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Thomas A. Nevins

Representability for some moduli stacks of framed sheaves

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Abstract. We prove that certain moduli functors (and stacks) for framed torsion-free sheaves on complex projective surfaces are represented by schemes.

1. Introduction

Moduli problems for various kinds of framed sheaves have been studied and used in many settings (see, for example, [Tha94], [Bra91], [Nak94]), and there is a good general theory of moduli for semistable framed sheaves, thanks to the work of Huybrechts and Lehn ([HL95a], [HL95b]). By contrast, there seem to be only a few examples in which the *full* moduli functor for framed sheaves (without conditions of semistability) is known to be represented by a scheme. In this paper, we prove a representability theorem for the full moduli functors of framed torsion-free sheaves on projective surfaces under certain conditions.

Let S denote a smooth, connected complex projective surface, and let $D \subset S$ denote a smooth connected complete curve in S. Fix a vector bundle E on D. An *E-framed torsion-free sheaf on S* is a pair $(\mathcal{E}, \phi)$ consisting of a torsion-free sheaf $\mathcal{E}$ on S and an isomorphism $\phi : \mathcal{E}\big|_D \to E$; the isomorphism ϕ is called an *E-framing* of $\mathcal{E}$. An *isomorphism* of E-framed torsion-free sheaves on S is an isomorphism of the underlying torsion-free sheaves on S that is compatible with the framings. Let $\mathrm{TF}_S(E)$ denote the moduli functor for isomorphism classes of E-framed torsion-free sheaves on S. The reader should note that in the work of Huybrechts–Lehn the framing ϕ need *not* be an isomorphism; as a consequence of our more restrictive definition, the moduli functors that we study have no hope of being proper.

Suppose the vector bundle E satisfies

$$H^0\left(D, \mathrm{End}\, E \otimes N_{D/S}^{-k}\right) = 0 \tag{1}$$

for all $k \geq 1$; here $N_{D/S}$ is the normal bundle of D in S. If $D \subset S$ is an arbitrary curve, there may be very few such bundles. However, if D is smooth and has positive

Th. A. Nevins: Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1109, USA. e-mail: nevins@umich.edu

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self-intersection in S, then $N_{D/S}^{-1}$ is a negative line bundle on D, and consequently this condition on E is an open condition which is satisfied by all semistable vector bundles on D.

Theorem 1. *Suppose that S is a smooth, connected complex projective surface and $D \subset S$ is a smooth connected complete curve. Suppose, in addition, that E is a vector bundle on D that satisfies Condition (1) for all $k \geq 1$. Then the functor* $\mathrm{TF}_S(E)$ *is represented by a scheme.*

In the proof of Theorem 1 we work in the slightly more general setting of a family of vector bundles on D, parametrized by a scheme U, that satisfies Condition (1) for all $k \geq 1$ at every point of U. Note also that the reader who is familiar with the language of stacks may restate Theorem 1 in the following form: over the substack of its target that parametrizes vector bundles on D that satisfy Condition (1) for all $k \geq 1$, the fibers of the restriction morphism from the moduli stack of torsion-free sheaves on S that are locally free along D to the moduli stack of vector bundles on D are schemes.

Functors of the type we study here arose naturally (in some special cases) in the representation-theoretic constructions of Nakajima; Theorem 1 demonstrates that the existence of the fine moduli schemes used by Nakajima is a much more general phenomenon, one which we hope can be exploited more widely in the study of sheaves on noncompact surfaces. The new ingredient in our proof of Theorem 1 is the use of formal geometry along the curve D; in particular, the techniques used here are completely different from those of [HL95a], [HL95b], and make no use of geometric invariant theory (GIT). Although Lehn ([Leh93]) has, under some conditions on the curve D and the bundle E along the curve, proven that the full moduli functors for vector bundles on S with framing along D by E are represented by *algebraic spaces*, from the point of view of the usual GIT techniques it is perhaps surprising that there is a fine moduli *scheme* (a much stronger fact) for all framed sheaves: indeed, there can be framed sheaves that are not semistable for *any* polarization.

2. Affine bundles over Bun$^{\circ}$ *(D)*

In this section we construct the fundamental affine bundles $\mathbf{A}_n$ (for n in the range $1 \leq n < \infty$) over U that we will use to embed the functor $\mathrm{TF}_S(E)$ in a scheme. The construction of these bundles and the description of the universal properties they possess must be well known (cf. [Gri66], in which the relevant cohomology groups are discussed), but the author does not know a suitable reference.

Fix a surface S, a curve D in S, a scheme U, and a vector bundle E on $D \times U$ as in Theorem 1. Let $D^{(n)}$ (that is, D with structure sheaf $\mathcal{O}_S/I_D^{n+1}$, $0 \leq n < \infty$) denote the nth order neighborhood of D in S.

Definition 2. *Let $\mathcal{A}_n$ denote the moduli functor over U of isomorphism classes of triples $(\mathcal{E}, V \xrightarrow{f} U, \phi)$ consisting of*

1. *a vector bundle* $\mathcal{E}$ *on* $D^{(n)} \times V$,
2. *a morphism* $f : V \to U$, *and*
3. *an isomorphism* $\phi : \mathcal{E}\big|_{D\times V} \to (1_D \times f)^* E$.

Suppose that $\mathcal{E}$ is a vector bundle over $D^{(n)}$; then $\mathcal{E}$ has a canonical (decreasing) filtration as an $\mathcal{O}_{D^{(n)}}$-module with filtered pieces $F_j\mathcal{E} = I_D^j\mathcal{E}$, where I_D is the ideal of $D \subset D^{(n)}$. By its construction, this filtration is preserved by any endomorphism of the vector bundle $\mathcal{E}$, and moreover $F_j\mathcal{E}/F_{j+1}\mathcal{E} \cong N_{D/S}^{-j} \otimes \left(F_0\mathcal{E}/F_1\mathcal{E}\right)$ provided $0 \le j \le n$. Using these facts together with the exact sequence

$$0 \to \mathrm{Hom}(E, E \otimes N_{D/S}^{-n}) \to \mathrm{Hom}(\mathcal{E}, \mathcal{E}) \to \mathrm{Hom}\left(\mathcal{E}, \mathcal{E}\big|_{D^{(n-1)}}\right) \to 0$$

and condition (1), one may prove by induction on n that $\mathrm{End}(\mathcal{E}) \subseteq \mathrm{End}(\mathcal{E}\big|_D)$ and consequently that E-framed bundles on $D^{(n)}$ are rigid.

Evidently $\mathcal{A}_0 \cong U$; moreover, there are maps $\pi_{n+1} : \mathcal{A}_{n+1} \to \mathcal{A}_n$ for all $n \ge 0$.

Proposition 3. *Each* $\mathcal{A}_n$ *(*$n \ge 1$*) is represented by a scheme* $\mathbf{A}_n$ *that is an affine bundle over* $\mathbf{A}_{n-1}$.

Proof. Working inductively, it will suffice to construct an $\mathbf{A}_{n-1}$-scheme $\mathbf{A}_n$ that represents $\mathcal{A}_n$ and is an affine bundle over $\mathbf{A}_{n-1}$. Fix a universal bundle $E^{(n-1)}$ on $D^{(n-1)} \times \mathbf{A}_{n-1}$. For any scheme T, an element of $\mathcal{A}_n(T)$ determines a map $f : T \to \mathbf{A}_{n-1}$, and, if $(\mathcal{E}, \phi)$ is the given element of $\mathcal{A}_n(T)$, there is an isomorphism of $\mathcal{E}\big|_{D^{(n-1)}\times T}$ with $(1 \times f)^* E^{(n-1)}$ compatibly with the framings by E. But then, because E-framed bundles on $D^{(n-1)}$ are rigid, we find that $\mathcal{A}_n$ as a functor over $\mathbf{A}_{n-1}$ is isomorphic to the functor taking $f : T \to \mathbf{A}_{n-1}$ to the set of isomorphism classes of pairs $(\mathcal{E}, \phi)$ consisting of a bundle $\mathcal{E}$ on $D^{(n)} \times T$ together with an isomorphism ϕ of $\mathcal{E}\big|_{D^{(n-1)}\times T}$ with $(1 \times f)^* E^{(n-1)}$. We will refer to such a pair as an $E^{(n-1)}$-framed bundle.

Because the statement of the proposition is local on $\mathbf{A}_{n-1}$, we may assume that $\mathbf{A}_{n-1}$ is an affine scheme that is the spectrum of a local ring R. For simplicity, write $\mathcal{O} = \mathcal{O}_{D^{(n-1)}\times \mathbf{A}_{n-1}}$ and $\mathcal{O}' = \mathcal{O}_{D^{(n)}\times \mathbf{A}_{n-1}}$. The "change of rings" spectral sequence (see Chap. XVI, Section 5 of [CE56])

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathcal{O}}\left(\underline{\mathrm{Tor}}_q^{\mathcal{O}'}(E^{(n-1)}, \mathcal{O}), E(-nD)\right) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}'}\left(E^{(n-1)}, E(-nD)\right)$$

yields the exact sequence of terms of low degree

$$\begin{aligned} 0 \to \mathrm{Ext}^1_{\mathcal{O}}\left(E^{(n-1)}, E(-nD)\right) &\to \mathrm{Ext}^1_{\mathcal{O}'}\left(E^{(n-1)}, E(-nD)\right) \\ &\xrightarrow{\beta} \mathrm{Hom}\left(\underline{\mathrm{Tor}}_1^{\mathcal{O}'}(E^{(n-1)}, \mathcal{O}), E(-nD)\right) \to 0. \end{aligned} \quad (2)$$

Note that β is surjective since the next term in the sequence is $\mathrm{Ext}^2_{\mathcal{O}}\left(E^{(n-1)}, E(-nD)\right)$, which vanishes because D is one-dimensional. Using $\underline{\mathrm{Tor}}_1^{\mathcal{O}'}(E^{(n-1)}, \mathcal{O}) \cong E(-nD)$ one may check that there is a canonical element e of $\mathrm{Hom}\left(\underline{\mathrm{Tor}}_1^{\mathcal{O}'}(E^{(n-1)},\right.$

$\mathcal{O}), E(-nD)\big)$ such that $\beta^{-1}(e)$ is exactly the $\mathrm{Ext}^1_{\mathcal{O}}\left(E^{(n-1)}, E(-nD)\right)$-subtorsor of $\mathrm{Ext}^1_{\mathcal{O}'}\left(E^{(n-1)}, E(-nD)\right)$ that classifies 1-extensions

$$0 \to E(-nD) \to \mathcal{E} \to E^{(n-1)} \to 0$$

for which $\mathcal{E}$ is a locally free $\mathcal{O}'$-module. Now, Condition (1), together with Cohomology and Base Change, implies that the R-module $\mathrm{Ext}^1_{\mathcal{O}}\left(E^{(n-1)}, E(-nD)\right) \cong H^1\left(D \times \mathbf{A}_{n-1}, \mathrm{End}(E) \otimes N_{D/S}^{-n}\right)$ is projective, hence free. One can easily construct, moreover, a universal 1-extension over $D^{(n)} \times \mathbf{A}_{n-1} \times \beta^{-1}(e)$ (using, for example, an affine subspace of the Čech cocycles that maps isomorphically to $\beta^{-1}(e)$ to furnish gluing data). Because the exact sequence (2) and the element e are functorial under pullback along morphisms of affine schemes $\mathrm{Spec}\, R' \xrightarrow{f} \mathrm{Spec}\, R = \mathbf{A}_{n-1}$, this universal 1-extension induces a functorial bijection between the set $\beta_{R'}^{-1}(e)$ (the inverse image of the canonical element under the base-changed map β) and the set of isomorphism classes of pairs $(\mathcal{E}, \phi)$ consisting of a vector bundle $\mathcal{E}$ on $D^{(n)} \times \mathrm{Spec}\, R'$ and a framing $\phi : \mathcal{E}\big|_{D^{(n-1)} \times \mathrm{Spec}\, R'} \to (1 \times f)^* E^{(n-1)}$.

Consequently $\mathcal{A}_n$ is represented as a functor over $\mathbf{A}_{n-1}$ by the torsor over $\mathrm{Spec}\, \mathrm{Sym}^\bullet \mathrm{Ext}^1_{\mathcal{O}}\left(E^{(n-1)}, E(-nD)\right)$ defined by $\beta^{-1}(e)$, proving the proposition. □

3. Proof of Theorem 1

There is a compatible family of morphisms $F_n : \mathrm{TF}_S(E) \to \mathbf{A}_n$ given by restriction. Fix a $\mathrm{Spec}\, \mathbf{C}$-valued point of $\mathrm{TF}_S(E)$, that is, a point $u \in U$ together with an E_u-framed pair $(\mathcal{F}, \phi)$ on S. We will show that there is an open subfunctor Z of $\mathrm{TF}_S(E)$ that contains $(\mathcal{F}, \phi)$ and is represented by a scheme.

Fix a polarization H of S, and choose m sufficiently large that

1. $\mathcal{F} \otimes H^m$ is globally generated and
2. $H^1(\mathcal{F} \otimes H^m) = H^2(\mathcal{F} \otimes H^m) = 0$.

Further, fix n sufficiently large that the restriction map

$$H^0\left(\mathcal{F} \otimes H^m\right) \to H^0\left(\mathcal{F} \otimes H^m\big|_{D^{(n)}}\right)$$

is injective; it is possible to choose such an n because $\mathcal{F}$ is torsion-free. Finally, choose m' sufficiently large that $H^1\left(\mathcal{F} \otimes H^{m+m'}\big|_{D^{(n)}}\right) = 0$.

Next, let $Z \subseteq \mathrm{TF}_S(E)$ denote the open subfunctor parametrizing those triples $\left(W \xrightarrow{f} U, \mathcal{E}, \phi : \mathcal{E}\big|_{D \times W} \to (1 \times f)^* E\right)$ for which the family $\mathcal{E}$ satisfies the following conditions:

a. $\mathcal{E}_w \otimes H^m$ is globally generated for all $w \in W$,
b. $H^1(\mathcal{E}_w \otimes H^m) = H^2(\mathcal{E}_w \otimes H^m) = 0$ for all $w \in W$,
c. the map $H^0\left(\mathcal{E}_w \otimes H^m\right) \to H^0\left(\mathcal{E}_w \otimes H^m\big|_{D^{(n)}}\right)$ is injective for all $w \in W$, and
d. $H^1\left(\mathcal{E}_w \otimes H^{m+m'}\big|_{D^{(n)}}\right) = 0$ for all $w \in W$.

In the previous section we showed that there is a universal vector bundle $E^{(n)}$ on $D^{(n)} \times \mathbf{A}_n$. Fix an element of $Z(W)$; then the map $F_n(W) : W \to \mathbf{A}_n$ yields a vector bundle $(1 \times F_n)^* E^{(n)}$ on $D^{(n)} \times W$ together with an isomorphism

$$\mathcal{E}_W\big|_{D^{(n)} \times W} \xrightarrow{\phi_n} (1 \times F_n)^* E^{(n)};$$

here $\mathcal{E}_W$ denotes the torsion-free sheaf on $S \times W$ determined by the fixed element of $Z(W)$. Let p_W denote the projection $S \times W \to W$. Then by construction the sheaves $(p_W)_* \mathcal{E}_W \otimes H^m$, $(p_W)_* \mathcal{E}_W \otimes H^{m+m'}$, and $(p_W)_* \left(\mathcal{E}_W \otimes H^{m+m'}\big|_{D^{(n)} \times W} \right)$ are vector bundles on W, and, choosing a section s of $H^{m'}$ the zero locus of which has transverse intersection with D, there is a commutative diagram

$$\begin{tikzcd}
(p_W)_* \mathcal{E}_W \otimes H^m \arrow[r] \arrow[d, "\otimes s"] & (p_W)_* \left(\mathcal{E}_W \otimes H^m\big|_{D^{(n)} \times W} \right) \arrow[d, "\otimes s"] \\
(p_W)_* \mathcal{E}_W \otimes H^{m+m'} \arrow[r] & (p_W)_* \left(\mathcal{E}_W \otimes H^{m+m'}\big|_{D^{(n)} \times W} \right)
\end{tikzcd}$$

for which the vertical arrows (given by tensoring with s) and the top row are injective. Using ϕ_n, we may replace this diagram canonically with the diagram

$$\begin{tikzcd}
(p_W)_* (\mathcal{E}_W \otimes H^m) \arrow[r] \arrow[d, "\otimes s"] & (p_W)_* \left((1 \times F_n)^* E^{(n)} \otimes H^m \right) \arrow[d, "\otimes s"] \\
(p_W)_* \mathcal{E}_W \otimes H^{m+m'} \arrow[r] & (p_W)_* \left((1 \times F_n)^* E^{(n)} \otimes H^{m+m'} \right).
\end{tikzcd}$$

Now, by assumption (d) on W, we have

$$(p_W)_* \left((1 \times F_n)^* E^{(n)} \otimes H^{m+m'} \right) = F_n^* \left((p_{\mathbf{A}_n})_* (E^{(n)} \otimes H^{m+m'}) \right),$$

where $p_{\mathbf{A}_n} : D^{(n)} \times \mathbf{A}_n \to \mathbf{A}_n$ is the projection, and so finally we obtain the diagram of vector bundles

$$\begin{tikzcd}
(p_W)_* \mathcal{E}_W \otimes H^m \arrow[d, "\otimes s"] \arrow[dr, "r"] & \\
(p_W)_* \mathcal{E}_W \otimes H^{m+m'} \arrow[r] & F_n^* \left((p_{\mathbf{A}_n})_* (E^{(n)} \otimes H^{m+m'}) \right)
\end{tikzcd}$$

on W, where the diagonal map r and the map $\otimes s$ are injective. By construction, furthermore, the image of the morphism r is a vector subbundle of $F_n^* \Big((p_{\mathbf{A}_n})_* (E^{(n)} \otimes H^{m+m'}) \Big)$ and consequently determines a morphism $W \to \mathbf{Gr}$ over $\mathbf{A}_n$, where $\mathbf{Gr} \xrightarrow{q} \mathbf{A}_n$ denotes the relative Grassmannian for the vector bundle $(p_{\mathbf{A}_n})_* (E^{(n)} \otimes H^{m+m'})$ on $\mathbf{A}_n$, the fiber of which over $a \in \mathbf{A}_n$ parametrizes vector subspaces of $H^0(E_a^{(n)} \otimes H^{m+m'})$ that are of dimension $h^0(\mathcal{F} \otimes H^m)$.

We now construct a Quot-scheme over **Gr** that we will use to represent Z. We may pull back $(p_{\mathbf{A}_n})_*(E^{(n)} \otimes H^{m+m'})$ to **Gr** to obtain a vector bundle $q^*(p_{\mathbf{A}_n})_*(E^{(n)} \otimes H^{m+m'})$ on (an open subset of) **Gr**, with universal subbundle

$$\mathcal{U} \subset q^*(p_{\mathbf{A}_n})_*(E^{(n)} \otimes H^{m+m'})$$

of rank $h^0(\mathcal{F} \otimes H^m)$. If $p_{\mathbf{Gr}} : S \times \mathbf{Gr} \to \mathbf{Gr}$ denotes the projection to **Gr**, we obtain a bundle $p_{\mathbf{Gr}}^*\mathcal{U} \subset p_{\mathbf{Gr}}^* q^*(p_{\mathbf{A}_n})_*(E^{(n)} \otimes H^{m+m'})$ on $S \times \mathbf{Gr}$, as well as a quotient

$$p_{\mathbf{Gr}}^* q^*(p_{\mathbf{A}_n})_*(E^{(n)} \otimes H^{m+m'}) \to (1 \times q)^*(E^{(n)} \otimes H^{m+m'})$$

and subquotient $(1 \times q)^*(E^{(n)} \otimes H^m) \subset (1 \times q)^*(E^{(n)} \otimes H^{m+m'})$ that are sheaves on $S \times \mathbf{Gr}$ supported on $D^{(n)} \times \mathbf{Gr}$.

Consider the relative Quot-scheme $q' : \mathrm{Quot}_{S \times \mathbf{Gr}/S}(p_{\mathbf{Gr}}^*\mathcal{U}) \longrightarrow \mathbf{Gr}$ that parametrizes quotient sheaves for the family $p_{\mathbf{Gr}}^*\mathcal{U}$ on $S \times \mathbf{Gr}/S$. There is a universal quotient $(1 \times q')^* p_{\mathbf{Gr}}^*\mathcal{U} \to \mathcal{Q}$ on $S \times \mathrm{Quot}_{S \times \mathbf{Gr}/S}$, giving a diagram

$$\begin{array}{ccc} (1 \times q')^* p_{\mathbf{Gr}}^*\mathcal{U} & \longrightarrow & (1 \times q')^* p_{\mathbf{Gr}}^* q^*(p_{\mathbf{A}_n})_*(E^{(n)} \otimes H^{m+m'}) \\ \downarrow & & \downarrow \\ \mathcal{Q} & & (1 \times qq')^*(E^{(n)} \otimes H^m) \subset (1 \times qq')^*(E^{(n)} \otimes H^{m+m'}). \end{array} \tag{3}$$

There is a closed subscheme of $\mathrm{Quot}_{S \times \mathbf{Gr}/S}$ (see the proof of Theorem 1.6 of [Ser86]) that represents the subfunctor of those quotients the kernels of which project to zero in $(1 \times qq')^*(E^{(n)} \otimes H^{m+m'})$, and a closed subscheme $\mathcal{C}$ of that closed subscheme that represents the sub-subfunctor that parametrizes those quotients the images of which in $(1 \times qq')^*(E^{(n)} \otimes H^{m+m'})$ actually lie in the subsheaf $(1 \times qq')^*(E^{(n)} \otimes H^m)$. $\mathcal{C}$ then represents the functor of quotients of $p_{\mathbf{Gr}}^*\mathcal{U}$ that map to $(1 \times qq')^*(E^{(n)} \otimes H^m)$—that is, it is exactly the closed subscheme over which Diagram (3) extends to

$$\begin{array}{ccc} (1 \times q')^* p_{\mathbf{Gr}}^*\mathcal{U} & \longrightarrow & (1 \times q')^* p_{\mathbf{Gr}}^* q^*(p_{\mathbf{A}_n})_*(E^{(n)} \otimes H^{m+m'}) \\ \downarrow & & \downarrow \\ \mathcal{Q} & \longrightarrow & (1 \times qq')^*(E^{(n)} \otimes H^m) \subset (1 \times qq')^*(E^{(n)} \otimes H^{m+m'}). \end{array} \tag{4}$$

Restricting further to an open subscheme $\mathcal{C}^\circ$ of $\mathcal{C}$, we may assume that, over $\mathcal{C}^\circ$, the map $\mathcal{Q}\big|_{D^{(n)} \times \mathcal{C}^\circ} \to (1 \times qq')^*(E^{(n)} \otimes H^m)$ is an isomorphism, that $\mathcal{Q}$ is a family of torsion-free sheaves on S, and that conditions (a) through (d) are satisfied.

By construction the morphism $W \to \mathbf{Gr}$ lifts to a morphism $W \to \mathcal{C}^\circ$; this construction thus determines a morphism of functors $Z \to \mathcal{C}^\circ$. Similarly, there is a forgetful morphism $\mathcal{C}^\circ \to Z$. Finally, it is clear from the construction that these two morphisms of functors are inverses of each other, as desired. $\square$

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