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Nonlinear Filtering with Fractional Brownian Motion

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Abstract. Our objective is to study a nonlinear filtering problem for the observation process perturbed by a Fractional Brownian Motion (FBM) with Hurst index $\frac{1}{2} < H < 1$. A reproducing kernel Hilbert space for the FBM is considered and a "fractional" Zakai equation for the unnormalized optimal filter is derived.

Key Words. Nonlinear filtering, Fractional Brownian motion, Reproducing kernel Hilbert space, Stochastic differential equations.

AMS Classification. 60H20, 60G15, 60G35.

1. Introduction

The goal of nonlinear filtering theory is to estimate a signal process $(X_t)(0 \leq t \leq T)$ observed in the presence of an additive noise. Consider a complete probability space $(\Omega, \mathcal{F}, P)$ and a family $(\mathcal{F}_t)_{t\geq 0}$ of right-continuous increasing P-complete sub-σ-fields of $\mathcal{F}$. Let $X = (X_t, t \in [0, T])$ be a measurable, $\mathcal{F}_t$-adapted stochastic process with values in a complete separable metric space S. The simplest model for the observation process (Y_t) is given by

$$Y_t = \int_0^t h(X_s)\, ds + B_t, \qquad 0 \leq t \leq T, \tag{1}$$

where (B_t) is a standard Brownian Motion (BM), and $h \in C(S)$ satisfies

$$\int_0^T h^2(X_s)(\omega)\, ds < \infty \qquad (P\text{-a.s.}). \tag{2}$$

The classical model (1)–(2) can be written in the following form:

$$Y_t(\omega) = F_t(X(\omega)) + B_t(\omega), \qquad t \in [0, T], \tag{3}$$

where $\{F_t(X(\omega)), t \in [0,T]\}$ is (P-a.s.) an element of the Reproducing Kernel Hilbert Space (RKHS) of the BM on $[0,T]$.

The objective of this paper is to investigate a nonlinear filtering problem in the case when an additive observation noise exhibits a certain long-range dependence structure. Namely, let $(B_t^H, t \in [0,T])$ be a Fractional Brownian Motion (FBM) with (fixed) Hurst index $\frac{1}{2} < H < 1$ and let $\mathcal{H}(B^H)$ stand for the RKHS of $(B_t^H, t \in [0,T])$. We study the following analogue of the model (3):

$$Y_t(\omega) = F_t(X(\omega)) + B_t^H(\omega), \qquad t \in [0,T], \tag{4}$$

where $\{F(X(\omega)\} \in \mathcal{H}(B^H)$ for almost all ω.

In Section 2 we present important properties of the RKHS $\mathcal{H}(B^H)$ and give a more explicit form to the observation model (4). Here we use the results on $\mathcal{H}(B^H)$ obtained by Barton and Poor [1]. In Section 3 we derive a Bayes' formula for the optimal filter for the observation model with FBM noise and state a corresponding "fractional" Zakai-type equation for the unnormalized conditional expectation.

Let us give a few comments on other types of filtering models considered in the literature. Interesting results on linear filtering with FBM were obtained by Kleptsyna et al. in [5]–[7] and Le Breton in [8]. As far as nonlinear theory is concerned, Coutin and Decreusefond in [2] considered a nonlinear filtering model where both the signal and the observation are solutions of a stochastic differential equation driven by a multidimensional FBM. A nonlinear filtering model with FBM in the signal process (and a Brownian component driving the observation process) was also investigated by Kleptsyna et al. in [4].

2. RKHS $\mathcal{H}(B^H)$ and the Observation Process Model

Let us fix a complete probability space $(\Omega, \mathcal{F}, P)$ on which all random processes are defined. For a given $H \in (\frac{1}{2}, 1)$, let $B^H = (B_t^H, t \in [0,T])$ be an FBM with Hurst index H. Namely, B^H has the following properties:

(i) B^H is a Gaussian process with continuous sample paths and stationary increments.

(ii) $B_0^H = 0$, $EB_t^H = 0$ for all $t \geq 0$ and the covariance kernel is given by

$$R_H(t_1,t_2) := EB_{t_1}^H B_{t_2}^H = \frac{c^2}{2}\{|t_1|^{2H} + |t_2|^{2H} - |t_1 - t_2|^{2H}\}, \tag{5}$$

where

$$c^2 := \mathrm{Var}(B_1^H) = -\frac{\Gamma(2-2H)\cos(\pi H)}{\pi H(2H-1)}.$$

It is well known that B^H is self-similar with self-similarity index H, B^H is not a semimartingale and (since $H \in (\frac{1}{2}, 1)$) has a long-range dependence structure, i.e.

$$\sum_{n=0}^{\infty} \mathrm{Cov}(B_1^H, B_{n+1}^H - B_n^H) = \infty.$$

Let $\mathcal{H}(B^H)$ denote the RKHS of $B^H = (B_t^H, t \in [0,T])$. Then $\mathcal{H}(B^H)$ satisfies the following conditions:

(i) $\mathcal{H}(B^H)$ is a Hilbert space of real-valued functions on $[0,T]$.
(ii) $\forall t \in [0,T]$, $R_H(\cdot, t) \equiv \mathrm{Cov}(B_\cdot^H, B_t^H) \in \mathcal{H}(B^H)$.
(iii) $\forall g \in \mathcal{H}(B^H)$, $\langle g(\cdot), R_H(\cdot, t)\rangle_{\mathcal{H}(B^H)} = g(t)$.

Specifically, by Theorem 4.4 in [1], for $\frac{1}{2} < H < 1$, $\mathcal{H}(B^H)$ consists of functions of the form

$$g(t) = \int_0^t g^*(s)\gamma_H(s,t)\,ds, \qquad t \in [0,T], \tag{6}$$

where $g^* \in L^2([0,T])$,

$$\gamma_H(s,t) := \frac{s^{1/2-H}}{\Gamma(H-\frac{1}{2})}\int_s^t \tau^{H-1/2}(\tau - s)^{H-3/2}\,d\tau, \tag{7}$$

and $\forall g_1, g_2 \in \mathcal{H}(B^H)$,

$$\langle g_1, g_2\rangle_{\mathcal{H}(B^H)} = \int_0^T g_1^*(s)g_2^*(s)\,ds. \tag{8}$$

Note that any g from $\mathcal{H}(B^H)$ has a derivative almost everywhere in $[0,T]$. For almost all $t \in [0,T]$, the relationship (6) between g and g^* can be inverted:

$$g^*(t) = t^{H-1/2}\frac{d}{dt}\left(\int_0^t k(t,\tau)\frac{\partial}{\partial\tau}g(\tau)\,d\tau\right), \tag{9}$$

where

$$k(t,\tau) := \frac{1}{\Gamma(\frac{3}{2}-H)}(t-\tau)^{1/2-H}\tau^{1/2-H}. \tag{10}$$

The observation model (4) can then be written as

$$Y_t(\omega) = \int_0^t h(X_s(\omega))\gamma_H(s,t)\,ds + B_t^H(\omega), \qquad t \in [0,T], \tag{11}$$

where

$$\gamma_H(s,t) = \frac{s^{1/2-H}}{\Gamma(H-\frac{1}{2})}\int_s^t \tau^{H-1/2}(\tau-s)^{H-3/2}\,d\tau,$$

and $h \in C(S)$ satisfies

$$\int_0^T h^2(X_s)(\omega)\,ds < \infty \qquad (P\text{-a.s.}). \tag{12}$$

Let us also assume the following condition:

$$(X_t) \text{ is independent of } (B_t^H). \tag{13}$$

3. Bayes' Formula and a "Fractional" Zakai-Type Equation

Let $(\tilde{\mathcal{F}}_t^X)$ be the natural family of X, and let $(\tilde{\mathcal{F}}_t^{B^H})$ be the natural family of B^H. Since X is independent of B^H, we can assume that (X_t) is defined on $(\tilde{\Omega}^X, \tilde{\mathcal{F}}^X, (\tilde{\mathcal{F}}_t^X), \tilde{P}^X)$, and (B_t^H) is defined on $(\tilde{\Omega}^{B^H}, \tilde{\mathcal{F}}^{B^H}, (\tilde{\mathcal{F}}_t^{B^H}), \tilde{P}^{B^H})$, where $\Omega = \tilde{\Omega}^X \times \tilde{\Omega}^{B^H}$, $\mathcal{F} = \tilde{\mathcal{F}}^X \times \tilde{\mathcal{F}}^{B^H}$, $P = \tilde{P}^X \times \tilde{P}^{B^H}$. Define $\mathcal{F}_t^X = \tilde{\mathcal{F}}_t^X \times \{\emptyset, \tilde{\Omega}^{B^H}\}$, $\mathcal{F}_t^{B^H} = \{\emptyset, \tilde{\Omega}^X\} \times \tilde{\mathcal{F}}_t^{B^H}$ and put $\mathcal{F}_t^Y = \sigma\{Y_s, 0 \le s \le t\}$. Let P^Y be the restriction of P to $\mathcal{F}_T^Y$.

Theorem 1. *Assume the conditions of the model* (11)–(13). *Then for any integrable and $\tilde{\mathcal{F}}_T^X$-measurable function f, we have that* (P^Y-a.s.)

$$E[f \mid \mathcal{F}_t^Y](\omega) = \frac{\int_{\tilde{\Omega}^X} f(u') \exp\{\alpha_{u'}(t)(\omega)\} \tilde{P}^X(du')}{\int_{\tilde{\Omega}^X} \exp\{\alpha_{u'}(t)(\omega)\} \tilde{P}^X(du')}, \tag{14}$$

where

$$\alpha_{u'}(t)(\omega) = \int_0^t h(X_s(u')) s^{H-1/2}\, d\left(\int_0^s k(s,\tau)\, dY_\tau(\omega)\right) - \frac{1}{2}\int_0^t [h(X_s(u'))]^2\, ds, \tag{15}$$

and

$$k(s,\tau) = \frac{1}{\Gamma(\frac{3}{2} - H)} (s-\tau)^{1/2-H} \tau^{1/2-H}. \tag{16}$$

Proof. The proof is based on the reference probability method. Let $Q(\cdot, \omega)$ be a version of the conditional probability relative to $\mathcal{F}_T^X$ on the σ-field $\mathcal{F}_t^Y$, i.e.

$$Q(A, \omega) = E(1_A \mid \mathcal{F}_T^X)(\omega) \qquad (P\text{-a.s.}), \tag{17}$$

$\forall A \in \mathcal{F}_t^Y$. Then, $\forall \omega' = (u', v') \in \Omega$, $A \in \mathcal{F}_t^Y$,

$$Q(A, \omega') = \delta_{u'} \times \tilde{P}^{B^H}(A), \tag{18}$$

where $\delta_{u'}$ is a probability measure on $\tilde{\mathcal{F}}_T^X$ with total mass concentrated at $\{u'\}$.

Moreover, under the law $\delta_{u'} \times \tilde{P}^{B^H}$, (B_t^H) is an FBM and

$$Y_t(\omega) = \int_0^t \gamma_H(s,t) h(X_s(u'))\, ds + B_t^H(\omega) \qquad \text{(a.s.)}, \tag{19}$$

$\forall t \in [0, T]$. Define $\hat{\mathcal{F}}_t^Y = \mathcal{F}_t^Y \vee \{\text{all } Q(\cdot, \omega')\text{-null sets in } \mathcal{F}_T^Y\}$. Then $B_t^H(\omega)$ and $Y_t(\omega)$ are both $\hat{\mathcal{F}}_t^Y$-adapted and, under $Q(\cdot, \omega')$, are FBMs with mean functions zero and $\beta(t; u')$, respectively, where

$$\beta(t; u') = \int_0^t \gamma_H(s,t) h(X_s(u'))\, ds, \qquad t \in [0, T]. \tag{20}$$

Let

$$\tilde{\Omega}_0^X := \left\{ u \in \tilde{\Omega}^X \colon \int_0^T h^2(X_s(u))\, ds < \infty \right\},$$

then $\tilde{P}^X(\tilde{\Omega}_0^X) = 1$ and $\forall u \in \tilde{\Omega}_0^X, \beta(\cdot\,; u) \in \mathcal{H}(B^H)$. Let us denote by $\mathcal{H}(B^{H|t})$ the RKHS of B^H restricted to $[0, t]$. Then, $\forall u \in \tilde{\Omega}_0^X, \beta(\cdot\,; u)$ viewed as a function on $[0, t]$ belongs to $\mathcal{H}(B^{H|t})$. Let $\Omega_0 = \tilde{\Omega}_0^X \times \tilde{\Omega}^{B^H}$. Fix $\omega' = (u', v') \in \Omega_0$. Since $Q \circ (B^H)^{-1}(\cdot, \omega') \equiv Q(\cdot, \omega') \circ (B^H)^{-1}$ and $Q \circ Y^{-1}(\cdot, \omega') \equiv Q(\cdot, \omega') \circ Y^{-1}$ are Gaussian measures on $C[0, t]$ with the common covariance kernel

$$R_H(s_1, s_2) = \frac{c^2}{2}\{|s_1|^{2H} + |s_2|^{2H} - |s_1 - s_2|^{2H}\}$$

and mean functions 0 and $(\beta(s\,; u'), 0 \le s \le t)$ from $\mathcal{H}(B^{H|t})$, by Theorem 5A in [9], the two measures are mutually absolutely continuous and the Radon–Nikodym derivative is given by

$$\frac{dQ \circ Y^{-1}}{dQ \circ (B^H)^{-1}}(Y) = \exp\{\langle Y, \beta(\cdot; u')\rangle_t - \tfrac{1}{2}\|\beta(\cdot; u')\|^2_{\mathcal{H}(B^{H|t})}\}, \tag{21}$$

where

$$\langle Y, \beta(\cdot; u')\rangle_t \equiv \langle \beta(\cdot; u') + B^{H|t}, \beta(\cdot; u')\rangle_t := \|\beta(\cdot; u')\|^2_{\mathcal{H}(B^{H|t})} + \varphi(\beta(\cdot; u')),$$

and φ is the congruence satisfying the following conditions:

(i) $\varphi\colon \mathcal{H}(B^{H|t}) \to L^2(B^{H|t})$.
(ii) $\varphi(R_H(\cdot, s)) = B_s^H$, $\forall s \in [0, t]$.
(iii) $E[\varphi(g)] = 0$, $\mathrm{Cov}[\varphi(g_1), \varphi(g_2)] = \langle g_1, g_2\rangle_{\mathcal{H}(B^{H|t})}$.

Consider an orthogonal increment process (Z_t) given by

$$Z_t = \int_0^t k(t, \tau)\, dB_\tau^H, \qquad t \in [0, T], \tag{22}$$

with the kernel $k(t, \tau)$ defined by (10). Then one can show (see [1]) that

$$\langle B^{H|t}, g\rangle_t := \varphi(g) = \int_0^t g^*(s) s^{H-1/2}\, dZ_s, \qquad \forall g \in \mathcal{H}(B^{H|t}).$$

For $\beta(\cdot; u') \in \mathcal{H}(B^{H|t})$, the function $h(X_\cdot(u'))$ plays the role of $\beta^*(\cdot; u')$. Thus,

$$\varphi(\beta(\cdot; u')) = \int_0^t h(X_s(u')) s^{H-1/2}\, dZ_s$$

and

$$\|\beta(\cdot; u')\|^2_{\mathcal{H}(B^{H|t})} = \int_0^t [h(X_s(u'))]^2\, ds.$$

Let

$$\alpha_{u'}(t)(\omega) := \langle Y(\omega), \beta(\cdot, u')\rangle_t - \tfrac{1}{2}\|\beta(\cdot, u')\|^2_{\mathcal{H}(B^{H|t})}. \tag{23}$$

Then

$$\alpha_{u'}(t) = \int_0^t h(X_s(u'))s^{H-1/2}\, dZ_s + \frac{1}{2}\int_0^t [h(X_s(u'))]^2\, ds, \tag{24}$$

where

$$Z_t = \int_0^t k(t,s)\, dB_s^H = \int_0^t k(t,s)\, dY_s - \int_0^t k(t,s)\frac{\partial}{\partial s}\beta(s,u')\, ds,$$

and, in view of (9),

$$Z_t = \int_0^t k(t,s)\, dY_s - \int_0^t s^{1/2-H} h(X_s(u'))\, ds.$$

Therefore, we obtain that (24) has the following form:

$$\begin{aligned}\alpha_{u'}(t)(\omega) = &\int_0^t h(X_s(u'))s^{H-1/2}\, d\left(\int_0^s k(s,\tau)\, dY_\tau(\omega)\right)\\ &- \frac{1}{2}\int_0^t [h(X_s(u'))]^2\, ds.\end{aligned} \tag{25}$$

Consider the measure $\lambda_{\omega'}$ on $\mathcal{F}_t^Y$ given by

$$d\lambda_{\omega'}(\cdot) = \exp\{-\alpha_{u'}(t)\}\, dQ(\cdot, \omega'). \tag{26}$$

Under $Q(\cdot, \omega')$,

$$\int_0^t h(X_s(u'))s^{H-\frac{1}{2}}\, dZ_s \sim \mathcal{N}\left(0, \int_0^t [h(X_s(u'))]^2\, ds\right).$$

Then $\lambda_{\omega'}$ is a probability measure on $\mathcal{F}_t^Y$ and Y is a mean-zero FBM under $\lambda_{\omega'}$. Also $\lambda_{\omega_1'} = \lambda_{\omega_2'}$ on $\mathcal{F}_t^Y$, i.e. it does not depend on ω'. Let us call it just λ. By Lemma 11.3.3 in [3] (λ-a.s.),

$$\exists q(\omega, \omega') = \frac{dQ(\cdot, \omega')}{d\lambda}(\omega) = \exp\{\alpha_{u'}(t)(\omega)\},$$

which is $(\mathcal{F}_t^Y \times \mathcal{F}_T^X)$-measurable, and for $\tilde{\mathcal{F}}_T^X$-measurable and integrable function f,

$$E[f \mid \mathcal{F}_t^Y](\omega) = \frac{\int_{\tilde{\Omega}^X} f(u')\exp\{\alpha_{u'}(t)(\omega)\}\tilde{P}^X(du')}{\int_{\tilde{\Omega}^X}\exp\{\alpha_{u'}(t)(\omega)\}\tilde{P}^X(du')} \qquad (P^Y\text{-a.s.}). \qquad \square$$

Remark 1. Bayes' formula for the model (11)–(13) could be obtained directly from the classical Bayes' formula with the help of the following integral representation of the FBM (Theorem 4.5 in [1]):

$\exists (B_t, 0 \le t \le T)$, a standard BM, such that $\forall t \in [0, T]$,

$$B_t^H(\omega) = \int_0^t \gamma_H(s,t)\, dB_s(\omega) \qquad (P\text{-a.s.}). \tag{27}$$

Namely, let

$$\tilde{Y}_t(\omega) := \int_0^t h(X_s(\omega))\, ds + B_t(\omega) \qquad (P\text{-a.s.}) \tag{28}$$

for $t \in [0, T]$. Then

$$Y_t(\omega) = \int_0^t \gamma_H(s,t)\, d\tilde{Y}_s(\omega), \qquad 0 \le t \le T.$$

We can invert the above relationship between Y and $\tilde{Y}$ (P-a.s.),

$$\tilde{Y}_t = \int_0^t s^{H-1/2}\, d\left(\int_0^s k(s,\tau)\, dY_\tau\right) \tag{29}$$

and note that $\mathcal{F}_t^Y = \mathcal{F}_t^{\tilde{Y}}$ for all t. Thus, $E[f \mid \mathcal{F}_t^Y](\omega) = E[f \mid \mathcal{F}_t^{\tilde{Y}}]$ almost surely, and using a Bayes' formula for the classical model (28) and equation (29), one obtains the desired result.

Theorem 2. *Assume that (X_t) is an S-valued Markov process with the generator L with domain $\mathcal{D}$. Moreover, assume that the paths of (X_t) are progressively measurable, and $E_P \int_0^T |f(X_s)|^2\, ds < \infty$ for all $f \in \mathcal{D}_0$, where $\mathcal{D}_0$ consists of all $f\colon S \to \mathbb{R}$ such that f_1 defined by $f_1(s,x) := f(x)$ belong to $\mathcal{D}$. For $f \in \mathcal{D}_0$ let us put $(L_t f)(x) := (Lf_1)(t,x)$. For the observation model* (11)–(13) *define*

$$\sigma_t(f, Y)(\omega) = \int_{\tilde{\Omega}^X} f(u') \exp\{\alpha_{u'}(t)(\omega)\} \tilde{P}^X(du'), \tag{30}$$

where

$$\begin{aligned}\alpha_{u'}(t)(\omega) = &\int_0^t h(X_s(u')) s^{H-1/2}\, d\left(\int_0^s k(s,\tau)\, dY_\tau(\omega)\right) \\ &- \frac{1}{2}\int_0^t \left[h(X_s(u'))\right]^2 ds.\end{aligned} \tag{31}$$

Then for all $f \in \mathcal{D}_0$, $\sigma_t(f, Y)$ satisfies the following Zakai-type equation:

$$d\sigma_t(f,Y) = \sigma_t(L_t f, Y)\, dt + \sigma_t(hf, Y) t^{H-1/2} d\left[\int_0^t k(t,s)\, dY_s\right] \tag{32}$$

with

$$k(t,s) = \frac{1}{\Gamma(\frac{3}{2} - H)} (t-s)^{1/2-H} s^{1/2-H}.$$

Proof. The equation follows immediately from the Zakai equation for the classical observation model (28) and Remark 1. □

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