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Stochastic Vorticity and Associated Filtering Theory

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Abstract. The focus of this work is on a two-dimensional stochastic vorticity equation for an incompressible homogeneous viscous fluid. We consider a signed measure-valued stochastic partial differential equation for a vorticity process based on the Skorohod–Ito evolution of a system of N randomly moving point vortices. A nonlinear filtering problem associated with the evolution of the vorticity is considered and a corresponding Fujisaki–Kallianpur–Kunita stochastic differential equation for the optimal filter is derived.

Key Words. Nonlinear filtering, Stochastic vorticity, Systems of stochastic differential equations, Signed measure-valued SPDE.

AMS Classification. 60G35, 60H20, 60H15.

1. Introduction

Experimental results often suggest that the nature of certain hydrodynamical phenomena calls for their stochastic formulation. High sensitivity to initial conditions and to perturbations, interplay of large numbers of degrees of freedom, and presence of conditions, under which existing microscopic perturbations get amplified to macroscopic scales, give rise to unsteady and chaotic flows. Thus, in many cases a natural approach to modeling of chaotic behavior in fluids is given via stochastic partial differential equations (SPDEs) of motions.

The focus of the current paper is on the stochastic modeling of the motion of a homogeneous viscous incompressible flow in $\mathbb{R}^2$ and the solution of an associated nonlinear filtering problem.

Section 2 presents a signed measure-valued stochastic Navier–Stokes equation in its vorticity formulation, obtained from a system of interacting point vortices driven by Skorohod–Ito SDEs. This stochastic vorticity model, introduced in [1], is rather general, contains both continuous (Brownian) and jump (Poisson) components and allows for a discontinuous (in time) displacement of point vortices. When the Poisson component is disregarded, the model reduces to a continuous stochastic vorticity model suggested by Kotelenez in [7].

Section 3 is devoted to our work on a filtering problem associated with the stochastic vorticity model. We present a nonlinear filtering theory for a stochastic vorticity process in the case when the vorticity is generated by a system of randomly moving point vortices (whose precise positions are unknown and are observed subject to an independent random noise).

However, before proceeding to stochastic modeling of the vorticity we briefly state the classical equations of hydrodynamics. Newton's second law of motion, applied to a fluid particle in an incompressible homogeneous viscous flow in $\mathbb{R}^2$, gives rise to a well-known Navier–Stokes model:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \frac{1}{\rho}\nabla p = \nu \Delta u, \tag{1}$$

$$\nabla \cdot u = 0, \tag{2}$$

where velocity $u = u(t, x)$ and pressure $p = p(t, x)$ are unknown variables of interest, ρ is a constant density and ν is a kinematic viscosity. Let the domain occupied by the fluid be $\mathcal{D} = \mathbb{R}^2$. Consider the following boundary condition:

$$u(t, x) \to 0 \qquad \text{as} \quad |x| \to \infty. \tag{3}$$

Then the Navier–Stokes model (1)–(3) with the domain $\mathcal{D}$ has an equivalent vorticity formulation given by

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega = \nu \Delta \omega, \tag{4}$$

where one can deduce the velocity field u from the vorticity $\omega = \omega(t, x)$ via a Biot–Savart law:

$$u(t, x) = \int K(x - y)\omega(t, y)\, dy \tag{5}$$

with the kernel K given by

$$K(x - y) := \nabla_x^{\perp} g(|x - y|) \equiv \begin{pmatrix} \partial_{x^2} \\ -\partial_{x^1} \end{pmatrix} \left(-\frac{1}{2\pi} \log|x - y| \right), \tag{6}$$

$$g(|x|) := -\frac{1}{2\pi} \log|x|. \tag{7}$$

Note that in $\mathbb{R}^3$ vorticity equals $curl(u)$ (half of the angular velocity) and describes how the fluid is rotating. In the case of just two spatial dimensions with $u = (u^1, u^2, 0)$ and $u^i = u^i(x^1, x^2)$, vorticity is scalar and is given by

$$\omega(t, x) = \partial_{x^1} u^2(t, x) - \partial_{x^2} u^1(t, x). \tag{8}$$

2. Stochastic Vorticity Model

One popular approach to stochastic modeling of viscous flows is to consider a stochastic Navier–Stokes equation, obtained by adding a random force (usually in the form of white noise) to the classical Navier–Stokes equation, which should account for various neglected effects and external disturbances. Interesting results in that direction can be found in [2], [8], [3] and [4]. However, in our opinion it is highly desirable to obtain a stochastic Navier–Stokes model where the stochastic component enters the model intrinsically, reflecting the chaotic behavior that is often observed in fluids even when no apparent external random forces are present.

The main idea behind a stochastic vorticity equation is to view the evolution of vorticity in the flow as an evolution of a system of randomly moving "point vortices", i.e. "particles" which carry concentrations of vorticity (while the rest of the flow is irrotational), and then analyze the resulting "mezoscopic" stochastic vorticity equation for an underlying microscopic model of randomly moving vortices. Namely, consider a system of N point vortices, where each vortex has an associated vorticity intensity $a_i \in \mathbb{R}$, $i = 1, \dots, N$ ($a_i > 0$ corresponds to the rotation in the counterclockwise direction, $a_i < 0$ corresponds to the rotation in the clockwise direction).

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a stochastic basis with right continuous filtration. All our stochastic processes are assumed to live on Ω, be $\mathcal{F}_t$-adapted and $(dP \times dt)$-measurable, where dt is the Lebesgue measure on $[0, \infty)$. Suppose the position of the ith vortex for $t \in [0, T]$ satisfies the following Skorohod–Ito type stochastic integral equation:

$$x_i(t) = \xi_i + \int_0^t u_{\varepsilon,s}(x_i(s))\,ds + \sigma \int_0^t \int_{\mathbb{R}^2} \Gamma(x_i(s), v) W(ds\,dv) + \theta \int_0^{t+} \int_\Lambda h(t, x_i(s-), \lambda)\tilde{\mathcal{N}}(ds\,d\lambda), \tag{9}$$

where

$$u_{\varepsilon,t}(x) = \sum_{j=1}^N a_j K_\varepsilon(x - x_j(t)), \qquad \forall x \in \mathbb{R}^2, \tag{10}$$

and the following conditions hold:

(a) $W(t, v) = (W^1(t, v), W^2(t, v))^T$, W^1 and W^2 are given independent Brownian sheets on $\mathbb{R}^+ \times \mathbb{R}^2$ with mean zero, variance $t|A|$, where A is a Borel set in $\mathbb{R}^2$ with finite Lebesgue measure $|A|$.

(b) $(\Lambda, \mathcal{E})$ is a measurable space; $\mathcal{N}$ is a given Poisson random measure on $\mathbb{R}^+ \times \Lambda$, independent of W, with a characteristic measure Π on Λ; $\tilde{\mathcal{N}}(t, B) = \mathcal{N}(t, B) - t\Pi(B)$ is the compensated Poisson random measure.

(c) For $0 < \varepsilon < 1$, $K_\varepsilon(x - y)$ is a regularized Biot–Savart kernel, i.e.

$$K_\varepsilon(x - y) = \begin{pmatrix} \partial_{x^2} \\ -\partial_{x^1} \end{pmatrix} g_\varepsilon(|x - y|),$$

where g_ε is at least a twice continuously differentiable approximation to $g(|x|) := (-1/2\pi)\log|x|$, with bounded derivatives up to order 2, satisfying $|g'_\varepsilon(\tau)| \leq |g'(\tau)|$ and $|g''_\varepsilon(\tau)| \leq |g''(\tau)|$ for $\tau > 0$.

(d) $\sigma^2/2 = \nu$ (viscosity), θ is a fixed nonnegative constant.

(e) $h(t, x, \lambda)$: $\mathbb{R}^+ \times \mathbb{R}^2 \times \Lambda \longrightarrow \mathbb{R}^2$ is a measurable function, such that there is a constant $D_1 > 0$ such that $\forall t \geq 0, \forall x \in \mathbb{R}^2$,

$$\int_\Lambda |h(t, x, \lambda)|^2 \Pi(d\lambda) \leq D_1, \qquad \int_\Lambda |h(t, x, \lambda) - h(t, y, \lambda)|^2 \Pi(d\lambda) \leq D_1 |x - y|^2.$$

(f)

$$\Gamma(x, v) := \begin{pmatrix} \hat{\Gamma}(x, v) & 0 \\ 0 & \hat{\Gamma}(x, v) \end{pmatrix}, \qquad \forall x, v \in \mathbb{R}^2,$$

where the correlation functions $\hat{\Gamma}$: $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^+$ are defined to be bounded Borel-measurable functions, symmetric in x, v, satisfying, for some constant $D_2 > 0$,

$$\int \hat{\Gamma}^2(x, v)\, dv = 1, \qquad \int_{\mathbb{R}^2} (\hat{\Gamma}(x, v) - \hat{\Gamma}(y, v))^2\, dv \leq D_2 |x - y|^2, \qquad \forall x, y \in \mathbb{R}^2.$$

(g) $\{\xi_i\}_{i=1}^N$ are square integrable $\mathcal{F}_0$-measurable random variables independent of W and $\mathcal{N}$.

Consider the empirical vorticity signed measure generated by the above system of point vortices:

$$\tilde{\omega}_t^N(dx) = \sum_{i=1}^N a_i \delta_{x_i(t)}(dx), \tag{11}$$

where $\delta_c(\cdot)$ stands for the Dirac point measure of mass 1 concentrated at the point $c \in \mathbb{R}^2$ and defined for any Borel set in the plane, and $\{x_i(t)\}_{t \geq 0}$, $i = 1, \ldots, N$, is the unique strong solution to the system (9)–(10).

Note that (10) represents a "regularized" Biot–Savart law, since $\forall x \in \mathbb{R}^2$,

$$u_{\varepsilon,t}(x) = \int K_\varepsilon(x - y) \tilde{\omega}_t^N(dy). \tag{12}$$

The regularization is needed in view of the singularity of the original Biot–Savart kernel K at zero.

For $a \in \mathbb{R}$, let $\mathcal{M}(a)$ denote the space of all Borel signed measures $\tilde{\mu}$ on $\mathbb{R}^2$ such that $\tilde{\mu}(\mathbb{R}^2) = a$. Let $C_b^m(\mathbb{R}^2, \mathbb{R})$ be the set of bounded and Lebesgue integrable functions from $\mathbb{R}^2$ into $\mathbb{R}$, which have bounded, continuous and Lebesgue integrable derivatives up to order m. $L_2(\Omega; D([0, T], \mathcal{M}(a)))$ denotes the space of square-integrable $\mathcal{M}(a)$-valued cadlag stochastic processes.

Also let $\langle \cdot, \cdot \rangle$ be the standard scalar product on $L_2(\mathbb{R}^2, dr)$, and for any finite signed Borel measure $\tilde{\mu}$ on $\mathbb{R}^2$,

$$\langle \tilde{\mu}, f \rangle = \int f(x) \tilde{\mu}(dx).$$

We will need the following result from [1]:

Theorem 1. *Let* $(\tilde{\omega}_t^N)_{t \in [0,T]}$ *be the empirical signed measure-valued process* (11) *associated with the evolution* (9)–(10) *of N point vortices and* $\sum_{i=1}^N a_i = a$. *Then, in* *the space* $L_2(\Omega; D([0,T], \mathcal{M}(a)))$, $(\tilde{\omega}_t^N)_{t\in[0,T]}$ *is the unique solution of the following stochastic "jump-type" vorticity equation*: *for all* $f \in C_b^3(\mathbb{R}^2, \mathbb{R})$, $\forall t \in (0,T)$,

$$\begin{cases} d\langle\tilde{\omega}_t^N, f\rangle = \langle\tilde{\omega}_t^N, u_{\varepsilon,t}\cdot\nabla f + \nu\Delta f\rangle\, dt \\ \qquad + \int_\Lambda \langle\tilde{\omega}_t^N, f(\cdot+\theta h(t,\cdot,\lambda)) - f(\cdot) - \theta h(t,\cdot,\lambda)\cdot\nabla f\rangle \Pi(d\lambda)\, dt \\ \qquad + \sigma\sum_{j=1}^{2}\int_{\boldsymbol{R}^2}\langle\tilde{\omega}_t^N, \hat{\Gamma}(\cdot,v)\partial_j f\rangle W^j(dt\,dv) \\ \qquad + \int_\Lambda \langle\tilde{\omega}_{t-}^N, f(\cdot+\theta h(t,\cdot,\lambda)) - f(\cdot)\rangle\tilde{\mathcal{N}}(dt\,d\lambda), \\ u_{\varepsilon,t}(x) = \int K_\varepsilon(x-y)\tilde{\omega}_t^N(dy) \qquad (\text{"regularized Biot–Savart law"}) \end{cases} \tag{13}$$

with the initial condition: $\tilde{\omega}_t^N = \sum_{i=1}^N a_i\delta_{\xi_i}$ $(a.s.)$.

Remark. if $\nu = \sigma^2/2 = 0$ and $\theta = 0$, (13) reduces to

$$\begin{cases} d\langle\tilde{\omega}_t^N, f\rangle = \langle\tilde{\omega}_t^N, u_{\varepsilon,t}\cdot\nabla f\rangle\, dt, \\ u_{\varepsilon,t}(x) = \int K_\varepsilon(x-y)\tilde{\omega}_t^N(dy), \end{cases}$$

which is a weak form of the (deterministic) "regularized" Euler equation in its vorticity formulation for an inviscid incompressible fluid in $\mathbb{R}^2$.

If $\nu > 0$, $\theta = 0$, (13) reduces to a continuous stochastic vorticity model of Kotelenez [7].

3. Filtering Problem Associated with the Evolution of a System of N Randomly Moving Point Vortices

We consider a stochastic vorticity process generated by a system of N point vortices whose motion is not observed directly. The positions of the vortices $x_i(t)$, $t \in [0,T]$, follow the Skorohod–Ito evolution given by (9)–(10). For notational convenience we denote by

$$X(t) := (x_1^1(t), x_1^2(t), x_2^1(t), x_2^2(t), \dots, x_N^1(t), x_N^2(t))^T$$

a $2N$-dimensional vector of the positions of our N vortices at time t.

Theorem 2. *Suppose* $(X(t))_{t\in[0,T]}$ *is the solution to the system* (9)–(10) *for which assumptions* (a)–(g) *are fulfilled. Then* $X(t)$ *is an* $\mathbb{R}^{2N}$*-valued Markov process and its transition probability can be defined by*

$$P(t,b,s,A) = P(X_{b,t}(s)\in A), \qquad \forall A \in \mathcal{B}(\mathbb{R}^{2N}), \quad b \in \mathbb{R}^{2N},$$

where $0 \le t < s \le T$ *and* $X_{b,t}(s)$ *satisfies*

$$X_{b,t}(s) = b + \int_t^s \overset{\circ}{\mathcal{K}} (X_{b,t}(\tau))\, d\tau + \sigma \int_t^s \int_{\mathbb{R}^2} \overset{\circ}{\Gamma} (X_{b,t}(\tau), v) \overset{\circ}{W} (d\tau\, dv)$$
$$+ \theta \int_t^{s+} \int_\Lambda \overset{\circ}{H} (\tau, X_{b,t}(\tau-), \lambda) \tilde{\mathcal{N}}(d\tau\, d\lambda)$$

and for any $c = (c_1^1, c_1^2, \dots, c_N^1, c_N^2)^T \in \mathbb{R}^{2N}$, $c_i \equiv (c_i^1, c_i^2)$, $i = 1, \dots, N$,

$$\overset{\circ}{\mathcal{K}} (c) := \left(\sum_{k=1}^N a_k K_\varepsilon^1(c_1 - c_k), \sum_{k=1}^N a_k K_\varepsilon^2(c_1 - c_k), \dots, \right.$$
$$\left. \sum_{k=1}^N a_k K_\varepsilon^1(c_N - c_k), \sum_{k=1}^N a_k K_\varepsilon^2(c_N - c_k) \right)^T,$$

$$\overset{\circ}{\Gamma} (c, v) := \begin{pmatrix} \Gamma(c_1, v) & 0 & \cdots & 0 \\ 0 & \Gamma(c_2, v) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \Gamma(c_N, v) \end{pmatrix},$$

which is a $2N \times 2N$ *matrix* (*with* $\Gamma(c_i, v)$ *being a* 2×2 *diagonal* (*sub*)*matrix*),

$$\overset{\circ}{W} := (W^1, W^2, \dots, W^1, W^2)^T,$$

and

$$\overset{\circ}{H} (t, c, \lambda) := (h^1(t, c_1, \lambda), h^2(t, c_1, \lambda), \dots, h^1(t, c_N, \lambda), h^2(t, c_N, \lambda))^T$$

are $2N$*-dimensional processes*.

By the infinitesimal generator of a family of transition probabilities $P(t, b, s, A)$ we mean the operator L_s defined on functions $F(b)$ ($b \in \mathbb{R}^{2N}$) by

$$(L_s F)(b) = \lim_{\substack{t, t' \to s \\ t < s < t'}} \frac{\int_{\mathbb{R}^{2N}} F(y) P(t, b, t', dy) - F(b)}{t' - t}. \tag{14}$$

The domain $\mathcal{D}(L)$ of the generator consists of all continuous functions $F(b)$ for which the limit (14) exists for each $s \in [0, T]$, $b \in \mathbb{R}^{2N}$.

Theorem 3. *Under the conditions of Theorem* 1, *let* L_s *be the infinitesimal generator for the Markov process* $X(t)$, *then* $\forall F \in C_b^2(\mathbb{R}^{2N}, \mathbb{R}) \subset \mathcal{D}(L)$, $\forall b = (b_1^1, b_1^2, \dots, b_N^1, b_N^2)^T \in \mathbb{R}^{2N}$, $b_i \equiv (b_i^1, b_i^2)^T$,

$$(L_s F)(b) = \nu \Delta F(b) + \sum_{\substack{i,k=1 \\ i \ne k}}^N \left\{ a_k K_\varepsilon(b_i - b_k) \cdot \nabla_i F(b) \right.$$
$$\left. + \nu \left(\int_{\mathbb{R}^2} \hat{\Gamma}(b_i, v) \hat{\Gamma}(b_k, v)\, dv \right) \Delta_{ik} F(b) \right\}$$

$$+ \int_\Lambda \left[F(b + \theta \overset{\circ}{H}(s, b, \lambda)) - F(b) - \theta \sum_{i=1}^{N} h(s, b_i, \lambda) \cdot \nabla_i F(b) \right] \Pi(d\lambda), \tag{15}$$

where

$$\nabla_i := \left(\frac{\partial}{\partial b_i^1}, \frac{\partial}{\partial b_i^2} \right), \qquad \Delta_{ik} := \frac{\partial^2}{\partial b_i^1 \, \partial b_k^1} + \frac{\partial^2}{\partial b_i^2 \, \partial b_k^2},$$

$$\Delta = \sum_{i=1}^{N} \Delta_{ii} = \sum_{i=1}^{N} \sum_{l=1}^{2} \frac{\partial^2}{(\partial b_i^l)^2}.$$

Proof. The proof of Theorems 2 and 3 is analogous to the argument given in Theorems 1 and 2 in Part II, Chapter 2.9, pp. 288–291, of [6]. □

Consider a system of N "noisy" observed vortices:

$$y_i(t) = \int_0^t \sum_{k=1}^{N} a_k \varphi_i(x_k(s)) \, ds + B_i(t), \qquad t \in [0, T], \tag{16}$$

where $\varphi_i \colon \mathbb{R}^2 \to \mathbb{R}^2$, $\forall i, k \in \{1, \ldots, N\}$, $\varphi_i(x_k(\omega))$ is (t, ω)-measurable,

$$\int_0^T E|\varphi_i(x_k(s))|^2 \, ds < \infty \tag{17}$$

and $B(t) = (B_1(t), \ldots, B_N(t))$ is a $2N$-dimensional $\mathcal{G}$-Brownian motion independent of $X = (X(t)) = (x_1(t), \ldots, x_N(t))$, where $\mathcal{G}_t \subseteq \mathcal{F}_t^Y$ and $(Y(t))_{t\in[0,T]} = (y_1(t), \ldots, y_N(t))_{t\in[0,T]}$, or, equivalently,

$$y_i(t) = \int_0^t \langle \tilde{\omega}_s^N, \varphi_i \rangle \, ds + B_i(t), \tag{18}$$

where $\tilde{\omega}_t^N$ is the unobserved vorticity process satisfying (13). For $i = 1, \ldots, N$, let

$$\Psi_i(t) := y_i - \int_0^t E(\langle \tilde{\omega}_s^N, \varphi_i \rangle \mid \mathcal{F}_s^Y) \, ds. \tag{19}$$

Then $\Psi(t) = (\Psi_1(t), \ldots, \Psi_N(t))$ is a $2N$-dimensional $\mathcal{F}_t^Y$-Brownian motion. Taking into account the Markov structure of the signal process X given in Theorems 2 and 3 and using a classical Fujisaki–Kallianpur–Kunita (FKK) equation obtained in [5] for the observation model (18) we obtain the following result:

Theorem 4. *Let $(Y_t)_{t\in[0,T]} = (y_1(t), \ldots, y_N(t))_{t\in[0,T]}$ be a system of N observed noisy vortices satisfying model* (18). *Then the analogue of the FKK filtering equation for the* *unobserved vorticity process* $\tilde{\omega}_t^N$ *is given by*: *for all* $f \in C_b^3(\mathbb{R}^2, \mathbb{R})$,

$$\begin{aligned}
&E(\langle \tilde{\omega}_t^N, f\rangle \mid \mathcal{F}_t^Y) \\
&\quad = E(\langle \tilde{\omega}_0^N, f\rangle \mid \mathcal{F}_t^Y) + \int_0^t E(\langle \tilde{\omega}_s^N, u_{\varepsilon,s} \cdot \nabla f + \nu \Delta f\rangle \mid \mathcal{F}_s^Y)\, ds \\
&\qquad + \int_0^t E\Bigg(\int_\Lambda \langle \tilde{\omega}_s^N, f(\cdot + \theta h(s,\cdot,\lambda)) - f(\cdot) \\
&\qquad\qquad - \theta h(s,\cdot,\lambda) \cdot \nabla f\rangle \Pi(d\lambda) \mid \mathcal{F}_s^Y\Bigg)\, ds \\
&\qquad + \sum_{k=1}^N \int_0^t \{E(\langle \tilde{\omega}_s^N, f\rangle \langle \tilde{\omega}_s^{N,\varepsilon}, \varphi_k\rangle \mid \mathcal{F}_s^Y) \\
&\qquad\qquad - E(\langle \tilde{\omega}_s^N, f\rangle \mid \mathcal{F}_s^Y) E(\langle \tilde{\omega}_s^N, \varphi_k\rangle \mid \mathcal{F}_s^Y)\} \cdot d\Psi_k(s), \qquad (20)
\end{aligned}$$

where

$$u_{\varepsilon,t}(x) := \int_{\mathbb{R}^2} K_\varepsilon(x - v)\tilde{\omega}_t^N(dv).$$

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