SECULAR MOTION IN A 2ND DEGREE AND ORDER-GRAVITY FIELD WITH NO ROTATION

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Abstract. The motion of a particle about a non-rotating 2nd degree and order-gravity field is investigated. Averaging conditions are applied to the particle motion and a qualitative analysis which reveals the general character of motion in this system is given. It is shown that the orbit plane will either be stationary or precess about the body's axis of minimum or maximum moment of inertia. It is also shown that the secular equations for this system can be integrated in terms of trigonometric, hyperbolic or elliptic functions. The explicit solutions are derived in all cases of interest.

Key words: dipole gravity, secular motion, averaging methods, analytical solution

1. Introduction

This paper shows that the secular motion of a satellite about a non-rotating, 2nd degree and order-gravity field is integrable in terms of trigonometric, hyperbolic, and elliptic functions. We show that the orbit inclination and longitude of the ascending node become time periodic in general and cause the orbit plane to precess about the body's minimum or maximum moment of inertia. The argument of periapsis is also shown to be time periodic, although its period does not, in general, coincide with the period of the inclination and node. Explicit solutions for all the averaged orbital elements are derived for cases of interest.

The analysis has application to understand the dynamics of spacecraft and natural particles about slowly rotating asteroids and comets. While the majority of asteroids have rotation periods on the order of 24 h or less, there are a significant number of asteroids that have rotation periods substantially longer than this – in many cases long enough to make them essentially 'stationary' in inertial space over the time periods of orbits close to them [7, 8, 10]. This result also provides some basic insight into motion about general non-rotating gravity fields, such as the investigation of orbital dynamics about a non-rotating lineal mass distribution reported in [11].

By restricting our study to the 2nd degree and order-gravity field we introduce an approximation, but also incorporate the main perturbation that acts on an orbiter in this system. Previous analyses of orbital dynamics about realistic asteroid shapes have established that the major perturbations acting on the orbiter are due to the

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2nd degree and order-gravity field [13, 14]. The inclusion of higher order gravity coefficients can be an important component, but will act on the already-perturbed system consisting of the central mass plus the 2nd degree and order-gravity field. Similarly, the influence of solar gravity on the orbiter is an important consideration, but can be neglected to first order for orbital motion sufficiently close to the body.

The use of averaging to approximate the dynamics of the system leads to our main result. By using averaging we implicitly make several assumptions about the system that may restrict the validity and applicability of our result. Averaging is generally applied to systems where the perturbing force is sufficiently small so that, over one orbital period, the deviations of the true trajectory from the Keplerian trajectory are relatively small. This perturbation can generally be related to the magnitude of the perturbing acceleration as compared to the central-body attraction. For the general 2nd degree and order-gravity field perturbation this ratio is always less than unity for orbits outside of the circumscribing sphere of the central body. Thus, formally, we can view the application of the averaging approach as valid in this system.

The reason we apply averaging to the system is to isolate the ‘main effect’ of the gravity-field perturbation, that is, to find the secular motion of the orbit as a function of time. From a qualitative view, the secular effect is the most important effect as it describes the long-term dynamics of the orbit about the body. From an analytical point of view, any higher order theory that would be applied to the study of orbital motion in this case would generally start from the perturbed, secular solution (as has been done classically with higher-order analytical investigations of the main satellite problem [2, 5, 9]).

The reduction of the averaged system to quadratures is not too surprising, but it is an interesting point as it has been recently shown that the general problem of orbit dynamics about a 2nd degree and order-gravity field is non-integrable through meromorphic integrals [12]. For our case, the averaging effectively removes two degrees of freedom, resulting in a one degree of freedom problem with an energy integral defined. This naturally leads to a quadrature of the system. What is interesting is that this quadrature can be performed in terms of elementary trigonometric and elliptic functions.

2. Problem Formulation

The potential of the 2nd degree and order field, parameterized by the gravity coefficients C_{20} and C_{22}, is

$$U_{20+22} = \frac{\mu}{r^3}\left[C_{20}\left(1 - \frac{3}{2}\cos^2\delta\right) + 3C_{22}\cos^2\delta\,\cos 2\lambda\right], \tag{1}$$

where r is the particle radius from the center of the body, δ is the particle latitude and λ is the particle longitude. By representing the 2nd degree and order-gravity field by the two coefficients C_{20} and C_{22} we implicitly assume a principal axis coordinate frame. For definiteness we will specify the frame so that $C_{20} \leqslant 0$ and $C_{22} \geqslant 0$, implying that the principal moments of inertia are ordered as $I_{xx} \leqslant I_{yy} \leqslant I_{zz}$ with principal axes x, y, and z. The longitude is measured counterclockwise from the x-axis in the x-y plane and latitude is measured from the x-y plane towards the z-axis. In terms of orbit elements these angles are defined as

$$\sin\delta = \sin i \sin u, \tag{2}$$

$$\tan\lambda = \frac{\sin\Omega\cos u + \cos\Omega\sin u\cos i}{\cos\Omega\cos u - \sin\Omega\sin u\cos i}, \tag{3}$$

$$u = \omega + f, \tag{4}$$

where i is the inclination, ω is the argument of periapsis, Ω is the longitude of the ascending node, and f is the true anomaly. We note that the central body is not rotating, and hence, time does not explicitly appear in the equation for the longitude λ.

The gravity coefficients can be derived from the principal moments of inertia of the body (normalized by the body mass)

$$C_{20} = -\tfrac{1}{2}\left(2I_{zz} - I_{xx} - I_{yy}\right), \tag{5}$$

$$C_{22} = \tfrac{1}{4}\left(I_{yy} - I_{xx}\right). \tag{6}$$

Then the parameter σ can be defined as

$$\sigma = \frac{I_{yy} - I_{xx}}{I_{zz} - I_{xx}}, \tag{7}$$

where we note that it lies in the interval [0 : 1] for any mass distribution. A value of $\sigma = 0$ denotes a body with rotational symmetry about the z-axis ($I_{yy} = I_{xx}$) and a value of $\sigma = 1$ denotes a body with rotational symmetry about the x-axis ($I_{yy} = I_{zz}$). The gravity coefficients can be rewritten as

$$C_{20} = -\tfrac{1}{2}\left(I_{zz} - I_{xx}\right)(2 - \sigma), \tag{8}$$

$$C_{22} = \frac{(I_{zz} - I_{xx})}{4}\sigma, \tag{9}$$

and the perturbing potential expressed as

$$U_{20+22} = \frac{\mu(I_{zz} - I_{xx})}{r^3}\left[-\frac{2-\sigma}{2}\left(1 - \frac{3}{2}\cos^2\delta\right) + \frac{3}{4}\sigma\cos^2\delta\cos(2\lambda)\right]. \tag{10}$$

The equations of motion have an energy integral of the form

$$J = \frac{1}{2}v^2 - \frac{\mu}{r} - U_{20+22}, \tag{11}$$

$$= -\frac{\mu}{2a} - U_{20+22}, \tag{12}$$

where a is the osculating semi-major axis of the orbit. This integral can be generalized to a Jacobi integral by addition of centrifugal terms should the central body be rotating [13].

3. Averaged Lagrange Equations

To formulate our approach to this problem we average the perturbing potential over the mean anomaly M

$$\bar{U}_{20+22} = \frac{1}{2\pi} \int_0^{2\pi} U_{20+22} \, \mathrm{d}M \tag{13}$$

to find

$$\bar{U}_{20+22} = \frac{\mu(I_{zz} - I_{xx})}{2a^3(1-e^2)^{\frac{3}{2}}} \left[-\frac{2-\sigma}{2} \left(\frac{3}{2} \sin^2 i - 1 \right) + \frac{3}{4} \sigma \sin^2 i \cos 2\Omega \right]. \tag{14}$$

Before being substituted into the Lagrange-Planetary Equations (see [3], p. 289, for example) a few points should be made. First, as with most averaged potential theories, the semi-major axis will be constant. Also, due to the absence of the argument of periapsis ω in the averaged potential, the eccentricity e will be constant.

The energy integral can be re-evaluated for the averaged case. Performing the averaging on the integral yields

$$\bar{J} = -\frac{\mu}{2a} - \bar{U}_{20+22} \tag{15}$$

and we can immediately note that $\bar{U}_{20+22}$ must be constant on average, specifically that the terms within the square brackets in Equation 14 are constant. Simplifying the expression we find a new form of the integral

$$C = \sin^2 i \left(1 - \sigma \cos^2 \Omega \right). \tag{16}$$

This quantity is conserved among the averaged orbital elements only and not among the unaveraged elements.

Next, evaluating the Lagrange Planetary equations for the inclination, longitude of the ascending node, argument of periapsis, and mean motion yields

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{1}{2} B\sigma \sin i \; \sin 2\Omega, \tag{17}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = -B \cos i \left(1 - \sigma \cos^2 \Omega \right), \tag{18}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{B}{2} \left(5C - 4 + \sigma + 2\sigma \cos^2 \Omega \right), \tag{19}$$

$$\frac{\mathrm{d}M}{\mathrm{d}t} = n \left[1 - \frac{B}{2} \sqrt{a^3(1-e^2)} \, (3C - 2 + \sigma) \right], \tag{20}$$

where

$$B = \frac{3n(I_{zz} - I_{xx})}{2a^2(1-e^2)^2}, \tag{21}$$

$$n = \sqrt{\frac{\mu}{a^3}}. \tag{22}$$

It is important to note that the mean anomaly rate in Equation 20 is a constant, implying that it can be used to define a new, effective semi-major axis value. Integrating this equation over one period of mean anomaly yields

$$2\pi = n\left[1 - \frac{B}{2}\sqrt{a^3(1-e^2)}\,(3C - 2 + \sigma)\right]\tilde{T}, \tag{23}$$

where $\tilde{T}$ is the new period of motion and is explicitly equal to

$$\tilde{T} = \frac{T}{1 - \frac{1}{2}B\sqrt{a^3(1-e^2)}\,(3C - 2 + \sigma)}, \tag{24}$$

$$T = \frac{2\pi}{n}, \tag{25}$$

where T is the unperturbed orbit period. We can define a new semi-major axis for our system as $\tilde{a}$ from the relation $\tilde{T} = 2\pi\tilde{a}^{3/2}/\sqrt{\mu}$ to find

$$\tilde{a} = a[1 - \tfrac{1}{2}B\sqrt{a^3(1-e^2)}(3C - 2 + \sigma)]^{-2/3}. \tag{26}$$

We can then redefine the constant B using this new semi-major axis.

We also note that the argument of periapsis does not appear in any of the right-hand sides of Equations 17–19 and thus can be solved by quadrature once solutions for i and Ω are found. Finally, we see that the equations for the inclination i and node Ω (Equations 17 and 18, respectively) are coupled with each other, however, Equation 16 can be used to decouple the equations from each other. We will do so later when we explicitly integrate these equations.

When the inclination of the orbit is equal to 0° or 180° the longitude of the ascending node and argument of periapsis are combined into the longitude of periapsis, defined as $\tilde{\omega}_\pm = \Omega \pm \omega$, where the $+$ sign is used for orbits with an inclination of 0° and the $-$ sign for orbits with an inclination of 180°. The general equation of motion for $\tilde{\omega}_\pm$ is defined as

$$\frac{\mathrm{d}\tilde{\omega}_\pm}{\mathrm{d}t} = \mp\frac{B}{2}\,[5C - 2 + \sigma] \pm B\,(1 \mp \cos i)\left[1 - \sigma\cos^2\Omega\right], \tag{27}$$

where the second term containing Ω disappears in $\dot{\tilde{\omega}}_+$ when $i = 0°$, and disappears in $\dot{\tilde{\omega}}_-$ when $i = 180°$. Inspection of both cases shows that the longitude of periapsis changes at a constant rate for equatorial orbits.

Once we find explicit relations for Ω, i, and ω we will use them to express the unit vector normal to the orbit plane, $\mathbf{h}$, and the unit vector that lies along the longitude of the ascending node, $\mathbf{n}$, defined as

$$\mathbf{h} = \begin{bmatrix} \sin i \sin\Omega \\ -\sin i \cos\Omega \\ \cos i \end{bmatrix}, \tag{28}$$

$$\mathbf{n} = \begin{bmatrix} \cos\Omega \\ \sin\Omega \\ 0 \end{bmatrix}. \tag{29}$$

From these two vectors we can construct the (normalized) eccentricity vector $\mathbf{e}$ as

$$\mathbf{e} = \cos\omega\mathbf{n} + \sin\omega\,(\mathbf{h}\times\mathbf{n})\,. \tag{30}$$

This set of vectors provide a different way to express the secular motion of the orbit and orbit plane.

4. Qualitative Analysis

Before we explicitly integrate Equations 17–19 the averaged integral of motion in Equation 16 can be used to understand the qualitative motion of a particle in this system.

First, let us note by inspection of Equation 16 that the constant C lies in the interval [0, 1], as it is the product of two quantities that lie within this same interval. Then we can understand the relationship between changes in i and Ω by plotting the contour curves for different values of C in this interval. Given our single parameter σ we need only view three different qualitative situations: $\sigma = 0$, $0 < \sigma < 1$, and $\sigma = 1$. The case $\sigma = 0$ yields i constant for all values of Ω, which just recovers the classic result that the inclination is constant on average for the C_{20}-only potential field. In Figure 1 we show the contour lines generated for a generic case of $0 < \sigma < 1$ and in Figure 2 we show the case for $\sigma = 1$. Motion in the averaged system will follow the contour lines.

4.1. MOTION FOR $0 < \sigma < 1$

For $C = 0$ the inclination is constrained to equal 0 or 180°. In this case the orbit plane is fixed and only the argument of periapsis will have a secular change

$$\frac{\mathrm{d}\tilde{\omega}_\pm}{\mathrm{d}t} = \pm B\left(1 - \frac{\sigma}{2}\right), \tag{31}$$

where the $+$ is for $i = 0°$ and the $-$ is for $i = 180°$. Conversely, for $C = 1$ the integral demands that $i = 90°$ and $\Omega = \pm 90°$, yielding a stationary value of Ω and i, with the argument of periapsis having a constant secular decrease

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{B}{2}\,(1+\sigma)\,. \tag{32}$$

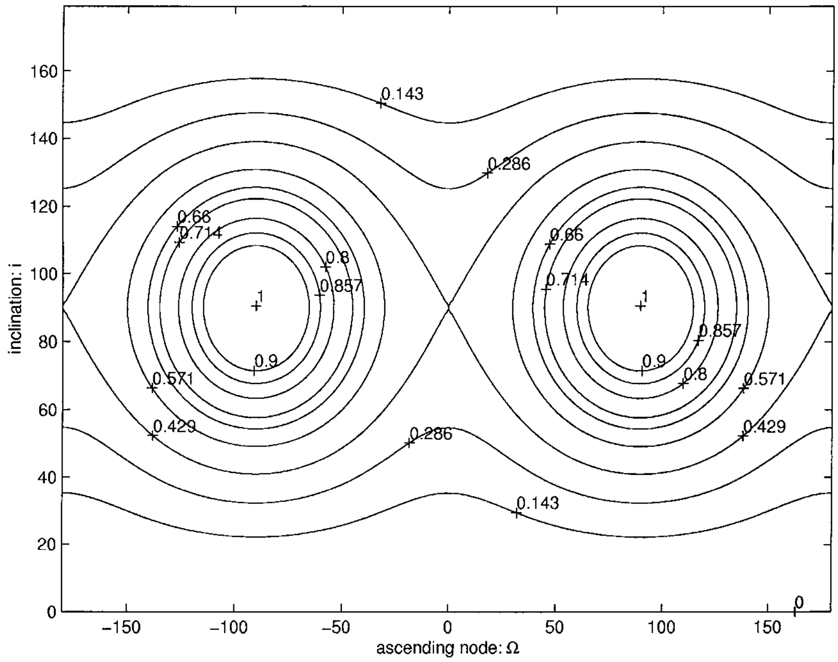

Figure 1. Contour plots of the averaged integral Equation 16 as a function of inclination i and node Ω for motion in the averaged system for the case of $\sigma = 0.571$. The motion follows all contour lines in the clock-wise direction.

For values of C not equal to 0 or 1 we find, in general, a time-periodic variation in the orbit plane (see Figure 1). For values of C sufficiently close to 0 the orbit plane will precess about the z-axis of the body (the maximum moment of inertia). For values of C sufficiently close to 1 the orbit plane will precess about the x-axis (the minimum moment of inertia). In both cases, the orbit precession occurs in the clock-wise direction, as measured from the orbit normal. For both of these general cases the argument of periapsis will be driven by a time-periodic differential equation, in general. Depending on the parameters C and σ this variation may even switch between clockwise and counter clockwise within one secular period of motion. A general rule, however, is that when C is near zero the argument of periapsis will decrease on average and when C is near 1 the argument of periapsis will increase on average.

As C decreases from 1 or increases from 0 we see that there must be a boundary between precession about the x and z axes. In the contour plots of Ω and i this appears as a heteroclinic connection between two equilibrium points located at $i = 90°$ and $\Omega = 0, 180°$. These equilibrium points correspond to an energy of $C = 1 - \sigma$ and are hyperbolic, with their stable and unstable manifolds forming heteroclinic connections to the other equilibrium point. These manifolds serve as a separatrix between the two modes of orbit normal precession.

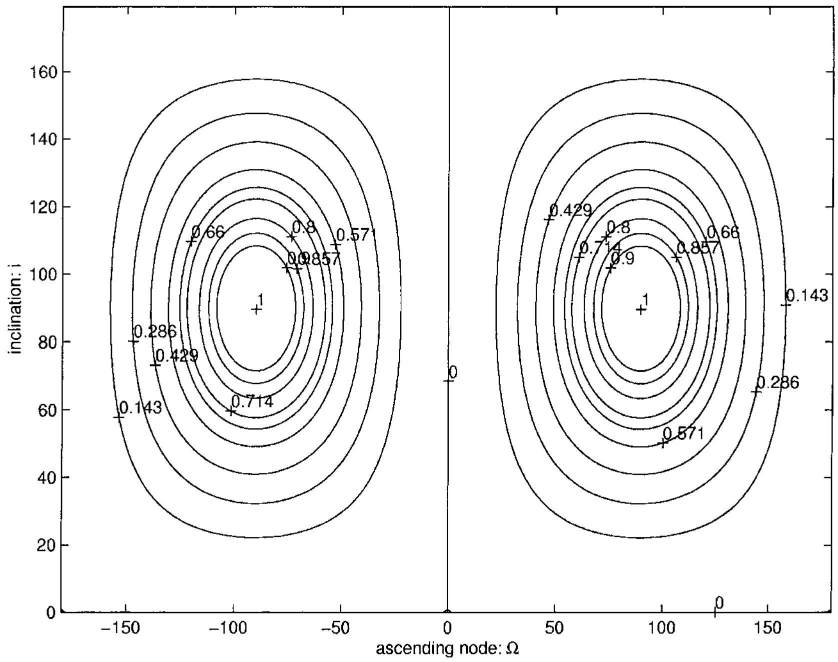

Figure 2. Contour plots of the averaged integral Equation 16 as a function of inclination i and node Ω for motion in the averaged system for the case of $\sigma = 1$. The motion follows all contour lines in the clock-wise direction.

4.2. MOTION FOR $\sigma = 1$

For $\sigma = 1$ we note that the orbit normal is either stationary or precesses about the x-axis (the minimum moment of inertia, see Figure 2). Motion in this case corresponds to orbital motion about a prolate body. We note that Ω is now constrained to lie in one of the intervals $(-180°, 0)$ or $(0, 180°)$. Exceptions to this occur if the orbit is in the equatorial plane or has $\Omega = 0, 180°$. If the orbit is in the equatorial plane the situation is similar to the case when $\sigma < 1$. If the orbit has an initial value of $\Omega = 0, 180°$ we see, from Equations 17 and 18, that the inclination and node are frozen at their given values and from Equation 16 that $C = 0$. Thus, the vertical lines in the contour plot correspond to a locus of equilibrium points. Essentially, the unstable equilibrium point and manifolds for $\sigma < 1$ degenerate into a locus of fixed points for $\sigma = 1$.

4.3. ADDITIONAL CONSIDERATIONS

The contours in Figures 1 and 2 can be mapped onto the surface of a sphere. To show this, first identify the $\Omega = 180°$ and $\Omega = -180°$ lines to form a cylinder. Then the two lines $i = 0$ and $i = 180°$ can be identified with themselves, respectively, to form a topological sphere. Performing these identifications brings to light a connection between the geometry of the polhodes of the torque-free rotation of a rigid body (cf [6], pg 391) and the geometry of our current analysis. The principal axes are similarly aligned in each case with the x and z-axes surrounded by level curves, and the y-axis lying at the intersection of hyperbolic manifolds. Thus we have the interesting result that motion in our averaged system has qualitative similarities to the rigid-body rotation of a torque-free body, even though the underlying equations of motion are significantly different. Additionally, the contours maintain their similarity in the case of a rotationally symmetric inertia tensor about either the x or z-axes.

5. Analytical Solutions

Given a qualitative understanding of motion for the averaged system we now proceed to integrate the equations for i, Ω, and ω explicitly. We will first discuss a number of special cases in which the motion can be expressed in terms of elementary functions, and then progress to the cases which can be solved in terms of elliptic functions. In the discussion we state the elliptic functions and integrals with minimal definition. See the Appendix for a brief definition of the necessary functions for this analysis.

Before proceeding we discard the case of $C = 1$ as this just corresponds to the stable equilibrium points ($i = 90°$, $\Omega = \pm 90°$) which exist in both cases of interest, $0 < \sigma < 1$ and $\sigma = 1$. Similarly we dispose of the case $C = 0$ as this just corresponds to the inclination being equal to 0 or 180° and the longitude of periapsis $\tilde{\omega}_{\pm}$ increasing at a constant rate of $\pm B(1 - \sigma/2)$, where the $+$ signifies $i = 0$ and the $-$ signifies $i = 180°$. If $\sigma = 1$ we also note that $C = 0$ will allow a frozen orbit with $\Omega = 0$, 180° and i at an arbitrary value. Thus we can constrain the constant C to the interval $0 < C < 1$. The remainder of the discussion considers the cases $\sigma = 0$, $0 < \sigma < 1$, and $\sigma = 1$ separately. We include the case $\sigma = 0$ for completeness.

5.1. ANALYTICAL SOLUTION FOR $\sigma = 0$

As noted above, this corresponds to the classical example of secular motion about a body with C_{20} only ([4], pg 345). From Equation 16 we see that $\sin i = \sqrt{C}$, and is constant. The solutions for Ω and ω can be immediately found as: $\Omega = \mp\sqrt{1 - C}B(t - t_o)$ and $\omega = \frac{1}{2}(4 - 5C)B(t - t_o)$, the $\pm$ in Ω denoting whether the initial inclination is less than or greater than 90°. Using this solution, the orbit normal and node vectors are found to be

$$\mathbf{h} = \begin{bmatrix} \mp\sqrt{C}\sin\left(\sqrt{1 - C}B(t - t_o)\right) \\ -\sqrt{C}\cos\left(\sqrt{1 - C}B(t - t_o)\right) \\ \pm\sqrt{1 - C} \end{bmatrix}, \tag{33}$$

$$\mathbf{n} = \begin{bmatrix} \cos\left(\sqrt{1-C}B(t-t_o)\right) \\ \mp \sin\left(\sqrt{1-C}B(t-t_o)\right) \\ 0 \end{bmatrix}. \tag{34}$$

This corresponds to the precession of the orbit plane at a constant rate about the z-axis (the maximum moment of inertia), in the clock-wise direction relative to the orbit normal.

5.2. ANALYTICAL SOLUTION FOR $\sigma = 1$

In this case Equations 17, 18, and 16 simplify to

$$\frac{\mathrm{d}i}{\mathrm{d}t} = B \sin i \sin \Omega \cos \Omega, \tag{35}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = -B \cos i \sin^2 \Omega, \tag{36}$$

$$C = \sin^2 i \sin^2 \Omega. \tag{37}$$

Define a new variable $v = \cot \Omega$, noting that $1 + v^2 = 1/\sin^2 \Omega$ and

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -(1+v^2)\frac{\mathrm{d}\Omega}{\mathrm{d}t}, \tag{38}$$

$$= B \cos i. \tag{39}$$

Then $\cos i$ can be solved from Equation 37 to find

$$\cos i = (\pm)_i \sqrt{1-C}\sqrt{1 - \frac{Cv^2}{1-C}}, \tag{40}$$

the sign of $(\pm)_i$ is equal to the sign of $\cos i$, and hence is positive when the inclination is less than 90°, negative when greater than 90°, and indeterminant when equal to 90°. Rewriting the equation for $\mathrm{d}v/\mathrm{d}t$ and separating the variables yields the differential equation

$$\frac{\mathrm{d}v}{\sqrt{1 - \frac{Cv^2}{1-C}}} = (\pm)_i B\sqrt{1-C}\,\mathrm{d}t. \tag{41}$$

Perform a final change of variable to $u = \sqrt{C/(1-C)}v$ to find

$$\frac{\mathrm{d}u}{\sqrt{1-u^2}} = (\pm)_i B\sqrt{C}\,\mathrm{d}t. \tag{42}$$

For an initial condition we choose a value of inclination equal to 90° as the motion we are investigating will always pass through this value. Then we find that

$u(t_o) = \pm 1$ and we choose the positive sign so that $\cot\Omega(t_o) = \sqrt{(1-C)/C}$, meaning that Ω begins in either quadrants I or III. From Equation 35 we note that the inclination will initially increase, leading to the negative root for $\cos i$. Evaluating the separated differential equation we find

$$\arcsin(u) = \frac{\pi}{2} - B\sqrt{C}(t - t_o). \tag{43}$$

Completing the solution, evaluating $\cos i$, and performing the quadrature for ω yields

$$\cot\Omega = \sqrt{\frac{1-C}{C}}\cos[B\sqrt{C}(t - t_o)], \tag{44}$$

$$\cos i = -\sqrt{1-C}\sin[B\sqrt{C}(t - t_o)], \tag{45}$$

$$\omega - \omega_o = -\frac{B}{2}(5C - 1)(t - t_o) + \arctan[\sqrt{C}\tan(B\sqrt{C}(t - t_o))]. \tag{46}$$

The solution holds identically when Ω is shifted by $\pm 180°$, generating both families in the contour plots for this case.

Evaluting the orbit normal vector with this solution yields

$$\mathbf{h} = \begin{bmatrix} (\pm)_i\sqrt{C} \\ -(\pm)_i\sqrt{1-C}\cos[\sqrt{C}B(t - t_o)] \\ -\sqrt{1-C}\sin[\sqrt{C}B(t - t_o)] \end{bmatrix}, \tag{47}$$

$$\mathbf{n} = \frac{(\pm)_\Omega}{\sqrt{\cos^2[\sqrt{C}B(t - t_o)] + C\sin^2[\sqrt{C}B(t - t_o)]}} \begin{bmatrix} \sqrt{1-C}\cos[\sqrt{C}B(t - t_o)] \\ \sqrt{C} \\ 0 \end{bmatrix}, \tag{48}$$

where $(\pm)_\Omega$ is positive for an initial node in quadrant I and negative for an initial node in quadrant III. Here we note that the projection of the orbit normal on the x axis remains constant, and the amplitude of the projection on the y-z plane is constant. Thus we see that this case reduces to a uniform precession of the orbit plane about the axis of symmetry, analogous to the case of $\sigma = 0$, except now applying to a prolate body.

5.3. ANALYTICAL SOLUTION FOR $0 < \sigma < 1$

Now the general form of the equations for $\mathrm{d}i/\mathrm{d}t$, $\mathrm{d}\Omega/\mathrm{d}t$, and C from Equations 17, 18, and 16 hold, respectively. Define a new variable $s = \tan\Omega$ with $1 + s^2 = 1/\cos^2\Omega$ and $\mathrm{d}s/\mathrm{d}\Omega = 1 + s^2$. Rewriting the differential equation for s yields

$$\frac{\mathrm{d}s}{\mathrm{d}t} = -(\pm)_i B\sqrt{\left[1 - C - \sigma + (1 - C)s^2\right]\left(1 - \sigma + s^2\right)}, \tag{49}$$

where the $(\pm)_i$ is defined as before.

For notational convenience we also define two constant parameters

$$m^2 = 1 - \sigma, \tag{50}$$

$$\theta = \frac{1 - C - \sigma}{1 - C}, \tag{51}$$

where $0 < m^2 < 1$, and $-\infty < \theta < 1$ for the cases of interest.

Equation 49 can then be separated into

$$\frac{\mathrm{d}s}{\sqrt{\left(\theta + s^2\right)\left(m^2 + s^2\right)}} = \mp\sqrt{1 - C}B\,\mathrm{d}t. \tag{52}$$

The three cases $\theta < 0$, $\theta = 0$, and $\theta > 0$ will yield three different fundamental solutions, corresponding to precession about the x-axis (minimum moment of inertia), motion on the separatrix, and precession about the z-axis (maximum moment of inertia), respectively.

5.3.1. *Motion Along the Separatrix ($\theta = 0$)*

We first deal with the special case of $\theta = 0$, which is related to motion on the manifolds of the unstable equilibrium points. We noted before that at the unstable equilibrium point the energy has a value $C = 1 - \sigma$, and hence $\theta = 0$ and motion along the manifolds of this point will also have this parameter value. For this case Equation 52 simplifies to

$$\frac{\mathrm{d}s}{s\sqrt{1 + s^2/m^2}} = -(\pm)_i B\sqrt{\sigma(1 - \sigma)}\,\mathrm{d}t. \tag{53}$$

For an initial value of Ω we will choose $\pm 90°$ as each separatrix will pass through this value, leading to $s_o = \pm\infty$. In this subsection we denote $(\pm)_\Omega$ as positive for an initial $\Omega = 90°$ and negative for an initial $\Omega = -90°$. Integrating the equation explicitly yields

$$\tan\Omega = \frac{(\pm)_i\sqrt{1 - \sigma}}{\sinh(u_f)}, \tag{54}$$

$$u_f = B\sqrt{\sigma(1 - \sigma)}(t - t_o). \tag{55}$$

At time $t = t_o$ motion commences at $\Omega = \pm 90°$. For the positive sign the node moves into quadrants I and III approaching limiting values of 0° and 180° degrees, respectively. For the negative sign the node moves into quadrants II and IV approaching limiting values of 180° and 0° degrees, respectively.

Solving for $\sin i$ and performing the quadrature for ω yields

$$\sin i = \frac{\sqrt{\cosh^2(u_f) - \sigma}}{\cosh(u_f)}, \tag{56}$$

$$\begin{aligned} \omega - \omega_o = &-\frac{B}{2}(1 - 2\sigma)(t - t_o) + \\ &+ \arctan \frac{\cosh u_f(\cosh u_f + \sinh u_f) - \sigma}{\sqrt{\sigma(1-\sigma)}} \\ &- \arctan \sqrt{\frac{1-\sigma}{\sigma}}. \end{aligned} \tag{57}$$

The orbit vectors for this solution becomes

$$\mathbf{h} = \frac{1}{\cosh u_f} \begin{bmatrix} (\pm)_\Omega \sqrt{1-\sigma}, \\ -(\pm)_\Omega (\pm)_i \sinh u_f \\ (\pm)_i \sqrt{\sigma} \end{bmatrix}, \tag{58}$$

$$\mathbf{n} = \frac{(\pm)_\Omega}{\sqrt{\cosh^2 u_f - \sigma}} \begin{bmatrix} (\pm)_i \sinh u_f \\ \sqrt{1-\sigma} \\ 0 \end{bmatrix}. \tag{59}$$

5.3.2. *Precession About the x-axis ($\theta < 0$)*

When $C + \sigma > 1, \theta < 0$ and the orbit plane will precess about the x-axis (minimum moment of inertia). Defining $\nu^2 = -\theta$, we rewrite Equation 52 as

$$-(\pm)_i \sqrt{1 - C} B \, \mathrm{d}t = \frac{\mathrm{d}s}{\sqrt{(m^2 + s^2)(s^2 - \nu^2)}}. \tag{60}$$

We choose our initial value to lie at an inclination of 90°, since all librational motion will pass through this value, at which point it can be shown that $s_o = \pm \nu$. For an initial point we choose the positive root, corresponding to Ω starting in either quadrant I or III. We see then that the inclination will initially increase, giving a definite sign to our integral

$$\sqrt{1 - C} B(t - t_o) = \int_\nu^s \frac{\mathrm{d}s}{\sqrt{(m^2 + s^2)(s^2 - \nu^2)}} \tag{61}$$

which can be solved in terms of elliptic functions as

$$\sqrt{(1-C)(m^2+\nu^2)}B(t-t_o) = \mathrm{cn}^{-1}\left(\frac{\nu}{s}, \frac{m}{\sqrt{m^2+\nu^2}}\right). \tag{62}$$

Specifically solving for $\tan\Omega$, $\cos i$, and ω yields

$$\tan\Omega = \sqrt{\frac{1-n_L}{n_L}}\frac{1}{\mathrm{cn}\left(u_f, k_L\right)}, \tag{63}$$

$$\cos i = -\sqrt{n_L - k_L^2}\frac{\mathrm{sn}\left(u_f, k_L\right)}{\mathrm{dn}\left(u_f, k_L\right)}, \tag{64}$$

$$\begin{aligned}\omega - \omega_o = &-\frac{B}{2}(5C-4+\sigma)(t-t_o) + \\ &+\sqrt{\frac{\sigma}{C}}\left[(1-n_L)\Pi(u_f, n_L, k_L) - F(u_f, k_L)\right],\end{aligned} \tag{65}$$

$$u_f = \sqrt{\sigma C}B(t-t_o), \tag{66}$$

$$k_L = \sqrt{\frac{(1-\sigma)(1-C)}{\sigma C}}, \tag{67}$$

$$n_L = \frac{1-C}{\sigma}, \tag{68}$$

where F is the incomplete elliptic integral of the first kind and Π is the incomplete elliptic integral of the third kind. The orbit vectors are found to be

$$\mathbf{h} = \frac{1}{\mathrm{dn}(u_f, k_L)}\begin{bmatrix} (\pm)_\Omega\sqrt{1-n_L} \\ -(\pm)_\Omega\sqrt{n_L}\mathrm{cn}\left(u_f, k_L\right) \\ -\sqrt{n_L - k_L^2}\mathrm{sn}\left(u_f, k_L\right) \end{bmatrix}, \tag{69}$$

$$\mathbf{n} = \frac{(\pm)_\Omega}{\sqrt{1-n_L\mathrm{sn}^2\left(u_f, k_L\right)}}\begin{bmatrix} \sqrt{n_L}\mathrm{cn}\left(u_f, k_L\right) \\ \sqrt{1-n_L} \\ 0 \end{bmatrix} \tag{70}$$

The period of the secular motion for the orbit normal is

$$T_L = \frac{4}{\sqrt{\sigma C}B}K(k_L) \tag{71}$$

where K is the complete elliptic integral of the first kind.

Note that when $\sigma \to 1$ the parameter $k_L \to 0$. In this case it can be shown that the above results reduce to the case given in Equations 47 and 48.

5.3.3. *Precession About the z-axis ($\theta > 0$)*
When $C+\sigma < 1$, Ω will precess about the z-axis (maximum moment of inertia). In this case $0 < \theta < m^2$ and we recover Equation 52. Now we choose our initial value at $\Omega = 0$ as all solutions will pass through this point, yielding $s_o = 0$. Evaluating the integral yields

$$-(\pm)_i\sqrt{1-C}B(t-t_o) = \int_0^s \frac{\mathrm{d}s}{\sqrt{(s^2+m^2)(s^2+\theta)}} \tag{72}$$

which can be solved for in terms of elliptic functions as

$$-(\pm)_i\sqrt{1-C}mB(t-t_o) = \mathrm{tn}^{-1}\left(\frac{s}{\sqrt{\theta}}, \sqrt{\frac{m^2-\theta}{m^2}}\right). \tag{73}$$

Solving for $\tan\Omega$, $\sin i$, and ω yields

$$\tan\Omega = -(\pm)_i\sqrt{1-n_C}\,\mathrm{tn}\left(u_f, k_C\right), \tag{74}$$

$$\sin i = \frac{k_C}{\sqrt{n_C}}\frac{\sqrt{1-n_C\mathrm{sn}^2\left(u_f, k_C\right)}}{\mathrm{dn}\left(u_f, k_C\right)}, \tag{75}$$

$$\begin{aligned}\omega - \omega_o = &-\frac{B}{2}(5C-4+\sigma)(t-t_o) + \\ &+\sqrt{\frac{1-C}{1-\sigma}}\left[(1-n_C)\Pi(u_f, n_C, k_C) - F(u_f, k_C)\right],\end{aligned} \tag{76}$$

$$u_f = \sqrt{(1-C)(1-\sigma)}B(t-t_o), \tag{77}$$

$$k_C = \sqrt{\frac{\sigma C}{(1-C)(1-\sigma)}}, \tag{78}$$

$$= \frac{1}{k_L}, \tag{79}$$

$$n_C = \frac{\sigma}{1-C}, \tag{80}$$

$$= \frac{1}{n_L}. \tag{81}$$

The orbit vectors are expressed as

$$\mathbf{h} = \frac{1}{\sqrt{n_C}\,\mathrm{dn}\left(u_f, k_C\right)} \begin{bmatrix} -(\pm)_i k_C \sqrt{1-n_C}\,\mathrm{sn}\left(u_f, k_C\right) \\ -k_C \mathrm{cn}\left(u_f, k_C\right) \\ -(\pm)_i \sqrt{n_C - k_C^2} \end{bmatrix}, \tag{82}$$

$$\mathbf{n} = \frac{1}{\sqrt{1 - n_C \mathrm{sn}^2\left(u_f, k_C\right)}} \begin{bmatrix} \mathrm{cn}\left(u_f, k_C\right) \\ -(\pm)_i \sqrt{1-n_C}\,\mathrm{sn}\left(u_f, k_C\right) \\ 0 \end{bmatrix}. \tag{83}$$

The period of the secular motion of the orbit normal is then

$$T_C = \frac{4}{\sqrt{(1-C)(1-\sigma)B}} K(k_C). \tag{84}$$

Note that when $\sigma \to 0$ the parameters k_C and n_C go to zero. In this case it can be shown that the above results reduce to the case given in Equations 33 and 34.

6. Conclusions

The analysis explicitly derives the closed form solutions for averaged orbital motion about a non-rotating 2nd degree and order-gravity field. The orbit plane is seen to have three different motions in general, precession about the asteroid minimum moment of inertia, precession about the asteroid maximum moment of inertia, and motion along the separatrix between these two regions of motion. The qualitative motion of the orbit plane has analogy with the rotational motion of a torque-free rigid body. In the future this result will be applied to understanding of motion about a slowly rotating body, specifically to the case when the ratio of orbital period to rotation period is small.

A.1. Appendix

A number of different elliptic functions and integrals are used in the discussion of this paper. Below we give the basic definitions of the functions and integrals needed for our analytical solution. See [1] for a comprehensive review and definition of elliptic functions and integrals.

A.1.1. ELLIPTIC FUNCTIONS

The basic elliptic functions used are the elliptic sine and cosine functions, denoted as sn and cn, respectively. These are functions of their arguments, u, and of a parameter denoted as k. In their general form we will denote them as $\mathrm{sn}(u, k)$ and $\mathrm{cn}(u, k)$. If $k = 0$ they degenerate to the trigonometric sine and cosine functions, and if $k = 1$ they degenerate to the hyperbolic tangent and secant functions, respectively. As with the trigonometric functions, they have an amplitude constraint denoted as $\mathrm{sn}^2(u, k) + \mathrm{cn}^2(u, k) = 1$. We will only consider these functions with real arguments of u. Then they are periodic with period $4\mathrm{K}(k)$, where K is the complete elliptic integral of the first kind. A related function is denoted as $\mathrm{dn}(u, k)$ and is defined from the relation $\mathrm{dn}^2(u, k) + k^2\mathrm{sn}^2(u, k) = 1$. Also, tn is defined as sn/cn. The functions dn and tn have periods of $2K$.

A.1.2. ELLIPTIC INTEGRALS

We require two incomplete elliptic integral definitions for our solution, that of the first and third kind. The usual definitions of these integrals are given as

$$\mathrm{F}(\phi, k) = \int_0^{\phi} \frac{\mathrm{d}\theta}{\sqrt{1 - k^2 \sin^2\theta}}, \tag{85}$$

$$\Pi(\phi, n, k) = \int_0^{\phi} \frac{\mathrm{d}\theta}{(1 + n\sin^2\theta)\sqrt{1 - k^2\sin^2\theta}}. \tag{86}$$

With these definitions, the complete elliptic integral of the first kind is then $K(k) = \mathrm{F}(\pi/2, k)$.

For the specific application we give a different definition of the argument, related to argument used in the elliptic functions. To give a clearer understanding we change the integrands from trigonometric to elliptic functions using the transformation $\sin\theta = \mathrm{sn}(u)$, with the differential transformation being computed as: $\cos\theta \mathrm{d}\theta = \mathrm{cn}(u)\mathrm{dn}(u)\,\mathrm{d}u$, or since $\cos\theta = \mathrm{cn}(u)$, $\mathrm{d}\theta = \mathrm{dn}(u)\,\mathrm{d}u$. The integrals then become

$$\mathrm{F}(u_f, k) = \int_0^{u_f} \mathrm{d}u, \tag{87}$$

$$= \mathrm{sn}^{-1}(u_f), \tag{88}$$

$$\Pi(u_f, n, k) = \int_0^{u_f} \frac{\mathrm{d}u}{(1 + n\mathrm{sn}^2(u, k))}. \tag{89}$$

We assume this particular definition in the text, as it clearly shows these integrals to be monotonically increasing functions of u_f. To transform to the usual form with argument of ϕ we can use $\phi_f = \arcsin(\mathrm{sn}(u_f))$.

We always assume that $0 < k < 1$. Furthermore, for the evaluation of the elliptic integrals of the third kind, we note that since $0 < k^2 < n < 1$, they can be classified as the circular case of that integral.

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