easily solved. From (2.12), and using the first-order equations (2.21) and the integrability relations (2.24), we find that η simply satisfies $d\,\eta = 0$. In other words, in the frame we are using the covariantly constant spinors have constant components, and satisfy the projection conditions (2.24). In fact we can reverse the logic, and *derive* the first-order equations (2.21) by requiring the existence of a covariantly-constant spinor η subject to the additional assumption that η has constant components. Equation (2.24) is then a further consequence. Moreover, the 2-form

$$\bar{\eta}\,\Gamma_{ab}\,\eta \tag{2.25}$$

is covariantly constant and may be normalised so that it squares to -1; in other words, it gives us the Kähler form.

2.4. Kähler form. From now on, we define a new radial coordinate r related to t by $dt = h\,dr$, where h can be chosen for convenience, and a prime will mean a derivative with respect to r. Thus the metric is now written as

$$ds^2 = h^2\,dr^2 + a^2\,\sigma_i^2 + b^2\,\tilde{\sigma}_i^2 + c^2\,\nu^2\,, \tag{2.26}$$

and the vielbein is

$$e^0 = h\,dr\,, \qquad e^i = a\,\sigma_i\,, \qquad e^{\tilde{i}} = b\,\tilde{\sigma}_i\,, \qquad e^{\tilde{0}} = c\,\nu\,. \tag{2.27}$$

It is easy to see that the Kähler form mentioned above is given by

$$J = -e^0 \wedge e^{\tilde{0}} + e^i \wedge e^{\tilde{i}} = -h\,c\,dr \wedge \nu + a\,b\,\sigma_i \wedge \tilde{\sigma}_i\,. \tag{2.28}$$

The closure of J follows from $(a\,b)' = h\,c$, which can be seen from the first-order equations (2.20). Further checking, using the spin connection (2.12), shows that J is indeed covariantly constant. Again, the logic could be reversed, and by requiring the existence of a covariantly-constant 2-form that squares to -1, one could derive the first-order equations (2.21).

From this, it follows that we can introduce a holomorphic tangent-space basis of complex 1-forms ϵ^α as follows:

$$\epsilon^0 \equiv -e^0 + \mathrm{i}\,e^{\tilde{0}}\,, \qquad \epsilon^i = e^i + \mathrm{i}\,e^{\tilde{i}}\,. \tag{2.29}$$

In terms of this, we have that the Kähler form is

$$J = \tfrac{\mathrm{i}}{2}\,\epsilon^\alpha \wedge \bar{\epsilon}^{\bar{\alpha}}\,, \tag{2.30}$$

and so it is manifestly of type $(1, 1)$ (one barred, one unbarred, complex index).

By looking at how other forms are expressed in terms of the complex holomorphic basis ϵ^α, we can see how they decompose into type (p, q) pieces, where p and q count the number of holomorphic and anti-holomorphic basis 1-forms in each term.

2.5. Explicit solutions for Ricci-flat Stenzel metrics. Here, we shall construct the explicit solutions to the first-order equations (2.20), for arbitrary n. This gives the class of Ricci-flat metrics on complete non-compact manifolds of dimension $d = 2n + 2$, as constructed by Stenzel. Starting from (2.20), and changing to the new radial coordinate r related to t by $dt = h\,dr$, we first make the coordinate gauge choice $h = c$. The first-order equations then give

$$\begin{aligned} &\alpha' - \beta' = -2\sinh(\alpha - \beta)\,, \qquad \alpha' + \beta' = e^{-\alpha-\beta+2\gamma}\,, \\ &\gamma' + \tfrac{1}{2}n\,(\alpha' + \beta') = n\,\cosh(\alpha - \beta)\,. \end{aligned} \tag{2.31}$$

The first equation gives $e^{\alpha-\beta} = \coth r$, the third gives $e^{\alpha+\beta} = k\,e^{-2\gamma/n}\,\sinh 2r$, where k is a constant, and then the second can be solved explicitly for γ. It is advantageous to introduce a function $R(r)$, defined by

$$R(r) \equiv \int_0^r (\sinh 2u)^n\,du\,. \tag{2.32}$$

Choosing $k = (n+1)^{-1/n}$ without loss of generality, the solution is then given by

$$\begin{aligned} a^2 &\equiv e^{2\alpha} = R^{1/(n+1)} \coth r \,, \\ b^2 &\equiv e^{2\beta} = R^{1/(n+1)} \tanh r \,, \\ h^2 = c^2 &\equiv e^{2\gamma} = \frac{1}{n+1} R^{-n/(n+1)} (\sinh 2r)^n \,, \end{aligned} \tag{2.33}$$

with the Ricci-flat metric taking the form

$$ds^2 = c^2\, dr^2 + c^2\, \nu^2 + a^2\, \sigma_i^2 + b^2\, \tilde{\sigma}_i^2 \,. \tag{2.34}$$

The integral (2.32) can be evaluated in general, in terms of a hypergeometric function:

$$R = \frac{2^n}{n+1} (\sinh r)^{n+1}\ {}_2F_1\left[\tfrac{1}{2}(1+n), \tfrac{1}{2}(1-n), \tfrac{1}{2}(3+n); -\sinh^2 r\right] . \tag{2.35}$$

For each n the result is expressible in relatively simple terms; for the first few values of n one has

$$\begin{aligned} n=1:&\ \ R = \sinh^2 r \,, \\ n=2:&\ \ R = \tfrac{1}{8}(\sinh 4r - 4r) \,, \\ n=3:&\ \ R = \tfrac{2}{3}(2+\cosh 2r)\ \sinh^4 r \,, \\ n=4:&\ \ R = \tfrac{1}{64}(24r - 8\sinh 4r + \sinh 8r) \,, \\ n=5:&\ \ R = \tfrac{2}{15}(19 + 18\cosh 2r + 3\cosh 4r)\ \sinh^6 r \,, \\ n=6:&\ \ R = \tfrac{1}{384}(-120r + 45\sinh 4r - 9\sinh 8r + \sinh 12r) \,. \end{aligned} \tag{2.36}$$

Note that when n is odd, one can always change to a new radial variable $z = \sinh r$ in terms of which the metric can be written using rational functions.

It is evident from (2.35) that at small r we shall have

$$R \sim \frac{2^n}{n+1} r^{n+1} \,, \tag{2.37}$$

and consequently, the metric near $r = 0$ takes the form

$$ds^2 \sim \left(\frac{2^n}{n+1}\right)^{1/(n+1)} \left[dr^2 + r^2\, \tilde{\sigma}_i^2 + \sigma_i^2 + \nu^2\right] . \tag{2.38}$$

Thus the radial coordinate runs from $r = 0$, where the metric approaches $\mathbb{R}^{n+1} \times S^{n+1}$ with an S^{n+1} "bolt," to the asymptotic region at $r = \infty$. Note that the S^{n+1} bolt at $r = 0$ is a Lagrangian submanifold; in other words, the Kähler form (2.28) vanishes when restricted to it.

When $n = 1$, the 4-dimensional metric is the Eguchi-Hanson instanton [11]. When $n = 2$, the 6-dimensional metric is the "deformed" conifold solution found by Candelas and de la Ossa [13]. For arbitrary n, the solutions were first obtained by Stenzel [10].

3. Harmonic Forms

3.1. Harmonic (p, q)-forms in $2(p + q)$ dimensions. Here, we present a general construction of harmonic forms in the "middle dimension," namely $(n + 1)$-forms in the $2(n + 1)$-dimensional Stenzel manifolds. These can be further refined as (p, q) forms where p and q denote the numbers of holomorphic and antiholomorphic indices on the form, and $p + q = n + 1$.

We begin by making the following ansatz for the (p, q) harmonic form:

$$\begin{aligned} G_{(p,q)} &= f_1\, \epsilon_{i_1\cdots i_{q-1} j_1\cdots j_p}\, \bar\epsilon^0 \wedge \bar\epsilon^{i_1} \wedge \cdots \wedge \bar\epsilon^{i_{q-1}} \wedge \epsilon^{j_1} \wedge \cdots \wedge \epsilon^{j_p} \\ &\quad + f_2\, \epsilon_{i_1\cdots i_{p-1} j_1\cdots j_q}\, \epsilon^0 \wedge \epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_{p-1}} \wedge \bar\epsilon^{j_1} \wedge \cdots \wedge \bar\epsilon^{j_q}\,, \end{aligned} \tag{3.1}$$

where f_1 and f_2 are functions of r. It is easy to see that the epsilon tensors cause each term in each sum to be a product of complex vielbeins in distinct subspaces each of complex dimension one,[6] and from this it follows that the Hodge dual is given by

$$*G_{(p,q)} = \mathrm{i}^{\,p-q}\, G_{(p,q)}\,. \tag{3.2}$$

Since $G_{(p,q)}$ is an eigenstate under $*$, it follows that the condition for harmonicity reduces to $dG_{(p,q)} = 0$.

It is useful first to note that from the expressions for the vielbeins in the Stenzel metrics, we can rewrite (3.1), up to an irrelevant constant factor, as

$$\begin{aligned} G_{(p,q)} &= f_1\, \epsilon_{i_1\cdots i_{q-1} j_1\cdots j_p}\, (dr + \mathrm{i}\,\nu) \wedge \bar h^{i_1} \wedge \cdots \wedge \bar h^{i_{q-1}} \wedge h^{j_1} \wedge \cdots \wedge h^{j_p} \\ &\quad + f_2\, \epsilon_{i_1\cdots i_{p-1} j_1\cdots j_q}\, (dr - \mathrm{i}\nu) \wedge h^{i_1} \wedge \cdots \wedge h^{i_{p-1}} \wedge \bar h^{j_1} \wedge \cdots \wedge \bar h^{j_q}\,, \end{aligned} \tag{3.3}$$

where

$$h^i \equiv \sigma_i\, \cosh r + \mathrm{i}\,\tilde\sigma_i\, \sinh r\,. \tag{3.4}$$

It is easy also to verify that

$$dh^i = \tfrac{1}{2}(\tanh r + \coth r)\,(dr - \mathrm{i}\,\nu) \wedge h^i + \tfrac{1}{2}(\tanh r - \coth r)\,(dr - \mathrm{i}\,\nu) \wedge \bar h^i\,. \tag{3.5}$$

Imposing $dG_{(p,q)} = 0$, we now find that the functions f_1 and f_2 satisfy the equations

$$\begin{aligned} f_1' + f_2' + 2(p\, f_1 + q\, f_2)\, \coth r = 0\,, \\ f_1' - f_2' + 2(p\, f_1 - q\, f_2)\, \tanh r = 0\,. \end{aligned} \tag{3.6}$$

These equations can be solved in terms of hypergeometric functions, to give

$$\begin{aligned} f_1 &= c_1\, q\, {}_2F_1\left[\tfrac{1}{2}p,\, \tfrac{1}{2}(q+1),\, \tfrac{1}{2}(p+q)+1;\, -(\sinh 2r)^2\right] \\ &\quad + c_2\, (\sinh 2r)^{-p-q}\, {}_2F_1\left[\tfrac{1}{2}(1-p),\, -\tfrac{1}{2}q,\, 1-\tfrac{1}{2}(p+q);\, -(\sinh 2r)^2\right], \\ f_2 &= -c_1\, p\, {}_2F_1\left[\tfrac{1}{2}q,\, \tfrac{1}{2}(p+1),\, \tfrac{1}{2}(p+q)+1;\, -(\sinh 2r)^2\right] \\ &\quad + c_2\, (\sinh 2r)^{-p-q}\, {}_2F_1\left[\tfrac{1}{2}(1-q),\, -\tfrac{1}{2}p,\, 1-\tfrac{1}{2}(p+q);\, -(\sinh 2r)^2\right], \end{aligned} \tag{3.7}$$

where c_1 and c_2 are arbitrary constants. Note that for any specific choice of the integers p and q these expressions reduce to elementary functions of r, so the occurrence of hypergeometric functions here is just an artefact of writing formulae valid for all p and q.

[6] There are no factors such as $\epsilon^1 \wedge \bar\epsilon^1$, for example. This also shows that these (p, q)-forms are entirely perpendicular to the Kähler form $J = \frac{\mathrm{i}}{2}\, \epsilon^\alpha \wedge \bar\epsilon^{\bar\alpha}$.

3.2. L^2-normalisable harmonic (p, p)-forms in $4p$ dimensions. In the special case where $p = q$, the above construction gives an harmonic (p, p)-form in the middle dimension of a Stenzel manifold of dimension $4p$. In this case, we find that with c_2 taken to be zero, the functions f_1 and f_2 in (3.7) become

$$f_1 = -f_2 = \frac{p\, c_1}{(\cosh r)^{2p}}\,, \tag{3.8}$$

and so the harmonic (p, p)-form $G_{(p,p)}$ is given by

$$\begin{aligned} G_{(p,p)} = {} & \frac{1}{(\cosh r)^{2p}}\, \epsilon_{i_1\cdots i_{p-1} j_1\cdots j_p}\, \bar{\epsilon}^0 \wedge \bar{\epsilon}^{i_1} \wedge \cdots \wedge \bar{\epsilon}^{i_{p-1}} \wedge \epsilon^{j_1} \wedge \cdots \wedge \epsilon^{j_p} \\ & - \frac{1}{(\cosh r)^{2p}}\, \epsilon_{i_1\cdots i_{p-1} j_1\cdots j_p}\, \epsilon^0 \wedge \epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_{p-1}} \wedge \bar{\epsilon}^{j_1} \wedge \cdots \wedge \bar{\epsilon}^{j_q}\,, \end{aligned} \tag{3.9}$$

(after scaling out an irrelevant constant factor.) It therefore has magnitude given by

$$|G_{(p,p)}|^2 = \frac{\text{constant}}{(\cosh r)^{4p}}\,. \tag{3.10}$$

Since the $2(n+1)$-dimensional Stenzel metric has $\sqrt{g} = \frac{1}{n+1}\,(\sinh 2r)^n$, and $n = 2p-1$ here, it follows that this harmonic form is L^2-normalisable (see footnote 2).

One can also express this normalisable harmonic form in terms of the original real vielbein basis. Doing so, we find

$$\begin{aligned} G_{(p,p)} = {} & \frac{1}{(\cosh r)^{n+1}} \\ & \times \sum_{s=0}^{m} \frac{m!}{s!\,(m-s)!} \Big[\epsilon_{i_1\cdots i_{2s+1} j_1\cdots j_{2m-2s}}\, e^{\tilde{0}} \wedge e^{i_1} \wedge \cdots \wedge e^{i_{2s+1}} \wedge e^{\tilde{j}_1} \wedge \cdots \wedge e^{\tilde{j}_{2m-2s}} \\ & \quad + \epsilon_{i_1\cdots i_{2m-2s} j_1\cdots j_{2s+1}}\, e^0 \wedge e^{i_1} \wedge \cdots \wedge e^{i_{2m-2s}} \wedge e^{\tilde{j}_1} \wedge c \cdots \wedge e^{\tilde{j}_{2s+1}} \Big]\,, \end{aligned} \tag{3.11}$$

where m is defined by $n = 2m + 1 = 2p - 1$. In fact another way to obtain the middle-dimension harmonic form when n is odd is to write down an ansatz of the form (3.11), with a different function of r for each term in the sum, and then impose closure. This leads to a recursive system of first-order differential equations for the functions, whose only solution giving an L^2 harmonic form is (3.11).

3.3. Non-normalisable harmonic (p, q)-forms. We saw above that the special case of a harmonic (p, p)-form in a Stenzel manifold of dimension $4p$ yields the simple expression (3.9) for an L^2-normalisable form. It is not hard to see that for any case other than $p = q$, the construction in Sect. 3.1 always gives harmonic (p, q)-forms that are not L^2-normalisable. A divergence in the integral of $|G_{(p,q)}|^2$ at $r = 0$ is avoided if the constant c_2 in (3.7) is chosen to be zero, but the integral diverges at large r unless $p = q$ (which can only occur in dimensions that are a multiple of 4, since in general the dimension is $2(p + q)$). In fact the degree of divergence becomes larger as $|p - q|$ becomes larger.

It follows from the above discussion that the "most nearly normalisable" harmonic (p, q)-form in a Stenzel manifold of dimension $4N + 2$ will be for the case $(p, q) = (N+1, N)$ (or its complex conjugate). One then finds that with $c_2 = 0$ the term involving f_2 dominates at large r, and that

$$|G_{(N+1,N)}|^2 \sim \frac{1}{(\sinh 2r)^{2N}} . \tag{3.12}$$

Since we have $\sqrt{g} = (\sinh 2r)^n/(n+1)$, and $n = 2N$ here, it follows that the harmonic $(N + 1, N)$-form is marginally not L^2 normalisable, and the integral of $|G_{(2N+1)}|^2$ diverges as the logarithm of the proper distance, at large radius.

Our findings for (p, q) middle-dimension harmonic forms, and especially, the fact that only in dimensions $4N$ can there exist L^2 harmonic forms, are consistent with the general discussion in Sect. 2.1.

3.4. Canonical form, and special Lagrangian submanifold. If we take $q = 0$, implying that $p = n + 1$ in the $2(n + 1)$-dimensional Stenzel manifold, then with $c_2 = 0$ we see from (3.7) that f_1 vanishes, while f_2 becomes a constant. This gives the so-called canonical form, of type $(n + 1, 0)$:

$$G_{(n+1,0)} = \epsilon^0 \wedge \epsilon^1 \wedge \cdots \wedge \epsilon^n . \tag{3.13}$$

It is easily verified that this is covariantly constant. From (3.3) and (3.4) we see that it restricts to

$$-\mathrm{i}\, \nu \wedge \sigma_1 \wedge \cdots \wedge \sigma_n \tag{3.14}$$

on the S^{n+1} bolt at $r = 0$. Thus $\Re(G_{(4)})$ restricted to the bolt vanishes. We have already seen that the Kähler form vanishes on the bolt, and so it follows that the bolt is a Special Lagrangian Submanifold. Hence it is a calibrated submanifold, and volume-minimising in its homology class; in other words, it is a supersymmetric cycle.

4. Applications: Resolved M2-Branes and D3-Branes

The sequence of Stenzel metrics begins with $n = 1$, which is the 4-dimensional Eguchi-Hanson metric. It admits a normalisable harmonic self-dual 2-form. It was shown in [6] that this can be used to smooth out the singularities in the heterotic 5-brane and in the dyonic string, including the singularity that is associated with the negative tension contribution in the dyonic string. The resolved solutions are smooth and supersymmetric, and have well-defined ADM masses. We refer the reader to [6] for details.

In this section, we review the construction of the deformed fractional D3-brane of [2], which uses the 6-dimensional Stenzel metric. We also construct a new resolved M2-brane using the 8-dimensional Stenzel metric. Both solutions are smooth and supersymmetric. The D3-brane does not have a well-defined ADM mass, whilst the M2-brane does.

4.1. Fractional D3-brane using the 6-dimensional Stenzel metric. The standard D3-brane can be deformed when the six-dimensional transverse space admits a harmonic self-dual 3-form. In the notation we shall use here, the general solution is given by [6]

$$\begin{aligned} d\hat{s}_{10}^2 &= H^{-1/2}\, dx^\mu\, dx^\nu\, \eta_{\mu\nu} + H^{1/2}\, ds_6^2\,, \\ F_{(5)} &= d^4x \wedge dH^{-1} + \hat{*} dH \qquad F_{(3)} = F_{(3)}^{\rm RR} + {\rm i}\, F_{(3)}^{\rm NS} = m\, G_{(3)}\,, \end{aligned} \tag{4.1}$$

where ds_6^2 is any six-dimensional Ricci-flat Kähler metric that admits a non-trivial complex harmonic self-dual 3-form $*G_{(3)} = {\rm i}\, G_{(3)}$, and $\hat{*}$ and $*$ are Hodge duals with respect to $d\hat{s}_{10}^2$ and ds_6^2 respectively. The function H satisfies that

$$\Box H = -\tfrac{1}{12} m^2\, |G_{(3)}|^2\,, \tag{4.2}$$

where $\Box$ is the scalar Laplacian in the 6-dimensional transverse space.

In [2], a particular fractional D3-brane was constructed where the six-dimensional Stenzel metric was used for the transverse ds_6^2, and we shall now review this solution. After making trivial redefinitions (including $r \longrightarrow r/2$) in order to adjust the conventions to those of [2], and taking $n = 2$, the solution found in Sect. 2.5 for the Stenzel metric becomes

$$\begin{aligned} &h^2 = \frac{1}{3K^2}\,, \quad a^2 = 2K\, \cosh^2(r/2)\,, \quad b^2 = 2K\, \sinh^2(r/2)\,, \quad c^2 = \frac{4}{3K^2}\,, \\ &\text{where} \quad K = \frac{(\sinh 2r - 2r)^{1/3}}{2^{1/3}\, \sinh r}\,, \end{aligned} \tag{4.3}$$

and the metric is then given by (2.26) with i running over 2 values. The Stenzel manifold is smooth, complete and non-compact, with r running from $r = 0$ to $r = \infty$.

In these conventions, the general result (3.7) yields a harmonic (2, 1) form

$$G_{(2,1)} = \frac{2(r\, \coth r - 1)}{\sinh^2 r}\, \bar{\epsilon}^0 \wedge \epsilon^1 \wedge \epsilon^2 - \frac{(\sinh 2r - 2r)}{2\sinh^3 r}\, \epsilon^0 \wedge (\epsilon^1 \wedge \bar{\epsilon}^2 + \bar{\epsilon}^1 \wedge \epsilon^2)\,. \tag{4.4}$$

This can be recognised as the self-dual harmonic 3-form constructed in [2], by noting that it can be expressed as

$$G_{(3)} = \omega_{(3)} - {\rm i} * \omega_{(3)}\,, \tag{4.5}$$

where

$$\omega_{(3)} = g_1\, e^0 \wedge e^1 \wedge e^2 + g_2\, e^0 \wedge e^{\tilde{1}} \wedge e^{\tilde{2}} + g_3\, e^{\tilde{0}} \wedge \Big(e^1 \wedge e^{\tilde{2}} - e^2 \wedge e^{\tilde{1}}\Big)\,, \tag{4.6}$$

and

$$g_1 = \frac{\sinh r - r}{\sinh r\, \sinh^2(r/2)}\,, \quad g_2 = \frac{\sinh r + r}{\sinh r\, \cosh^2(r/2)}\,, \quad g_3 = \frac{2(r\, \coth r - 1)}{\sinh^2 r}\,. \tag{4.7}$$

Calculating the norm of $G_{(3)}$, one obtains the result

$$\begin{aligned} |G_{(3)}|^2 &= 12 g_1^2 + 12 g_2^2 + 24 g_3^2 \\ &= \frac{96}{\sinh^6 r} \Big((3 + 2\sinh^2 r)\, r^2 - 3r\, \sinh 2r + 3\sinh^2 r + \sinh^4 r\Big)\,. \end{aligned} \tag{4.8}$$

Since for the metric we have $\sqrt{g} = \frac{2}{3}\sinh^2 r$, it follows that $G_{(3)}$ is not L^2 normalisable; it does not fall off sufficiently rapidly at large r.

It was argued in [2] that the self-dual harmonic 3-form was of type (2, 1), and then in [3, 4] arguments were presented that would show that the deformed D3-brane solution built using $G_{(3)}$ would be supersymmetric. Our explicit proof that $G_{(3)}$ is of type (2, 1) thus demonstrates the supersymmetry of the solution.

Because the L^2 norm of $G_{(3)}$ converges for small r, but diverges for large r, it follows that the function H is regular at small r, but does not fall off fast enough at large r to have a well-defined ADM mass. In fact, H has the large-r asymptotic behaviour given in (1.1).

4.2. Fractional M2-brane using the 8-dimensional Stenzel metric. As a consequence of the Chern-Simons modification to the equation of the motion of the 3-form potential in $D = 11$ supergravity, namely

$$d{*}\hat{F}_{(4)} = \tfrac{1}{2} F_{(4)} \wedge F_{(4)} \,, \tag{4.9}$$

it is possible to construct a deformed M2-brane, given by [9, 21, 6]

$$\begin{aligned} d\hat{s}_{11}^2 &= H^{-2/3}\, dx^\mu\, dx^\nu\, \eta_{\mu\nu} + H^{1/3}\, ds_8^2 \,, \\ F_{(4)} &= d^3x \wedge dH^{-1} + m\, G_{(4)} \,, \end{aligned} \tag{4.10}$$

where $G_{(4)}$ is the harmonic self-dual 4-form in the Ricci-flat transverse space ds_8^2, and the function H satisfies

$$\Box H = -\tfrac{1}{48} m^2\, G_{(4)}^2 \,. \tag{4.11}$$

Warped reductions of this type, were also discussed in [22–24].

In this section, we shall construct a deformed M2-brane using the 8-dimensional Stenzel metric for the transverse ds_8^2. In this case, the index i on σ_i and $\tilde{\sigma}_i$ in the metric (2.26) runs over 3 values. The Ricci-flat solution coming from the first-order equations (2.20) is given by

$$\begin{aligned} &a^2 = \tfrac{1}{3}(2+\cosh 2r)^{1/4}\, \cosh r \,, \qquad b^2 = \tfrac{1}{3}(2+\cosh 2r)^{1/4}\, \sinh r\, \tanh r \,, \\ &h^2 = c^2 = (2+\cosh 2r)^{-3/4}\, \cosh^3 r \,, \end{aligned} \tag{4.12}$$

with the metric then given by (2.26). The radial coordinate runs from $r = 0$ to $r = \infty$, and the metric lives on a smooth complete non-compact manifold.

In terms of the vielbein basis (2.27), we find from (3.11) that the following is an L^2-normalisable self-dual harmonic 4-form (of type (2, 2)):

$$\begin{aligned} G_{(4)} = {} & \frac{3}{\cosh^4 r}\left[e^{\tilde{0}} \wedge e^1 \wedge e^2 \wedge e^3 + e^0 \wedge e^{\tilde{1}} \wedge e^{\tilde{2}} \wedge e^{\tilde{3}}\right] \\ & + \frac{1}{2\cosh^4 r}\, \epsilon_{ijk}\left[e^0 \wedge e^i \wedge e^j \wedge e^{\tilde{k}} + e^{\tilde{0}} \wedge e^i \wedge e^{\tilde{j}} \wedge e^{\tilde{k}}\right] . \end{aligned} \tag{4.13}$$

We can easily see that

$$|G_{(4)}|^2 = \frac{360}{\cosh^8 r} \,. \tag{4.14}$$

The 8-dimensional Stenzel manifold can be used as the transverse space to construct the deformed M2-brane. The solution is given by

$$\begin{aligned} ds_{11}^2 &= H^{-2/3}\left(-dt^2 + dx_1^2 + dx_2^2\right) + H^{1/3}\, ds_8^2\,, \\ F_{(4)} &= dt \wedge dx_1 \wedge dx_2 \wedge dH^{-1} + m\, G_{(4)}\,. \end{aligned} \tag{4.15}$$

All the equations of motions are satisfied provided that

$$\Box H = -\tfrac{1}{48} m^2\, |G_{(4)}|^2\,, \tag{4.16}$$

where $\Box$ is the scalar Laplacian in the 8-dimensional transverse space. Since we have $\sqrt{g} = \frac{1}{216}\sinh^3(2r)$, assuming that H depends only on r, we have

$$\left(h^{-2}\,\sqrt{g}\,H'\right)' = -\frac{5m^2\,(\sinh 2r)^3}{144\cosh^8 r}\,. \tag{4.17}$$

The first integration can be performed straightforwardly, giving

$$H' = \frac{h^2}{\sqrt{g}}\left(\beta + \frac{7m^2\,\cosh 2r}{72\cosh^4 r}\right)\,, \tag{4.18}$$

where β is an arbitrary integration constant. In order for the solution regular at $r = 0$, we must have

$$\beta = -\frac{5m^2}{72}\,. \tag{4.19}$$

It is easier to perform the next integration by making a coordinate redefinition,

$$2 + \cosh 2r = y^4\,. \tag{4.20}$$

In terms of y, with β given in (4.19), the function H is then given by

$$\begin{aligned} H &= -\frac{15m^2}{\sqrt{2}}\int \frac{dy}{(y^4-1)^{5/2}} \\ &= c_0 - \frac{5m^2\,(5y^5 - 7y)}{4\sqrt{2}(y^4-1)^{3/2}} + \frac{25m^2}{4\sqrt{2}}\,F\left(\arcsin\left(\frac{1}{y}\right)|-1\right)\,, \end{aligned} \tag{4.21}$$

where c_0 is an integration constant, and $F(\phi|m)$ is the incomplete elliptic integral of the first kind,

$$F(\phi|m) \equiv \int_0^\phi (1 - m\,\sin^2\theta)^{-1/2}\, d\theta\,. \tag{4.22}$$

It is easy to verify that the function H is regular for r running from 0 to infinity. For $r = 0$, H is just a constant. At large r, the function H behaves as

$$H = c_0 + \frac{640m^2}{2187\rho^6} - \frac{20480\,2^{1/3}\,m^2}{28431\,3^{2/3}\,\rho^{26/3}} + \cdots\,, \tag{4.23}$$

where ρ is the proper distance, defined by $h\,dr = d\rho$. Thus the M2-brane has no singularity, and it has a well-defined ADM mass.

It is worth commenting further on the choice (4.19) for the integration constant β. The solution to (4.16) has two integration constants β and c_0, which originate from the fact that one can add to H any solution H_0 of the homogeneous equation

$$\Box H_0 = 0\,. \tag{4.24}$$

However, the solution for H_0 has a singularity at small distance, and so it requires an external delta-function source at the singularity. For the usual M2-brane with flat transverse space, the divergence of H signals a breakdown of our coordinate system, and the delta function corresponds to a smooth horizon. However, if, as in the present case, the transverse space is a smooth non-flat manifold, the singularity is real, and not just a coordinate artefact. In the case when there is a $|G_{(4)}|^2$ source, it is possible to find a smooth everywhere-bounded positive function H. In this case there is no breakdown of our coordinate system, nor is there a naked singularity, and we get a complete non-singular solution without an horizon.[7] The L^2 normalisability then guarantees finiteness of the ADM mass, as may be easily seen by integrating (4.16). Thus our choice for the constant β in (4.19) ensures that our solution is not only smooth and non-singular, but it is also free of any horizon, and is a rigorous supergravity solution. Any other choice of the constant β would give a solution that was singular, requiring an external source at the spacetime singularity on the horizon.

Let us now consider the supersymmetry of the deformed solution. From the $D = 11$ supersymmetry transformations, it follows that if any supersymmetry is to be preserved, the harmonic 4-form must satisfy:

$$\delta\psi_a = \tfrac{1}{288}\,(G_{bcde}\,\Gamma_{abcde} - 8G_{abcd}\,\Gamma_{bcd})\,\eta = 0\,. \tag{4.25}$$

Multiplying by Γ^a, we deduce that the two terms separately must give zero, and in fact the supersymmetry condition can be reduced to [9, 23]

$$G_{abcd}\,\Gamma_{bcd}\,\eta = 0\,. \tag{4.26}$$

Now from (4.13), the vielbein components of the 4-form are given by

$$G_{\tilde{0}ijk} = 3u\,\epsilon_{ijk}\,, \quad G_{0\tilde{i}\tilde{j}\tilde{k}} = 3u\,\epsilon_{ijk}\,, \quad G_{0ij\tilde{k}} = u\,\epsilon_{ijk}\,, \quad G_{\tilde{0}i\tilde{j}\tilde{k}} = u\,\epsilon_{ijk}\,, \tag{4.27}$$

where $u \equiv 1/\cosh^4 r$. Substituting into (4.26), we see that taking $a = 0$, i, $\tilde{i}$ and $\tilde{0}$ respectively, we obtain the following conditions that must be satisfied if there is to be preserved supersymmetry:

$$\begin{aligned}
a = 0:&\ \ \epsilon_{ijk}\,\Big(\Gamma_{\tilde{i}\tilde{j}\tilde{k}} + \Gamma_{ij\tilde{k}}\Big)\,\eta = 0\,,\\
a = i:&\ \ \epsilon_{ijk}\,\Big(3\Gamma_{\tilde{0}jk} + 2\Gamma_{0j\tilde{k}} + \Gamma_{\tilde{0}\tilde{j}\tilde{k}}\Big)\,\eta = 0\,,\\
a = \tilde{i}:&\ \ \epsilon_{ijk}\,\Big(3\Gamma_{0\tilde{j}\tilde{k}} + 2\Gamma_{\tilde{0}\tilde{j}k} + \Gamma_{0jk}\Big)\,\eta = 0\,,\\
a = \tilde{0}:&\ \ \epsilon_{ijk}\,\Big(\Gamma_{ijk} + \Gamma_{i\tilde{j}\tilde{k}}\Big)\,\eta = 0\,.
\end{aligned} \tag{4.28}$$

It is now a simple matter to show, using the integrability conditions (2.24) which we already established, that Eq. (4.28) are satisfied, for both of the covariantly-constant spinors on the Stenzel 8-manifold. In other words, turning on the deforming flux from the harmonic 4-form $G_{(4)}$ does not lead to any further breaking of supersymmetry, and so the resolved M2-brane preserves $\frac{1}{4}$ of the original supersymmetry.

[7] If the transverse space is a cone, and the harmonic function depends only on the radial coordinate, the singularity in H_0 corresponds to AdS_4 times the base of the cone.

5. Ricci-Flat Kähler Metrics on $\mathbb{C}^k$ Bundles

There are many possible ansätze that one can adopt for constructing classes of Ricci-flat metrics. A classic procedure is to look for metrics of cohomogeneity one, in which there are level surfaces composed of homogeneous manifolds, with arbitrary functions of radius parameterising homogeneous deformations of these surfaces.[8] The conditions for Ricci-flatness then reduce to ordinary second-order differential equations for these functions. If one is lucky, the equations are solvable and the solutions include ones that describe metrics on smooth complete manifolds. Indeed, the Stenzel construction that we studied in Sect. 2 is an example of this type. In cases where there are Ricci-flat solutions with special holonomy, such as hyper-Kähler, Kähler or the G_2 and Spin(7) exceptional cases, we have always found that first-order equations, derivable from a superpotential, can be constructed. All solutions of these satisfy the second-order equations, but the converse is not necessarily true.

In this section we study another general class of metrics of cohomogeneity one, where the level surfaces are taken to be $U(1)$ bundles over a product of N Einstein-Kähler manifolds, which would typically themselves be homogeneous. We then introduce $(N+1)$ arbitrary functions of the radial coordinate r, parameterising the volumes of the N base-space factors, and the length of the $U(1)$ fibres. Following the familiar pattern, we then calculate the curvature, derive the second-order equations for Ricci-flatness, and then look for a first-order system coming from a superpotential. Having done this, we are able to solve the equations and obtain complete non-compact Ricci-flat Kähler metrics.

The Ricci-flat solutions that we obtain here are such that the metric coefficient for one of the factors in the base space goes to zero at $r = 0$, as does the coefficient in the $U(1)$ fibre direction. This implies that this particular factor in the base space must be a complex projective space $\mathbb{CP}^m$, so that $r = 0$ can become the origin of spherical polar coordinates on $\mathbb{R}^{2k}$, where $k = m + 1$. If we write the base space as $\mathcal{M} = \mathbb{CP}^m \times \widetilde{\mathcal{M}}$, where $\widetilde{\mathcal{M}}$ denotes the product of the remaining Einstein-Kähler manifolds in the base, then the total manifold has the topology of a $\mathbb{C}^k$ bundle over $\widetilde{\mathcal{M}}$. The manifold has a bolt with the topology $\widetilde{\mathcal{M}}$ at $r = 0$. Global considerations impose constraints on the possible choices for the other Einstein-Kähler base space factors. See [12] for a detailed discussion.

Our principal focus will be on the case where all the Einstein-Kähler factors in the base space are taken to be complex projective spaces $\mathbb{CP}^{m_i}$. It is shown in [12] that if the metric coefficient for the $\mathbb{CP}^{m_1}$ factor is the one that goes to zero at $r = 0$, then regularity at $r = 0$ implies that the other $\mathbb{CP}^{m_i}$ factors must be such that

$$m_1 + 1 = \mathrm{gcd}(m_1 + 1, m_2 + 1, m_3 + 1, \dots)\,, \tag{5.1}$$

where gcd denotes the greatest common divisor of its arguments. The special case of just two factors, with the first being the trivial zero-dimensional manifold $\mathbb{CP}^0$, and $\widetilde{\mathcal{M}}$ being $\mathbb{CP}^m$, gives a well-known sequence of Ricci-flat manifolds on the complex line bundle over $\mathbb{CP}^m$. The $m = 1$ case is the Eguchi-Hanson instanton. We obtain an L^2-normalisable harmonic $(m+1)$-form for all the $\mathbb{C}^{m+1}$ bundles over $\mathbb{CP}^m$ where m is odd.

The special case of two factors $\mathbb{CP}^{m_1} \times \mathbb{CP}^{m_2}$ with $m_1 = m_2 = 1$, for which the base space is $S^2 \times S^2$ and the topology of the total space is a $\mathbb{C}^2$ bundle over $\mathbb{CP}^1$, is the

[8] See [25, 26] for a general discussion of such metrics.

6-dimensional "small resolution" of the conifold discussed in [13], and more recently in [5], as an alternative to the "deformation" of the conifold. We shall study this in some detail, and show that the non-normalisable harmonic 3-form used in [5] to construct a fractional D3-brane gives a non-supersymmetric solution. We shall also consider three other special cases in some detail, giving 8-dimensional examples where the base space is $S^2 \times \mathbb{CP}^2$ or $S^2 \times S^2 \times S^2$,[9] construct L^2-normalisable harmonic 4-forms in two of these manifolds, and use them to build further supersymmetric deformed M2-branes.

5.1. Curvature calculations, and superpotential. To begin with, since it illustrates most of the key features, we shall consider the case of a base space that is the product of just two factors, comprising Einstein-Kähler spaces of real dimensions n and $\tilde{n}$. In the next subsection, we shall present the general results for a product of N Einstein-Kähler spaces.

We make the following ansatz for metrics of cohomogeneity one whose level surfaces are $U(1)$ bundles over products of two Einstein-Kähler base spaces:

$$d\hat{s}^2 = dt^2 + a^2\, ds^2 + b^2\, d\tilde{s}^2 + c^2\, \sigma^2, \tag{5.2}$$

where a, b and c are functions of the radial coordinate t, ds^2 and $d\tilde{s}^2$ are Einstein-Kähler spaces of real dimensions n and $\tilde{n}$ respectively, and

$$\sigma = d\psi + A + \widetilde{A}\,. \tag{5.3}$$

The potentials A and $\widetilde{A}$, living in ds^2 and $d\tilde{s}^2$ respectively, have field strengths $F = dA$ and $\widetilde{F} = d\widetilde{A}$, given by $F = p\, J$, $\widetilde{F} = q\, \widetilde{J}$, where J and $\widetilde{J}$ are the Kähler forms on ds^2 and $d\tilde{s}^2$. Furthermore, we assume cosmological constants λ and $\tilde{\lambda}$ for the two spaces, so

$$R_{ij} = \lambda\, \delta_{ij}\,, \qquad \widetilde{R}_{ab} = \tilde{\lambda}\, \delta_{ab}\,, \qquad F_{ik}\, F_{jk} = p^2\, \delta_{ij}\,, \qquad \widetilde{F}_{ac}\, \widetilde{F}_{bc} = q^2\, \delta_{ab}\,. \tag{5.4}$$

Note that there is a considerable redundancy in the use of constants here, since λ and $\tilde{\lambda}$ could be absorbed into rescalings of the functions a and b. It is advantageous to keep all the constants λ, $\tilde{\lambda}$, p and q unfixed for now, since the choice of how to specify them most conveniently depends on what choice one makes for the Einstein-Kähler metrics in the base space.

In the orthonormal basis

$$\hat{e}^0 = dt\,, \qquad \hat{e}^{\tilde{0}} = c\,\sigma\,, \qquad \hat{e}^i = a\, e^i\,, \qquad \hat{e}^a = b\, e^a\,, \tag{5.5}$$

we find that the non-vanishing components of the Ricci tensor are

$$\begin{aligned}
\hat{R}_{00} &= -n\,\frac{\ddot{a}}{a} - \tilde{n}\,\frac{\ddot{b}}{b} - \frac{\ddot{c}}{c}\,,\\
\hat{R}_{\tilde{0}\tilde{0}} &= -n\,\frac{\dot{a}\,\dot{c}}{a\,c} - \tilde{n}\,\frac{\dot{b}\,\dot{c}}{b\,c} - \frac{\ddot{c}}{c} + \frac{n\,p^2\,c^2}{4a^4} + \frac{\tilde{n}\,q^2\,c^2}{4b^4}\,,\\
\hat{R}_{ij} &= -\left(\frac{\ddot{a}}{a} + \frac{\dot{a}\,\dot{c}}{a\,c} + (n-1)\,\frac{\dot{a}^2}{a^2} + \tilde{n}\,\frac{\dot{a}\,\dot{b}}{a\,b} - \frac{\lambda}{a^2} + \frac{p^2\,c^2}{2a^4}\right)\delta_{ij}\,,\\
\hat{R}_{ab} &= -\left(\frac{\ddot{b}}{b} + \frac{\dot{b}\,\dot{c}}{b\,c} + (\tilde{n}-1)\,\frac{\dot{b}^2}{b^2} + n\,\frac{\dot{a}\,\dot{b}}{a\,b} - \frac{\tilde{\lambda}}{b^2} + \frac{q^2\,c^2}{2b^4}\right)\delta_{ab}\,.
\end{aligned} \tag{5.6}$$

[9] Note that the choice $S^2 \times \mathbb{CP}^2$ for the base actually violates the condition (5.1) for regularity when one of the factors collapses at the origin. This means that the metric actually has orbifold-like singularities at $r = 0$. We shall discuss a way to avoid this difficulty later.

From this, after introducing the new radial variable η defined by $dt = c\, a^n\, b^{\tilde{n}}\, d\eta$, we find that the conditions for Ricci-flatness can be derived from the Lagrangian $L = T - V$, where

$$\begin{aligned} T &= n\,\alpha'\,\gamma' + \tilde{n}\,\beta'\,\gamma' + n\,\tilde{n}\,\alpha'\,\beta' + \tfrac{1}{2}n(n-1)\,\alpha'^2 + \tfrac{1}{2}\tilde{n}(\tilde{n}-1)\,\beta'^2\,,\\ V &= \tfrac{1}{8}n\,p^2\,e^{(2n-4)\,\alpha+2\tilde{n}\,\beta+4\gamma} + \tfrac{1}{8}\tilde{n}\,q^2\,e^{2n\,\alpha+(2\tilde{n}-4)\,\beta+4\gamma}\\ &\quad -\tfrac{1}{2}n\,\lambda\,e^{(2n-2)\,\alpha+2\tilde{n}\,\beta+2\gamma} - \tfrac{1}{2}\tilde{n}\,\tilde{\lambda}\,e^{2n\,\alpha+(2\tilde{n}-2)\,\beta+2\gamma}\,, \end{aligned} \tag{5.7}$$

together with the requirement that $T + V$ vanishes. Here, a prime means a derivative with respect to η.

Defining $\alpha^i = (\alpha, \beta, \gamma)$ as usual, we find that the Lagrangian can be written as $L = \frac{1}{2}g_{ij}\,(d\alpha^i/d\eta)\,(d\alpha^j/d\eta) + \frac{1}{2}g^{ij}\,\partial W/\partial\alpha^i\,\partial W/\partial\alpha^j$, where the superpotential is given by

$$W = \tfrac{1}{4}n\,p\,e^{(n-2)\,\alpha+\tilde{n}\,\beta+2\gamma} + \tfrac{1}{4}\tilde{n}\,q\,e^{n\,\alpha+(\tilde{n}-2)\,\beta+2\gamma} + k\,e^{n\,\alpha+\tilde{n}\,\beta} \tag{5.8}$$

and the various constants must be chosen so that

$$k = \frac{\lambda}{p} = \frac{\tilde{\lambda}}{q}\,. \tag{5.9}$$

This leads to the first-order equations

$$\begin{aligned} \alpha' &= \tfrac{1}{2}p\,e^{(n-2)\,\alpha\,+\tilde{n}\,\beta+2\gamma}\,, \qquad \beta' = \tfrac{1}{2}q\,e^{n\,\alpha+(\tilde{n}-2)\,\beta+2\gamma}\,,\\ \gamma' &= -\tfrac{1}{4}n\,p\,e^{(n-2)\,\alpha+\tilde{n}\,\beta+2\gamma} - \tfrac{1}{4}\tilde{n}\,q\,e^{n\,\alpha+(\tilde{n}-2)\,\beta+2\gamma} + k\,e^{n\,\alpha+\tilde{n}\,\beta}\,. \end{aligned} \tag{5.10}$$

5.2. Solving the first-order equations. We proceed here by introducing a new radial variable r, defined by[10]

$$dr = e^{(n-1)\,\alpha+\tilde{n}\,\beta+2\gamma}\,d\eta\,. \tag{5.11}$$

The first-order equations (5.10) now become

$$\frac{d\alpha}{dr} = \tfrac{1}{2}p\,e^{-\alpha}\,, \quad \frac{d\beta}{dr} = \tfrac{1}{2}q\,e^{\alpha-2\beta}\,, \quad \frac{d\gamma}{dr} = -\tfrac{1}{4}n\,p\,e^{-\alpha} - \tfrac{1}{4}\tilde{n}\,q\,e^{\alpha-2\beta} + k\,e^{\alpha-2\gamma}\,. \tag{5.12}$$

The first can be solved at sight; the second can then be solved, and then using these results the third can be solved. After making an appropriate choice of integration constants, the result is

$$\begin{aligned} e^{2\alpha} &= \tfrac{1}{4}p^2\,r^2\,, \qquad e^{2\beta} = \tfrac{1}{4}p\,q\,(r^2+\ell^2)\,,\\ e^{2\gamma} &= \frac{k\,p\,r^2}{n+2}\left(1+\frac{r^2}{\ell^2}\right)^{-\tilde{n}/2}\,{}_2F_1\left[1+\tfrac{1}{2}n, -\tfrac{1}{2}\tilde{n}, 2+\tfrac{1}{2}n, -\frac{r^2}{\ell^2}\right], \end{aligned} \tag{5.13}$$

[10] Note that another choice is to take $dr = e^{n\,\alpha+(\tilde{n}-1)\,\beta+2\gamma}\,d\eta$; this will reverse the rôles of the two metrics ds^2 and $d\bar{s}^2$, with consequences that will become clear later.

where ℓ is a constant. The Ricci-flat metric is given by

$$d\hat{s}^2 = e^{2\alpha-2\gamma}\, dr^2 + e^{2\gamma}\, \sigma^2 + e^{2\alpha}\, ds^2 + e^{2\beta}\, d\tilde{s}^2 \,. \tag{5.14}$$

(Note that once one plugs in specific integer values for n and $\tilde{n}$, the hypergeometric function in the expression for $e^{2\gamma}$ becomes purely algebraic.)

At small r, we have

$$e^{2\alpha} = \tfrac{1}{4} p^2\, r^2 \,, \qquad e^{2\beta} \sim \tfrac{1}{4} p\, q\, \ell^2 \,, \qquad e^{2\gamma} \sim \frac{k\, p}{n+2}\, r^2 \,. \tag{5.15}$$

Bearing in mind that $k = \lambda/p$, we therefore find that near $r = 0$, the metric approaches

$$d\hat{s}^2 \sim \frac{(n+2)\, p^2}{4\lambda}\, dS^2 + \tfrac{1}{4} p\, q\, \ell^2\, d\tilde{s}^2 \,, \tag{5.16}$$

where

$$dS^2 = dr^2 + r^2 \left(\frac{4\lambda^2}{p^2\, (n+2)^2}\, \sigma^2 + \frac{\lambda}{n+2}\, ds^2 \right) . \tag{5.17}$$

Regularity at $r = 0$ therefore requires that the quantity enclosed in the parentheses be the unit $(n+1)$-sphere metric. This means in particular that ds^2 should be the standard Fubini-Study metric on $\mathbb{CP}^m$, where $n = 2m$. The canonical choice for the cosmological constant that gives a "unit" $\mathbb{CP}^m$ is in fact

$$\lambda = n + 2 \,, \tag{5.18}$$

and the Fubini-Study metric is then $ds^2 = d\Sigma_m^2$, where

$$d\Sigma_m^2 = F^{-1}\, d\bar{z}^a\, dz^a - F^{-2}\, \bar{z}^a\, z^b\, dz^a\, d\bar{z}^b \,, \tag{5.19}$$

and $F = 1 + \bar{z}^a\, z^a$. After setting $\lambda = n + 2$, we therefore find that

$$d\Omega^2 \equiv \frac{4}{p^2}\, \sigma^2 + ds^2 \,, \tag{5.20}$$

should be the unit $(2m+1)$-sphere metric. The $\widetilde{A}$ term in σ is irrelevant here, and so regularity demands that

$$d\Omega^2 = \left(d\psi + \frac{2}{p}\, A \right)^2 + ds^2 \tag{5.21}$$

must be the unit $(n+1)$-sphere. Recalling that we originally required that $dA = p\, J$, where J is the Kähler form on ds^2, we see that this means that regularity requires that the potential B in $d\Omega^2 = (d\psi + B)^2 + ds^2$ should give $dB = 2J$. This is precisely what one finds in the description of S^{2m+1} as the Hopf fibration over $\mathbb{CP}^m$. More detailed discussions of the regularity conditions at $r = 0$ are discussed in [12]: In order for the fibre coordinate to have the correct periodicity, it must be that the other Einstein-Kähler factor must impose no further restriction above those implied by the $\mathbb{CP}^m$ itself. For example, if the other factor is $\mathbb{CP}^{\tilde{m}}$, then it must be that [12]

$$m + 1 = \gcd(m+1, \tilde{m}+1) \,. \tag{5.22}$$

We can summarise the above results as follows. We have found that the Ricci-flat metric given by (5.13) and (5.14) can be regular at $r = 0$, if the n-dimensional Einstein-Kähler metric ds^2 is taken to be the Fubini-Study metric on $\mathbb{CP}^m$, with $n = 2m$. Furthermore the Einstein-Kähler manifold for the metric $d\tilde{s}^2$ must satisfy certain topological conditions [12], which reduce to (5.22) if it is $\mathbb{CP}^{\tilde{m}}$. Then $r = 0$ is a regular region in the manifold, corresponding to a bolt whose topology is that of the Einstein-Kähler manifold with metric $d\tilde{s}^2$. For $r > 0$, we have level surfaces that are $U(1)$ bundles over the product of the two Einstein-Kähler spaces whose metrics are ds^2 and $d\tilde{s}^2$.

Of course the constants p and q must be chosen appropriately, to be commensurate with the periodicity of the fibre coordinate ψ. For example, if one takes the base space to be the product $\mathbb{CP}^m \times \mathbb{CP}^{\tilde{m}}$, and chooses the canonical values $\lambda = 2(m+1)$ and $\tilde{\lambda} = 2(\tilde{m}+1)$ for the cosmological constants so as to give unit Fubini-Study metrics, then, after taking into account the relation (5.9), we may without loss of generality take $p = m+1$, $q = \tilde{m}+1$. The fibre coordinate z must then have period 2π, implying that the $U(1)$ bundle over $\mathbb{CP}^m \times \mathbb{CP}^{\tilde{m}}$ is simply-connected, or $2\pi/s$, where s is any integer, in which case the bundle space is not simply connected.[11] Thus when we consider $\mathbb{CP}^m \times \mathbb{CP}^{\tilde{m}}$ base spaces, we shall typically make the choices

$$\lambda = 2(m+1)\,, \quad \tilde{\lambda} = 2(\tilde{m}+1)\,, \quad p = m+1\,, \qquad q = \tilde{m}+1\,. \tag{5.23}$$

We can, of course, consider instead the situation where the rôles of the two metrics ds^2 and $d\tilde{s}^2$ are interchanged, as mentioned in the footnote above. Everything goes through, *mutatis mutandis*, in exactly the same way as described above. It will now be the metric $d\tilde{s}^2$ that is required to be the Fubini-Study metric on $\mathbb{CP}^{\tilde{m}}$, with $\tilde{n} = 2\tilde{m}$.

Substituting the first-order equations (5.10) back into the expressions for the curvature 2-forms, we can read off the integrability conditions $\hat{R}_{ABCD}\,\Gamma_{CD}\,\eta = 0$ for the existence of covariantly-constant spinors. These conditions give

$$(\Gamma_{0i} + J_{ij}\,\Gamma_{\tilde{0}j})\,\eta = 0\,, \qquad (\Gamma_{0a} + \widetilde{J}_{ab}\,\Gamma_{\tilde{0}b})\,\eta = 0\,. \tag{5.24}$$

The spinors that satisfy these conditions are the expected complex pair of covariantly-constant spinors in the Ricci-flat Kähler metrics.

It is straightforward to establish that the Kähler form is given by

$$\hat{J} = \hat{e}^0 \wedge \hat{e}^{\tilde{0}} + e^{2\alpha}\,J + e^{2\beta}\,\widetilde{J}\,, \tag{5.25}$$

or, in other words, the vielbein components $\hat{J}_{AB}$ are given by $\hat{J}_{0\tilde{0}} = 1$, $\hat{J}_{ij} = J_{ij}$, $\hat{J}_{ab} = \widetilde{J}_{ab}$. (As in our discussion of the Stenzel metrics, we could again instead *derive* the first-order equations (5.10) by requiring that (5.25) be covariantly constant.)

It can be useful to obtain complex holomorphic coordinates z^μ for the Kähler metrics on the $\mathbb{C}^k$ bundle spaces. This can be done by solving the holomorphicity conditions $\big(\delta^N_M + \mathrm{i}\,J_M{}^N\big)\,\partial_M\,z^\mu = 0$. It is most convenient to do this in the orthonormal frame, for which we therefore need the basis of vector fields dual to the vielbein (5.5). Using the r coordinate of (5.14), it is given by

$$\begin{aligned}
&\hat{E}_0 = e^{\gamma-\alpha}\,\frac{\partial}{\partial r}\,, \qquad \hat{E}_{\tilde{0}} = e^{-\gamma}\,\frac{\partial}{\partial\psi}\,,\\
&\hat{E}_i = e^{-\alpha}\,\left(E_i - A_i\,\frac{\partial}{\partial\psi}\right)\,, \qquad \hat{E}_a = e^{-\beta}\,\left(\widetilde{E}_a - \widetilde{A}_a\,\frac{\partial}{\partial\psi}\right)\,,
\end{aligned} \tag{5.26}$$

[11] See, for example, [27] for a detailed discussion. It is also shown in [27] that these specific $U(1)$ bundles over $\mathbb{CP}^m \times \mathbb{CP}^{\tilde{m}}$ admit Killing spinors when the scalings are chosen so that the metric is Einstein.

where E_i and $\widetilde{E}_a$ are the vielbeins inverse to e^i and $\tilde{e}^a$ on ds^2 and $d\tilde{s}^2$. From this we obtain the following holomorphicity conditions:

$$\begin{aligned} \left(\frac{\partial}{\partial\rho} + \mathrm{i}\,\frac{\partial}{\partial\psi}\right) z^\mu &= 0\,, \\ (E_i + \mathrm{i}\,J_{ij}\,E_j)\,z^\mu - (A_i + \mathrm{i}\,J_{ij}\,A_j)\,\frac{\partial z^\mu}{\partial\psi} &= 0\,, \\ (\widetilde{E}_a + \mathrm{i}\,\widetilde{J}_{ab}\,\widetilde{E}_b)\,z^\mu - (\widetilde{A}_a + \mathrm{i}\,\widetilde{J}_{ab}\,\widetilde{A}_b)\,\frac{\partial z^\mu}{\partial\psi} &= 0\,, \end{aligned} \tag{5.27}$$

where we have defined the new radial coordinate ρ by

$$d\rho = e^{\alpha-2\gamma}\,dr\,. \tag{5.28}$$

It is evident therefore that the holomorphic coordinates for the Kähler metrics ds^2 and $d\tilde{s}^2$ themselves can be used as holomorphic coordinates in the total manifold. It therefore remains to find one more complex coordinate z^0, constructed from the additional real coordinates ρ and ψ. The first equation in (5.27) shows that z^0 should be a function of $\rho + \mathrm{i}\,\psi$. Noting that the vector potentials A_i and $\widetilde{A}_a$ can be written in terms of the Kähler functions K and $\widetilde{K}$ on ds^2 and $d\tilde{s}^2$ as

$$A_i = p\,J_i{}^j\,\partial_j\,K\,, \qquad \widetilde{A}_a = q\,\widetilde{J}_a{}^b\,\partial_b\,\widetilde{K}\,, \tag{5.29}$$

it therefore follows that the extra complex coordinate can be taken to be

$$z^0 = e^{\rho+\mathrm{i}\,\psi+p\,K+q\,\widetilde{K}}\,. \tag{5.30}$$

We conclude this subsection with a number of explicit examples. Our first two make use of an $S^2 \times \mathbb{CP}^2$ base space. These will actually still have orbifold-like singularities at $r = 0$, as we discussed above, since (5.22) is not satisfied. However, they are still of interest for the purposes of constructing deformed M2-brane solutions, since the remaining singularities are rather mild ones. We shall discuss this, and a complete resolution of the singularities, later.

$\mathbb{C}^2/\mathbb{Z}_2$ bundle over $\mathbb{CP}^2$. A particular class of examples would be to take the base space to be $S^2 \times \mathbb{CP}^2$, in which case we get 8-dimensional Ricci-flat Kähler metrics. Note that there are two distinct types of solution; one of them has a $\mathbb{CP}^2$ bolt at $r = 0$, whilst the other has instead an S^2 bolt.

Consider first the case with the $\mathbb{CP}^2$ bolt; with our form of the solution where the untilded metric is singled out as the one whose coefficient goes to zero at $r = 0$, we therefore take ds^2 to be the S^2 metric, and $d\tilde{s}^2$ to be the $\mathbb{CP}^2$ metric. From our general results, after making the conventional choices (5.23), *i.e.* $\lambda = 4$, $\tilde{\lambda} = 6$, $p = 2$, $q = 3$ here, the Ricci-flat Kähler 8-metric is then given by (5.14),

$$e^{2\alpha} = r^2\,, \qquad e^{2\beta} = \tfrac{3}{2}(r^2 + \ell^2)\,, \qquad e^{2\gamma} = \frac{r^2\,(3r^4 + 8\ell^2\,r^2 + 6\ell^4)}{6(r^2+\ell^2)^2}\,. \tag{5.31}$$

(Note that the unit $\mathbb{CP}^1$ is actually a 2-sphere of radius $\frac{1}{2}$.) Thus the Ricci-flat Kähler metric is

$$d\hat{s}_8^2 = U^{-1}\, dr^2 + r^2\, U\, \sigma^2 + \tfrac{1}{4} r^2\, (d\theta^2 + \sin^2\theta\, d\phi^2) + \tfrac{3}{2}(r^2 + \ell^2)\, d\Sigma_2^2\,, \qquad (5.32)$$

where

$$\sigma = d\psi - \tfrac{1}{2}\cos\theta\, d\phi + \widetilde{A}\,, \qquad U = \frac{3r^4 + 8\ell^2 r^2 + 6\ell^4}{6(r^2+\ell^2)^2}\,. \qquad (5.33)$$

Here the maximum allowed periodicity for ψ is $(\Delta\psi)_{\max} = \pi$ (see, for example, [28]), and $d\widetilde{A} = 3\widetilde{J}$, where $\widetilde{J}$ is the Kähler form on the unit $\mathbb{CP}^2$ metric $d\Sigma_2^2$, given in (5.19). If ψ had the period 2π, then $U(1)$ fibres over S^2 would describe S^3, and the metric would approach $\mathbb{R}^4 \times \mathbb{CP}^2$ locally near $r = 0$. Instead, we get the lens space $S^3/\mathbb{Z}_2$, and the metric therefore approaches $(\mathbb{R}^4/\mathbb{Z}_2) \times \mathbb{CP}^2$; the 8-manifold is a $\mathbb{C}^2/\mathbb{Z}_2$ bundle over $\mathbb{CP}^2$. We could, of course, replace $\mathbb{CP}^2$ by the standard Einstein-Kähler metric on $S^2 \times S^2$ in this metric, in which case the fibre coordinate would be allowed to have the period 2π needed for complete regularity at $r = 0$. In fact this would give a special case of a more general class of Ricci-flat Kähler metrics on $\mathbb{C}^2$ bundles over $S^2 \times S^2$, which we shall construct in Sect. 5.3.

$\mathbb{C}^3/\mathbb{Z}_3$ bundle over $\mathbb{CP}^1$. The other possibility using $S^2 \times \mathbb{CP}^2$ in the base space is to interchange the rôles of the S^2 and $\mathbb{CP}^2$ factors, so that now ds^2 is the $\mathbb{CP}^2$ metric, and $d\tilde{s}^2$ is the S^2 metric. It is convenient to refer to this therefore as a $\mathbb{CP}^2 \times S^2$ base, with the understanding that it is always the first factor whose metric coefficient goes to zero at $r = 0$. For this example, it is therefore convenient to choose the constants so that $\lambda = 6$, $\tilde{\lambda} = 4$, $p = 3$ and $q = 2$. The resulting Ricci-flat Kähler 8-metric is then

$$d\hat{s}_8^2 = U^{-1}\, dr^2 + \tfrac{9}{4} r^2\, r^2\, U\, \sigma^2 + \tfrac{9}{4} r^2\, d\Sigma_2^2 + \tfrac{3}{2}(r^2 + \ell^2)\, d\Sigma_1^2\,, \qquad (5.34)$$

where in this case we have

$$\sigma = d\psi + A + \widetilde{A}\,, \qquad U = \frac{3r^2 + 4\ell^2}{9(r^2+\ell^2)}\,, \qquad (5.35)$$

and $dA = 3J$, $d\widetilde{A} = 2\widetilde{J}$. The metrics $d\Sigma_2^2$ and $d\Sigma_1^2$ are the unit metrics on $\mathbb{CP}^2$ and $\mathbb{CP}^1$ respectively, given by (5.19), and J and $\widetilde{J}$ are their respective Kähler forms. (Note that $d\Sigma_1^2 = \frac{1}{4} d\Omega_2^2 = \frac{1}{4}(d\theta^2 + \sin^2\theta\, d\phi^2)$.) The maximum allowed periodicity for ψ is again $(\Delta\psi)_{\max} = \pi$, while the period that would be need for the $U(1)$ bundle over $\mathbb{CP}^2$ to describe S^5 would be $\Delta\psi = 3\pi$. This means that we instead get the lens space $S^5/\mathbb{Z}_3$, and so near $r = 0$ the metric approaches $\mathbb{R}^6/Z_3 \times S^2$; the 8-manifold is a $\mathbb{C}^3/\mathbb{Z}_3$ bundle over S^2 (or $\mathbb{CP}^1$).

Complex line bundle over $\mathbb{CP}^m$. Another possibility is to take one factor in the product base manifold to be trivial, and the other to be $\mathbb{CP}^m$ (or any other Einstein-Kähler manifold). The case where $m = 1$ is Eguchi-Hanson; for general m the corresponding Ricci-flat Kähler metrics were constructed in [29], and also in [30]. Since we shall make use of one of these examples later, we shall summarise the general results here. By taking

$p = n = \lambda = 0, q = 1, \tilde{\lambda} = \tilde{n}+2 = 2m+2$, and setting $\alpha = 0$, the first-order equations (5.10) can be solved to give the $2(m+1)$-dimensional Ricci-flat Kähler metric

$$d\hat{s}^2 = U^{-1}\, dr^2 + 4r^2\, U\, \sigma^2 + r^2\, d\Sigma_m^2\,, \tag{5.36}$$

where r here is related to the r variable in (5.10) by $r \longrightarrow r^2$, the function U is given here by

$$U = 1 - \Big(\frac{r_0}{r}\Big)^{2m+2}\,, \tag{5.37}$$

with r_0 being a constant, and $d\Sigma_m^2$ is the metric on the unit Fubini-Study metric on $\mathbb{CP}^m$, given in (5.19). Note that $\sigma = d\psi + \widetilde{A}$ here, where $d\widetilde{A} = J$, the Kähler form on the $\mathbb{CP}^m$. The radial coordinate r runs from $r = r_0$, where the metric approaches $\mathbb{R}^2 \times \mathbb{CP}^m$, to infinity. Topologically, the manifold is a $\mathbb{C}^1$ bundle over $\mathbb{CP}^m$.

For future reference we note that it is very easy to solve for an L^2-normalisable (anti)-self-dual harmonic form in the middle dimension, when m is odd. It is given by

$$G_{(m+1)} = \frac{1}{r^{2m+2}}\left[r^{m-1}\,\hat{e}^0 \wedge \hat{e}^{\tilde{0}} \wedge J^{(m-1)/2} - \frac{2}{m+1}\, r^{m+1}\, J^{(m+1)/2}\right]. \tag{5.38}$$

Note that the factors of r within the square brackets just convert each power of the Kähler form J on $\mathbb{CP}^m$ into a 2-form of unit magnitude in the metric $d\hat{s}^2$, *i.e.* $r^2\, J = \frac{1}{2} J_{ab}\, \hat{e}^a \wedge \hat{e}^b$. Thus each term within the square brackets is just a constant times a wedge product of hatted vielbeins. The magnitude of $G_{(m+1)}$ is therefore given by

$$|G_{(m+1)}|^2 = \frac{\text{constant}}{r^{4m+4}}\,, \tag{5.39}$$

and so the L^2-normalisability is manifest.

5.3. General results for N Einstein-Kähler factors in the base space. As we indicated above, the construction of the previous subsection has a straightforward generalisation to the case where we have N Einstein-Kähler factors in the base space,

$$\mathcal{M} = M_1 \times M_2 \times \cdots \times M_N\,, \tag{5.40}$$

with real dimensions n_i and metrics ds_i^2. Thus we write

$$d\hat{s}^2 = dt^2 + \sum_{i=1}^{N} a_i^2\, ds_i^2 + c^2\, \sigma^2\,, \tag{5.41}$$

where

$$\sigma = d\psi + \sum_i A^i\,, \tag{5.42}$$

where $dA^i = p_i\, J^i$, and J^i is the Kähler form on the factor M_i in the base manifold. By comparing with the previous subsection, our notation here and in what follows should be self-evident.

We find that the Ricci tensor for $d\hat{s}^2$ has components

$$\begin{aligned}
\hat{R}_{00} &= -\sum_i n_i \frac{\ddot{a}_i}{a} - \frac{\ddot{c}}{c}\,,\\
\hat{R}_{\tilde{0}\tilde{0}} &= -\sum_i n_i \frac{\dot{a}_i\,\dot{c}}{a_i\,c} - \frac{\ddot{c}}{c} + \sum_i \frac{n_i\,p_i^2\,c^2}{4a_i^4}\,,\\
\hat{R}_{a_i b_i} &= -\left(\frac{\ddot{a}_i}{a_i} + \frac{\dot{a}_i\,\dot{c}}{a_i\,c} - \frac{\dot{a}_i^2}{a_i^2} + \frac{\dot{a}_i}{a_i}\sum_j n_j \frac{\dot{a}_j}{a_j} - \frac{\lambda_i}{a_i^2} + \frac{p_i^2\,c^2}{2a_i^4}\right)\delta_{a_i b_i}\,.
\end{aligned} \tag{5.43}$$

Defining $a_i = e^{\alpha_i}$, $c = e^{\gamma}$, the conditions for Ricci-flatness can be derived from the Lagrangian

$$L = \tfrac{1}{2}\sum_{i,j} n_i\,n_j\,\alpha_i'\,\alpha_j' - \tfrac{1}{2}\sum_i n_i\,\alpha_i'^2 + \sum_i n_i\,\alpha_i'\,\gamma' - V\,, \tag{5.44}$$

where

$$V = \tfrac{1}{8}\sum_i n_i\,p_i^2\,e^{2\mu_i} - \tfrac{1}{2}\sum_i n_i\,\lambda_i\,e^{2\mu_i+2\alpha_i-2\gamma}\,, \tag{5.45}$$

with

$$\mu_i \equiv 2\gamma - 2\alpha_i + \sum_j n_j\,\alpha_j\,. \tag{5.46}$$

The primes denote derivatives with respect to η, defined by

$$dt = e^{\sum_i n_i\,\alpha_i+\gamma}\,d\eta\,. \tag{5.47}$$

Defining $\alpha_0 = \gamma$, and indices $a = (0, i)$, the Lagrangian (5.44) can be written as $L = \frac{1}{2}g_{ab}\,(d\alpha^a/d\eta)\,(d\alpha^b/d\eta) - V$, with $g_{ij} = n_i\,n_j - n_i\,\delta_{ij}$, $g_{0i} = n_i$, $g_{00} = 0$. This has the inverse

$$g^{ij} = \frac{1}{D}\,, \qquad g^{0i} = \frac{1}{D}\,, \qquad g^{00} = \frac{1}{D} - 1\,, \tag{5.48}$$

where $D = \sum_i n_i$ is the total dimension of the base space. It is then straightforward to show that the potential V can be written in terms of a superpotential W, as $V = -\frac{1}{2}g^{ab}\,(\partial W/\partial\alpha^a)\,(\partial W/\partial\alpha^b)$, where

$$W = \tfrac{1}{4}\sum_i n_i\,p_i\,e^{\mu_i} + k\,e^{\sum_i n_i\,\alpha_i}\,, \tag{5.49}$$

provided that the constants p_i and λ_i satisfy

$$\lambda_i = k\,p_i\,. \tag{5.50}$$

It follows that the following first-order equations imply Ricci-flatness:

$$\alpha_i' = \tfrac{1}{2}p_i\,e^{\mu_i}\,, \qquad \gamma' = k\,e^{\sum_i n_i\,\alpha_i} - \tfrac{1}{4}\sum_i n_i\,p_i\,e^{\mu_i}\,. \tag{5.51}$$

We can solve these by defining a new radial coordinate[12] r:

$$dr = e^{\mu_1+\alpha_1}\, d\eta\,, \tag{5.52}$$

which leads to

$$\frac{d\alpha_i}{dr} = \tfrac{1}{2}\, p_i\, e^{\alpha_1-2\alpha_i}\,, \qquad \frac{d\gamma}{dr} = k\, e^{\alpha_1-2\gamma} - \tfrac{1}{4}\sum_i n_i\, p_i\, e^{\alpha_1-2\alpha_i}\,. \tag{5.53}$$

The equation for α_1 can be solved immediately, and then those for the remaining α_i can be integrated. We find

$$e^{2\alpha_i} = \tfrac{1}{4} p_1\, p_i\, \Big(r^2 + \ell_i^2\Big)\,, \tag{5.54}$$

where $\ell_1 = 0$ and the other ℓ_i are constants of integration. Defining $\tilde{\gamma} \equiv \gamma + \frac{1}{2}\sum_i n_i\,\alpha_i$ in an intermediate step, and $x \equiv r^2$, the equation for γ can be solved to give

$$e^{2\gamma} = \tfrac{1}{2} p_1\, k \prod_i \Big(x + \ell_i^2\Big)^{-n_i/2} \int_0^x dy \prod_j \Big(y + \ell_j^2\Big)^{n_j/2}\,. \tag{5.55}$$

The integration is elementary, giving an expression for $e^{2\gamma}$ as a rational function of x for any given choice of the integers n_i, but the general expression for arbitrary dimensions n_i requires the use of hypergeometric functions. In terms of the r coordinate, the metric is given by

$$d\hat{s}^2 = e^{2\alpha_1-2\gamma}\, dr^2 + \sum_i e^{2\alpha_i}\, ds_i^2 + e^{2\gamma}\,\sigma^2\,. \tag{5.56}$$

The analysis of the structure of the Ricci-flat metrics proceeds in a fashion that is analogous to that of the previous section. The radial coordinate runs from $r = 0$, where the metric functions $e^{2\alpha_1}$ and $e^{2\gamma}$ vanish, to $r = \infty$. Regularity at $r = 0$ requires that the Einstein-Kähler metric ds_1^2 on the factor M_1 in the base space (5.40) be the Fubini-Study metric on $\mathbb{CP}^{m_1}$, where $n_1 = 2m_1$, so that $r = 0$ becomes the origin of spherical polar coordinates on $\mathbb{R}^{n_1+2}$. Even though the other metric functions $e^{2\alpha_i}$ for $i \geq 2$ are non-zero for the entire range $0 \leq r \leq \infty$, there are again topological restrictions on the choice of Einstein-Kähler manifolds for these factors, stemming from the requirement that the $U(1)$ fibre coordinate should have the periodicity needed for the $U(1)$ bundle over $\mathbb{CP}^{m_1}$ to be the $(2m_1+1)$-sphere and not a lens space. (See [12] for a detailed discussion: If, for example, all the other factors are complex projective spaces $\mathbb{CP}^{m_i}$, then they must satisfy (5.1).) Topologically, the manifold on which the metric $d\hat{s}^2$ is then defined is a $\mathbb{C}^k$ bundle over the product of the remaining base-space factors $M_2 \times M_3 \times \cdots \times M_N$, where $k = \frac{1}{2}n_1 + 1$.

[12] We single out the $i = 1$ factor in the base space purely as a matter of convention; there is no loss of generality, since we have not yet specified the choices for these factors.

Arguments analogous to those of the previous subsection show that the Kähler form for the metric $d\hat{s}^2$ is given by

$$\hat{J} = \hat{e}^0 \wedge \hat{e}^{\tilde{0}} + \sum_i a_i^2 \, J^i \,, \tag{5.57}$$

where J^i denotes the Kähler form on the i^{th} factor in the product of Einstein-Kähler manifolds (5.40) in the base space. The two covariantly-constant spinors will satisfy the integrability conditions

$$(\Gamma_{0a_i} + J_{a_i b_i} \, \Gamma_{\tilde{0}b_i}) \, \eta = 0 \,, \tag{5.58}$$

where $J_{a_i b_i}$ are the vielbein components of the Kähler form J^i. It is straightforward to see, generalising the discussion of Sect. 5.2, that for holomorphic complex coordinates on the total space we can use the holomorphic coordinates of the various factors M_i in the base space $\mathcal{M}$, together with the additional complex coordinate z^0 given by

$$z^0 = e^{\rho + \mathrm{i}\,\psi + \sum_i p_i \, K_i} \,, \tag{5.59}$$

where K_i denotes the Kähler function on the factor M_i, and ρ is defined by $d\rho = e^{\alpha_1 - 2\gamma} \, dr$.

Let us present one explicit example of the more general Ricci-flat Kähler solutions:

$\mathbb{C}^2$ *bundle over* $\mathbb{CP}^1 \times \mathbb{CP}^1$. Consider the case where we take the base space to be $S^2 \times S^2 \times S^2$, so $n_1 = n_2 = n_3 = 2$. Then we find

$$\begin{aligned} e^{2\alpha_1} &= \tfrac{1}{4} p_1^2 \, r^2 \,, \qquad e^{2\alpha_2} = \tfrac{1}{4} p_1 \, p_2 \left(r^2 + \ell_2^2\right) , \qquad e^{2\alpha_3} = \tfrac{1}{4} p_1 \, p_3 \left(r^2 + \ell_3^2\right) , \\ e^{2\gamma} &= \frac{p_1 \, k \, r^2 \left[\ell_2^2 \, \ell_3^2 + \frac{2}{3}\left(\ell_2^2 + \ell_3^2\right) r^2 + \frac{1}{2} r^4\right]}{4 \left(r^2 + \ell_2^2\right) \left(r^2 + \ell_3^2\right)} \,, \end{aligned} \tag{5.60}$$

and after making convenient choices $p_i = 1$, $\lambda_i = 1$ for the constants, the metric is given by

$$d\hat{s}_8^2 = U^{-1} \, dr^2 + \tfrac{1}{4} r^2 \, U \, \sigma^2 + \tfrac{1}{4} r^2 \, d\Omega_1^2 + \tfrac{1}{4} \left(r^2 + \ell_2^2\right) d\Omega_2^2 + \tfrac{1}{4} \left(r^2 + \ell_3^2\right) d\Omega_3^2 \,, \tag{5.61}$$

where

$$U = \frac{3r^4 + 4\left(\ell_2^2 + \ell_3^2\right) r^2 + 6 \ell_2^2 \, \ell_3^2}{6 \left(r^2 + \ell_2^2\right) \left(r^2 + \ell_3^2\right)} \,, \tag{5.62}$$

$d\Omega_1^2$, $d\Omega_2^2$ and $d\Omega_3^2$ are metrics on three unit 2-spheres, and in an obvious notation we have

$$\sigma = d\psi + \cos\theta_1 \, d\phi_1 + \cos\theta_2 \, d\phi_2 + \cos\theta_3 \, d\phi_3 \,, \tag{5.63}$$

where ψ has period 4π. The metric approaches $\mathbb{R}^4 \times S^2 \times S^2$ at $r = 0$, with an $S^2 \times S^2$ bolt; topologically, the manifold is a $\mathbb{C}^2$ bundle over $S^2 \times S^2$ (or $\mathbb{CP}^1 \times \mathbb{CP}^1$).

6. More Fractional D3-Branes and Deformed M2-Branes

6.1. The resolved fractional D3-brane.

6.1.1. Harmonic 3-form on the $\mathbb{C}^2$ bundle over $\mathbb{CP}^1$. This is a special case of the construction Sect. 5, in which the base space is taken to be just $S^2 \times S^2$. It gives a complete non-compact manifold that provides a "small resolution" of the singular conifold [13]. The metric can be written in the form [5]

$$ds_6^2 = \frac{r^2+6\ell^2}{r^2+9\ell^2}\,dr^2 + \tfrac{1}{9}\left(\frac{r^2+9\ell^2}{r^2+6\ell^2}\right) r^2\,\sigma^2 + \tfrac{1}{6}r^2\,d\Omega_2^2 + \tfrac{1}{6}\left(r^2+6\ell^2\right)\,d\widetilde{\Omega}_2^2\,, \tag{6.1}$$

where

$$\begin{aligned} d\Omega_2^2 &= d\theta^2 + \sin^2\theta\,d\phi^2\,, \qquad d\widetilde{\Omega}_2^2 = d\tilde{\theta}^2 + \sin^2\tilde{\theta}\,d\tilde{\phi}^2\,, \\ \sigma &= d\psi + \cos\theta\,d\phi + \cos\tilde{\theta}\,d\tilde{\phi}\,, \end{aligned} \tag{6.2}$$

and ℓ is a constant. The radial coordinate runs from $r = 0$ to $r = \infty$. Near $r = 0$, the metric smoothly approaches flat R^4 times a 2-sphere of radius ℓ, while at large r the metric describes the cone with level surfaces that are the $U(1)$ bundle over $S^2 \times S^2$. (We are using the notation of [5] here; it corresponds in our notation to taking $p = q = \lambda = \tilde{\lambda} = 1$, and then sending $r \longrightarrow \sqrt{\frac{2}{3}}\,r$ and $\ell \longrightarrow 2\ell$.)

From (5.25) we see that a holomorphic basis of 1-forms is

$$\epsilon^0 = -e^0 + \mathrm{i}\,e^{\tilde{0}}\,, \qquad \epsilon^1 = e^1 + \mathrm{i}\,e^2\,, \qquad \epsilon^2 = e^3 + \mathrm{i}\,e^4\,, \tag{6.3}$$

where

$$\begin{aligned} &e^0 = h\,dr\,, \qquad e^5 = c\,\sigma\,, \qquad e^1 = a\,d\theta\,, \qquad e^2 = a\,\sin\theta\,d\phi\,, \\ &e^3 = b\,d\tilde{\theta}\,, \qquad e^4 = b\,\sin\tilde{\theta}\,d\tilde{\phi}\,, \end{aligned} \tag{6.4}$$

and a, b and c and h are the metric coefficients in (6.1), given by

$$a^2 = \tfrac{1}{6}r^2\,, \qquad b^2 = \tfrac{1}{6}(r^2+\ell^2)\,, \qquad c^2 = \tfrac{1}{9}\left(\frac{r^2+9\ell^2}{r^2+6\ell^2}\right) r^2\,, \qquad h^2 = \frac{r^2+6\ell^2}{r^2+9\ell^2}\,. \tag{6.5}$$

It can be useful also to obtain complex coordinates z^μ compatible with the complex structure. Solving the conditions $\left(\delta_a^b + \mathrm{i}\,J_a{}^b\right)\partial_b\,z^\mu = 0$, we are led to the following choice of complex coordinates:

$$z_1 = \tan\tfrac{1}{2}\theta\,e^{\mathrm{i}\,\phi}\,, \qquad z_2 = \tan\tfrac{1}{2}\tilde{\theta}\,e^{\mathrm{i}\,\tilde{\phi}}\,, \qquad z_3 = \sin\theta\,\sin\tilde{\theta}\,e^{\rho - \mathrm{i}\,\psi}\,, \tag{6.6}$$

where ρ is related to r by $3h^2\,dr = r\,d\rho$, *i.e.* $e^\rho = r^2\,(r^2+9\ell^2)^{1/2}$. The complex vielbein basis given in (6.3) then takes the form

$$\epsilon^0 = c\,\left(\cos\theta\,\frac{dz_1}{z_1} + \cos\tilde{\theta}\,\frac{dz_2}{z_2} - \frac{dz_3}{z_3}\right)\,, \quad \epsilon^1 = a\,\sin\theta\,\frac{dz_1}{z_1}\,, \quad \epsilon^2 = b\,\sin\tilde{\theta}\,\frac{dz_2}{z_2}\,, \tag{6.7}$$

which shows that it is indeed holomorphic. Note that the complex coordinate z_3 is related to the z^0 coordinate of the general discussion in Sect. 5.2 by simple coordinate transformations.

There is a complex self-dual 3-form, satisfying $*G_{(3)} = \mathrm{i}\, G_{(3)}$, given by

$$G_{(3)} = \frac{1}{c\, a^2}\,(e^5 \wedge e^1 \wedge e^2 - \mathrm{i}\, e^0 \wedge e^3 \wedge e^4) - \frac{1}{c\, b^2}\,(e^5 \wedge e^3 \wedge e^4 - \mathrm{i}\, e^0 \wedge e^1 \wedge e^2)\,. \tag{6.8}$$

From this, it follows that $G_{(3)}$ is given by

$$G_{(3)} = -f_1\, \bar\epsilon^0 \wedge (\epsilon^1 \wedge \bar\epsilon^1 + \epsilon^2 \wedge \bar\epsilon^2) + f_2\, \epsilon^0 \wedge (\epsilon^1 \wedge \bar\epsilon^1 - \epsilon^2 \wedge \bar\epsilon^2)\,, \tag{6.9}$$

where

$$f_1 \equiv \frac{1}{4c\, a^2} - \frac{1}{4c\, b^2}\,, \qquad f_2 \equiv \frac{1}{4c\, a^2} + \frac{1}{4c\, b^2}\,. \tag{6.10}$$

Thus we see that $G_{(3)}$ in general has (2, 1) and (1, 2) pieces. It would become pure (2, 1) if f_1 vanished. This would happen only if the scale parameter ℓ were set to zero, since then a and b become equal. In this limit, the metric reverts to the original unresolved conifold. The (1, 2) piece does, of course, go to zero faster than the (2, 1) piece as r tends to infinity in the resolved metric. Thus the harmonic 3-form $G_{(3)}$ becomes "asymptotically pure" at large distances.

This 3-form was used to construct a fractional D3-brane in [5]. Owing to the (marginal) non-normalisability of the 3-form at large distance, it follows that the solution has a logarithmic correction to the D3-brane metric function H at large proper distance, as in (1.1). The solution also has a repulsion type of singularity owing to the non-normalisability of $G_{(3)}$ at small distance. In the next subsection, we shall address the issue of supersymmetry.

6.1.2. The issue of supersymmetry in the Pando Zayas-Tseytlin D3-brane. In the general discussions of supersymmetry for fractional D3-branes in [3, 4], it is argued that the deformed solution will only be supersymmetric if the complex self-dual harmonic 3-form is purely of type (2, 1). In fact, it was argued in [3, 4] that the self-duality of the 3-form already implied that it could contain only (2, 1) and (0, 3) pieces, and in [4] it was proved that the presence of a (0, 3) term would imply that there would be no supersymmetry. Since we have found that the self-dual harmonic 3-form in the resolved D3-brane solution of [5] has both (2, 1) and (1, 2) pieces, it is appropriate first to discuss why the (1, 2) piece can in fact be present. After that, we shall discuss its implications for supersymmetry.

The general statement about the duality of (p, q)-forms in six-dimensional Kähler spaces is as follows. One must distinguish between (2, 1) or (1, 2)-forms that are perpendicular to the Kähler form, $G_{abc}\, J^{ab} = 0$, and those that are parallel, $G_{abc} = K_{[a}\, J_{bc]}$. Denoting these by $(p, q)_\perp$ and $(p, q)_\parallel$, we then have, in an obvious notation,

$$\begin{aligned} *(2,1)_\perp &= \mathrm{i}\,(2,1)_\perp\,, & *(2,1)_\parallel &= -\mathrm{i}\,(2,1)_\parallel\,,\\ *(1,2)_\perp &= -\mathrm{i}\,(1,2)_\perp\,, & *(1,2)_\parallel &= \mathrm{i}\,(1,2)_\parallel\,,\\ *(0,3) &= \mathrm{i}\,(0,3)\,, & *(3,0) &= -\mathrm{i}\,(3,0)\,. \end{aligned} \tag{6.11}$$

We can indeed verify by inspection of (6.9) that the first term is of type $(1,2)_{\parallel}$, and the second term is of type $(2,1)_{\perp}$. This is therefore compatible with the fact that $G_{(3)}$ is self-dual, $*G_{(3)} = \mathrm{i}\, G_{(3)}$.[13]

Now let us turn to the question of supersymmetry. It is shown in [3, 4] that in the Majorana basis of [31], the criterion for unbroken supersymmetry for fractional D3-branes is that in addition to the usual requirements of the standard D3-brane, the harmonic self-dual 3-form should satisfy

$$G_{abc}\,\Gamma^{abc}\,\eta = 0\,, \qquad G_{abc}\,\Gamma^{abc}\,\eta^* = 0\,, \tag{6.12}$$

where η is covariantly-constant in the six-dimensional Ricci-flat Kähler metric. The Majorana basis implies that the ten-dimensional Dirac matrices $\hat{\Gamma}_A$ with spatial indices are symmetric and real, while the Dirac matrix with the timelike index is antisymmetric and real. (These are the conventions of [31], modified to our notation where the metric signature is mostly positive.) In terms of a $4+6$ decomposition, we shall have

$$\hat{\Gamma}_\mu = \gamma_\mu \otimes \mathbb{1}\,, \qquad \hat{\Gamma}_m = \gamma_5 \otimes \Gamma_m\,, \tag{6.13}$$

where $\gamma_5 = \frac{\mathrm{i}}{4!}\,\epsilon^{\mu\nu\rho\sigma}\,\gamma_{\mu\nu\rho\sigma}$ is antisymmetric and imaginary, and the Dirac matrices Γ_m in the six-dimensional space are also antisymmetric and imaginary. We also have that the chirality operator $\Gamma_7 = \frac{\mathrm{i}}{6!}\,\epsilon^{a_1\cdots a_6}\,\Gamma_{a_1\cdots a_6}$ is imaginary and antisymmetric, while $\hat{\Gamma}_{11}$ is symmetric and real. Note that because Γ_7 is imaginary in the Majorana basis, this means that η^* has the opposite chirality to η.

We can now see that if the harmonic self-dual 3-form is written as

$$G_{(3)} = G^{\mathrm{Re}}_{(3)} + \mathrm{i}\, G^{\mathrm{Im}}_{(3)}\,, \tag{6.14}$$

where $G^{\mathrm{Re}}_{(3)}$ and $G^{\mathrm{Im}}_{(3)}$ are both real, then the criterion for supersymmetry is equivalent to

$$G^{\mathrm{Re}}_{abc}\,\Gamma_{abc}\,\eta = 0\,, \qquad G^{\mathrm{Im}}_{abc}\,\Gamma_{abc}\,\eta = 0\,. \tag{6.15}$$

Expressing the conditions in this form has the advantage that it is now independent of the choice of basis for the Dirac matrices. In particular, substituting (6.9) into (6.15), and making use of the conditions $\Gamma_{12}\,\eta = \Gamma_{\tilde{1}\tilde{2}}\,\eta = -\Gamma_{0\tilde{0}}\,\eta$ satisfied by the covariantly-constant spinor η (see [6]), we arrive at the conclusion that the resolved D3-brane solution of [5], using the Ricci-flat metric on the $\mathbb{C}^2$ bundle over $\mathbb{CP}^1$ is not supersymmetric, since f_1 is non-zero. This is consistent with the fact that the $(1,2)_{\parallel}$ piece in $G_{(3)}$ is non-vanishing. One can also demonstrate the breaking of supersymmetry by a direct substitution of $G_{(3)}$ into (6.12) in the Majorana basis.

6.2. Harmonic 4-form for $\mathbb{C}^2/\mathbb{Z}_2$ and $\mathbb{C}^2$ bundles over $\mathbb{CP}^2$, and smooth M2-branes. Let us now consider examples where the 8-dimensional Ricci-flat solution is obtained by taking the level surfaces to be the $U(1)$ bundle over $S^2 \times \mathbb{CP}^2$. First, we shall choose the case where the bolt at $r = 0$ is $\mathbb{CP}^2$, so the metric is given by (5.32); by our general

[13] Note that there would be no such harmonic form of type $(1,2)_{\parallel}$ in a compact Calabi-Yau 3-fold since it would require the existence of a harmonic $(0,1)$-form $K_{\tilde{\alpha}}$, which is excluded by the fact that the first cohomology group $H^1(Z)$ vanishes. However, in a non-compact manifold, where furthermore the harmonic forms are not being required to be L^2-normalisable, such arguments break down.

arguments in Sect. 2.1, we can expect that a harmonic 4-form should exist for this manifold. Of course in this case there is still an orbifold-type singularity in the metric near the origin, as we discussed earlier. Afterwards, we shall present a completely regular generalisation of the solution.

Making a natural ansatz for a self-dual harmonic 4-form that is invariant under the isometry group, we obtain equations that admit the simple solution

$$G_{(4)} = \frac{\ell^2}{(r^2+\ell^2)^3}\left[e^{2\beta}\,\hat{e}^0\wedge\hat{e}^{\tilde{0}}\wedge\widetilde{J} - 2e^{2\alpha}\,\hat{e}^0\wedge\hat{e}^{\tilde{0}}\wedge J + e^{2\alpha+2\beta}\,J\wedge\widetilde{J} - e^{4\beta}\,\widetilde{J}\wedge\widetilde{J}\right], \tag{6.16}$$

where J is the Kähler form (*i.e.* volume form) on S^2, and $\widetilde{J}$ is the Kähler form on $\mathbb{CP}^2$. Note that $\frac{1}{2}\widetilde{J}\wedge\widetilde{J}$ is the volume form on $\mathbb{CP}^2$. We therefore find

$$G_{(4)}^2 = \frac{288\ell^4}{(r^2+\ell^2)^6}\,, \tag{6.17}$$

from which it follows that the harmonic 4-form $G_{(4)}$ is L^2 normalisable.

By making a canonical choice for the vielbeins and Kähler structures on S^2 and $\mathbb{CP}^2$, we may write $J = e^1\wedge e^2$, $\widetilde{J} = \tilde{e}^{\tilde{1}}\wedge\tilde{e}^{\tilde{2}} + \tilde{e}^{\tilde{3}}\wedge\tilde{e}^{\tilde{4}}$. It then follows from (5.25) that a holomorphic vielbein basis for the 8-dimensional metric is

$$\epsilon^0 = \hat{e}^0 + \mathrm{i}\,\hat{e}^{\tilde{0}}\,,\quad \epsilon^1 = \hat{e}^1 + \mathrm{i}\,\hat{e}^2\,,\quad \epsilon^2 = \hat{e}^{\tilde{1}} + \mathrm{i}\,\hat{e}^{\tilde{2}}\,,\quad \epsilon^3 = \hat{e}^{\tilde{3}} + \mathrm{i}\,\hat{e}^{\tilde{4}}\,, \tag{6.18}$$

and the Kähler form is given by

$$\hat{J} = \tfrac{\mathrm{i}}{2}\,(\epsilon^0\wedge\bar{\epsilon}^0 + \epsilon^1\wedge\bar{\epsilon}^1 + \epsilon^2\wedge\bar{\epsilon}^2 + \epsilon^3\wedge\bar{\epsilon}^3)\,. \tag{6.19}$$

The harmonic 4-form (6.16) can then be rewritten as

$$\begin{aligned} G_{(4)} = -\frac{\ell^4}{4(r^2+\ell^2)^3}\Big[&\epsilon^0\wedge\bar{\epsilon}^0\wedge\epsilon^2\wedge\bar{\epsilon}^2 + \epsilon^0\wedge\bar{\epsilon}^0\wedge\epsilon^3\wedge\bar{\epsilon}^3 - 2\epsilon^0\wedge\bar{\epsilon}^0\wedge\epsilon^1\wedge\bar{\epsilon}^1 \\ &-\epsilon^1\wedge\bar{\epsilon}^1\wedge\epsilon^2\wedge\bar{\epsilon}^2 + 2\epsilon^2\wedge\bar{\epsilon}^2\wedge\epsilon^3\wedge\bar{\epsilon}^3 + \epsilon^1\wedge\bar{\epsilon}^1\wedge\epsilon^3\wedge\bar{\epsilon}^3\Big]\,, \end{aligned} \tag{6.20}$$

which shows that it is a (2, 2)-form. Furthermore, it satisfies $G_{abcd}\,\hat{J}_{ab} = 0$, and so it is perpendicular to the Kähler form. In the notation we used earlier, it is therefore a 4-form of type $(2,2)_\perp$.

Solving Eq. (4.16) for the function H in the deformed M2-brane (4.16), we first find that

$$r^3\,(3r^4 + 8\ell^2 r^2 + 6\ell^4)\,H' = \beta + \frac{3\,m^2\,\ell^4\,(3r^2+\ell^2)}{(r^2+\ell^2)^3}\,. \tag{6.21}$$

If the constant of integration β is chosen to be $\beta = -3\,m^2$, then the solution for H is non-singular at $r = 0$. Explicitly, we find

$$H = 1 - \frac{3m^2\,(3r^2+2\ell^2)}{2\ell^4\,(r^2+\ell^2)^2} + \frac{27\sqrt{2}\,m^2}{4\ell^6}\,\arctan\left[\frac{\sqrt{2}\,\ell^2}{3r^2+4\ell^2}\right]. \tag{6.22}$$

This tends to a constant at small r, and at large r it has the asymptotic form

$$H \sim 1 + \frac{m^2}{6r^6} - \frac{m^2\,\ell^2}{3r^8} + \frac{19m^2\,\ell^4}{90r^{10}} + \cdots . \tag{6.23}$$

The asymptotic behaviour is best analysed using the proper distance ρ, defined by $e^{\alpha-\gamma}\,dr = d\rho$, as the radial coordinate. The coordinates ρ and r are related by

$$r \sim \tfrac{1}{\sqrt{2}}\left(\rho - \frac{2\ell^2}{3\rho} + \frac{8\ell^6}{45\,\rho^5} + \cdots\right). \tag{6.24}$$

Thus in terms of ρ, the function H behaves as follows in the asymptotic region:

$$H \sim 1 + \frac{4m^2}{3\rho^6} - \frac{416m^2\,\ell^4}{45\rho^{10}} + \cdots . \tag{6.25}$$

As discussed in Sect. 4.2, the condition for supersymmetry of the deformed M2-brane is that the harmonic 4-form should satisfy

$$G_{ABCD}\,\Gamma_{BCD}\,\eta = 0\,, \tag{6.26}$$

where η is covariantly constant in the 8-dimensional transverse metric. From the integrability conditions (5.24) for η, and the form of the harmonic 4-form (6.16), it is straightforward to show that (6.26) is satisfied, and so this deformed M2-brane solution is supersymmetric.

As we mentioned earlier, the maximum periodicity $(\Delta\psi)_{\max} = \pi$ of the ψ coordinate on the $U(1)$ fibres, implied by the requirement of regularity of the principal orbits, means that near $r = 0$ the $U(1)$ bundle over S^2 gives $S^3/\mathbb{Z}_2$ rather than S^3, and so there is an orbifold singularity at $r = 0$. This can be avoided if we replace $\mathbb{CP}^2$ by $S^2 \times S^2$, since then ψ can have period 2π instead. Since this is just a special case of more general $\mathbb{C}^2$ bundles over $S^2 \times S^2$ that we discuss in the next section, we shall not consider this further here.

It is worth noting, however, that there does exist a generalisation of the metrics on complex bundles over $S^2 \times \mathbb{CP}^2$ that avoids the orbifold singularity. The solution is obtained, along with wide classes of more general related examples, in [12]. The investigation of these generalisations was motivated by a construction of a six-dimensional example in [32]. Here, we shall just quote the result for the new eight-dimensional metric over $S^2 \times \mathbb{CP}^2$, referring to [12] for explicit details. It is given by

$$d\hat{s}^2 = e^{-2\gamma}\,dr^2 + e^{2\gamma}\,\sigma^2 + 2\left(r + \ell_1^2\right)\,d\Sigma_1^2 + 3\left(r + \ell_2^2\right)\,d\Sigma_2^2\,, \tag{6.27}$$

where as usual $d\Sigma_m^2$ denotes the Fubini-Study metric on the unit $\mathbb{CP}^m$. The function γ is given by

$$e^{2\gamma} = \frac{4\ell_1^2\,r\,\left(r^2 + 3\ell_2^2\,r + 3\ell_2^4\right) + r^2\,\left(3r^2 + 8\ell_2^2\,r + 6\ell_2^2\right)}{3\left(r + \ell_1^2\right)\left(r + \ell_2^2\right)^2}\,. \tag{6.28}$$

When $\ell_1 = 0$ this reduces, after simple coordinate transformations, to the metric given in (5.32). However, when ℓ_1 is non-zero the entire $S^2 \times \mathbb{CP}^2$ remains uncollapsed at $r = 0$. The regularity conditions at $r = 0$ now imply that ψ should have period π, which is the same as the value dictated by regularity of the principal orbits.

It is easy to repeat the analysis of the harmonic 4-form for this metric. One finds that there is again a normalisable self-dual form, given by

$$G_{(4)} = \frac{\ell_2^2}{\left(r^2+\ell_2^2\right)^3}\left[e^{2\beta}\,\hat{e}^0\wedge\hat{e}^{\tilde{0}}\wedge\widetilde{J} - 2e^{2\alpha}\,\hat{e}^0\wedge\hat{e}^{\tilde{0}}\wedge J + e^{2\alpha+2\beta}\,J\wedge\widetilde{J} - e^{4\beta}\,\widetilde{J}\wedge\widetilde{J}\right], \tag{6.29}$$

where J is the Kähler form on $\mathbb{CP}^1$, and $\widetilde{J}$ is the Kähler form on $\mathbb{CP}^2$. It is again of type $(2,2)_\perp$, and so it will give a supersymmetric deformed M2-brane solution.

The solution for H is now given by

$$\begin{aligned} H = c_0 &+ \frac{12m^2\,\left(3r+\ell_1^2+2\ell_2^2\right)}{\ell_2^2\,\left(\ell_2^2-4\ell_1^1\right)\left(r+\ell_2^2\right)^2} \\ &- \frac{108m^2}{\ell_2^2\,\left(\ell_2^2-4\ell_1^2\right)}\sum_{i=1}^{3}\frac{2\ell_1^2\,\log(r-r_i)+r_i\,\log(r-r_i)}{9r_i^2+8\left(\ell_1^2+2\ell_2^2\right)\,r_i+6\ell_2^2\,\left(\ell_2^2+2\ell_1^2\right)}\,, \end{aligned} \tag{6.30}$$

where r_i are the three roots of the cubic expression in $e^{2\gamma}$. The function H approaches a positive constant at $r=0$, and at large r it becomes

$$\begin{aligned} H = c_0 &+ \frac{4m^2\,\left(2\ell_1^2+\ell_2^2\right)}{3\ell_2^2\,r^3} - \frac{4m^2\,\left(2\ell_1^2+\ell_2^2\right)\left(\ell_1^2+2\ell_2^2\right)}{3\ell_2^2\,r^4} \\ &+ \frac{4m^2\,\left(32\ell_1^6+72\ell_1^4\,\ell_2^2+120\ell_1^2\,\ell_2^4+19\ell_2^6\right)}{45\ell_2^2\,r^5}\,. \end{aligned} \tag{6.31}$$

In terms of proper distance ρ, defined by $e^{-\gamma}\,dr = d\rho$, we have

$$H = c_0 + \frac{256m^2}{3\rho^6} - \frac{106496m^2\,\ell_2^2}{45\rho^{10}}\,. \tag{6.32}$$

Note that when $\ell_1 = 1/2\ell_2$, the solution becomes rather simpler:

$$H = c_0 + \frac{m^2\,\left(20r^2+55\ell_2^2\,r+23\ell_2^4\right)}{10\left(r+\ell_2^2\right)^5}\,. \tag{6.33}$$

6.3. Harmonic 4-form for $\mathbb{C}^2$ bundle over $\mathbb{CP}^1\times\mathbb{CP}^1$, and smooth M2-brane. We can also construct a harmonic self-dual 4-form for the 8-dimensional metric with the $S^2\times S^2\times S^2$ base space, which we obtained in (5.61). The natural self-dual ansatz is

$$\begin{aligned} G_{(4)} = \hat{e}^0\wedge\hat{e}^{\tilde{0}}\wedge[&e^{2\alpha_1}\,f_1\,\Omega_1 + e^{2\alpha_2}\,f_2\,\Omega_2 + e^{2\alpha_3}\,f_3\,\Omega_3] \\ &- e^{2\alpha_1+2\alpha_2}\,f_3\,\Omega_1\wedge\Omega_2 - e^{2\alpha_1+2\alpha_3}\,f_2\,\Omega_1\wedge\Omega_3 - e^{2\alpha_2+2\alpha_3}\,f_1\,\Omega_2\wedge\Omega_3\,. \end{aligned} \tag{6.34}$$

If we let $x\equiv r^2$, then the equations that follow from $dG_{(4)}=0$ are

$$\begin{aligned} x\,f_1 + \left(x+\ell_2^2\right)\,f_2 &= \left(x\,\left(x+\ell_2^2\right)\,f_3\right)'\,, \\ x\,f_1 + \left(x+\ell_3^2\right)\,f_3 &= \left(x\,\left(x+\ell_3^2\right)\,f_2\right)'\,, \\ \left(x+\ell_2^2\right)\,f_2 + \left(x+\ell_3^2\right)\,f_3 &= \left(\left(x+\ell_2^2\right)\left(x+\ell_3^2\right)\,f_1\right)'\,, \end{aligned} \tag{6.35}$$

where a prime means d/dx here.

It is straightforward to solve these equations. By choosing the integration constants appropriately we can obtain a solution which gives a self-dual harmonic 4-form that is normalisable, namely

$$f_1 = \frac{2\ell_2^2\,\ell_3^2 + \left(\ell_2^2+\ell_3^2\right) r^2}{\left(r^2+\ell_2^2\right)^2\,\left(r^2+\ell_3^2\right)^2}\,, \quad f_2 = -\frac{\ell_3^2}{\left(r^2+\ell_2^2\right)\,\left(r^2+\ell_3^2\right)^2}\,, \quad f_3 = -\frac{\ell_2^2}{\left(r^2+\ell_3^2\right)\,\left(r^2+\ell_2^2\right)^2}\,. \tag{6.36}$$

One can see that in the special case where $\ell_3 = \ell_2$, the solution reduces to the one found in (6.20). This is not surprising, since then the final $S^2 \times S^2$ factors in $S^2 \times S^2 \times S^2$ become an Einstein-Kähler 4-manifold, and the equations arising from solving for the harmonic 4-form reduce to those that we had to solve previously for the $S^2 \times \mathbb{CP}^2$ base space.

Using the harmonic 4-form given by (6.34) and (6.36), we can construct another completely regular deformed M2-brane. It is easily seen from (6.34) that the magnitude of $G_{(4)}$ will be given by

$$|G_{(4)}|^2 = 48\left(f_1^2 + f_2^2 + f_3^2\right)\,. \tag{6.37}$$

From the expression (5.61) for the metric, we find that $\Box\, H = -\frac{1}{48}\, m^2\, |G_{(4)}|^2$ becomes

$$\left(\sqrt{g}\, U\, H'\right)' = -\tfrac{1}{48}\, m^2\, \sqrt{g}\, |G_{(4)}|^2\,, \tag{6.38}$$

where $\sqrt{g} = r^3\left(r^2+\ell_2^2\right)\left(r^2+\ell_3^2\right)/128$. The first integration gives rise to

$$H' = \frac{1}{\sqrt{g}\, U}\left\{\beta + \frac{m^2}{256\left(\ell_2^2-\ell_3^2\right)}\left[\frac{\ell_2^6}{\left(r^2+\ell_2^2\right)^2} - \frac{\ell_3^6}{\left(r^2+\ell_3\right)^2}\right]\right\}\,. \tag{6.39}$$

The singularity at $r = 0$ is avoided by choosing $\beta = -m^2/256$. Then we find that the function H is given by

$$H = c_0 - \frac{3m^2\,\left(\ell_2^2+\ell_3^2+3r^2\right)}{2\left(2\ell_3^2-\ell_2^2\right)\left(2\ell_2^2-\ell_3^2\right)\left(r^2+\ell_2^2\right)\left(r^2+\ell_3^2\right)} + \frac{27m^2\sqrt{2}}{4\left(2\ell_3^2-\ell_2^2\right)^{3/2}\,\left(2\ell_2^2-\ell_3^2\right)^{3/2}}\,\arctan\left[\frac{\sqrt{2\left(2\ell_3^2-\ell_2^2\right)\left(2\ell_2^2-\ell_3^2\right)}}{3r^2+2\left(\ell_2^2+\ell_3^2\right)}\right]\,. \tag{6.40}$$

The coordinate r runs from 0 to infinity, and the function H is finite and positive definite. For small r, H approaches a constant, and for large r, it behaves as

$$H \sim c_0 + \frac{m^2}{6r^6} - \frac{m^2\,\left(\ell_2^2+\ell_3^2\right)}{6r^8} + \frac{m^2\,\left(7\ell_2^4+5\ell_2^2\,\ell_3^2+7\ell_3^4\right)}{90r^{10}} + \cdots\,. \tag{6.41}$$

As usual, it is helpful to express the asymptotic behaviour in terms of proper distance ρ, defined by $dr/\sqrt{U} = d\rho$. The r and ρ coordinates are related, at large r, by

$$r \sim \frac{1}{\sqrt{2}}\left(\rho - \frac{\ell_2^2+\ell_3^2}{3\rho} + \frac{\left(\ell_2^2-\ell_3^2\right)^2}{6\rho^3} + \cdots\right)\,. \tag{6.42}$$

In terms of ρ, H has the following large-distance behaviour:

$$H \sim c_0 + \frac{4m^2}{3\rho^6} - \frac{32m^2\,\left(4\ell_2^4 + 5\ell_2^2\,\ell_3^2 + 4\ell_3^4\right)}{45\rho^{10}} + \cdots . \tag{6.43}$$

This deformed M2-brane is therefore completely regular, and it has a well-defined ADM mass. It is again supersymmetric.

Note that this solution for H reduces to the solution (6.22) if the parameters ℓ_2 and ℓ_3 are set equal, as would be expected in the light of our earlier discussion. It is interesting also to note that the solution (6.40) becomes especially simple if the parameters satisfy $\ell_2^2 = 2\ell_3^2$ or $\ell_3^2 = 2\ell_2^2$. Choosing the first of these two equivalent cases, we then find that the solution can be written as

$$H = c_0 + \frac{m^2\,\left(r^2 + 4\ell_3^2\right)}{6\left(r^2 + \ell_3^2\right)\,\left(r^2 + 2\ell_3^2\right)^3} . \tag{6.44}$$

6.4. Deformed M2-brane on the complex line bundle over $\mathbb{CP}^3$. At the end of Sect. 5.3 we described the $2(m+1)$-dimensional Ricci-flat Kähler metrics on the complex line bundles over $\mathbb{CP}^m$, and we obtained an L^2-normalisable (anti)-self-dual harmonic $(m+1)$-form for each case when m is odd. In particular, we can take $m = 3$, and consider the 8-dimensional complex line bundle over $\mathbb{CP}^3$. The metric is given in (5.36), and the harmonic 4-form can be read off from (5.38). Equation (4.16) for the M2-brane metric function H can be straightforwardly solved in this case, giving

$$H = c_0 + \frac{m^2\, r_0^6}{6\, r^6} . \tag{6.45}$$

(We have made an appropriate choice for the normalisation of the harmonic 4-form.) Since the radial coordinate r runs from r_0 to infinity, it follows that again we have a completely non-singular deformed M2-brane.

In terms of the proper radial distance ρ defined by $U^{-1/2}\, dr = d\rho$ for this metric, the asymptotic large-distance behaviour of the function H in the corresponding resolved M2-brane is easily seen to be

$$H = 1 + \frac{m^2\, r_0^6}{\rho^6} - \frac{m^2\, r_0^{14}}{14\rho^{14}} + \cdots . \tag{6.46}$$

It should be noted that this solution is not supersymmetric. This can be shown by substituting $G_{(4)}$ directly into the supersymmetry condition $G_{ABCD}\,\Gamma_{BCD}\,\eta = 0$, and making use of the integrability conditions (5.24), which reduce here to just $(\Gamma_{0a} + J_{ab}\,\Gamma_{\tilde{0}b})\,\eta = 0$. One finds that the only solution to all these conditions is $\eta = 0$. Alternatively, we may observe that although the harmonic $(m+1)$-form constructed in (5.38) is of type $\left(\frac{1}{2}(m+1), \frac{1}{2}(m+1)\right)$, it is not perpendicular to the Kähler form $\hat{J} = \hat{e}^0 \wedge \hat{e}^{\tilde{0}} + r^2\, J$, when the odd integer m is greater than 1. In particular, the harmonic 4-form in the complex line bundle over $\mathbb{CP}^3$ is of type (2, 2) but does not satisfy $G_{ABCD}\,\hat{J}_{CD} = 0$, and, as shown in [9], the vanishing of this quantity is another way of expressing the criterion for supersymmetry.

6.5. Deformed M2-brane on an 8-manifold of Spin(7) holonomy. Recently a resolved M2-brane was constructed using a Ricci-flat 8-manifold of Spin(7) holonomy [6]. We shall summarise the key features of that solution here, in order to allow a comparison with the resolved M2-branes using Ricci-flat Kähler 8-manifolds (which have $SU(4)$ holonomy) that we have obtained in this paper. The metric for the Spin(7) manifold, which is an $\mathbb{R}^4$ bundle over S^4, is given by

$$ds_8^2 = \left(1 - \frac{a^{10/3}}{r^{10/3}}\right)^{-1} dr^2 + + \tfrac{9}{100}\, r^2 \left(1 - \frac{a^{10/3}}{r^{10/3}}\right) (\sigma_i - A^i)^2 + \tfrac{9}{20} r^2\, d\Omega_4^2\,, \tag{6.47}$$

where σ_i are left-invariant 1-forms on $SU(2)$, $d\Omega_4^2$ is the metric on the unit 4-sphere, and A^i is the $SU(2)$ Yang-Mills instanton on S^4 [33, 34]. The Yang-Mills field strengths $F^i_{\alpha\beta}$ satisfy the algebra of the imaginary unit quaternions, $F^i_{\alpha\gamma}\, F^j_{\gamma\beta} = -\delta_{ij}\, \delta_{\alpha\beta} + \epsilon_{ijk}\, F^k_{\alpha\beta}$. A normalisable anti-self-dual harmonic 4-form was found in [6], with orthonormal components given by

$$G_{0ijk} = 6f\, \epsilon_{ijk} \quad G_{\alpha\beta\gamma\delta} = -6f\, \epsilon_{\alpha\beta\gamma\delta}\,, \quad G_{ij\alpha\beta} = f\, \epsilon_{ijk}\, F^k_{\alpha\beta}\,, \quad G_{0i\alpha\beta} = -f\, F^i_{\alpha\beta}\,, \tag{6.48}$$

where $f = r^{-14/3}$.

The deformed M2-brane is given by (4.16), with [6]

$$\begin{aligned} H = c_0 - \frac{40000 m^2}{729 a^{16/3}\, r^{2/3}} & \left[9 - \left(\frac{a}{r}\right)^{10/3} + \frac{3\left(1 - \frac{a^2}{r^2}\right)}{1 - \left(\frac{a}{r}\right)^{10/3}}\right] \\ + \frac{32000\sqrt{2\sqrt{5}}\, m^2}{243 a^6} & \left[(\sqrt{5} - 1)\, \arctan\left(\frac{\sqrt{5} + 1 + 4\left(\frac{a}{r}\right)^{10/3}}{\sqrt{2\sqrt{5}\left(\sqrt{5} - 1\right)}}\right)\right. \\ & \left. + (\sqrt{5} + 1)\, \arctan\left(\frac{\sqrt{5} - 1 + 4\left(\frac{a}{r}\right)^{10/3}}{\sqrt{2\sqrt{5}\left(\sqrt{5} + 1\right)}}\right)\right]. \end{aligned} \tag{6.49}$$

At large r, H has the asymptotic form

$$H = c_0 + \frac{2\, 10^5\, m^2}{3^7\, r^6} - \frac{28\, 10^4\, a^{4/3}\, m^2}{2673\, r^{22/3}} + \cdots\,. \tag{6.50}$$

In terms of the proper distance ρ, the asymptotic behaviour of the first two terms in H is the same as in the r coordinate.

The supersymmetry of the solution was not discussed in [6], but has since been demonstrated in [24]. Here, we note that another simple proof of supersymmetry can be given by making use of the results in [34] on the integrability conditions for the covariantly-constant spinor in the Spin(7) manifold. These are all encapsulated in the equations

$$4\Gamma_{0i}\, \eta + F^i_{\alpha\beta}\, \Gamma_{\alpha\beta}\, \eta = 0\,. \tag{6.51}$$

It useful also to note that these imply other equations, including

$$\Gamma_{0i}\eta = \tfrac{1}{2}\epsilon_{ijk}\Gamma_{jk}\eta\,, \quad F^i_{\alpha\beta}\Gamma_{0i\beta}\eta = 3\Gamma_\alpha\eta\,, \quad \epsilon_{ijk}\Gamma_{0ijk}\eta = 6\eta\,. \tag{6.52}$$

Using these equations, and the expressions given in (6.48) for the components of the harmonic 4-form, it is now elementary to verify that $G_{abcd}\Gamma_{bcd}\eta = 0$, and hence that the single supersymmetry allowed by the Spin(7) holonomy is preserved in the deformed solution.

7. Conclusions and Comments on Dual Field Theories

The purposes of this paper were manifold. Our first motivation was purely formal. We have provided an explicit construction of self-dual harmonic forms for a class of complete non-compact Ricci-flat Kähler manifolds in $2(n+1)$ real dimensions. Specifically, we focused on the Stenzel metrics [10]. These spaces have $SO(n+2)$ isometry, with level surfaces corresponding to $SO(n+2)/SO(n)$ coset spaces. The degenerate orbit ("bolt") corresponds to the base space $S^{n+1} \equiv SO(n+2)/SO(n+1)$. (The $n = 1$ case is the Eguchi-Hanson instanton, and the $n = 2$ case was first constructed by Candelas and de la Ossa [13] as the deformed conifold.) For these manifolds we provided an explicit construction of all the the harmonic, self-dual, middle dimension forms. Specifically, the solution for the harmonic (p, q)-forms in $p + q = 2(n + 1)$ dimensions reduces to finding the solution to two coupled first-order differential equations, which we solved explicitly.

Interestingly, the (p, p)-form (which implies n is odd) is proportional to $(\cosh r)^{-2p}$ and thus turns out to be L^2-normalisable. On the other hand all the other (p, q)-forms (for n odd or even) are not L^2 normalisable, with the degree of divergence increasing with the value $|p - q|$.

We also gave a construction of another general set of complete Ricci-flat metrics, whose homogeneous level surfaces are $U(1)$ bundles over a product of N Einstein-Kähler base spaces. The regularity of the solution implies that one of the base spaces has to be $\mathbb{CP}^m$ with its Fubini-Study metric, while the other Einstein-Kähler factors are restricted by topological considerations. For example, if they are complex projective spaces $\mathbb{CP}^{m_i}$, then they must satisfy (5.1) The total space is topologically a $\mathbb{C}^{m+1}$ bundle over the remaining base-space factors. (The 6-dimensional example where there are just two S^2 factors appeared in [13] and was further discussed in [5]; the metric has level surfaces that are the 5-manifold known as $T^{1,1}$, which is a $U(1)$ bundle over $S^2 \times S^2$.) We discussed explicit examples, and constructed normalisable harmonic 4-forms for two 8-dimensional cases, where the base spaces are $S^2 \times \mathbb{CP}^2$ and $S^2 \times S^2 \times S^2$, and harmonic $(m + 1)$-forms for all the cases with $\mathbb{CP}^m$ as base space, for all odd m.

These formal constructions of self-dual harmonic forms turn out to have intriguing applications in the study of deformed p-brane configurations whose transverse spaces are non-compact Ricci-flat manifolds. In particular, the fractional $D3$-brane found in [2] provides the non-singular gravity dual of $N = 1$ super Yang-Mills theory in four dimensions. A generalisation to a number of deformed p-brane configurations with odd or even dimensional Ricci-flat transverse spaces was recently given in [6]. The systematic construction of the middle-dimension harmonic forms for the Stenzel spaces, as well as the generalisations given in Sect. 5 allowed us to provide another set of regular gravity solutions corresponding in particular to deformed M2-branes with 8-dimensional transverse Ricci-flat spaces. We constructed two examples using Ricci-flat Kähler 8-manifolds, and in each case the deformed M2-branes are supported by $(2, 2)$-harmonic

forms that are normalisable, and so the M2-branes are regular everywhere. In both cases, as well as for the case of the M2-brane on the Spin(7) manifold that was constructed in [6], the solutions are supersymmetric. This should be contrasted with the 6-dimensional Ricci-flat Kähler metric on the $\mathbb{C}^2$ bundle over $\mathbb{CP}^1$, which has a harmonic form with both (1, 2) and (2, 1) contributions. Consequently, we show that the fractional D3-brane using this metric is not supersymmetric.

The deformed M2-branes that we constructed in this paper, and the previously-known fractional D3-branes, provide supergravity duals to field theories with less than maximal supersymmetry. In fact, the lower-dimensional conformal symmetry associated with the AdS/CFT correspondence can be broken by the extra contributions to the "harmonic" function H of these resolved branes. Indeed, in all the known fractional D3-branes the function H has a universal asymptotic logarithmic modification, given by (1.1), owing to the (marginal) non-normalisability of the complex harmonic self-dual 3-forms in six-dimensions. This implies that the geometry no longer has an AdS_5 background, and consequently the dual four-dimensional Yang-Mills field theory has no conformal symmetry. General mathematical arguments imply that for any six-dimensional Ricci-flat Kähler metric with an asymptotically conical structure, complex harmonic 3-forms will necessarily be non-normalisable.

By contrast, deformed M2-branes have a richer structure, with a larger range of possibilities for the asymptotic behaviour. At large distance the modification to H takes the form

$$H = c_0 + \frac{Q}{\rho^6}\left(1 + \frac{c}{\rho^\gamma} + \cdots\right). \tag{7.1}$$

For our Ricci-flat Kähler examples constructed in this paper γ takes the values $\frac{8}{3}$, 4 for supersymmetric M2-branes, and 8 for the non-supersymmetric solution, whilst for the Spin(7) example in [6], which is supersymmetric, we have $\gamma = \frac{4}{3}$. (The constant c is negative in all cases.) Thus in all these examples we have $\gamma > 0$, implying that the breaking of the conformal symmetry of the 3-dimensional field theories is much milder. In fact after dropping the constant 1 in the function H, the solutions are all asymptotically $AdS_4 \times M_7$ at large r.[14]

The resolved M2-brane and dyonic string solutions can reduce on the compact level surfaces of the transverse spaces to give rise to domain walls that are asymptotically AdS. The asymptotically AdS geometry is supported, from the viewpoint of the dimensionally-reduced theory, by a non-trivial (and possibly massive) scalar potential that has a fixed point. Thus these geometries describe the renormalisation group flows of the corresponding dual field theories. However, they are very different from those associated with continuous distributed brane configurations [35–41]. Notably, there are fewer supersymmetries in our resolved brane solutions than there are in the distributed brane solutions, which do not break further supersymmetry. Furthermore, the solutions we obtained in this paper are completely free of singularities, while the distributed branes in general have singularities, including naked ones. Finally, while the distributed brane configurations are naturally dual to the Coulomb branch of the corresponding dual field theory, the resolved M2-branes we obtained here, which are coincident rather than distributed, are related to the Higgs branch.

[14] Similarly, the resolved dyonic string using the Eguchi-Hanson metric, which was constructed in [6], has $H \sim c_0 + Q\,\rho^{-2} - c\,\rho^{-6} + \cdots$, in terms of large proper distance ρ. As a consequence, the solution with $c_0 = 0$ is also asymptotically AdS_3 [6].

In [6], a second deformed M2-brane with Spin(7) holonomy supported by a harmonic 4-form of the opposite duality was also explicitly constructed. In this case the 4-form is non-normalisable at large r, and as a consequence, the modification to the function H in (7.1) has a negative value of γ, namely $\gamma = -\frac{4}{3}$. Thus unlike the deformed M2-branes we discussed above, this solution will not approach AdS_4 spacetime, and the corresponding three-dimensional field theory dual would have no conformal symmetry. An analogous solution with marginally non-normalisable large-distance behaviour appears to be absent for the dyonic string with an Eguchi-Hanson transverse space, which is perhaps consistent with the more central rôle of conformal symmetry in two dimensional field theories.

In general a deformed p-brane solution has a reduced number of supersymmetries, or none at all.[15] In order for the solution to be free of (naked) singularities, the relevant harmonic form has to be normalisable at small proper distance. If the harmonic form is also normalisable at large proper distance, the solution may become asymptotically AdS in the decoupling limit, describing the renormalisation group flow of the Higgs branch of the corresponding less-supersymmetric dual conformal field theory. If, on the other hand, the harmonic form is non-normalisable at large distance, then the correction terms to the function H will break the AdS structure completely, and the dual field theory will have no conformal symmetry.

There are clearly open avenues to be investigated along the formal directions, by constructing harmonic forms not only in the middle dimension, and for other types of Ricci-flat even-dimensional manifolds, such as hyper-Kähler ones, as well as odd-dimensional ones. In particular, the construction of harmonic forms in other than the middle-dimension may prove to be useful in the study of a larger class of deformed branes, thus providing gravity dual candidates for a larger class of models. Another intriguing question relates to the many exact Ricci-flat metrics that we obtained in this paper. It would be interesting to see how these are related to the general investigation of the integrability of the Einstein equations for cohomogeneity one metrics contained in [26].

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[15] In all the examples that we have studied, turning on the flux from the harmonic form either breaks all the supersymmetry, or else it preserves all the supersymmetry that still remains after replacing the flat transverse metric by the more general complete Ricci-flat metric.

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