Conjugate free convection over a vertical slender hollow cylinder embedded in a porous medium

I. Pop, T.-Y. Na

Abstract A numerical study of the steady conjugate free convection over a vertical slender, hollow circular cylinder with the inner surface at a constant temperature and embedded in a porous medium is reported. The governing boundary layer equations for the fluid-saturated porous medium over the cylinder along with the one-dimensional heat conduction equation for the cylinder are cast into dimensionless form, by using a non-similarity transformation. The resulting non-similarity equations with their corresponding boundary conditions are solved by using the Keller box method. Emphasis is placed on the effects caused by the wall conduction parameter, p, and calculations have covered a wide range of this parameter. Heat transfer results including the temperature profiles, the interface temperature profiles and the local Nusselt number are presented.

List of symbols

F reduced stream function for $p = 0$ (non-conjugate problem)
f reduced stream function for $p \neq 0$
g acceleration due to gravity
h_x local heat transfer coefficient
k thermal conductivity
k_m effective thermal conductivity of the saturated porous medium
K permeability of saturated porous medium
L length of the cylinder
Nu_x local Nusselt number
p conjugate conduction parameter defined in Eq. (18)
$\dot{q}$ heat transfer
r radial coordinate
r_i, r_o inner and outer radii of the hollow cylinder
Ra modified Rayleigh number
Ra_x modified local Rayleigh number
T temperature
T_b temperature at the inner surface of the hollow cylinder
T_s temperature of the hollow cylinder

ΔT reference temperature
u_c reference velocity
v_r velocity component in r-direction
v_x velocity component in x-direction
x coordinate along the axis
y independent variable defined as $y = r - r_o$

Greek symbols

α effective thermal diffusivity of the saturated porous medium
β thermal expansion coefficient
ξ transformed variable defined in Eq. (23)
η non-similarity variable
θ dimensionless temperature
ν kinematic viscosity
ψ stream function

Subscripts

w wall condition
s solid
∞ ambient condition

Superscripts

-    dimensionless variables
*   transformed variables
′   differentiation with respect to η

Ioan Pop
Faculty of Mathematics
University of Cluj, R-3400 Cluj, CP 253, Romania

Tsung-Yen Na (✉)
Dept. of Mechanical Engineering
University of Michigan-Dearborn
Dearborn, MI, 48128, USA

1
Introduction

The heat transfer mode in porous media has great practical importance in geophysics and energy related engineering problems. These include the utilization of geothermal energy, the control of pollutant spread in ground-water, the design of nuclear reactors, compact heat exchangers, solar power collectors, high performance insulation for buildings, food processing, casting and welding of a manufacturing process, etc. A detailed review of the subject of convective flow in porous media, including an exhaustive list of references, was recently performed by Nield and Bejan[1], and Ingham and Pop [2].

In many practical problems the information on the interfacial temperature is essential because the heat transfer characteristics are mainly determined by the temperature differences between the bulk flow and the interface. These are usually referred to as conjugate heat transfer problems, and they have many practical applications, particularly those related to energy conservation in buildings. Conju-

gate heat transfer from vertical or horizontal surfaces as well as horizontal cylinders embedded in porous media have recently been extensively investigated [3–11], and a detailed literature survey was most recently provided by Kimura et al. [12]. However, the problem of conjugate heat transfer over a vertical cylinder in a porous medium has not yet received any attention. Therefore, the present study proposes a mathematical model to investigate the conjugate problem of free convection over the outside surface of a vertical slender, hollow circular cylinder which is embedded in a porous medium. The temperature of the inner surface of the cylinder is kept at a constant value T_b and the temperature of the outer surface is determined by the conjugate solution of the energy equation of the solid and the boundary layer equations of the fluid-saturated porous medium. The approximation of the one-dimensional conduction term (transversal) in the energy equation of the cylinder is taken into account because the aspect ratio r_o/L (outer radius/length) is sufficiently small in this study. Due to close coupling for this problem, the boundary layer equations outside the cylinder and the one-dimensional heat conduction equation for the hollow cylinder must simultaneously be solved. These equations being non-similar are solved by a very efficient finite-difference method known as Keller-box scheme [13]. This solution methodology has been well established in a recent paper by Na [14] for the corresponding problem of a viscous (non-porous) fluid. However, it is worth mentioning that neither the surface temperature nor the heat flux through the wall of the hollow cylinder is known a priori. Thus, an iterative procedure would have to be applied. The effects of the conduction parameter, p, on the heat transfer characteristics are discussed.

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Basic equations
The geometry considered is a vertical slender, hollow circular cylinder with inner and outer radii r_i and r_o,
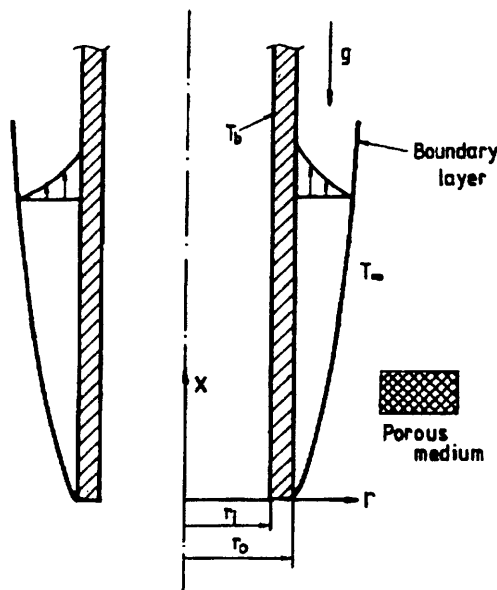

Fig. 1. Physical model and coordinate system

respectively, which embedded in a fluid-saturated porous medium as shown in Fig. 1. The inner surface of the cylinder is held at a constant temperature T_b, while the temperature of the ambient fluid is T_∞, where $T_b > T_\infty$. The porous material is isotropic and homogeneous, and the fluid is incompressible. Under these assumptions, the governing boundary layer equations for the conjugate free convection over a vertical slender, hollow circular cylinder embedded in a porous medium based on Darcy–Boussinesq approximation are:

for the fluid-porous medium

$$\frac{\partial(rv_x)}{\partial x} + \frac{\partial(rv_r)}{\partial r} = 0 \tag{1}$$

$$v_x = \frac{g\beta K}{v}(T - T_\infty) \tag{2}$$

$$v_x \frac{\partial T}{\partial x} + v_r \frac{\partial T}{\partial r} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{3}$$

for the solid cylinder

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_s}{\partial r}\right) + \frac{\partial^2 T_s}{\partial x^2} = 0 \tag{4}$$

If Eq. (4) is normalized by introducing the dimensionless variables

$$\bar{r} = \frac{r}{r_o}, \quad \bar{x} = \frac{x}{L}, \quad \theta_s = \frac{(T_s - T_\infty)}{\Delta T} \tag{5}$$

Where $\Delta T = T_b - T_\infty$, we then get

$$\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(\bar{r}\frac{\partial\theta_s}{\partial \bar{r}}\right) + \left(\frac{r_o}{L}\right)^2\frac{\partial^2\theta_s}{\partial\bar{x}^2} = 0 \tag{6}$$

Assuming that $r_o/L \ll 1$, then the axial conduction temperature, i.e., $\partial^2 T_s/\partial x^2$, can be neglected, so that Eq. (6) becomes

$$\frac{\partial}{\partial r}\left(r\frac{\partial\theta_s}{\partial r}\right) = 0 \tag{7}$$

subject to

$$r = r_i : T_s = T_b; \quad r = r_o : T_s = T(x, r_o) \tag{8}$$

The solution of Eq. (7) along with (8) is

$$T_s = T_b + (T(x, r_o) - T_b)\frac{\ln(r/r_i)}{\ln(r_o/r_i)} \tag{9}$$

On the other hand, we have that at the surface of the cylinder the heat flux from the solid and fluid-porous medium interface are equal, i.e.,

$$r = r_o : -k_s\frac{\partial T_s(r_o)}{\partial r} = -k_m\frac{\partial T(x, r_o)}{\partial r} \tag{10}$$

which, on using (9), becomes

$$y = 0 : -k_s\frac{T(x, r_o) - T_b}{r_o \ln(r_o/r_i)} = -k_m\frac{T(x, r_o)}{\partial y} \tag{11}$$

where y is defined as $y = r - r_o$. The rest of the boundary conditions for Eqs. (1) to (3) are

$y = 0 : v_r(x, 0) = 0;$

$y \to \infty : v_x(x, \infty) = 0, \ T(x, \infty) = T_\infty$ (12)

Further, we introduce the following dimensionless variables

$$\bar{x} = \frac{x}{r_o}, \ \bar{y} = \frac{y}{r_o}\sqrt{\text{Ra}}, \ \bar{v}_x = \frac{v_x}{u_c}, \ \bar{v}_r = \frac{v_r}{u_c}\sqrt{\text{Ra}}$$

$$\theta = \frac{T - T_\infty}{\Delta T}, \ u_c = \frac{gK\beta\Delta T}{v}$$ (13)

where $\text{Ra} = u_c r_o / \alpha$ is the modified Rayleigh number for a porous medium. Equations (1) to (3) become

$$\frac{\partial}{\partial \bar{x}}(\bar{r}\bar{v}_x) + \frac{\partial}{\partial \bar{y}}(\bar{r}\bar{v}_r) = 0$$ (14)

$$\bar{v}_x = \theta$$ (15)

$$\bar{v}_x \frac{\partial \theta}{\partial \bar{x}} + \bar{v}_r \frac{\partial \theta}{\partial \bar{y}} = \frac{1}{\bar{r}}\frac{\partial}{\partial \bar{y}}\left(\bar{r}\frac{\partial \theta}{\partial \bar{y}}\right)$$ (16)

subject to the boundary conditions

$$\bar{v}_r(\bar{x}, 0) = 0, \ \theta(\bar{x}, 0) - 1 = p\frac{\partial \theta(\bar{x}, 0)}{\partial \bar{y}};$$

$$\bar{v}_x(\bar{x}, \infty) = 0, \ \theta(\bar{x}, \infty) = 0$$ (17)

where p is the conjugate conduction parameter given by

$$p = \frac{k_m}{k_s}\text{Ra}^{1/2}\ln\frac{r_o}{r_i}$$ (18)

Let us next introduce the Mangler transformation

$$dx^* = d\bar{x}, \ dy^* = \bar{r}d\bar{y}$$ (19)

and the stream function, $\bar{\psi}$, defined by

$$\bar{r}\bar{v}_x = \frac{\partial \bar{\psi}}{\partial \bar{y}}, \ \bar{r}\bar{v}_r = -\frac{\partial \bar{\psi}}{\partial \bar{x}}$$ (20)

Equations (15) and (16), with $\theta = \partial \bar{\psi}/\partial y^*$, now become

$$\frac{\partial \bar{\psi}}{\partial y^*}\frac{\partial^2 \bar{\psi}}{\partial x^*\partial y^*} - \frac{\partial \bar{\psi}}{\partial x^*}\frac{\partial^2 \bar{\psi}}{\partial y^{*2}} = \frac{\partial}{\partial y^*}\left\{\left(1 + \frac{2y^*}{\sqrt{\text{Ra}}}\right)\frac{\partial^2 \bar{\psi}}{\partial y^{*2}}\right\}$$

(21)

subject to the boundary conditions

$$\bar{\psi}(x^*, 0) = 0, \ \frac{\partial \bar{\psi}(x^*, 0)}{\partial y^*} - 1 = p\frac{\partial^2 \bar{\psi}(x^*, 0)}{\partial y^{*2}},$$

$$\frac{\partial \bar{\psi}(x^*, \infty)}{\partial y^*} = 0$$ (22)

To get Eq. (21) amenable for numerical integration, we introduce the following transformation

$$f(\xi, \eta) = \frac{\bar{\psi}}{x^{*2/3}}, \ \eta = \frac{y^*}{x^{*1/3}}, \ \xi = \frac{2x^{*1/3}}{\sqrt{\text{Ra}}}$$ (23)

We then obtain

$$\{(1 + \xi\eta)f''\}' + \frac{2}{3}ff'' - \frac{1}{3}f'^2 = \frac{1}{3}\xi\left(f'\frac{\partial f'}{\partial \xi} - f''\frac{\partial f}{\partial \xi}\right)$$

(24)

subject to the boundary conditions

$$f(\xi, 0) = 0, \ \frac{\sqrt{\text{Ra}}}{2}\xi f'(\xi, 0) - 1 = pf''(\xi, 0);$$

$$f'(\xi, \infty) = 0$$ (25)

where primes denote differentiation with respect to η.

The dimension less temperature on the surface of the cylinder can be written as

$$\theta(\bar{x}, 0) = \frac{\sqrt{\text{Ra}}}{2}\xi f'(\xi, 0)$$ (26)

where

$$\bar{x} = \frac{x}{r_o} = \left\{\frac{\xi\sqrt{\text{Ra}}}{2}\right\}^3$$ (27)

Let us consider the effect of conduction on the rate of heat transfer. First, the local rate of heat transfer is given by

$$\dot{q}_w(x) = -k_m\frac{\partial T(x, 0)}{\partial y} = k_m\frac{\Delta T}{r_o}\text{Ra}^{1/2}[-f''(\xi, 0)]$$ (28)

On the other hand, the overall local heat transfer coefficient, h_x, is given by

$$h_x = \frac{\dot{q}_w(x)}{\Delta T}$$ (29)

and the local Nusselt number at the outer surface of the hollow cylinder is defined as

$$\text{Nu}_x = \frac{h_x x}{k_m}$$ (30)

Using (28), (29) and (30), we get

$$\frac{\text{Nu}_x}{\text{Ra}_x^{1/2}} = \begin{cases} -F''(\xi, 0) & \text{for } p = 0 \\ \bar{x}^{1/2}[-f''(\xi, 0)] & \text{for } p \neq 0 \end{cases}$$ (31)

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Results and discussion

We first notice from this study that for a hollow cylinder, where the temperature on the inner surface ($r = r_i$) is maintained at a constant value of T_b, the temperature on the outer surface of the cylinder ($r = r_o$) given by Eq. (26) depends on the coupled solution of the conduction across the cylinder and the natural convection of the fluid-saturated porous medium over the cylinder. Intuitively, if the heat conductivity of the hollow cylinder is very large, its temperature can be expected uniform at T_b and therefore a solution of the natural convection based on a constant wall temperature will be acceptable. An inspection of the definition of the conduction parameter p reveals that the same conclusion is true if the effective thermal conductivity of the saturated porous medium is very small and/or if the thickness of the hollow cylinder is very thin.

Equation (24) subject to the boundary conditions (25) has been solved numerically using the Keller box method as described by Na [14]. Solutions are generated for a range of values of the axial non-dimensional coordinate $\bar{x}$ and three values of the conjugate convection parameter

$p = 0.0$ (non-conjugate problem), 1.0 and 2.0. It should be noted that the solution of the basic case $p = 0.0$, where the cylinder's wall temperature is assumed to be uniform at T_b, corresponds to the problem of free convection from an isothermal vertical cylinder embedded in a porous medium first considered by Minkowycz and Cheng [15].

Fig. 2–4 show the non-dimensional temperature profiles within the boundary layer plotted against η for some values of $\bar{x}$ when $p = 0.0, 1.0$ and 2.0. These figures show that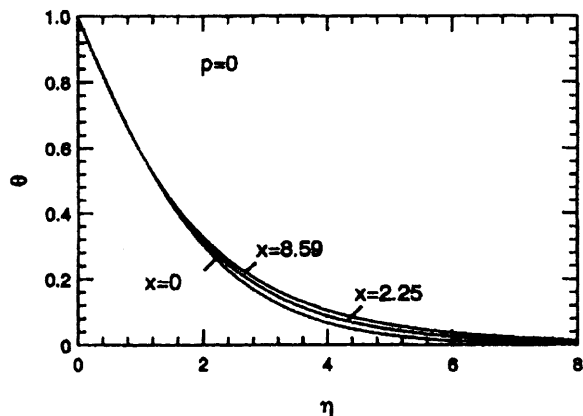 the temperature profiles increase with the increase of the axial distance x. This fact is in agreement with the results found by Pop and Merkin [10] for the corresponding problem of a vertical flat plate embedded in a porous medium.

The variation of the non-dimensional temperature at the outer surface of the cylinder $\theta(\bar{x}, 0)$, plotted against $\bar{x}$, is depicted in Fig. 5 for $p = 0.0$ (basic case), 1.0 and 2.0, where the basic case is represented by a straight line as resulted from Eq. (17). values of $\theta(\bar{x}, 0)$ are also given in Table 1 for future reference. It is seen from this figure and Table 1 that $\theta(\bar{x}, 0)$ decreases as p increases. For $p \neq 0$ this temperature is lowest at the leading edge and increases

Fig. 2. Temperature profiles for $p = 0.0$ (non-conjugate problem)

Table 1. Representative solutions for $\theta(\bar{x}, 0)$

$\bar{x}$	$p = 0.0$	$p = 1.0$	$p = 2.0$
0.0000	1.0000	0.0000	0.0000
0.0039	1.0000	0.1829	0.1199
0.0366	1.0000	0.3396	0.2326
0.1434	1.0000	0.4712	0.3368
0.3951	1.0000	0.5788	0.4314
0.8995	1.0000	0.6654	0.5160
1.8156	1.0000	0.7336	0.5904
3.3754	1.0000	0.7871	0.6546
5.9119	1.0000	0.8285	0.7094
9.8984	1.0000	0.8608	0.7554
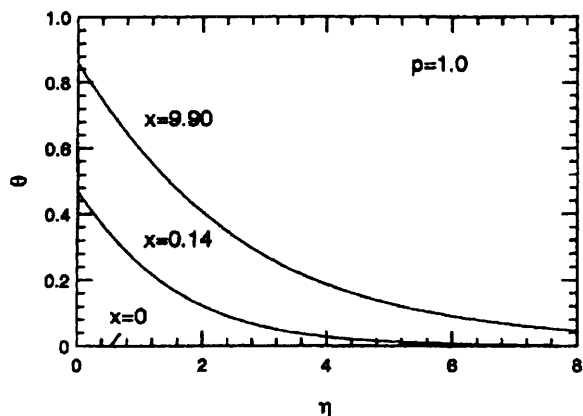
Fig. 3. Temperature profiles for $p = 1.0$
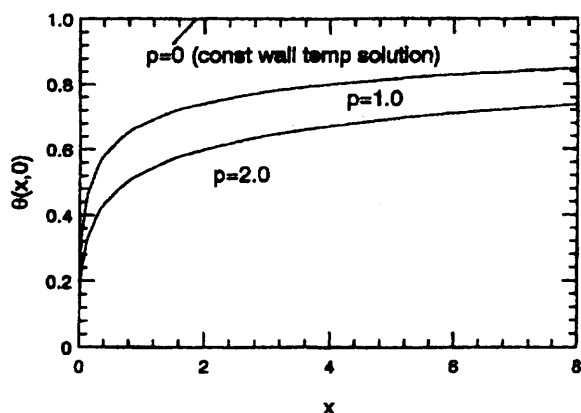
Fig. 5. Variation of $\theta(\bar{x}, 0)$ with p

Fig. 4. Temperature profiles for $p = 2.0$

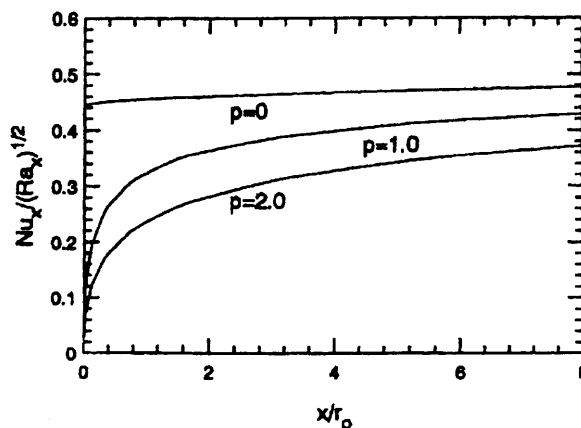**Fig. 6.** Variation of $\mathrm{Nu}_x / \mathrm{Ra}_x^{1/2}$ with p

with $\bar{x}$ until it approaches the constant value of the basic case at very large $\bar{x}$. Finally, we show in Fig. 6 the effect of the parameter p on the local Nusselt number as given by Eq. (31). Again, the basic solution ($p = 0.0$) is included into this figure. It is worth mentioning that for $p = 0.0$ and $\bar{x} = 0$, we found $-f''(0, 0) = .44375$, while Minkowycz and Cheng [15] reported the value of 0.4440, which shows that the present result is in excellent agreement with that from [15]. Further, it is to be noticed from Fig. 6 that the heat flux rate at the outer surface of the cylinder increases to the basic case ($p = 0.0$). This conclusion is very important for practical applications.

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