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The isotropy groups of bounded homogeneous domains are nontrivial

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Abstract. It will be shown in this paper that the automorphism group of a bounded homogeneous domain D in $\mathbb{C}^n$ can never act freely on D. An equivalent statement is that the isotropy groups of bounded homogeneous domains always contain at least two elements.

1. Notation

The cardinality of a set S will be denoted by $|S|$. The automorphism group of a complex manifold M will be denoted by Aut M, the isotropy group of a point $p \in M$ will be denoted by $\mathrm{Iso}_p\ M$. Since the isotropy groups of a homogeneous manifold are isomorphic, the actual choice of the point $p \in M$ will usually not be of importance. To reflect this fact, we will suppress the index p if a statement holds for arbitrary $p \in M$.

2. Introduction

A complex manifold M is homogeneous by definition iff for all $z, w \in M$ there exists *at least one* automorphism φ of M with $\varphi(z) = w$.

This definition gives rise to the following natural question: does there exist a homogeneous complex manifold (or to mention some other interesting cases: compact complex manifold, domain, bounded domain) M that is minimal in the sense that for all $z, w \in M$ there exists *exactly one* automorphism φ of M with $\varphi(z) = w$? Obviously, this is the case if and only if $\mathrm{Iso}_p\ M = \{id\}$ for one and thus all $p \in M$. In other words: for arbitrary $z, w \in M$ there is always exactly one $\varphi \in \mathrm{Aut}\ M$ with $\varphi(z) = w$ iff Aut M acts freely on M. So the question that arises is:

Question 1. Does there exists a homogeneous complex manifold, compact complex manifold, domain or bounded domain M such that Aut M acts freely on M?

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The goal of this paper is to prove that the answer to this question is “no” in the case of bounded homogeneous domains. The proof given here is – besides using deep theorems by Pijateckii–Sapiro, Gindikin, Vinberg and Rothaus – quite simple.

We will prove the following theorem:

Theorem 1. *The automorphism group of a bounded homogeneous domain D can never act freely on D. In other words:* $|\operatorname{Iso} D| > 1$ *for every bounded homogeneous domain D.*

It should be mentioned right at the beginning that it is only impossible for the *full* automorphism group of a bounded homogeneous domain $D \subset \mathbb{C}^n$ to act freely on D. It is possible in certain cases that a subgroup of $\operatorname{Aut} D$ acts transitively and freely on D. A trivial example of this will be discussed in section 4.

The proof of the theorem uses the well known result of Pijateckii–Sapiro, Gindikin and Vinberg ([GPVS63]) that every bounded homogeneous domain is biholomorphically equivalent to a Siegel domain of the second kind, as well as an inductive construction of O. S. Rothaus ([Rot66]) of homogeneous regular cones by representations of lower dimensional cones sitting in the boundary.

We will include the definition of Siegel domains of the second kind and the results of Rothaus in this paper for the convenience of the reader.

Definition. *A set $C \subset \mathbb{R}^n$ is a* **cone***, if $x \in C \Leftrightarrow \lambda x \in C$ holds for all $\lambda \in \mathbb{R}^+$. A cone is called* **regular***, if it is nonempty, open, convex and does not contain an entire line. Note that it follows from convexity that $x, y \in C \Rightarrow x + y \in C$ holds for regular cones. Let from now on C denote a regular cone in $\mathbb{R}^n$.*
The **Siegel domain of the first kind** *over $C \subset \mathbb{R}^n$ is the tube domain* $\{z \in \mathbb{C}^n : \operatorname{Im} z \in C\}$.
*A C-***hermitian form** *is a mapping $H : \mathbb{C}^k \times \mathbb{C}^k \to \mathbb{C}^n$ with the following properties:*

(i) H *is $\mathbb{C}$-linear in the first argument*
(ii) $H(z, w) = \overline{H(w, z)}$
(iii) $H(z, z) \in \overline{C}$
(iv) $H(z, z) = 0 \Leftrightarrow z = 0$

Note that this definition coincides with the usual definition of a hermitian form if $C = \mathbb{R}^+$.

The *Siegel domain of the second kind over C with C-hermitian form H* is now defined to be $\{(z, w) \in \mathbb{C}^{n+k} : \operatorname{Im} z - H(w, w) \in C\}$.

Definition. *A cone $C \in \mathbb{R}^n$ is called* **homogeneous***, iff a subgroup of* $\operatorname{Gl}(n, \mathbb{R})$ *acts transitively on C.*

The following theorem of Pijateckii–Sapiro, Gindikin and Vinberg is of fundamental importance for the study of bounded homogeneous domains:

Theorem (Pijateckii–Sapiro, Gindikin and Vinberg). *Let D be a bounded homogeneous domain. Then D is biholomorphically equivalent to a Siegel domain of the second kind over a regular homogeneous cone.*

Let $\mathrm{Sym}(n, \mathbb{R})$ denote the set of all $n \times n$ symmetric matrices.

Definition. *Let $C \subset \mathbb{R}^n$ be a regular homogeneous cone. A* **representation of degree** *ρ of C is a linear mapping $R : \mathbb{R}^n \to \mathrm{Sym}(\rho, \mathbb{R}), v \mapsto R(v)$, such that*

(i) $R(v)$ is positive definite for $v \in C$
(ii) there exists a transitive group Q of automorphisms of C, such that for each $q \in Q$ there exists a $\rho \times \rho$-matrix p with $R(qv) = pR(v)p^t$ for all $v \in \mathbb{R}^n$.

Then the set

$$\{(t, u, v) \in \mathbb{R}^{n+\rho+1} : t \in \mathbb{R}, u \in \mathbb{R}^\rho, v \in C,\ t > u^t [R(v)]^{-1} u\}$$

is called the **extension** *of C by the representation R.*

Rothaus proved the following theorem:

Theorem (O. S. Rothaus).

(i) *An extension of a homogeneous regular cone is once again a homogeneous and regular cone.*
(ii) *Every homogeneous regular cone arises by a finite number of extensions starting from the cone $\mathbb{R}^+$.*

3. The size of isotropy groups of regular homogeneous cones

Before we can proof our main theorem we will have to analyze the size of isotropy groups of homogeneous regular cones. Let $\mathrm{Aut}\ C$ denote the linear automorphism group of a cone C, and let $\mathrm{Iso}_p\ C$ denote the isotropy group of $p \in C$. We will again use the notation $\mathrm{Iso}\ C$ if the choice of p is irrelevant. The following proposition is the equivalent of our main theorem in the case of homogeneous cones:

Proposition 1. *Let C be a regular homogeneous cone in $\mathbb{R}^n$ with $n > 1$. Then $|\mathrm{Iso}\ C| > 1$.*

Proof. Let C be a regular homogeneous cone in $\mathbb{R}^n$ with $n > 1$. The theorem of O. S. Rothaus guarantees the existence of regular homogeneous cones C_j in $\mathbb{R}^{n_j}$, $1 \leq j \leq r$, and representations R_j of dimensions k_j, $1 \leq j \leq r-1$ with the following properties:
$C_1 = \mathbb{R}^+$, $C_r = C$ and C_{j+1} arises from C_j by the representation R_j.
First we want to study the last step in this process: C arises from $\tilde{C} := C_{r-1} \subset \mathbb{R}^m$ by the representation $R := R_{r-1}$ of dimension $k := k_{r-1}$

$$\Rightarrow C = \{(t, u, v) : t \in \mathbb{R},\ u \in \mathbb{R}^k,\ v \in \tilde{C},\ t > u^t R^{-1}(v)u\}.$$

Obviously, we have $(t, u, v) \in C \Leftrightarrow (t, -u, v) \in C$; it follows that $A : (t, u, v) \mapsto (t, -u, v)$ is in Aut C.
Let q be an arbitrary point in $\tilde{C}$ and let $p := (1, 0, q)$. Then p is in C, $Ap = p$ and $A \in \mathrm{Iso}_p\ C$. Since $A \neq id$ if $k > 0$, it follows that $|\,\mathrm{Iso}\ C| > 1$ if $k > 0$.
If $k = 0$ we have $C = \{(t, v) : t \in \mathbb{R},\ v \in \tilde{C},\ t > 0\} = \mathbb{R}^+ \times \tilde{C}$. If $\tilde{A} \in \mathrm{Aut}_q\ \tilde{C}$ and $p := (1, q)$ then $p \in C$ and $A : (t, v) \mapsto (t, \tilde{A}v) \in \mathrm{Iso}_p\ C$. It follows that $|\,\mathrm{Iso}\ C| > 1$, if $|\,\mathrm{Iso}\ \tilde{C}| > 1$. Inductive use of this argument yields the following:
$|\,\mathrm{Iso}\ C| > 1$ if $k_s > 0$ and $k_{s+1} = \cdots = k_{r-1} = 0$.
It remains to study the case $k_1 = \cdots = k_{r-1} = 0$. In this case we have $C = \left(\mathbb{R}^+\right)^n$, and $\mathrm{Iso}_p\ C$, $p = (1, \ldots, 1)$, consists of the $n!$ mappings $(x_1, \ldots, x_n)^t \mapsto \mathbf{P} \cdot (x_1, \ldots, x_n)^t$, $\mathbf{P}$ a $n \times n$ permutation matrix. Consequently, $|\,\mathrm{Iso}\ C| > 1$, since $n > 1$.
We have thus proved that $|\,\mathrm{Iso}\ C| > 1$ in every possible case. □

Remark. The inductive construction of Rothaus is not well behaved with respect to isotropy groups: if

$$C_1 \xrightarrow{R_1} C_2 \xrightarrow{R_2} \ldots \xrightarrow{R_{r-2}} C_{r-1} \xrightarrow{R_{r-1}} C_r$$

is a sequence of representations, the size of the isotropy groups Iso C_j is in general not increasing with j. The method of Rothaus only guarantees that a certain subgroup of Aut C_j – which still acts transitively on C_j – extends to a transitive group of automorphisms of C_{j+1}. The information obtained about the groups Iso C_j can therefore in general not be used to gain information about Iso C_{j+1} (a noteworthy exception is the case $\dim R_j = 0$, where $C_{j+1} = \mathbb{R}^+ \times C_j$ and $|\,\mathrm{Iso}\ C_{j+1}| \geq |\,\mathrm{Iso}\ C_j|$). This is the reason why the only piece of information about the sequence of representations we used in the case $k_r > 0$ was the structure of the map R_{r-1}. A different kind of description of homogeneous cones seems to be needed to get better estimates on the size of the isotropy groups.

4. Proof of the main theorem

Proposition 1 together with the theorem of Pijateckii–Sapiro, Gindikin and Vinberg will now enable us to prove theorem 1.

Proof (of theorem 1). Let D be a bounded homogeneous domain in $\mathbb{C}^m$. By the theorem of Pijateckii–Sapiro, Gindikin and Vinberg we can assume that the domain D is a Siegel domain of the second kind.
Let $D = \{(z, w) \in \mathbb{C}^{n+k} : \operatorname{Im} z - H(w, w) \in C\}$ with $n + k = m$, C a regular cone in $\mathbb{R}^n$, $H : \mathbb{C}^k \times \mathbb{C}^k \to \mathbb{C}^n$ a C–hermitian form.
Consider the mappings

$$f_\theta : \mathbb{C}^{n+k} \to \mathbb{C}^{n+k},\ (z, w) \mapsto (z, e^{i\theta} w),\ \theta \in \mathbb{R}$$

Then the following holds:

$$\begin{aligned}(z, w) \in D &\Leftrightarrow \operatorname{Im} z - H(w, w) \in C \Leftrightarrow \operatorname{Im} z - H\left(e^{i\theta} w, e^{i\theta} w\right) \in C \\ &\Leftrightarrow f_\theta(z, w) \in D\end{aligned}$$

Consequently, $f_\theta \in \operatorname{Aut} D$. Furthermore, if $z \in \mathbb{C}^n$ is arbitrary with $\operatorname{Im} z \in C$, it follows that $p := (z, 0) \in D$ and $f_\theta(p) = p$. Thus $f_\theta \in \operatorname{Iso}_p D$.
Since $\{f_\theta : \theta \in \mathbb{R}\}$ is a one dimensional torus of mappings if $k > 0$, it follows that $|\operatorname{Iso} D| = \infty$ if $k > 0$.
Consider now the case $k = 0$ and $n > 1$. In this case D is the Siegel domain of the first kind $D = \{z \in \mathbb{C}^n : \operatorname{Im} z \in C\}$. Let p be an arbitrary point in C. It follows from proposition 1 that there is a linear mapping $id \neq A : \mathbb{R}^n \to \mathbb{R}^n$ with $A \in \operatorname{Iso}_p C$. Consider now the map A as a map from $\mathbb{C}^n$ to $\mathbb{C}^n$. Then

$$z = x + iy \in D \Leftrightarrow y \in C \Leftrightarrow Ay \in C \Leftrightarrow Ax + iAy = A(x + iy) \in D$$

Thus $A \in \operatorname{Aut} D$. Furthermore, $ip \in D$ and $A(ip) = ip$. Thus $id \neq A \in \operatorname{Iso}_{ip} D$.
The last remaining case is $k = 0$ and $n = 1$. But in this case D is just the upper half plane with $\operatorname{Iso} D \cong S^1$.
It follows that $|\operatorname{Iso} D| > 1$ in all cases, and the theorem is proved. □

The obvious (open) question that arises is the following:

Question 2. Does there exist a bounded homogeneous domain D in $\mathbb{C}^n$ with $|\operatorname{Iso} D| = 2$?

Note that there is a trivial example of a homogeneous *unbounded* domain with this property: Let $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Then $Iso_1\, \mathbb{C}^* = \{id, z \mapsto 1/z\}$, and $\{z \mapsto az : a \in \mathbb{C}^*\}$ acts transitively on $\mathbb{C}^*$. A possible example of a bounded homogeneous domain D with this property is necessarily more complicated; its dimension has to be at least 5 for the following reason: D would have to be biholomorphically equivalent to a Siegel domain of the first kind, since it has been shown in the proof of theorem 1 that $|\,\mathrm{Iso}\, D| = \infty$ otherwise. But homogeneous Siegel domains of the first kind are biholomorphically equivalent to bounded symmetric domains in all dimensions $n \leq 4$, and the isotropy groups of bounded symmetric domains are always infinite (see [Hel78] for a complete description of automorphism and isotropy groups of bounded symmetric domains).

It remains to find an example of a bounded homogeneous domain D such that a subgroup of the full automorphism group acts transitively and freely on D. This is not a difficult task; we will give only the most basic example here:

Example. There is a group of automorphisms of the polydisc $P_n(0; 1)$ that acts transitively and freely on $P_n(0; 1)$.

The easiest way to see this is to use the realization of $P_n(0; 1)$ as the unbounded Siegel domain of the first kind $H^n := H \times \cdots \times H$, where H is the upper half plane in $\mathbb{C}$. The set

$$\{z = (z_1, \ldots, z_n) \mapsto (a_1 z_1 + b_1, \ldots, a_n z_n + b_n) : a_j > 0,\ b_j \in \mathbb{R}\ \forall j\}$$

is obviously a group of automorphisms of H^n that acts transitively and freely on H^n.

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