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Department of Mathematics

Technical Report No. 2

STABLE MATCHING OF DIFFERENCE SCHEMES

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ORA Project 027750

under contract with:

OFFICE OF NAVAL RESEARCH
DEPARTMENT OF THE NAVY
CONTRACT NO. N00014-67-A-0181-0023
WASHINGTON, D.C.

administered through:

OFFICE OF RESEARCH ADMINISTRATION   ANN ARBOR

February 1972

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STABLE MATCHING OF DIFFERENCE SCHEMES*

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ABSTRACT

Approximations that result from the natural matching of two stable dissipative difference schemes across a coordinate line are shown to be stable. The basic idea is to reformulate the matching scheme consistent to an equivalent initial boundary value problem and verify the algebraic conditions for stability of such systems. An interesting comparison to the above result is the case of redefinition of a scheme at a single point. In particular, we show that some unstable perturbations do not upset the stability of the Lax-Wendroff scheme.

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* This research was supported by O. N. R. Contract No. NR044-377.

+ This research was performed while the author was a Visiting Lecturer at Tel-Aviv University, Tel-Aviv, Israel.

INTRODUCTION

A local regridding procedure for finite difference approximations, referred to as a mesh refinement scheme, was analyzed by the author in [1], [2] for hyperbolic partial differential equations. For parabolic equations the mesh refinement technique has been analyzed by Osher [10], Varah [11] and Ciment and Sweet [3]. The basic idea in the mesh refinement technique is to employ one underlying finite difference scheme consistent to a given partial differential equation on what are essentially two different grid patterns. In the above works it has been shown how to relate a given difference scheme across neighboring uneven mesh points in a stable and computationally efficient manner.

The mesh refinement technique was proposed to help in solving problems where the solution has large gradients, discontinuities, or generally what we call frontal effects in a localized region. In general, use of a mesh refinement scheme for problems with moving frontal effects will necessitate the following additional work:

1) Prediction of direction of frontal motion.
2) Interpolation and storage (regridding) of the pertinent variables and data in advance of the frontal effect.
3) Special treatment for solving the resulting algebraic equations when implicit schemes are used (as in the parabolic case).

Inspite of the above extra work, in cases where the frontal effect itself causes changes in the grid domain, the mesh refinement scheme can be very effective [4] .

Another approach for treating problems with frontal effects is what we call the matching scheme technique. Here the strategy is to leave the grid uniform, wherever at all possible, but to locally change the underlying finite difference shceme to suit the local phenomenon. The matching technique, in the case of moving frontal effects, would also need feature (1) above, but would not suffer from the other two drawbacks. Thus, it is to be expected that when applicable, the matching scheme technique should be computationally more efficient than the mesh refinement technique for moving frontal effects. In the future we hope to report on numerical experiments comparing both these techniques.

This paper considers the L_2 stability of an approximation to a pure initial value problem which results from the natural matching of two difference schemes across a coordinate line. As in our treatment of mesh refinements [1], [2], stability is analyzed by reformulating the matching scheme as a difference approximation to an initial boundary value problem for a system of partial differential equations. General sufficient conditions for the stability of such hyperbolic systems have been given by Kreiss [6] , [7] , and Osher

[9]. Using analogous results for parabolic systems obtained by Varah [11], we verify that a large class of matched parabolic schemes are L_2 stable.

As an interesting comparison to the above result, we examine the case of redefinition of a difference scheme at a single point. In particular we show that even some unstable perturbations do not upset the stability of the Lax-Wendroff scheme. It is obvious from our presentation that our approach can be used to study the stability of a given dissipative difference scheme with a local perturbation or a discontinuity.

I. Formulation of Problem.

Consider the Cauchy problem for the following hyperbolic equation

$$(1.1) \qquad u_t = A u_x + B u_y + C u_z \; ; \; u(x, y, z, 0) = \varphi(x, y, z)$$

in the region $-\infty < x, y, z < \infty$, $t \geq 0$. For simplicity, we take u and φ scalar functions and A constant and $B = C \equiv 0$. Observe that following Kreiss [6] and the author [1], our results will hold for A,B,C Lipschitz continuous diagonal matrices if the difference schemes are also diagonal.

Consider a set of uniform mesh points (x_j, t_n) where

$$x_j = j \Delta x \quad ; \quad j = 0, \pm 1 \pm 2 \cdots \; ; \; t_n = n \Delta t \; ; \quad n = 0, 1, 2 \cdots$$

On this grid represent two different difference approximations consistent to (1.1), (with $B = C \equiv 0$) as

$$V_j^{n+1} = Q_1 V_j^n \equiv \sum_{\alpha=-r_1}^{p_1} C_\alpha^{(1)} V_{j+\alpha}^n \tag{1.2a}$$

$$U_j^{n+1} = Q_2 U_j^n \equiv \sum_{\alpha=-r_2}^{p_2} C_\alpha^{(2)} U_{j+\alpha}^n \tag{1.2b}$$

where $C_\alpha^{(i)} = C_\alpha^{(i)}(b)$ with $b = A \frac{\Delta t}{\Delta x}$, and where r_i and p_i are positive integers.

By the <u>matching of Q_1 and Q_2</u>, we mean the difference approximation Q defined by

$$Z_j^{n+1} = Q Z_j^n \equiv \begin{cases} Q_1 Z_j^n & j = 1, 2, 3, \cdots \\ Q_2 Z_j^n & j = 0, -1, 2, \cdots \end{cases}$$

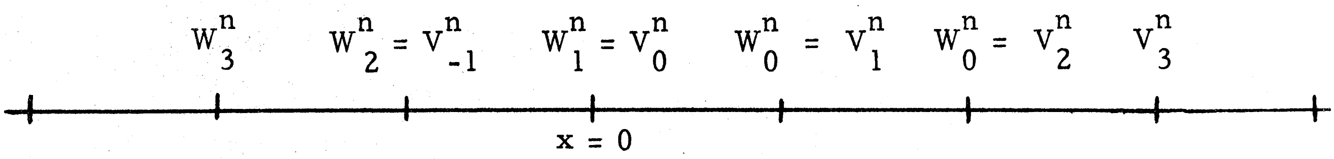
Our main result is that Q is L_2 stable if both Q_1 and Q_2 are stable and dissipative. For simplicity of notation we have considered explicit schemes, but it will be obvious from our proof that the same result holds for analogous implicit schemes.

We treat this analytically by using an equivalent formulation as an initial boundary value problem. It is convenient to obtain this by reflecting and shifting the difference functions used on the left hand side. Defining $W_j^n \equiv U_{-j+1}^n$ gives

$$W_j^{n+1} = Q_2' W_j^n \equiv \sum_{\alpha=-p_2}^{r_2} C_\alpha^{(2)}(-b)\, W_{j+\alpha}^n$$

as an approximation to $u(-(j-1)\Delta x,\ (n+1)\Delta t)$. Since V_j^n denotes an approximation to $u(j\Delta x,\ n\Delta t)$, near the interface the following overlapping identification of grid points is implied (see diagram)

$$\begin{cases} W_{-j+1}^n = V_j^n & j = 1, 2, \cdots p_2 \\ V_{-j+1}^n = W_j^n & j = 1, 2, \cdots r_1 \end{cases}$$



(Case $p_2 = 2$, $r_1 = 1$)

We complete the reformulation of this matching scheme as an equivalent system of difference equations for an initial boundary value problem by introducing the matrix notation

$$Z_j^n = \begin{bmatrix} V_j^n \\ \\ W_j^n \end{bmatrix} \qquad \begin{matrix} j = 0, \pm 1, \pm 2, \cdots \quad . \\ \\ n = 0, 1, 2, \cdots \quad . \end{matrix}$$

Then the precise expression of the matching scheme Q is

(1.3a) $$Z_j^{n+1} = \begin{bmatrix} Q_1 & 0 \\ \\ 0 & Q_2' \end{bmatrix} Z_j^n \; ; \qquad \begin{matrix} j = 1, 2, \cdots \\ \\ n = 0, 1, 2 \cdots \end{matrix}$$

with boundary conditions

(1.3b) $$Z_{-j}^n = \begin{bmatrix} 0 & 1 \\ \\ 1 & 0 \end{bmatrix} Z_{j+1}^n \; ; \quad j = 0, 1, \cdots, r-1$$

where $r \equiv \max [r_1, p_2]$. If $p_2 \neq r_1$ these boundary conditions identify some additional overlapping points, but this does not alter the algorithm. Now the matching scheme expressed in (1.3a, b) appears as an approximation to the system

(1.4) $$Z_t \equiv \begin{pmatrix} U \\ \\ W \end{pmatrix}_t = \begin{bmatrix} A & 0 \\ \\ 0 & -A \end{bmatrix} Z_x$$

with compatibility condition $U(0, t) = W(0, t)$.

Sufficient conditions for the L_2 stability of dissipative difference approximations to initial boundary value problems of the above type have been given by Kreiss [6], [7] and Osher [9]. Accordingly, because of the diagonal structure of our systems, to prove stability it suffices to show that there are no eigensolutions of (1.3a, b) of the form

(1.5) $$Z_j^n = \lambda^n g_j \ ; \quad \sum_{j=1}^{\infty} |g_j|^2 < \infty$$

for $\lambda \neq 1$, $|\lambda| \geq 1$, and furthermore, that $\lambda = 1$ is not a generalized (approximate) eigenvalue in the sense of Kreiss [6].

In the following it is assumed that the reader is familiar with the paper of Kreiss [6] or Corollary 1 of [2]. Briefly stated for our case, these say that the eigensolutions of interest are those where g_j are formed using the characteristic roots of (1.6) having absolute value less than one. Furthermore, as $\lambda \rightarrow 1$, those characteristic roots in the component of Z_j^n corresponding to the positive diagonal element of the leading matrix in (1.4) must have a limiting absolute value strictly less than one.

<u>Theorem.</u> Let Q_1 and Q_2 defined by (1.2) each be a stable and consistent dissipative approximation to (1, 1). Then the matching of Q_1 and Q_2 as defined in (1.3a, b) is a stable approximation.

Proof. To find all eigensolutions of the form (1.5) it is sufficient to find all characteristics roots $\tau_i(\lambda)$, $\mu_i(\lambda)$ such that

$$\lambda = \sum_{\alpha=-r_1}^{p_1} C_\alpha^{(1)}(b)\ \tau_i^\alpha \tag{1.6a}$$

$$\lambda = \sum_{\alpha=-p_2}^{r_2} C_\alpha^{(2)}(-b)\ \mu_i^\alpha \tag{1.6b}$$

with $|\tau_i(\lambda)| \leq 1$ and $|\mu_i(\lambda)| \leq 1$. For $\lambda \neq 1$, $|\lambda| \geq 1$, dissipativity implies the separation of the roots property [1], [6], that is, counting multiplicities there are only r_1 linearly independent characteristic roots τ_i (p_2 roots μ_i) such that $|\tau_i| < 1$ ($|\mu_i| < 1$). Furthermore, <u>strict inequality remains for either all τ_i or all μ_i roots as $\lambda \to 1$.</u>

All admissible eigensolutions can be expressed in the form

$$Z_j^n = \lambda^n \begin{bmatrix} \sum_{i=1}^{d_1} f_i(j)\ \tau_i^j \\ \sum_{i=1}^{d_2} h_i(j)\ \mu_i^j \end{bmatrix} ; \quad |\lambda| \geq 1, \quad \lambda \neq 1. \tag{1.7}$$

Here each distinct τ_i solves (1.6a) with multiplicity m_i and $f_i(j)$ is a polynomial of degree $m_i - 1$, and each distinct μ_i solves (1.6b) with multiplicity $\ell_i - 1$, and $h_i(j)$ is a polynomial of degree $\ell_i - 1$, and

$$\sum_{i=1}^{d_1} m_i = r_1 \; ; \quad \sum_{i=1}^{d_2} \ell_i = p_2 \; .$$

Substitute Z_j^n into the boundary conditions and find

$$\left.\begin{array}{ll} (1.8a) & \sum_{i=1}^{d_1} f_i(-j)\,\tau_i^{-j} + \sum_{i=1}^{d_2} h_i(j+1)\,\mu_i^{j+1} = 0 \\ & \\ (1.8b) & \sum_{i=1}^{d_1} f_i(j+1)\,\tau_i^{j+1} + \sum_{i=1}^{d_2} h_i(-j)\,\mu_i^{-j} = 0 \end{array}\right\} \quad j = 0, 1, \cdots, r-1 \, .$$

Now order the equations of group (a) from $r-1$ to 0 and those of group (b) from 0 to r to obtain

$$\sum_{i=1}^{d_1} f_i(j)\,\tau_i^{j} + \sum_{i=1}^{d_2} h_i(-j+1)\,[\mu_i^{-1}]^{j-1} = 0$$

vanishing at the $2r$ consecutive integers $j = -r+1, \cdots, -1, 0, 1, \cdots, r$. Despite the shift of indices, the set of difference functions

$$\left\{ f_i(j)\,\tau_i^{j} \right\}_{i=1}^{d_1} \cup \left\{ h_i(-j+1)\,[\mu_1^{-1}]^{j-1} \right\}_{i=1}^{d_2}$$

form a fundamental set for the linear homogeneous difference equation whose characteristic polynomial is

$$p(x) = \prod_{i=1}^{d_1} (x - \tau_i)^{m_i} \prod_{i=1}^{d_2} (x - \mu_i^{-1})^{\ell_i}$$

Observe that $p(x)$ is of degree $r_1 + p_2$ and the linear combinations (1.8) vanish at $2r$ consecutive points. Since $r_1 + p_2 \leq 2r$, for a non-trivial solution this can only occur (See Theorem 5.3 [5]), if there are repeated roots. Now the $\{\tau_i\}_{i=1}^{d_1}$ and $\{\mu_i^{-1}\}_{i=1}^{d_2}$ are separately sets of distinct roots by definition. Thus it must be for some i and m that $\tau_i = \mu_m^{-1}$. This in turn implies that $1 = \tau_i \mu_m$ which implies that $|\tau_i| = |\mu_m| = 1$. This establishes stability, since as remarked above all the roots of one group must be strictly less than one in absolute value even as $\lambda \to 1$.

We note that the same matching of two difference schemes with different mesh spacings is easily seen to be stable for the case $r_1 = p_2 = 1$ when $\Delta x_1/\Delta x_2 = 1/N$, and N is a positive integer.

II. Parabolic Case.

Recent results obtained by Varah [11] allow a similar analysis for the analogous matching scheme for second-order parabolic equations in one space dimension. For the case of matching diagonal parabolic difference schemes, our above analysis remains valid and indicates that there are no eigenvalues λ greater than one in modulus. Varah's sufficient condition can be expressed in the form that the determinant of essentially system (1.8) vanish of the order $\geq O(\sqrt{\lambda - 1})$ as $\lambda \to 1$. We found this hard to verify in general. However, if $p_1 = r_1 = p_2 = r_2 = 2$, after some simple operations the determinant in question is of Vandermonde form

$$\det SY = \det\{ (\tau_1^j, \mu_1^{-j+4}, \tau_2^j, \mu_2^{-j+4})_{j=0,1,2,3} \}$$

$$\doteq (\tau_1\mu_1 - 1)(\tau_1 - \tau_2)(\tau_1\mu_2 - 1)(1 - \tau_2\mu_1)(\mu_2 - \mu_1)(\tau_2\mu_2 - 1).$$

Varah shows that the separation of the roots for <u>parabolic schemes</u> is such that as $\lambda \to 1$

say, $\tau_1 = 1 + O(\sqrt{\lambda - 1})$ then $|\tau_2| \leq \rho < 1$

say, $\mu_1 = 1 + O(\sqrt{\lambda - 1})$ then $|\mu_2| \leq \rho < 1$.

Only the first term in the expression for the determinant can vanish while all others are bounded away from zero. The first term gives the desired estimate. Thus, most of the well known schemes can be matched in a stable manner.

III. Local Redefinition of a Difference Scheme

Consider the one dimensional case (1.1) with the Lax-Wendroff scheme [8] used at all points except $x = 0$. Then W_j^n denotes an approximation to $u(j\Delta x, n\Delta t)$ and is given by

$$W_j^{n+1} = Q(b)\, W_j^n \equiv \left(\frac{b^2 - b}{2}\right) W_{j-1}^n + (1 - b^2)\, W_j^n + \left(\frac{b^2 + b}{2}\right) W_{j+1}^n$$

for $j = \pm 1, \pm 2, \cdots$; $n = 0, 1, 2 \cdots$. Here $b = A\frac{\Delta t}{\Delta x}$ must be restricted by $|b| \leq 1$ for stability of the pure initial value problem. We consider a simple case of locally redefining our scheme by some 3-point approximation which is also consistent to (1.1) denoted by

$$W_0^{n+1} = \alpha_{-1} W_{-1}^n + \alpha_0 W_0^n + \alpha_1 W_1^n , \quad n = 0, 1, 2, \cdots \tag{3.1}$$

Introducing a reflection of our problem, we express this locally perturbed scheme as the system of difference equations

$$Z_j^{n+1} = \begin{bmatrix} Q(b) & 0 \\ 0 & Q(-b) \end{bmatrix} Z_j^n \qquad \begin{matrix} j = 1, 2, \cdots \\ n = 0, 1, 2, \cdots \end{matrix} \tag{3.2a}$$

with boundary conditions

(3.2b) $$Z_0^{n+1} = \sum_{i=-1}^{1} \alpha_i Z_i^n$$

where by consistency $\alpha_{-1} = \dfrac{(1 - b - \alpha_0)}{2}$ and $\alpha_1 = \dfrac{(1 + b - \alpha_0)}{2}$.

Here $$Z_j^n = \begin{bmatrix} W_j^n \\ \\ V_j^n \end{bmatrix} \sim Z(x, t) = \begin{bmatrix} u(x, t) \\ \\ u(-x, t) \end{bmatrix} ; \qquad \begin{matrix} j\Delta x = x \\ \\ n\Delta t = t . \end{matrix}$$

Now one needs to consider all eigensolutions of the form (1.5) for $\lambda \neq 1$, $|\lambda| \geq 1$. The separation of the roots property allows us to express these in the form

(3.3) $$Z_j^n = \lambda^n \begin{bmatrix} \beta_1 \tau_1^j \\ \\ \beta_2 \tau_2^j \end{bmatrix} \qquad \begin{matrix} j = 1, 2, \cdots \\ \\ n = 0, 1, \cdots \end{matrix}$$

where $|\tau_1| < 1$, $|\tau_2| < 1$ are the characteristic roots of

(3.4) $$\lambda = Q\,[(-1)^{i+1} b]\ \tau_i^j \qquad i = 1, 2 .$$

Observing that τ_2 satisfies the reflected difference equation of τ_1, one concludes that τ_2 can be expressed as the reciprocal of the separated

τ_1 root, i.e. the inadmissible τ_1 root that is outside the unit circle. Thus, by product of the roots rule

$$\tau_1 \frac{1}{\tau_2} = \frac{b-1}{b+1} .$$

Substitution of (3.3) into (3.2a, b) using this last relationship yields

$$(3.5) \qquad \lambda = \alpha_0 \left[1 - \frac{b}{b-1} \tau_2\right] .$$

The table below lists the results of our stability analysis of the locally perturbed Lax-Wendroff scheme for several local redefinitions. The instabilities of (4) are apparent by the geometrical arguments of C.F.L. For case (1), (3.5) implies $\lambda = 0$. Cases (2) and (3) are more interesting since the perturbation schemes are separately unstable. However, for (2) we find that $|\lambda|^2 = (1 + b^2)^{-1/2} < 1$ and for (3) a short calculation using (3.5) and (3.4) gives $|\lambda|^2 = (1 - \frac{1}{2} b^2)^2$. Since $|b| < 1$ this guarantees stability. For the locally perturbed scheme, we have been unable to find any general result comparable to our above theorem on matching schemes.

	α_0	Perturbation Scheme (3.1)	Redefined Scheme (3.2a, b)
(1)	0	stable	stable
(2)	1	unstable unless $b = 0(\Delta x)$	stable
(3)	$1-b^2/2$	always unstable	stable
(4)	$1-b$	stable for $1 > b > 0$	stable
		unstable for $b < 0$	unstable

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Unclassified

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Department of Mathematics University of Michigan Ann Arbor, Michigan 48103	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

Stable Matching of Difference Schemes

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report, January 1972

5. AUTHOR(S) (First name, middle initial, last name)

Melvyn Ciment

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
January, 1972	18	11

8a. CONTRACT OR GRANT NO. N00014-67-A-0181-0023 b. PROJECT NO. NR 044-377 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) Technical Report No. 2 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Mathematics Program Office of Naval Research Arlington, Virginia 22217

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DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6801

Unclassified

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Hyperbolic differential equations Mesh refinement Finite Differences Lax-Wendroff scheme						