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Lefschetz Motives and the Tate Conjecture*

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Abstract. A Lefschetz class on a smooth projective variety is an element of the $\mathbb{Q}$-algebra generated by divisor classes. We show that it is possible to define $\mathbb{Q}$-linear Tannakian categories of abelian motives using the Lefschetz classes as correspondences, and we compute the fundamental groups of the categories. As an application, we prove that the Hodge conjecture for complex Abelian varieties of CM-type implies the Tate conjecture for all Abelian varieties over finite fields, thereby reducing the latter to a problem in complex analysis.

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Introduction

Grothendieck mainly envisaged constructing categories of motives by using as correspondences all algebraic classes modulo an adequate equivalence relation. Unfortunately, we know little about algebraic classes and, hence, even less about these categories. In our present state of ignorance, categories of motives constructed using other correspondences, for example, those defined by Hodge classes, have proved to be more useful, and have played an important role, for example, in the theory of Shimura varieties.

In this article, we construct categories of motives using the algebraic classes we do understand, namely, those in the $\mathbb{Q}$-algebra generated by divisor classes, which I call *Lefschetz classes*. It is not obvious that there are sufficient of these to define a category of motives – for example, in general the direct image of a Lefschetz class is not Lefschetz – but this is proved in Milne 1999a for Lefschetz classes on Abelian varieties.

In the first section of this paper, I explain how to define a category $\mathbf{LMot}(k)$ of 'Lefschetz motives' over any field k. It is generated by the motives of Abelian varieties, and its morphisms are the correspondences defined by Lefschetz classes. It is a $\mathbb{Q}$-linear semisimple Tannakian category whose fundamental group has a description in terms of the simple isogeny classes of Abelian classes. For Abelian varieties of CM-type over $\mathbb{C}$ and for Abelian varieties over finite fields there are explicit classifications of the isogeny classes, which we use to make explicit our

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description of the fundamental groups (Sections 2 and 4). We also compute the homomorphisms of fundamental groups corresponding to the functor taking a Lefschetz motive of CM-type over $\mathbb{C}$ to the corresponding Hodge motive (Section 3) and the functor taking a Lefschetz motive of CM-type over $\mathbb{Q}^{\mathrm{al}}$ to its reduction over the algebraic closure $\mathbb{F}$ of a finite field (Section 5).

In the remaining two sections, we apply the theory to the Tate conjecture for Abelian varieties over finite fields. For an Abelian variety A over $\mathbb{F}$, there is a cycle class map into étale cohomology

$$\{\text{algebraic cycles on } A \text{ of codimension } r\} \to H^{2r}(A, \mathbb{Q}_\ell)(r),$$

$\ell \neq \operatorname{char}(\mathbb{F})$. The choice of a model A_0 of A over a finite subfield $\mathbb{F}_q$ of $\mathbb{F}$ determines an action of Gal $(\mathbb{F}/\mathbb{F}_q)$ on $H^{2r}(A, \mathbb{Q}_\ell(r))$. The Tate conjecture (Tate, 1965, Conjectures (a′) and 1) predicts that, for all r,

(0.1) the kernel of the cycle class map is the group of cycles numerically equivalent to zero, and its image spans the $\mathbb{Q}_\ell$-space

$$\mathcal{T}_\ell^{\,r}(A) \overset{\text{df}}{=} \bigcup_{A_0/\mathbb{F}_q} H^{2r}(A, \mathbb{Q}_\ell(r))^{\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)}.$$

Statement (0.1) for A implies the similar statements for any model A_0 of A over a finite field – specifically, it implies the statements denoted $E(A_0)$ and $T(A_0)$ in Tate 1994 and, hence, also the injectivity statement $I(A_0)$ and the equality of the order of the pole of the zeta function $Z(A_0, t)$ of A_0 at $t = q^{-r}$ with the rank of the group numerical equivalence classes of algebraic cycles of codimension r (ibid. Theorem 2.9).

Tate proved the conjecture for $r = 1$, and various authors have shown that, in some cases, $\mathcal{T}_\ell(A) =_{\text{df}} \oplus^r \mathcal{T}_\ell^{\,r}(A)$ consists of Lefschetz classes. However, Wei (1993) showed that, for a general simple isogeny class over $\mathbb{F}$, some power of an Abelian variety in the class supports an 'exotic' Tate class not in the $\mathbb{Q}_\ell$-algebra generated by divisor classes. Therefore, to prove Tate's conjecture, we need a new source of algebraic cycles. Up to isogeny, every Abelian variety over $\mathbb{F}$ lifts to an Abelian variety of CM-type in characteristic zero, and one possibility is to use the algebraic classes obtained by reduction from such a lifting, but without the Hodge conjecture, we know of very few algebraic classes on an Abelian variety of CM-type that are not already Lefschetz. We prove (Theorem 7.1):

> The Hodge conjecture for Abelian varieties of CM-type over $\mathbb{C}$ implies the Tate conjecture (0.1) for Abelian varieties over $\mathbb{F}$.

The proof makes use of Jannsen's theorem that the category of motives for numerical equivalence is semisimple (Jannsen, 1992).

Remark. (a) The proof of Theorem 7.1 does *not* show that every Tate class on an Abelian variety over $\mathbb{F}$ lifts to a Hodge class on an Abelian variety of CM-type, even up to isogeny. In fact, as Oort has pointed out, this is false. For a simple Abelian variety A over a field of characteristic zero,

$$E \subset \mathrm{End}(A) \otimes \mathbb{Q}, \quad E \text{ a field}, \qquad [E\colon \mathbb{Q}] = 2 \dim A \Rightarrow E \text{ is a CM-field},$$

whereas this is not true for Abelian varieties over fields of nonzero characteristic. Let $E \subset \mathrm{End}(A) \otimes \mathbb{Q}$ be a counterexample over $\mathbb{F}$, and let α generate E over $\mathbb{Q}$. Then the graph of α does not lift to any lifting of A to characteristic zero.

Rather, the proof uses the Tannakian formalism to show that there are sufficiently many algebraic classes conjecturally coming from Abelian varieties of CM-type and divisors to force the Tate conjecture to be true.

(b) For Abelian varieties of CM-type, the Hodge conjecture is known to be equivalent to the Tate conjecture (Pohlmann, 1968). Therefore Theorem 7.1 can be restated as follows: the Tate conjecture for Abelian varieties of CM-type over number fields implies the Tate conjecture for Abelian varieties over finite fields.

(c) To prove the Hodge conjecture for an Abelian variety A over $\mathbb{C}$, it suffices to construct enough vector bundles on A so that their Chern classes generate the $\mathbb{Q}$-algebra of Hodge classes. Because A is projective, it even suffices to construct the vector bundles analytically. Therefore, Theorem 7.1 reduces the proof of the Tate conjecture for Abelian varieties over finite fields to a problem in complex analysis.

Apart from the theory of Lefschetz motives developed in the first five sections, the proof of Theorem 7.1 uses one further crucial result (Theorem 6.1) concerning the relationship of the fundamental groups of various categories of motives.

In a later article (Milne, 1999b), I shall use Theorem 7.1 to construct a canonical category of 'motives' over $\mathbb{F}$ that

- has the 'correct' fundamental group, and equals the true category of motives if the Tate conjecture holds for Abelian varieties over $\mathbb{F}$,
- canonically contains the category of Abelian varieties up to isogeny as a *polarized* subcategory,

thereby resolving a problem that goes back to Grothendieck. The category of motives plays the same role in describing the points on Shimura varieties with coordinates in finite fields as Deligne's category of Hodge motives does for their points with coordinates in fields of characteristic zero (Milne, 1995, 1999b).

Notations and conventions. For a field k, k^{al} denotes an algebraic closure of k. Except in Section 6, $\Gamma = \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$.

Complex conjugation on $\mathbb{C}$ is denoted by ι. A *CM-field* is a field E algebraic over $\mathbb{Q}$ admitting a nontrivial involution ι_E such that $\rho \circ \iota_E = \iota \circ \rho$ for all homomorphisms $\rho\colon E \to \mathbb{C}$. The fixed field of ι_E is called the *real subfield* of E. The composite of all CM-subfields of $\mathbb{Q}^{\mathrm{al}}$ is again a CM-field, which we denote $\mathbb{Q}^{\mathrm{cm}}$.

An algebraic variety over a field k is a geometrically reduced (not necessarily connected) scheme of finite type over k.

In general, groups act on the left. The action of $\sigma \in \Gamma$ on a map $f: X \to Y$ from one Γ-set to a second (possibly with trivial action on one set), is defined by the rule:

$$(\sigma f)(x) = \sigma(f(\sigma^{-1}x)), \quad x \in X, \text{ i.e., } \sigma f = \sigma \circ f \circ \sigma^{-1}.$$

For a set (topological space) X, $\mathbb{Z}^X$ denotes the set of (locally constant) functions $f: X \to \mathbb{Z}$. When X is finite, we sometimes denote $\mathbb{Z}^X$ by $\mathbb{Z}[X]$ and an element f of $\mathbb{Z}^X$ by a sum $\sum_{x \in X} f(x)x$.

'Vector space' and 'representation' mean 'finite-dimensional vector space' and 'finite-dimensional representation'. For a vector space V over k, $\mathrm{GL}(V)$ denotes either the algebraic group or its k-rational points.

'Algebraic group' means 'affine algebraic group'. For such a group G, $G(K)$ is the set of points on G with coordinates in K, and G_K or $G_{/K}$ is $G \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, K$.

An algebraic group is of *multiplicative type* if it is commutative and its identity component is a torus, and an affine group scheme over a field is of *multiplicative type* if all of its algebraic quotients are. For such a group T over a field k, $X^*(T) =_{\mathrm{df}} \mathrm{Hom}(T_{/k^{\mathrm{al}}}, \mathbb{G}_{m/k^{\mathrm{al}}})$ denotes the group of characters of T and $X_*(T)$ the group of cocharacters. We often identify $X_*(T)$ with the dual $\mathrm{Hom}\,(X^*(T), \mathbb{Z})$ of $X^*(T)$.

For an algebraic group G over a field K (or product of fields) of finite degree over a field k, $(G)_{K/k} =_{\mathrm{df}} \mathrm{Res}_{K/k}(G)$ is the algebraic group over k obtained by restriction of scalars. For example, when K/k is separable, $(\mathbb{G}_m)_{K/k}$ is the torus with character group $\mathbb{Z}^{\mathrm{Hom}_k(K,k^{\mathrm{al}})}$.

Let $(G_i, t_i)_{i \in I}$ be a family of pairs consisting of an algebraic group G_i and a homomorphism $t_i: G_i \to \mathbb{G}_m$. We define the product $\prod_{i \in I}(G_i, t_i)$ of the family to be the pair (G, t) consisting of the largest subgroup of $\prod G_i$ on which the characters $(g_i)_{i \in I} \mapsto t_{i_0}(g_{i_0})$ agree and of the common restriction of these characters to G. It is universal with respect to the maps $(G, t) \to (G_i, t_i)$.

For Abelian varieties A and B, $\mathrm{Hom}^0(A, B) = \mathrm{Hom}(A, B) \otimes \mathbb{Q}$.

In general, our conventions concerning tensor categories are those of Deligne and Milne 1982. For a field k, a *k-linear tensor category* is an additive category **C** together with

(a) a bi-additive functor $\otimes$: **C** $\times$ **C** $\to$ **C** and associativity and commutativity constraints satisfying the usual axioms (ibid., p. 104);
(b) an identity object $\mathbb{1} = (U, u)$ and an isomorphism $k \to \mathrm{End}(U)$.

A *Tannakian subcategory* of a k-linear Tannakian category is a k-linear subcategory that is closed under the formation of sums, tensor products, subobjects, quotient objects, and duals. It is again a Tannakian category.

To signify that objects X and Y are isomorphic, we write $X \approx Y$; when a particular isomorphism is given (or there is a canonical or preferred isomorphism), we write $X \cong Y$. Also, $X =_{\mathrm{df}} Y$ means that X is defined to be Y, or that $X = Y$

by definition. When x is an element of a set X on which there is an equivalence relation, we sometimes use $[x]$ to denote the equivalence class containing x.

1. The Category of Lefschetz Motives

In this section we define the category of Lefschetz motives.

PRELIMINARIES

Let $\sim$ be an adequate equivalence relation on algebraic cycles, for example, rational equivalence (rat), homological equivalence with respect to some Weil cohomology theory (hom), or numerical equivalence (num). For a smooth projective variety X over a field k, $\mathcal{Z}^r(X)$ will denote the $\mathbb{Q}$-vector space with basis the irreducible subvarieties of X of codimension r, and $\mathcal{C}^r_\sim(X) = \mathcal{Z}^r(X)/\sim$. Then $\mathcal{C}_\sim(X) =_{\mathrm{df}} \oplus_r \mathcal{C}^r_\sim(X)$ becomes a graded $\mathbb{Q}$-algebra under intersection product. A regular map $\phi: X \to Y$ defines a homomorphism $\phi^*: \mathcal{C}_\sim(Y) \to \mathcal{C}_\sim(X)$ of graded $\mathbb{Q}$-algebras and a homomorphism $\phi_*: \mathcal{C}_\sim(X) \to \mathcal{C}_\sim(Y)$ of $\mathbb{Q}$-vector spaces (which is homogeneous of degree $\dim Y - \dim X$ if X and Y are equidimensional), related by the projection formula:

$$\phi_*(x) \cdot y = \phi_*(x \cdot \phi^* y), \quad x \in \mathcal{C}^r(X),\ y \in \mathcal{C}^s(Y).$$

We define $\mathcal{D}_\sim(X)$ to be the $\mathbb{Q}$-subalgebra of $\mathcal{C}_\sim(X)$ generated by $\mathcal{C}^1_\sim(X)$, i.e., by the divisor classes. The elements of $\mathcal{D}_\sim(X)$ are called the *Lefschetz classes* on X for the relation $\sim$. We list some properties of Lefschetz classes.

1.1. *For any regular map $\phi: X \to Y$, ϕ^* maps Lefschetz classes on Y to Lefschetz classes on X (for any adequate equivalence relation).*

Because ϕ^* is a homomorphism of graded $\mathbb{Q}$-algebras.

1.2. *For any n and any adequate equivalence relation, $\mathcal{D}_\sim(\mathbb{P}^n) = \mathbb{Q}[t]/(t^{n+1})$, where t denotes the class of any hyperplane in $\mathbb{P}^n$, and for any X,*

$$\mathcal{D}_\sim(X \times \mathbb{P}^n) \cong \mathcal{D}_\sim(X) \otimes \mathcal{D}_\sim(\mathbb{P}^n).$$

This follows from the similar statement with $\mathcal{D}$ replaced by $\mathcal{C}$.

Now let $\mathcal{V}(k)$ be the class of algebraic varieties over k whose connected components are products of projective spaces and varieties admitting the structure of an Abelian variety.

1.3. *For any variety X in $\mathcal{V}(k)$, the diagonal $\Delta_X \subset X \times X$ is a Lefschetz class (for any adequate equivalence relation).*

It suffices to prove this for the finest adequate equivalence relation, namely, rational equivalence. For an Abelian variety, there is an explicit expression of Δ_X as a

Lefschetz class in Scholl (1994), 5.9 (see also Milne (1999a), 5.10). To extend the statement to a product Abelian varieties and projective spaces, use 1.2.

Note that (1.1) and (1.3) imply that the graph Γ_ϕ of any regular map $\phi\colon X \to Y$ of varieties in $\mathcal{V}(k)$ is Lefschetz, because $\Gamma_\phi = (\phi \times \mathrm{id}_Y)^*(\Delta_Y)$.

1.4. *For any regular map $\phi\colon X \to Y$ of varieties in $\mathcal{V}(k)$, ϕ_* maps $\mathcal{D}_{\mathrm{num}}(X)$ into $\mathcal{D}_{\mathrm{num}}(Y)$.* See Milne (1999a), 5.5.

Let X and Y be varieties in $\mathcal{V}(k)$, and let $X = \coprod X_i$ be the decomposition of X into its equidimensional components. Then

$$\mathcal{D}_\sim(X \times Y) = \oplus_i \mathcal{D}_\sim(X_i \times Y),$$

and we set

$$\mathrm{LCorr}^m(X, Y) = \oplus_i \mathcal{D}_{\mathrm{num}}^{\dim X_i + m}(X_i \times Y).$$

The map

$$\alpha, \beta \mapsto \beta \circ \alpha =_{\mathrm{df}} p_{XZ*}(p_{XY}^* \alpha \cdot p_{YZ}^* \beta)$$

is a pairing

$$\mathrm{LCorr}^m(X, Y) \times \mathrm{LCorr}^n(Y, Z) \to \mathrm{LCorr}^{m+n}(X, Z).$$

Define $\mathcal{LCV}^0(k)$ to be the category whose objects are symbols hX, one for each $X \in \mathcal{V}(k)$, and whose morphisms are $\mathrm{Hom}(hX, hY) = \mathrm{LCorr}^0(X, Y)$. The transpose of the graph of a regular map $\phi\colon X \to Y$ defines an element $h\phi =_{\mathrm{df}} [{}^t\Gamma_\phi] \in \mathrm{LCorr}^0(Y, X)$, and h is a contravariant functor $\mathcal{V}(k) \to \mathcal{LCV}^0(k)$.

1.5. *For an Abelian variety A of dimension g, there are unique elements $p_i \in \mathrm{LCorr}^0(A, A)$ such that*

(a) $[\Delta_A] = p_0 + \cdots + p_g$
(b) $p_i \circ p_j = 0$ *if* $i \neq j$, *and* $p_i \circ p_i = p_i$;
(c) *for any integer n, $h(n_A) \circ p_i = n^i[\Delta_A] \circ p_i$, where n_A is the endomorphism of A 'multiplication by n'.*

This is proved in Scholl (1994), 5.2.

Now let $X \mapsto H^*(X)$ be a Weil cohomology theory (cf. the Appendix to Milne, 1999a), and write $H^{2*}(X)(*) = \oplus^r H^{2r}(X)(r)$. By assumption, there is given a homomorphism of graded $\mathbb{Q}$-algebras cl: $\mathcal{C}_{\mathrm{rat}}(X) \to \oplus H^{2*}(X)(*)$.

1.6. *For a Lefschetz class x on a variety $X \in \mathcal{V}(k)$, the following are equivalent:*

(a) $\mathrm{cl}(x) \cdot y = 0$ *for all cohomology classes* y;
(b) $x \cdot y = 0$ *for all algebraic classes* y;

(c) $x \cdot y = 0$ *for all Lefschetz classes* y.

Clearly (a) $\Rightarrow$ (b) $\Rightarrow$ (c), but (c) $\Rightarrow$ (a) is proved in Milne (1999a), 5.2.

In particular, $\mathcal{D}_{\text{hom}}$ is independent of the cohomology theory and equals $\mathcal{D}_{\text{num}}$. From now on, I drop the subscript. Thus '$\mathcal{D}(X)$' means '$\mathcal{D}_{\text{num}}(X)$', and 'Lefschetz class on X' means 'element of $\mathcal{D}(X)$'.

THE CATEGORY OF LEFSCHETZ MOTIVES

The category $\mathbf{LMot}(k)$ of Lefschetz motives is defined as follows. An object is a symbol $h(X, e, m)$ where X is a variety in $\mathcal{V}(k)$, e is an idempotent in $\text{LCorr}^0(X, X)$, and $m \in \mathbb{Z}$. If $h(X, e, m)$ and $h(Y, f, n)$ are two motives, then

$$\text{Hom}(h(X, e, m), h(Y, f, n)) = \{f \circ \alpha \circ e | \alpha \in \text{ LCorr}^{n-m}(X, Y)\}.$$

The composite of two morphisms of motives is their composite as correspondences.

Exactly as in the usual case (Scholl (1994), Section 1), one shows that $\mathbf{LMot}(k)$ is a $\mathbb{Q}$-linear pseudo-Abelian rigid tensor category, with

$$h(X, e, m) \oplus h(Y, f, m) = h(X \coprod Y, e \oplus f, m);$$

$$h(X, e, m) \otimes h(Y, f, n) = h(X \times Y, e \otimes f, m + n);$$

$$h(X, e, m)^\vee = h(X, {}^t e, d - m) \text{ if } X \text{ has pure dimension } d.$$

Moreover, $h(\mathbb{P}^n, \text{id}, 0) = 1 \oplus \mathbb{L} \oplus \cdots \oplus \mathbb{L}^{\otimes n}$ where $\mathbb{L} =_{\text{df}}$ (Spec k, id, -1). The proofs of these facts use 1.1–1.5.

Note that $hX \leftrightarrow h(X, \text{id}, 0)$ identifies $\mathcal{LCV}^0(k)$ with a full subcategory of $\mathbf{LMot}(k)$. Moreover, every motive is a direct sum of motives of the form $h(A, e, m)$ with A an Abelian variety.

From 1.5, we find that $\mathbf{LMot}(k)$ has a canonical $\mathbb{Z}$-grading for which $h(A, p_i, m)$ has weight $i - 2m$. This can be used to modify the commutativity constraint (Saavedra (1972), p. 365) to obtain the 'true' category of Lefschetz motives. The method of Jannsen (1992) shows that (1.6) implies that $\mathbf{LMot}(k)$ is a semisimple Abelian category. Finally, Deligne (1990), 7.1, implies that $\mathbf{LMot}(k)$ is Tannakian. In summary:

THEOREM 1.7. *The category* $\mathbf{LMot}(k)$ *is a semisimple* $\mathbb{Q}$*-linear Tannakian category endowed with a canonical* $\mathbb{Z}$*-grading* w *and a canonical (Tate) object* $\mathbb{T} = (\text{Spec}\, k, \text{id}, 1)$.

THE FUNDAMENTAL GROUP OF $\mathbf{LMot}(k)$

We now assume k to be algebraically closed, and we fix a Weil cohomology theory $X \mapsto H^*(X)$ with coefficient field Q. There is a unique fibre functor $\omega_H \colon \mathbf{LMot}(k) \to \mathbf{Vec}_Q$ such that $\omega_H(h(A)) = H^*(A)$ for all Abelian varieties A.

Let $H_1(A)$ be the linear dual of $H^1(A)$, and let $C(A)$ be the centralizer of $\mathrm{End}^0(A)$ in $\mathrm{End}\,(H_1(A))$. A polarization $\lambda\colon A \to A^\vee$ of A determines an involution

$$\alpha \mapsto \alpha^\dagger \overset{\mathrm{df}}{=} H_1(\lambda)^{-1} \circ H_1(\alpha^\vee) \circ H_1(\lambda)$$

of $\mathrm{End}\,(H_1(A))$ whose restriction to $C(A)$ is independent of the choice of λ. The *Lefschetz group* $L(A)$ of A is the algebraic subgroup of $\mathrm{GL}\,(H_1(A)) \times \mathbb{G}_m$ such that

$$L(A)(R) = \{(\gamma, c) \in (C(A) \otimes R)^\times \times R^\times | \gamma^\dagger\gamma = c\}$$

for all Q-algebras R (Milne (1999a), 4.3, 4.4). It is reductive (not necessarily connected), and $(\gamma, c) \mapsto c$ is a homomorphism $l(A)\colon L(A) \to \mathbb{G}_m$ rational over Q.

Let $h_1(A) = h^1(A)^\vee$, and let $\langle A\rangle^\otimes$ be the Tannakian subcategory of $\mathbf{LMot}(k)$ generated by $h_1(A)$ and $\mathbb{T}$. Because $h(A^r) \cong \bigwedge h^1(A^r)$ and $h_1(A^r) \cong h_1(A) \oplus \cdots \oplus h_1(A)$, $\langle A\rangle^\otimes$ contains $h(A^r)$ for all r. Let $\boldsymbol{\pi}(A)$ be the fundamental group of the Tannakian category $\langle A\rangle^\otimes$ (in the sense of Deligne (1990), 8.13).

PROPOSITION 1.8. *For every Abelian variety A, there is a canonical isomorphism $\omega_H(\boldsymbol{\pi}(A)) \cong L(A)$.*

Proof. We know (ibid. 8.13.1) that $\omega_H(\boldsymbol{\pi}(A)) \cong \underline{\mathrm{Aut}}^\otimes(\omega_H|\langle A\rangle^\otimes)$. Therefore, the action of $\boldsymbol{\pi}(A)$ on $h_1(A)$ and $\mathbb{T}$ identifies $\omega_H(\boldsymbol{\pi}(A))$ with the subgroup of $\mathrm{GL}(H_1(A)) \times \mathbb{G}_m$ fixing $\omega_H(\phi)$ for all morphisms ϕ of objects in $\langle A\rangle^\otimes$. On the other hand, $L(A)$ is the largest subgroup $\mathrm{GL}(H_1(A)) \times \mathbb{G}_m$ fixing all Lefschetz classes on A^r for all r (Milne, 1999a, 4.3). These two groups are equal. □

Let $(L, l) = \prod_B (L(B), l(B))$, where B runs over a set of representatives for the simple isogeny classes of Abelian varieties over k.

COROLLARY 1.9. *Let $\boldsymbol{\pi}$ be the fundamental group of* $\mathbf{LMot}(k)$. *Then $\omega_H(\boldsymbol{\pi})$ is canonically isomorphic to L.*

Proof. For any Abelian variety A, $(L(A), l(A)) \cong \prod_B (L(B), l(B))$ where B runs over a set of representatives for the simple isogeny factors of A (Milne, 1999a, 4.7). Therefore the corollary follows from the proposition by passing to the limit over A. □

Remark 1.10. Let A be an Abelian variety over k. For each Weil cohomology theory H we have a Lefschetz group $L(A)_H$, which is an algebraic group over the field of coefficients of H. Proposition 1.8 shows each $L(A)_H$ is a realization of $\boldsymbol{\pi}(A)$, which should therefore be considered as the archetype for all the Lefschetz groups of A. Unfortunately, $\boldsymbol{\pi}(A)$ is only an algebraic group in a Tannakian category and, hence, is a somewhat mysterious object. There are two situations in which $\boldsymbol{\pi}(A)$ can be identified with an algebraic group over $\mathbb{Q}$ in the usual sense. The first is when $k = \mathbb{C}$. Here there is a canonical Weil cohomology theory with coefficients in $\mathbb{Q}$, namely, the Betti cohomology, and so we can identify $\boldsymbol{\pi}(A)$ with the Betti Lefschetz group of A. The second is when A has 'many endomorphisms', which we now explain.

For any k-linear Tannakian category $\mathbf{T}$, the category $\mathbf{Vec}_k$ of finite-dimensional vector spaces over k can be identified with the full subcategory of $\mathbf{T}$ of objects on which $\boldsymbol{\pi}(\mathbf{T})$ acts trivially. If $\boldsymbol{\pi}(\mathbf{T}) = \mathrm{Sp}(R)$ is commutative, then the action of $\boldsymbol{\pi}(\mathbf{T})$ on R is trivial, and so $\boldsymbol{\pi}(\mathbf{T})$ is an affine group scheme in the Tannakian category $\mathbf{Vec}_k \subset \mathbf{T}$, i.e., it is an affine group scheme over k in the usual sense (cf. Milne, 1994, 2.37, p. 428).

A semisimple algebra R of finite degree over $\mathbb{Q}$ is a product of simple algebras, say, $R = R_1 \times \cdots \times R_m$, and the centre K_i of each R_i is a field. The reduced degree $[R\colon \mathbb{Q}]_{\mathrm{red}}$ of R over $\mathbb{Q}$ is $\sum [R_i\colon K_i]^{1/2} \cdot [K_i\colon \mathbb{Q}]$. For an Abelian variety A, $[\mathrm{End}^0(A)\colon \mathbb{Q}]_{\mathrm{red}} \leqslant 2 \dim A$, and when equality holds we say that A has *many endomorphisms*.

Let A be a simple Abelian variety with many endomorphisms, and let $C_0(A)$ be the centre of $\mathrm{End}^0(A)$. A Rosati involution on $\mathrm{End}^0(A)$ defines an involution on $C_0(A)$, which is independent of the choice of the Rosati involution. For any Weil cohomology theory H with coefficient field Q, the canonical map

$$C_0(A) \otimes_{\mathbb{Q}} Q \to C(A)$$

is an isomorphism – this follows easily from the definition of A's having many endomorphisms and the fact that $H_1(A)$ is a free $C_0(A) \otimes_{\mathbb{Q}} Q$-module (Milne, 1999a, 2.1). Therefore, $L(A) \cong L_0(A)_{/Q}$ where $L_0(A)$ is the algebraic group over $\mathbb{Q}$ such that

$$L_0(A)(R) = \{(\gamma, c) \in C_0(A)^\times \times R^\times | \gamma^\dagger \gamma = c\}$$

for all $\mathbb{Q}$-algebras R. This shows that $\boldsymbol{\pi}(A)$ is commutative (because its realizations are) and, hence, can be regarded as an algebraic group in the usual sense; moreover, the action of $L_0(A)$ on $h_1(A)$ identifies $L_0(A)$ with $\boldsymbol{\pi}(A)$.

In Sections 2 and 4, we consider two categories of Lefschetz motives generated by Abelian varieties with many endomorphisms. The remark shows that their fundamental groups are affine group schemes of multiplicative type in the usual sense. In each case, there is a classification of the isogeny classes and a description of the endomorphism algebra of each isogeny class, which allow us to compute the fundamental groups explicitly.

2. Lefschetz Motives of CM-Type

The theory of Abelian varieties of CM-type provides a classification of the simple isogeny classes of such varieties, which allows us to compute the fundamental group of the category of Lefschetz motives generated by Abelian varieties of CM-type.

Throughout this section, C is an algebraically closed field of characteristic zero and ι is an involution of C restricting to complex conjugation on every CM-subfield, and $\mathbb{Q}^{\mathrm{al}}$ is the algebraic closure of $\mathbb{Q}$ in C. Recall that $\mathbb{Q}^{\mathrm{cm}} \subset \mathbb{Q}^{\mathrm{al}}$ and that $\Gamma = \mathrm{Gal}(\mathbb{Q}^{\mathrm{al}}/\mathbb{Q})$.

ABELIAN VARIETIES OF CM-TYPE

Let E be a CM-field. A *CM-type* on E is a locally constant map φ: Hom $(E, \mathbb{Q}^{\mathrm{al}}) \to \mathbb{Z}$ such that $\varphi(\tau) \geqslant 0$ and $\varphi(\tau) + \varphi(\iota \circ \tau) = 1$ for all τ. A CM-type on E is said to be *primitive* if it is not the extension $\tau \mapsto \varphi_0(\tau|E_0)$ of a CM-type φ_0 on a proper subfield E_0. Every CM-type φ on E is the extension of a unique primitive CM-type.

A simple Abelian variety A over C is said to be of *CM-type* if $\mathrm{End}^0(A)$ is a field (necessarily CM) of degree $2 \dim A$ over $\mathbb{Q}$, and an arbitrary Abelian variety over C is said to be of *CM-type* if all its simple isogeny factors are of CM-type.

Let A be a simple Abelian variety over C of CM-type, and let $E = \mathrm{End}^0(A)$. Let i be the inclusion $\mathbb{Q}^{\mathrm{al}} \hookrightarrow C$. For $\tau \in \mathrm{Hom}(E, \mathbb{Q}^{\mathrm{al}})$, define $\varphi(\tau)$ to be 1 or 0 according as $i \circ \tau$ does, or does not, occur in the representation of E on the tangent space to A at 0.

PROPOSITION 2.1. *With the above notations, φ is a primitive CM-type on E, and the map $A \mapsto (E, \varphi)$ defines a bijection from the set of isogeny classes of simple Abelian varieties over C of CM-type to the set of isomorphism classes of pairs (E, φ) consisting of a CM-field of finite degree over $\mathbb{Q}$ and a primitive CM-type on the field.*

Proof. Suppose first that $C = \mathbb{C}$. Let φ be a CM-type on a CM-field E, and let $\Sigma = \{\tau | \varphi(\tau) = 1\}$. Define A_φ to be the Abelian variety over $\mathbb{C}$ such that $A_\varphi(\mathbb{C}) = \mathbb{C}^\Sigma/\Sigma(\mathcal{O}_E)$ where $\mathbb{C}^\Sigma = \mathrm{Hom}(\Sigma, \mathbb{C})$ and $\mathcal{O}_E$, the ring of integers in E, is embedded in $\mathbb{C}^\Sigma$ by $a \mapsto (\sigma a)_{\sigma \in \Sigma}$. Then $(E, \varphi) \mapsto A_\varphi$ provides an inverse to the map $A \mapsto (E, \varphi)$.

To extend the result to fields other than $\mathbb{C}$, use the following observation: let $C \hookrightarrow C'$ be an inclusion of algebraically closed fields of characteristic zero, and let (A, i) be an Abelian variety of CM-type (E, φ) over C'; then any specialization of (A, i) to C is again of CM-type (E, φ), and hence becomes isogenous to (A, i) over C'. $\square$

Let φ be a CM-type on a CM-field E. For each $\tau: E \to \mathbb{Q}^{\mathrm{al}}$ and $\sigma \in \Gamma$, define $\psi_\tau(\sigma) = \varphi(\sigma^{-1} \circ \tau)$. Then ψ_τ depends only on the restriction of σ to $\mathbb{Q}^{\mathrm{cm}}$, and ψ_τ, when regarded as a map Hom $(\mathbb{Q}^{\mathrm{cm}}, \mathbb{Q}^{\mathrm{al}}) \to \mathbb{Z}$, is a CM-type on $\mathbb{Q}^{\mathrm{cm}}$. Moreover, for any $\rho \in \Gamma$, $\psi_{\rho\circ\tau}(\sigma) = \psi_\tau(\rho^{-1} \circ \sigma) = (\rho\psi_\tau)(\sigma)$, and so, as τ runs over the embeddings $E \hookrightarrow \mathbb{Q}^{\mathrm{al}}$, ψ_τ runs over a Γ-orbit of CM-types on $\mathbb{Q}^{\mathrm{cm}}$.

PROPOSITION 2.2. *The map* $(E, \varphi) \mapsto \{\psi_\tau\}$ *defines a bijection from the set of isomorphism classes of pairs* (E, φ) *consisting of a CM-field of finite degree over* $\mathbb{Q}$ *and a primitive CM-type on the field to the set of* Γ*-orbits of CM-types on* $\mathbb{Q}^{\text{cm}}$.

Proof. We construct an inverse. For a CM-type ψ on $\mathbb{Q}^{\text{cm}}$, define Γ_ψ to be the stabilizer of ψ in Γ and E_ψ to be the fixed field of Γ_ψ. Let $\tau_0 \colon E_\psi \hookrightarrow \mathbb{Q}^{\text{al}}$ be the given embedding. Then any embedding $\tau \colon E_\psi \to \mathbb{Q}^{\text{al}}$ can be written $\tau = \sigma \circ \tau_0$ with $\sigma \in \Gamma$, and we define $\varphi_\psi(\tau) = \psi(\sigma^{-1})$. Then φ_ψ is a CM-type on E_ψ, and the map $\psi \mapsto (E_\psi, \varphi_\psi)$ gives the required inverse. □

The *reflex field* K of (E, φ) is defined to be the fixed field of the stabilizer of φ in Γ. Thus $\sigma \in \Gamma$ fixes K if and only if $\varphi(\sigma^{-1} \circ \tau) = \varphi(\tau)$ for all $\tau \colon E \to \mathbb{Q}^{\text{al}}$. For any $\tau \colon E \hookrightarrow C$, ψ_τ is the extension to $\mathbb{Q}^{\text{cm}}$ of a primitive CM-type on K.

The *reflex field* of a simple Abelian variety over C of CM-type is defined to be the reflex field of its associated CM-type.

PROPOSITION 2.3. *Let* K *be a CM-subfield of* C. *There is a natural one-to-one correspondence between the set of isogeny classes of simple Abelian varieties over* C *of CM-type whose reflex field is contained in* K *and the set of* Γ*-orbits of CM-types on* K.

Proof. When $K = \mathbb{Q}^{\text{cm}}$, this is an immediate consequence of the preceding two propositions. The remark following the definition of the reflex field of a CM-type allows one to extend it to an arbitrary CM-subfield of C. □

Remark 2.4. Let E be a CM-subfield of $\mathbb{Q}^{\text{al}}$, and let φ be a CM-type on E. Let K be the reflex field of (E, φ), and let $\psi = \psi_{\tau_0}$ where τ_0 is the given inclusion of E into $\mathbb{Q}^{\text{al}}$. Then $\psi(\sigma | K) = \varphi(\sigma^{-1} | E)$ for any $\sigma \in \Gamma$, and (K, ψ) is the reflex of (E, φ) in the classical sense (Shimura, 1971, p. 126).

THE FUNDAMENTAL GROUP OF THE CATEGORY OF LEFSCHETZ MOTIVES OF CM-TYPE

Fix a CM-field $K \subset \mathbb{Q}^{\text{al}}$ and define $\mathbf{LCM}^K(C)$ to be the Tannakian subcategory of $\mathbf{LMot}(C)$ generated by the motives of simple Abelian varieties over C of CM-type with reflex field contained in K. When $K = \mathbb{Q}^{\text{cm}}$, we omit the superscript. We fix a Weil cohomology theory $X \mapsto H^*(X)$ with coefficient field $\mathbb{Q}$, and write ω_H for the corresponding fibre functor on $\mathbf{LMot}(C)$ or its Tannakian subcategories.

For a Γ-orbit Ψ of CM-types on $\mathbb{Q}^{\text{cm}}$, define T^Ψ to be the torus over $\mathbb{Q}$ with character group

$$X^*(T^\Psi) = \frac{\{f \colon \Psi \to \mathbb{Z}\}}{\{f | f = \iota f \text{ and } \sum_\psi f(\psi) = 0\}}.$$

The element $\psi + \iota\psi$ of $X^*(T^\Psi)$ is independent of the choice of $\psi \in \Psi$ and is fixed by Γ. It therefore defines a homomorphism $t^\Psi \colon T^\Psi \to \mathbb{G}_m$ rational over $\mathbb{Q}$.

Let A^Ψ be a simple Abelian variety corresponding (as in Proposition 2.3) to the orbit Ψ. Although A^Ψ is defined only up to isogeny, its Lefschetz group $L(A^\Psi)$ with respect to $X \mapsto H^*(X)$ is well-defined up to a unique isomorphism.

PROPOSITION 2.5. *For any* Ψ, $(L(A^\Psi), l(A^\Psi)) = (T^\Psi, t^\Psi)$.

Proof. Choose a $\psi \in \Psi$. Let (E_ψ, φ_ψ) be as in the proof of Proposition 2.2, and let A^Ψ be the Abelian variety A_{φ_ψ} defined in the proof of Proposition 2.1. Then $L(A^\Psi)$ is the subtorus of $(\mathbb{G}_m)_{E_\psi/\mathbb{Q}}$ such that

$$L(A^\Psi)(\mathbb{Q}) = \{\alpha \in E_\psi^\times | \alpha \cdot \iota\alpha \in \mathbb{Q}^\times\}$$

and its canonical character $l(A^\Psi)$ sends α to $\alpha \cdot \iota\alpha$. Therefore, $X^*(L(A^\Psi))$ is the quotient of $\mathbb{Z}^{\mathrm{Hom}(E_\psi, \mathbb{Q}^{\mathrm{al}})}$ by the subgroup of f such that $f(\tau) = f(\iota\tau)$ for all $\tau\colon E_\psi \to \mathbb{Q}^{\mathrm{al}}$ and $\sum f(\tau) = 0$, and $l(A^\Psi)$ is represented by $\iota + 1$. By definition, Hom $(E_\psi, \mathbb{Q}^{\mathrm{al}}) = \Gamma/\Gamma_\psi$ where Γ_ψ is the group fixing ψ, and $\sigma \mapsto \sigma\psi$ is a bijection from Γ/Γ_ψ onto Ψ. This map identifies $X^*(L(A^\Psi))$ with $X^*(T^\Psi)$ and $l(A^\Psi)$ with t^Ψ. □

An Abelian variety of CM-type has many endomorphisms in the sense of Remark 1.10, and so the fundamental group of $\mathbf{LCM}^K(C)$ can be identified with an affine group scheme over $\mathbb{Q}$ in the usual sense.

THEOREM 2.6. *For any CM-subfield K of $\mathbb{Q}^{\mathrm{al}}$, the fundamental group (T^K, t^K) of $\mathbf{LCM}^K(C)$ is $\prod_\Psi (T^\Psi, t^\Psi)$, where the product is over the set of Γ-orbits of CM-types on K.*

Proof. For any Abelian variety A over C of CM-type, the fundamental group of $\langle A\rangle^\otimes$ is equal to the Lefschetz group of A, which is $\prod_B (L(B), l(B))$ where B runs over a set of representatives for the simple isogeny factors of A. When $[K\colon\mathbb{Q}] < \infty$, $\mathbf{LCM}^K(C) = \langle \prod_\Psi A^\Psi\rangle^\otimes$ where Ψ runs through the Γ-orbits of CM-types on K, and so

$$(T^K, t^K) = (L(\prod A^\Psi), l(\prod A^\Psi)) = \prod (L(A^\Psi), l(A^\Psi)) = \prod (T^\Psi, t^\Psi).$$

The case when $[K\colon\mathbb{Q}]$ is infinite follows by passing to the limit over the CM-subfields of K finite over $\mathbb{Q}$. □

3. The Functor from Lefschetz Motives of CM-Type to Hodge Motives

Certainly, a Lefschetz class on an Abelian variety over $\mathbb{C}$ is a Hodge class, and so there is a natural functor from the category of Lefschetz motives of CM-type to the category of Hodge motives of CM-type. We shall describe the homomorphism of fundamental groups defined by this functor.

In this section, $\mathbb{Q}^{\mathrm{al}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

HODGE STRUCTURES OF CM-TYPE

Let $\mathbb{S} = (\mathbb{G}_m)_{\mathbb{C}/\mathbb{R}}$. A *rational Hodge structure* is a vector space V over $\mathbb{Q}$ together with a homomorphism $h\colon \mathbb{S} \to \mathrm{GL}(V \otimes \mathbb{R})$ such that the resulting weight gradation is defined over $\mathbb{Q}$. *We always assume our Hodge structures are polarizable.* Let $\mu_h\colon \mathbb{G}_m \to \mathrm{GL}(V \otimes \mathbb{C})$, $\mu_h(z) = h_{\mathbb{C}}(z, 1)$, be the cocharacter associated with h. A Hodge structure (V, h) is said to be of *CM-type* if μ_h factors through $T_{/\mathbb{C}}$ for some subtorus T of $\mathrm{GL}(V)$. In this case the field of definition of μ_h is a finite extension of $\mathbb{Q}$ contained in $\mathbb{Q}^{\mathrm{cm}}$ called the *reflex field* of (V, h).

Let K be a CM-subfield of $\mathbb{Q}^{\mathrm{al}}$. The Hodge structures of CM-type with reflex field contained in K form a $\mathbb{Q}$-linear Tannakian category $\mathbf{Hod}^K_{\mathrm{cm}}$. The forgetful functor $(V, h) \mapsto V$ is a fibre functor for $\mathbf{Hod}^K_{\mathrm{cm}}$ whose automorphism group is the Serre group S^K. This is the (pro-)torus over $\mathbb{Q}$ with character group $X^*(S^K)$ equal to the set of locally constant functions $f\colon \mathrm{Hom}(K, \mathbb{Q}^{\mathrm{al}}) \to \mathbb{Z}$ such that $f(\tau) + f(\iota\tau)$ is independent of τ. Denote the given embedding $K \hookrightarrow \mathbb{Q}^{\mathrm{al}}$ by τ_0, and define μ^K to be the cocharacter $f \mapsto f(\tau_0)\colon X^*(S^K) \to \mathbb{Z}$ of S^K. For any Hodge structure (V, h) of CM-type with reflex field contained in K, there is a unique representation $\rho_h\colon S^K \to \mathrm{GL}(V)$ such that $\rho_{h\mathbb{C}} \circ \mu^K = \mu_h$. The functor $(V, h) \mapsto (V, \rho_h)$ is a tensor equivalence $\mathrm{Hod}^K_{\mathrm{cm}} \to \mathbf{Rep}_{\mathbb{Q}}(S^K)$.

The function $s^K\colon \mathrm{Hom}(K, \mathbb{Q}^{\mathrm{al}}) \to \mathbb{Z}$ sending each element to 1 is a character of S^K rational over $\mathbb{Q}$.

Here (and elsewhere), when $K = \mathbb{Q}^{\mathrm{cm}}$, we drop the superscript.

EXAMPLE 3.1. Let A be a simple Abelian variety over $\mathbb{C}$ of CM-type, and let $E = \mathrm{End}^0(A)$. The Betti homology group $H_1(A)$ is a rational Hodge structure, and its cocharacter μ_A factors through $(\mathbb{G}_m)_{E/\mathbb{Q}} \subset \mathrm{GL}(H_1(A))$. Therefore, $H_1(A)$ is of CM-type. We can regard μ_A as a cocharacter of $(\mathbb{G}_m)_{E/\mathbb{Q}}$ and, hence, as a homomorphism $X^*((\mathbb{G}_m)_{E/\mathbb{Q}}) \to \mathbb{Z}$, in which guise it is the $\mathbb{Z}$-linear extension of the CM-type φ of A. Therefore the reflex field K of the rational Hodge structure $H_1(A)$ is equal to the reflex field of A, and so Hom $(K, \mathbb{Q}^{\mathrm{al}}) = \Gamma/\Gamma_\varphi$ where Γ_φ is the stabilizer of φ in Γ.

The homomorphism ρ_h factors through $(\mathbb{G}_m)_{E/\mathbb{Q}}$, and we shall describe ρ_h: $S^K \to (\mathbb{G}_m)_{E/\mathbb{Q}}$ by giving its action on characters. For $\tau\colon \mathrm{Hom}(E, \mathbb{Q}^{\mathrm{al}}) \to \mathbb{Z}$, let ψ_τ be the homomorphism $\Gamma \to \mathbb{Z}$ defined in Section 2. Then ψ_τ factors through Γ/Γ_φ and lies in $X^*(S^K)$. The map $X^*(\rho_h)$ is $f \mapsto \sum_{\tau\colon E\hookrightarrow\mathbb{Q}^{\mathrm{al}}} f(\tau)\psi_\tau$. The characters of S^K acting on $H_1(A)$ are the ψ_τ.

CM-MOTIVES

We refer the reader to Deligne and Milne, (1982), Section 6, for the definition of the category of Hodge motives over a field of characteristic zero. Fix a CM-subfield K of $\mathbb{Q}^{\mathrm{al}}$, and let $\mathbf{CM}^K(\mathbb{C})$ be the Tannakian subcategory of the category of Hodge motives over $\mathbb{C}$ generated by the motives of Abelian varieties of CM-type with reflex field contained in K.

The Betti cohomology theory $X \mapsto H_B^*(X)$ defines a tensor functor ω_B: $\mathbf{CM}^K(\mathbb{C}) \to \mathrm{Hod}_{\mathrm{cm}}^K$.

THEOREM 3.2. *For any CM-field $K \subset \mathbb{C}$, ω_B defines an equivalence of tensor categories $\mathbf{CM}^K(\mathbb{C}) \to \mathbf{Hod}_{\mathrm{cm}}^K$.*

Proof. As we noted in (3.1), the reflex field of a simple Abelian variety A of CM-type is equal to the reflex field of the Hodge structure $\omega_B(h_1(A)) = H_1(A)$, and so ω_B does map $\mathbf{CM}^K(\mathbb{C})$ into $\mathbf{Hod}_{\mathrm{cm}}^K$. The functor is obviously fully faithful, and so it remains to prove that it is essentially surjective. It suffices to do this when K has finite degree over $\mathbb{Q}$. If A is a simple Abelian variety corresponding to the Γ-orbit Ψ of CM-types on K (as in 2.3), then the representation of S^K on $H_1(A)$ is a multiple of the simple representation of S^K with characters the elements of Ψ (3.1, last sentence), and the next lemma implies that the CM-types on K generate S^K, which completes the proof. □

LEMMA 3.3 *Let K be a CM-field of degree $2g$ over $\mathbb{Q}$, and let $\varphi = \tau_1 + \cdots + \tau_g$ be a CM-type on K. Define CM-types*

$$\varphi_i = \tau_i + \sum_{j \neq i} \iota\tau_j, \qquad \bar{\varphi} = \sum_{j=1}^{g} \iota\tau_j.$$

Then $\{\varphi_1, \ldots, \varphi_g, \bar{\varphi}\}$ is a basis for the $\mathbb{Z}$-module $X^*(S^K)$.

Proof. The elements of $X^*(S^K)$ are of the form $\sum_{i=1}^{g} m_i\tau_i + \sum_{i=1}^{g} n_i\iota\tau_i$ with $m_i + n_i = c$, where c is independent of i. But such an element equals $\sum_{i=1}^{g} m_i\varphi_i + (c - \sum_{i=1}^{g} m_i)\bar{\varphi}$. This shows that $\{\varphi_1, \ldots, \varphi_g, \bar{\varphi}\}$ spans $X^*(S^K)$, and it is obvious that it is linearly independent. □

For any field k, let $\mathbf{Isab}(k)$ be the category of Abelian varieties up to isogeny over k. Its objects are the Abelian varieties over k, and $\mathrm{Mor}(A, B) = \mathrm{Hom}^0(A, B)$.

COROLLARY 3.4. *The functor $A \mapsto H_1(A)$ defines an equivalence from the full subcategory of $\mathbf{Isab}(\mathbb{C})$ whose objects are Abelian varieties of CM-type with reflex field contained in K to the full subcategory of $\mathbf{Rep}(S)$ whose characters are CM-types on K.*

Proof. The two subcategories correspond under the equivalence in the theorem. □

THE FUNCTOR FROM LEFSCHETZ MOTIVES OF CM-TYPE TO HODGE MOTIVES OF CM-TYPE

Fix a CM-field $K \subset \mathbb{C}$. Since a Lefschetz class is a Hodge class, there is a tensor functor $\mathbf{LCM}^K(\mathbb{C}) \to \mathbf{CM}^K(\mathbb{C})$ sending $h(A, e, m)$ to $h(A, e, m)$ (e now regarded as a Hodge class). We describe the homomorphism $S^K \to T^K$ of fundamental groups that it defines.

For any Γ-orbit Ψ of CM-types on K, the map $f \mapsto \sum_{\psi\in\Psi} f(\psi)\psi\colon \mathbb{Z}^\Psi \to X^*(S^K)$ factors through $X^*(T^\Psi)$ and, hence, defines a homomorphism $\gamma^\Psi\colon S^K \to T^\Psi$. Its composite with t^Ψ is s^K, and so the t^Ψ define a homomorphism $\gamma^K\colon (S^K, s^K) \to \prod_\Psi (T^\Psi, t^\Psi)$.

PROPOSITION 3.5. *The homomorphism* $(S^K, s^K) \to (T^K, t^K)$ *defined by the tensor functor* **LCM**$(\mathbb{C}) \to$ **CM**$(\mathbb{C})$ *is* γ^K.

Proof. As we noted above, if A is a simple Abelian variety corresponding to the Γ-orbit Ψ of CM-types on K, then the representation of S^K on $H_1(A)$ has the elements of Ψ as its characters. The shows that the homomorphism $(S^K, s^K) \to (T^\Psi, t^\Psi)$ defined by the tensor functor $\langle A\rangle^\otimes \to$ **CM**$^K(\mathbb{C})$ is γ^Ψ. Therefore, the two homomorphisms $(S^K, s^K) \to (T^K, t^K)$ agree when composed with the projections $(T^K, t^K) \to (T^\Psi, t^\Psi)$, which implies that they are equal. $\square$

Remark 3.6. The homomorphism $\gamma^K\colon S^K \to T^K$ is injective. Indeed, its kernel is killed by every CM-type on K, but these generate $X^*(S^K)$.

Remark 3.7. The observation in the proof of Proposition 2.1 allows one to extend the results of this section from $\mathbb{C}$ to any algebraically closed field of characteristic zero.

4. Lefschetz Motives over $\mathbb{F}$

The theorems of Honda and Tate classify the isogeny classes of simple Abelian varieties over the algebraic closure $\mathbb{F}$ of a finite field, and the theorem of Tate shows that every Abelian variety over $\mathbb{F}$ has many endomorphisms and allows us to compute the Lefschetz group of each isogeny class. Thus, we are able to compute the fundamental group of the category of Lefschetz motives over $\mathbb{F}$.

In this section, $\mathbb{Q}^{\mathrm{al}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

WEIL NUMBERS AND ABELIAN VARIETIES

Let p be a prime number. An element π of a field algebraic over $\mathbb{Q}$ is said to be a *Weil* p^n*-number of weight* $-m$ if

(a) for all embeddings $\rho\colon \mathbb{Q}[\pi] \hookrightarrow \mathbb{C}$, $\rho(\pi)\cdot\iota\rho(\pi) = (p^n)^m$;
(b) for some N, $p^N\pi$ is an algebraic integer.

Condition (a) implies that $\pi \mapsto p^{nm}/\pi$ defines an involution (possibly trivial) ι' of $\mathbb{Q}[\pi]$ such that $\rho\circ\iota' = \iota\circ\rho$ for all embeddings $\rho\colon \mathbb{Q}[\pi] \hookrightarrow \mathbb{C}$. Therefore $\mathbb{Q}[\pi]$ is either a CM-field or is totally real.

Let $W(p^n)$ be the group of Weil p^n-numbers in $\mathbb{Q}^{\mathrm{al}}$. If $n|n'$, then $\pi \mapsto \pi^{\frac{n'}{n}}$ maps $W(p^n)$ into $W(p^{n'})$, and we define $W(p^\infty) = \varinjlim_n W(p^n)$. Thus an element of $W(p^\infty)$ is represented by an element of $W(p^n)$ for some n, and elements $\pi \in W(p^n)$ and $\pi' \in W(p^{n'})$ represent the same element of $W(p^\infty)$ if and only $\pi^{n'}$ and π'^n differ by a root of unity. We let $[\pi]$ denote the element of $W(p^\infty)$ represented by π.

There is a natural action of Γ on $W(p^\infty)$, and the *Weil-number torus* P is defined to be the pro-torus over $\mathbb{Q}$ with $X^*(P) = W(p^\infty)$.

Let $W_{1,+}(p^n)$ be the subset of $W(p^n)$ consisting of those π that are of weight -1 and are algebraic integers, and let $W_{1,+}(p^\infty) = \varinjlim W_{1,+}(p^n)$.

Let A be a simple Abelian variety over $\mathbb{F}$. Choose a model A_0 of A over a finite field $\mathbb{F}_{p^n}$ such that all endomorphisms of A are rational over $\mathbb{F}_{p^n}$, and let π be the Frobenius endomorphism of $A_0/\mathbb{F}_{p^n}$. According to Tate 1966, Theorem 2, $\mathbb{Q}[\pi]$ is the centre $Z(A)$ of $\mathrm{End}^0(A)$. For any embedding $\rho\colon Z(A) \hookrightarrow \mathbb{Q}^{\mathrm{al}}$, $\rho(\pi) \in W^K_{1,+}(p^n)$. The class $[\rho(\pi)]$ of $\rho(\pi)$ in $W^K_{1,+}(p^\infty)$ is independent of the choice of the model A_0, and as ρ runs over the embeddings of $Z(A)$ into $\mathbb{Q}^{\mathrm{al}}$, $[\rho(\pi)]$ runs over a Γ-orbit in $W^K_{1,+}(p^\infty)$.

PROPOSITION 4.1. *The map* $A \mapsto \{[\rho(\pi)] | \rho \in \mathrm{Hom}(Z(A), \mathbb{Q}^{\mathrm{al}})\}$ *defines a bijection from the set of isogeny classes of simple Abelian varieties over* $\mathbb{F}$ *to* $\Gamma\backslash W_{1,+}(p^\infty)$.

Proof. The injectivity follows from Tate (1966), Theorem 1, and the surjectivity from Honda (1968). □

THE FUNDAMENTAL GROUP OF **LMot**$(\mathbb{F})$

For a Γ-orbit Π in $W^K_{1,+}(p^\infty)$, define L^Π to be the torus over $\mathbb{Q}$ with character group

$$X^*(L^\Pi) = \frac{\{f\colon \Pi \to \mathbb{Z}\}}{\{f | f = \iota f \text{ and } \sum_{\pi\in\Pi} f(\pi) = 0\}}.$$

The element $\pi + \iota\pi$ of $X^*(L^\Pi)$ is independent of the choice of $\pi \in \Pi$ and is fixed by Γ. It therefore defines a homomorphism $l^\Pi\colon L^\Pi \to \mathbb{G}_m$ rational over $\mathbb{Q}$. Let A^Π be a simple Abelian variety over $\mathbb{F}$ corresponding (as in Proposition 4.1) to the orbit Π. Although A^Π is defined only up to isogeny, its Lefschetz group $L(A^\Pi)$ is well-defined up to a unique isomorphism.

LEMMA 4.2. *For any* Γ*-orbit* Π *of Weil numbers of weight* -1, $(L(A^\Pi), l(A^\Pi)) = (L^\Pi, l^\Pi)$.

Proof. The Lefschetz group $L(A^\Pi)$ of A^Π is the subtorus of $(\mathbb{G}_m)_{Z(A^\Pi)/\mathbb{Q}}$ such that $L(A^\Pi)(\mathbb{Q}) = \{\alpha \in Z(A^\Pi) | \alpha\cdot\iota\alpha \in \mathbb{Q}^\times\}$, and $l(A^\Pi)$ sends α to $\alpha\cdot\iota\alpha$ (cf. 1.10).

Choose a model for A^Π over a finite field whose Frobenius endomorphism π_0 generates $Z(A^\Pi)$ as a $\mathbb{Q}$-algebra. Then the bijection $\rho \mapsto [\rho(\pi_0)]\colon \mathrm{Hom}(Z(A^\Pi), \mathbb{Q}^{\mathrm{al}}) \to \Pi$ induces an isomorphism $X^*(L^\Pi) \to X^*(L(A^\Pi))$, which is independent of the choice of the model and maps l^Π to $l(A^\Pi)$. □

Because the Lefschetz group of an Abelian variety over $\mathbb{F}$ is commutative, so also is the fundamental group of **LMot**($\mathbb{F}$), which therefore may be identified with an affine group scheme over $\mathbb{Q}$ in the usual sense.

THEOREM 4.3. *The fundamental group of* **LMot**($\mathbb{F}$) *is* $\prod_{\Pi\in\Gamma\backslash W_{1,+}(p^\infty)}(L^\Pi, l^\Pi)$.

Proof. Combine Lemma 4.2 with Proposition 4.1 and Proposition 1.8. □

THE FUNDAMENTAL GROUP OF **LMot**$^K(\mathbb{F})$

For a Weil p^n-number π in a field K finite over $\mathbb{Q}$ and a prime w of K lying over p, define

$$f_\pi(w) = \frac{\mathrm{ord}_w(\pi)}{\mathrm{ord}_w(p^n)}[K_w\colon\mathbb{Q}_p].$$

Now let K be a CM-subfield of $\mathbb{C}$, finite and Galois over $\mathbb{Q}$. Define $W^K(p^n)$ to be the set of Weil p^n-numbers π in K such that $f_\pi(w) \in \mathbb{Z}$ for all $w|p$, and set $W^K(p^\infty) = \varinjlim W^K(p^n)$. It is a Γ-submodule of $W(p^\infty)$, and we define P^K to be the corresponding quotient of P. Let Y be the set of primes of K lying over p. The number $f_\pi(w)$ depends only on the class $[\pi]$ of π in $W^K(p^\infty)$, and so $[\pi] \mapsto f_\pi$ is a homomorphism from $W^K(p^\infty)$ to the set of functions $f\colon Y \to \mathbb{Z}$. This homomorphism is obviously injective, and the functions in the image have the property that $f(w) + f(\iota w)$ is an integer independent of $w \in Y$ and divisible by $[K_w\colon\mathbb{Q}_p]$. Later (5.1) we shall see that every f with this property is in the image.

Define $W^K_{1,+}(p^n)$ and $W^K_{1,+}(p^\infty)$ similarly. Then $[\pi] \mapsto f_\pi$ defines an isomorphism

$$W^K_{1,+}(p^\infty) \xrightarrow{\approx} \{f\colon Y \to \mathbb{Z} | f(w) + f(\iota w) = [K_w\colon\mathbb{Q}_p],\ f(w) \geqslant 0\}.$$

Let A be a simple Abelian variety over $\mathbb{F}$. According to Tate (1968/69), Théorème 1, the invariant of $\mathrm{End}^0(A)$ at a prime v of its centre $Z(A)$ is given by

$$\mathrm{inv}_v(\mathrm{End}^0(A)) = \frac{\mathrm{ord}_v(\pi)}{\mathrm{ord}_v(p^n)}[Z(A)_v\colon\mathbb{Q}_p] \quad (= f_\pi(v))$$

where $\pi \in Z(A)$ is the Frobenius endomorphism of a model $A_0/\mathbb{F}_{p^n}$ of A with the property that $\mathrm{End}^0(A_0) = \mathrm{End}(A)$. Therefore, for any embedding $\rho\colon Z(A) \hookrightarrow K$ and $w \in Y$,

$$\mathrm{inv}_w(\mathrm{End}^0(A) \otimes_{Z(A),\rho} K) = f_{\rho(\pi)}(w).$$

Consequently, $f_{\rho(\pi)}(w)$ is an integer for all $w|p$ if and only if K splits End$^0(A)$. Therefore, under the bijection in Proposition 4.1, $\Gamma\backslash W^K_{1,+}(p^\infty)$ corresponds to the set of isogeny classes of A's having the following property:

($*$) for all $\rho\colon Z(A) \hookrightarrow \mathbb{Q}^{\mathrm{al}}$, $\rho(Z(A)) \subset K$ and End$^0(A) \otimes_{Z(A),\rho} K$ is a matrix algebra over K.

For a fixed CM-field $K \subset \mathbb{Q}^{\mathrm{al}}$ of finite degree and Galois over $\mathbb{Q}$, let $\mathbf{LMot}^K(\mathbb{F})$ be the full subcategory of $\mathbf{LMot}(\mathbb{F})$ whose objects are direct sums of motives of the form $h(A, p, m)$ with A satisfying the condition ($*$). It is a Tannakian subcategory of $\mathbf{LMot}(\mathbb{F})$, whose fundamental group is $\prod_{\Pi\in\Gamma\backslash W^K_{1,+}(p^\infty)}(L^\Pi, t^\Pi)$.

THE MAP $\beta^K\colon P^K \to L^K$

The element $p \in K$ is a Weil p-number of weight -2. Its class $[p]$ in $W^K(p^\infty)$ is fixed under the action of Γ, and so defines homomorphism $p^K\colon P^K \to \mathbb{G}_m$ rational over $\mathbb{Q}$.

Let Π be a Γ-orbit in $W^K_{1,+}(p^\infty)$. The map $f \mapsto \prod_{\pi\in\Pi} \pi^{f(\pi)}\colon \mathbb{Z}^\Pi \to W^K(p^\infty)$ factors through $X^*(L^\Pi)$ and, hence, defines a homomorphism $\beta^\Pi\colon P^K \to L^\Pi$. This map sends p^K to l^Π and, hence, the family $(\beta^\Pi)_{\Pi\in W_{1,+}(p^\infty)}$ defines a homomorphism $\beta^K\colon (P^K, p^K) \to (L^K, l^K)$, which is injective because it corresponds to a surjective map on the character groups. On passing to the inverse limit over all $K \subset \mathbb{Q}^{\mathrm{cm}}$ finite and Galois over $\mathbb{Q}$, we obtain an injective homomorphism $\beta\colon (P, p) \to (L, l)$.

5. The Reduction Functor on Lefschetz Motives of CM-Type

Because an Abelian variety of CM-type has potential good reduction, for each prime w_0 of $\mathbb{Q}^{\mathrm{al}}$ there is a 'reduction' functor from the category of Lefschetz motives of CM-type over $\mathbb{Q}^{\mathrm{al}}$ to the category of Lefschetz motives over the algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$. Using the theorem of Shimura and Taniyama, we shall compute the map of the fundamental groups it defines.

Throughout this section $\mathbb{Q}^{\mathrm{al}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. We fix a prime w_0 of $\mathbb{Q}^{\mathrm{al}}$ lying over p, and denote its residue field by $\mathbb{F}$.

THE MAP $P \to S$

We review the construction of the map $P \to S$ that is conjecturally associated with the reduction of motives of CM-type.

Fix a CM-subfield K of $\mathbb{Q}^{\mathrm{al}}$ of finite degree and Galois over $\mathbb{Q}$ and a prime w_0 of K lying over p. Recall that $X^*(S^K)$ consists of the homomorphisms $g\colon \mathrm{Hom}(K, \mathbb{Q}^{\mathrm{al}}) \to \mathbb{Z}$ such that $g + \iota g$ is constant, and that the weight of g is $-g - \iota g$.

For $g \in X^*(S^K)$ and $a \in K$, define $g(a) = \prod_{\tau\colon K\to\mathbb{Q}^{\mathrm{al}}}(\tau g)^{g(\tau)} \in \mathbb{Q}^{\mathrm{al}}$. Then $g(a) \cdot \iota g(a) = \mathrm{Nm}_{K/\mathbb{Q}} a^{-\mathrm{wt}(g)}$. If a lies in the real subfield F of K, then $g(a) =$

$\mathrm{Nm}_{F/\mathbb{Q}}(a)^{-\mathrm{wt}(g)}$. Because the group of units of F has finite index in the group of units of K, this shows that g maps units in K to roots of unity.

Let ϖ generate the ideal $\mathfrak{P}_{w_0}^h$, where h is the order of the prime ideal $\mathfrak{P}_{w_0}$ corresponding to w_0 in the class group of K. According to the above remarks, $g(\varpi)$ is independent of the choice of ϖ up to a root of unity, and it is a Weil $p^{f(\frac{w_0}{p})h}$-number of weight $\mathrm{wt}(g)$. Moreover, for any prime w of K lying over p, $\mathrm{ord}_w(g(\varpi)) = h\sum_{\tau,\tau w_0=w} g(\tau)$. Therefore, with the notation of Section 4,

$$f_{g(\varpi)}(w) = \sum_{\tau w_0=w} g(\tau) \in \mathbb{Z}, \tag{5.1}$$

and so $g(\varpi) \in W^K(p^{f(w_0/p)})$. The class it represents in $W^K(p^\infty)$ is independent of the choice of ϖ, and so we have a homomorphism $g \mapsto [g(\varpi)]: X^*(S^K) \to W^K(p^\infty)$. We sometimes denote this map as $g \mapsto \pi(g)$. It commutes with the action of Γ, and so defines a homomorphism $\alpha^K: P^K \to S^K$.

LEMMA 5.1. *The maps*

$$X^*(S^K) \xrightarrow{g\mapsto[g(\varpi)]} W^K(p^\infty) \xrightarrow{[\pi]\mapsto f_\pi} \{f: Y \to \mathbb{Z} | f + \iota f \in [K_{w_0}: \mathbb{Q}_p]\mathbb{Z}\}$$

are surjective.

Proof. We know (Section 4) that the second map is injective, and so it suffices to prove that the composite map is surjective. But it sends $g \in X^*(S^K)$ to the map $f: Y \to \mathbb{Z}$ such that $f(w) = \sum_{\tau w=w_0} g(\tau)$. Choose a section s to the map $\tau \mapsto \tau w_0: \mathrm{Gal}(K/\mathbb{Q}) \to Y$ such that $s(\iota w) = \iota s(w)$, and define g so that $g(\tau)$ is $f(\tau w_0)$ or 0 according as τ is in the image of s or not. Then $g \mapsto f$. □

Remark 5.2. (a) The lemma shows that the homomorphism $\alpha^K: P^K \to S^K$ is injective. On passing to the limit over all K, we obtain an injective homomorphism $\alpha: P \to S$.

(b) The lemma shows that

$$[\pi] \mapsto f_\pi: X^*(P^K) \to \{f: Y \to \mathbb{Z} | f + \iota f \in [K_{w_0}: \mathbb{Q}_p]\mathbb{Z}\}$$

is an isomorphism.

(c) The homomorphism $\alpha^K: P^K \to S^K$ sends p^K to s^K.

THE REDUCTION OF ABELIAN VARIETIES

Let A be an Abelian variety over $\mathbb{Q}^{\mathrm{al}}$ of CM-type, and let A' be a model of A over a subfield L of $\mathbb{Q}^{\mathrm{al}}$ finite over $\mathbb{Q}$. After possibly replacing L by a larger field, A' will have good reduction at w_0 (Serre and Tate, 1968, Theorem 6). Let A'_0 be the reduction of A'. Then $A_0 =_{\mathrm{df}} A'_0 \times_{\mathrm{Spec}\,k(w_0)} \mathbb{F}$ is independent of all choices (up to a well-defined isomorphism) and $A \mapsto A_0$ is a functor $\mathbf{Isab}^{\mathrm{cm}}(\mathbb{Q}^{\mathrm{al}}) \to \mathbf{Isab}(\mathbb{F})$.

Now assume A to be simple, and let $E = \mathrm{End}^0(A)$. It is a CM-field, and the action of E on $\mathrm{Tgt}_0(A)$ defines a CM-type $\varphi\colon \mathrm{Hom}(E, \mathbb{Q}^{\mathrm{al}}) \to \mathbb{Z}$.

The centre $Z(A_0)$ of $\mathrm{End}^0(A_0)$ is a subfield of E. Let $\pi \in E$ be the Frobenius endomorphism of some model of A_0 over a finite subfield, say $\mathbb{F}_{p^n}$, of $\mathbb{F}$. Any two such π's represent the same class in $W(p^\infty)$. For any $\rho\colon E \hookrightarrow \mathbb{Q}^{\mathrm{al}}$, let $\rho^{-1}w_0$ be the valuation on E such that $|c|_{\rho^{-1}w_0} = |\rho c|_{w_0}$. According to the Theorem of Shimura and Taniyama (Tate, 1968/69, Lemme 5), for any prime $w|p$ of E,

$$f_\pi(w) \overset{\mathrm{df}}{=} \frac{\mathrm{ord}_w(\pi)}{\mathrm{ord}_w(p^n)}[E_w\colon \mathbb{Q}_p] = \sum_{\rho, \rho^{-1}w_0 = w} \varphi(\rho).$$

Let K be a CM-subfield of $\mathbb{Q}^{\mathrm{al}}$, finite and Galois over $\mathbb{Q}$, and large enough to contain all conjugates of E (and hence also the reflex field of (E, φ)). The choice of an embedding $\rho_0\colon E \hookrightarrow \mathbb{Q}^{\mathrm{al}}$ determines a Weil q-integer $\rho(\pi)$ of weight -1 in K and a CM-type ψ_{ρ_0} on K (see Section 2). From the inclusion $K \subset \mathbb{Q}^{\mathrm{al}}$, K acquires a valuation $w_0|p$, and we choose a $\varpi \in K$ such that $(\varpi) = \mathfrak{P}_{w_0}^h$. Then $\psi_{\rho_0}(\varpi) \overset{\mathrm{df}}{=} \prod_{\tau\colon K \to \mathbb{Q}^{\mathrm{al}}} (\tau\varpi)^{\psi_{\rho_0}(\tau)}$ is a Weil $p^{f(\frac{w_0}{p})h}$-integer of weight -1 in K.

PROPOSITION 5.3. *The Weil numbers $\rho_0(\pi)$ and $\psi_{\rho_0}(\varpi)$ represent the same element of $W^K(p^\infty)$.*

Proof. Because of the injectivity of the map $[\pi] \mapsto f_\pi$, it suffices to show that $f_{\rho_0(\pi)} = f_{\psi_{\rho_0}(\varpi)}$. From the Theorem of Shimura and Taniyama, we find that

$$f_{\rho_0(\pi)}(w) = [K_w\colon (\rho_0 E)_v] f_{\rho_0(\pi)}(v) \overset{\mathrm{S-T}}{=} [K_w\colon (\rho_0 E)_v] \sum_{\sigma^{-1}w_0 = v} \varphi(\sigma \circ \rho_0)$$

where v is the restriction of w to ρE and the sum is over the embeddings $\sigma\colon \rho E \to K$.

On the other hand, we know (5.1) that $f_{\psi_{\rho_0}(\varpi)} = \sum_{\tau w_0 = w} \psi_{\rho_0}(\tau)$ where τ runs over the elements of Gal $(K/\mathbb{Q})$. As $\psi_{\rho_0}(\tau) =_{\mathrm{df}} \varphi(\tau^{-1} \circ \rho_0)$, the two sums are equal. □

THE REDUCTION FUNCTOR

The functor $A \mapsto A_0$ extends to a functor

$$R\colon \mathbf{LCM}(\mathbb{Q}^{\mathrm{al}}) \to \mathbf{LMot}(\mathbb{F}), \quad h(A, e, m) \mapsto h(A_0, e_0, m).$$

Above, we defined a surjective homomorphism $f \mapsto \pi(f)\colon X^*(S^K) \to W^K(p^\infty)$ which sends CM-types on K to Weil integers of weight -1. Since the map is Γ-equivariant, to each Γ-orbit Ψ of CM-types it attaches a Γ-orbit $\Pi(\Psi)$ of Weil integers of weight -1 and a surjective Γ-equivariant homomorphism $\Psi \to \Pi(\Psi)$. This last map induces a surjective homomorphism

$$\sum_{\psi\in\Psi} f(\psi)\psi \mapsto \sum_{\psi\in\Psi} f(\psi)\pi(\psi) \colon X^*(T^\Psi) \to X^*(L^\Pi)$$

sending t^Ψ to l^Π and, hence, an injective homomorphism

$$\alpha'^\Psi \colon (L^{\Pi(\Psi)}, l^{\Pi(\Psi)}) \to (T^\Psi, t^\Psi).$$

On combining these maps for all Ψ, we obtain a injective homomorphism $\alpha'^K \colon (L^K, l^K) \to (T^K, t^K)$.

THEOREM 5.4. *The homomorphism $(L^K, l^K) \to (T^K, t^K)$ of fundamental groups defined by the reduction functor* $\mathbf{LCM}^K(\mathbb{Q}^{\mathrm{al}}) \to \mathbf{LMot}^K(\mathbb{F})$ *is α'.*

Proof. It suffices to check this on $\langle A\rangle^\otimes$ for A a simple Abelian variety of CM-type, but here it follows from Proposition 5.3.

6. The Serre and Lefschetz Groups Intersect in the Weil-Number Torus

In this section, $\mathbb{Q}^{\mathrm{al}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ and w_0 is a fixed prime of $\mathbb{Q}^{\mathrm{al}}$ lying over p.

Recall that we have defined affine group schemes of multiplicative type:

T :	$\mathbf{LCM}(\mathbb{Q}^{\mathrm{al}})$:	(Abelian varieties of CM-type over $\mathbb{Q}^{\mathrm{al}}$; Lefschetz classes)
S :	$\mathbf{CM}(\mathbb{Q}^{\mathrm{al}})$:	(Abelian varieties of CM-type over $\mathbb{Q}^{\mathrm{al}}$; Hodge classes)
L :	$\mathbf{LMot}(\mathbb{F})$:	(Abelian varieties over $\mathbb{F}$; Lefschetz classes)
P :	?$\mathbf{Mot}(\mathbb{F})$?	(Abelian varieties over $\mathbb{F}$; algebraic classes).

Each of T, S, L has been shown to be the fundamental group of the Tannakian category to its right, and it is conjectured that the same is true of P. We have defined injective homomorphisms as in the left-hand square and have shown that α' and γ correspond to the natural functors in the right hand square (conjecturally, the same is true of α and β):

$$\begin{array}{ccc} T & \xleftarrow{\gamma} & S \\ \uparrow{\scriptstyle\alpha'} & & \uparrow{\scriptstyle\alpha} \\ L & \xleftarrow{\beta} & P \end{array} \qquad \begin{array}{ccc} \mathbf{LCM}(\mathbb{Q}^{\mathrm{al}}) & \longrightarrow & \mathbf{CM}(\mathbb{Q}^{\mathrm{al}}) \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle R} \\ \mathbf{LMot}(\mathbb{F}) & \longrightarrow & ?\mathbf{Mot}(\mathbb{F})? \end{array}$$

This section is devoted to proving the following result.

THEOREM 6.1. *The diagram at left commutes, and identifies P with $L \cap S$ (intersection in T).*

Start of the proof of Theorem 6.1. Fix a CM-field $K \subset \mathbb{Q}^{\mathrm{al}}$ finite and Galois over $\mathbb{Q}$.

LEMMA 6.2. *The diagram*

$$\begin{tikzcd} T^K & S^K \arrow[l, "\gamma^K"'] \\ L^K \arrow[u, "\alpha'^K"] & P^K \arrow[l, "\beta^K"] \arrow[u, "\alpha^K"'] \end{tikzcd}$$

commutes.

Proof. We check this on the character groups. Let Ψ be a Γ-orbit of CM-types on K, and let $f \in \mathbb{Z}^{\Psi}$. Then f represents an element of $X^*(T^K)$, and its image in $X^*(P^K) =_{\mathrm{df}} W^K(p^{\infty})$ under either map in the diagram is $\prod_{\psi \in \Psi} \pi(\psi)^{f(\psi)}$. □

On passing to the limit over all $K \subset \mathbb{Q}^{\mathrm{cm}}$, we find that the diagram referred to in Theorem 6.1 commutes. To complete the proof of Theorem 6.1 we shall show that $P^K = S^K \cap L^K$ (inside T^K), or, equivalently, that

$$P^K \xrightarrow{\binom{\beta^K}{-\alpha^K}} L^K \times S^K \xrightarrow{(\alpha'^K\ \gamma^K)} T^K$$

is exact, for all sufficiently large $K \subset \mathbb{Q}^{\mathrm{cm}}$.

ALMOST CARTESIAN SQUARES

We say that a commutative square of Abelian groups

$$\begin{tikzcd} N' \arrow[r, "\gamma"] \arrow[d, "\alpha'"'] & N \arrow[d, "\alpha"] \\ M' \arrow[r, "\beta"] & M \end{tikzcd} \tag{6.1}$$

is *almost Cartesian* if all the maps are surjective and the map $N' \xrightarrow{\binom{\alpha'}{\gamma}} M' \times_M N$ is surjective, i.e., if

$$N' \xrightarrow{\binom{\alpha'}{\gamma}} M' \oplus N \xrightarrow{(\beta - \alpha)} M$$

is exact.

LEMMA 6.3. *For a square* (6.1) *in which all the maps are surjective, the following conditions are equivalent:*

(a) *the square is almost Cartesian;*
(b) *the map* $\operatorname{Ker}\gamma \to \operatorname{Ker}\beta$ *induced by* α' *is surjective;*
(c) *the map* $\operatorname{Ker}\alpha' \to \operatorname{Ker}\alpha$ *induced by* γ *is surjective.*

Proof. Assume (a). If $\beta(m') = 0$, then the pair $(m', 0)$ maps to 0 in M and, therefore, is the image of an $n' \in N'$, i.e., m' is the image of an element $n' \in \operatorname{Ker}(\gamma)$. Hence (b) holds.

Assume (b). Suppose $\beta(m') = \alpha(n)$. Choose n' such that $\gamma(n') = n$. Then $\alpha'(n') - m' \in \operatorname{Ker}(\beta)$, and so there exists an $x \in \operatorname{Ker}(\gamma)$ such that $\alpha'(x) = \alpha'(n') - m'$. Now

$$\alpha'(n' - x) = m', \qquad \gamma(n' - x) = n.$$

Hence (a) holds.

This proves the equivalence of (a) and (b), and the equivalence of (a) and (c) is proved symmetrically. □

LEMMA 6.4. (a) *Suppose the square* (6.1) *is almost Cartesian, and let* $N'' \subset \operatorname{Ker}(\gamma)$ *and* $M'' \subset \operatorname{Ker}(\beta)$ *be such that* $\alpha'(N'') \subset M''$. *Then*

$$\begin{tikzcd} N'/N'' \arrow[r, "\bar\gamma"] \arrow[d, "\bar\alpha'"'] & N \arrow[d, "\alpha"] \\ M'/M'' \arrow[r, "\bar\beta"] & M \end{tikzcd}$$

is almost Cartesian.

(b) *If both inner squares in the diagram*

$$\begin{tikzcd} N'' \arrow[r, "\gamma'"] \arrow[d, "\alpha''"'] & N' \arrow[r, "\gamma"] \arrow[d, "\alpha'"] & N \arrow[d, "\alpha"] \\ M'' \arrow[r, "\beta'"] & M' \arrow[r, "\beta"] & M \end{tikzcd}$$

are almost Cartesian, then so also is the outer square.

Proof. (a) The composite $\operatorname{Ker}(\alpha') \to \operatorname{Ker}(\bar\alpha') \to \operatorname{Ker}(\alpha)$ is surjective, and so therefore is $\operatorname{Ker}(\bar\alpha') \to \operatorname{Ker}(\alpha)$.

(b) Both maps $\operatorname{Ker}(\alpha'') \to \operatorname{Ker}(\alpha') \to \operatorname{Ker}(\alpha)$ are surjective, and so therefore is their composite. □

SOME LINEAR ALGEBRA

For $n \geqslant 1$ and $d \in \mathbb{Z}$, let $A(n, d)$ be the $2n \times 2n$ matrix

$$\begin{pmatrix} I_n & dE_n - I_n \\ dE_n - I_n & I_n \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix and E_n is the $n \times n$ matrix with all entries equal to 1.

PROPOSITION 6.5. *The matrix* $A(n,d)$ *is row equivalent over* $\mathbb{Z}$ *to*

$$\begin{pmatrix} I_n & dE_n - I_n \\ 0 & B \end{pmatrix}, \qquad B = \begin{pmatrix} 2d-nd^2 & 2d-nd^2 & \cdots & 2d-nd^2 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Proof. After a set of row operations to reduce the block at lower-left to zero, $A(n,d)$ becomes

$$\begin{pmatrix} I_n & dE_n - I_n \\ 0 & (2d-nd^2)E_n \end{pmatrix},$$

which is obviously row-equivalent to the desired matrix. □

COROLLARY 6.6. *Assume* $d(2-nd) \neq 0$. *Then the kernel of the map* $\mathbf{x} \mapsto A(n,d)\mathbf{x}\colon \mathbb{Z}^{2n} \to \mathbb{Z}^{2n}$ *is the set of vectors of the form*

$$(a_1,\dots,a_n,b_1,\dots,b_n), \quad a_i = b_i \text{ for } 1 \leqslant i \leqslant n, \quad \sum a_i = 0.$$

Proof. For an element $(x_1,\dots,x_{2n})$ of the kernel, we may assign arbitrary values, say $b_2,\dots,b_n$ to $x_{n+2},\dots,x_{2n}$. Then the $(n+1)^{\text{st}}$ equation becomes $x_{n+1} + \sum_{i=2}^n b_i = 0$, and so b_1 is determined by the equation $\sum_{i=1}^n b_i = 0$. Now the first n equations show that

$$x_i = -\sum_{j=n+1}^{2n} dx_{j+n} + x_{i+n} = b_i, \quad 1 \leqslant i \leqslant n.$$

This proves the statement. □

Completion of the proof of the Theorem 6.1. It suffices to prove that

$$\begin{array}{ccc} X^*(T^K) & \longrightarrow & X^*(S^K) \\ \downarrow & & \downarrow \\ X^*(L^K) & \longrightarrow & X^*(P^K) \end{array}$$

is almost Cartesian for all sufficiently large CM-fields $K \subset \mathbb{Q}^{\mathrm{al}}$ of finite degree over $\mathbb{Q}$. We shall in fact prove it under the assumption that K

– is finite and Galois over $\mathbb{Q}$,
– contains a quadratic imaginary extension Q of $\mathbb{Q}$ in which (p) splits,
– and is not equal to Q.

Thus $K = Q \cdot F$ with F totally real, and

$$\Gamma = \Gamma_0 \times \langle \iota \rangle, \qquad \begin{cases} \Gamma \overset{\text{df}}{=} \mathrm{Gal}(K/\mathbb{Q}) \\ \Gamma_0 \overset{\text{df}}{=} \mathrm{Gal}(K/Q) \cong \mathrm{Gal}(F/\mathbb{Q}). \end{cases}$$

As a subfield of $\mathbb{Q}^{\mathrm{al}}$, K acquires a prime w_0. Let $D = D(w_0) \subset \Gamma$ be the decomposition group of w_0. Because p splits in Q, $D \subset \Gamma_0$.

Write $\Gamma_0 = \{\tau_0 = 1, \ldots, \tau_{n-1}\}$, so that

$$\Gamma = \{\tau_0, \ldots, \tau_{n-1}, \iota\tau_0, \ldots, \iota\tau_{n-1}\}.$$

Let $d = (D\colon 1)$. We can assume that the τ_i have been numbered so that $D = \{\tau_0, \ldots, \tau_{d-1}\}$ and $\tau_i D = \tau_{d[\frac{i}{d}]} D$, i.e., $\tau_0 D = \cdots = \tau_{d-1} D$, $\tau_d D = \cdots = \tau_{2d-1} D$, etc.. In particular, $\{\tau_0, \tau_d, \ldots\}$ is a set of representatives for the cosets of D in Γ.

We shall use the map $\tau \mapsto \tau w_0$ to identify Γ/D with the set of primes of K lying over p. We have a commutative diagram (Lemma 5.1):

$$\begin{array}{ccc} X^*(S^K) & \xrightarrow{\text{natural inclusion}} & \mathbb{Z}[\Gamma] \\ \downarrow & & \downarrow \\ X^*(P^K) & \xrightarrow{\pi \mapsto f_\pi} & \mathbb{Z}[\Gamma/D]. \end{array}$$

The first vertical map is $X^*(\alpha^K)$, which maps f to $\pi(f)$, and the second is $\sum f(\tau)\tau \mapsto \sum f(\tau)(\tau D)$.

Let

$$\psi_i = \tau_i + \sum_{j \neq i} \iota\tau_i, \qquad \bar{\psi} = \sum \iota\tau_i.$$

Then $\psi_0, \ldots, \psi_{n-1}, \bar{\psi}$ form a basis for $X^*(S^K)$ (Lemma 3.3). As $\tau_i\psi_0 = \psi_i, \quad (\iota\tau_i)\psi_0 = \iota\psi_i$ we see that

$$\Psi \overset{\text{df}}{=} \{\psi_0, \ldots, \psi_{n-1}, \iota\psi_0, \ldots, \iota\psi_{n-1}\}$$

is a Γ-orbit in $X^*(S^K)$. Let $\pi_i = \pi(\psi_{id}) \in W^K_{1,+}(p^\infty)$. Then

$$\Pi \overset{\text{df}}{=} \{\pi_0, \ldots, \pi_{(n/d)-1}, \iota\pi_0, \ldots, \iota\pi_{(n/d-1}\}$$

is a Γ-orbit in $X^*(P^K)$.

LEMMA 6.7. *The diagram*

$$\begin{array}{ccc} X^*(T^{\Psi}) & \xrightarrow{X^*(\gamma)} & X^*(S^K) \\ {\scriptstyle X^*(\alpha')}\downarrow & & {\scriptstyle X^*(\alpha)}\downarrow \\ X^*(L^{\Pi}) & \xrightarrow{X^*(\beta)} & X^*(P^K) \end{array}$$

becomes almost Cartesian when the two groups at right are replaced by the images of the horizontal arrows.

Proof. We shall prove this by showing that the bottom map is injective. The map $\tau \mapsto \tau\pi_0$ defines a bijection $\Gamma/D \to \Pi$, and hence an isomorphism $\mathbb{Z}[\Gamma/D] \to \mathbb{Z}[\Pi]$. On combining this with the natural map $\mathbb{Z}[\Pi] \to X^*(L^{\Pi})$, we get the first map in the sequence

$$\mathbb{Z}[\Gamma/D] \to X^*(L^{\Pi}) \xrightarrow{X^*(\beta)} X^*(P^K) \to \mathbb{Z}[\Gamma/D].$$

The map at right sends π to the map $\sigma \mapsto f_\pi(\sigma w_0)$—it is injective (Section 4). Let $\sigma_i = \tau_{di}D$, $i = 0, \ldots, (n/d) - 1$, and let

$$\begin{aligned} \varpi_i \;=\; & \sigma_i + d\iota\sigma_0 + \cdots + d\iota\sigma_{i-1} + \\ & +(d-1)\iota\sigma_i + d\iota\sigma_{i+1} + \cdots + d\iota\sigma_{(n/d)-1} \in \mathbb{Z}[\Gamma/D]. \end{aligned}$$

Then the composite of the three maps in the sequence is $\epsilon \mapsto \epsilon\varpi_0\colon \mathbb{Z}[\Gamma/D] \to \mathbb{Z}[\Gamma/D]$. Since $\sigma_i\varpi_0 = \varpi_i$, $\iota\sigma_i\varpi_0 = \iota\varpi_i$, this composite map has matrix $A(n/d, d)$ relative to the basis $\{\sigma_0, \ldots, \sigma_{\frac{n}{d}-1}, \iota\sigma_0, \ldots, \iota\sigma_{\frac{n}{d}-1}\}$ of $\mathbb{Z}[\Gamma/D]$. Now Corollary 6.6 implies that the kernel of the composite map is $\{\sum a_i(\sigma_i + \iota\sigma_i) \mid \sum a_i = 0\}$, but this is also the kernel of the map $\mathbb{Z}[\Gamma/D] \to X^*(L^{\Pi})$. □

As $\tau_i\bar\psi = \bar\psi$ and $\iota\tau_i\bar\psi = \iota\bar\psi$, $\bar\Psi \overset{\text{df}}{=} \{\bar\psi, \iota\bar\psi\}$ is a Γ-orbit. Let $\bar\pi = \bar\psi$, and let $\bar\Pi = \{\bar\pi, \iota\bar\pi\}$.

LEMMA 6.8. *The diagram*

$$\begin{array}{ccc} X^*(T^{\bar\Psi}) & \xrightarrow{X^*(\gamma)} & X^*(S^K) \\ {\scriptstyle X^*(\alpha')}\downarrow & & {\scriptstyle X^*(\alpha)}\downarrow \\ X^*(L^{\bar\Pi}) & \xrightarrow{X^*(\beta)} & X^*(P^K) \end{array}$$

becomes almost Cartesian when the two groups at right are replaced by the images of the horizontal arrows.

Proof. As in the preceding lemma, one shows that the bottom arrow is injective. □

LEMMA 6.9. *The square*

$$\begin{array}{ccc} X^*(T^{\Psi}) \oplus X^*(T^{\bar{\Psi}}) & \to & X^*(S^K) \\ \downarrow & & \downarrow \\ X^*(T^{\Pi}) \oplus X^*(T^{\bar{\Pi}}) & \longrightarrow & X^*(P^K) \end{array}$$

is almost Cartesian.

Proof. Consider

$$\begin{array}{ccccc} X^*(T^{\Psi}) \oplus X^*(T^{\bar{\Psi}}) & \longrightarrow & \mathbb{Z}[\Gamma]\psi_0 \oplus \mathbb{Z}[\Gamma]\bar{\psi} & \longrightarrow & X^*(S^K) \\ \downarrow & & \downarrow & & \downarrow \\ X^*(T^{\Pi}) \oplus X^*(T^{\bar{\Pi}}) & \longrightarrow & \mathbb{Z}[\Gamma/D]\pi_0 \oplus \mathbb{Z}[\Gamma/D]\bar{\pi} & \longrightarrow & X^*(P^K). \end{array}$$

The left-hand square is almost Cartesian because it is a direct sum of almost cartesian squares, and so it remains to show that the right hand square is almost cartesian. The image of the top-right map contains $\psi_0, \dots, \psi_{n-1}, \bar{\psi}$ and, hence, is onto. Since the vertical maps are both onto, this shows that all the maps in the square are onto. The elements $\pi_0, \pi_1, \dots, \pi_{n-1}, \bar{\pi}$ of $X^*(P^K)$ are linearly independent. Therefore, an element $\psi = \sum a_i\psi_i + a\bar{\psi}$ of $X^*(S^K)$ maps to zero in $X^*(P^K)$ if and only if $a = 0$ and $\sum_{dj \leqslant i < d(j+1)} a_i = 0$ for $j = 0, \dots, (n/d) - 1$. The first condition implies that $\psi \in \mathbb{Z}[\Gamma]\psi_0$, and the second condition implies that it lies in the kernel of $\mathbb{Z}[\Gamma]\psi_0 \to \mathbb{Z}[\Gamma/D]\pi_0$. We can now apply Lemma 6.3. □

Let $I = \Gamma\backslash\{$ CM-types on $K\}$ and let $I' = \Gamma\backslash W^K_{1,+}(p^\infty)$.

LEMMA 6.10. *The square*

$$\begin{array}{ccc} \bigoplus_{\Phi\in I} X^*(T^{\Phi}) & \xrightarrow{\gamma} & X^*(S^K) \\ \downarrow{\scriptstyle \alpha''} & & \downarrow{\scriptstyle \alpha} \\ \bigoplus_{\Pi\in I'} X^*(T^{\Pi}) & \xrightarrow{\beta} & X^*(P^K) \end{array}$$

is almost Cartesian.

Proof. Since the maps are all surjective, it suffices to prove that the map Ker $(\alpha'') \to$ Ker(α) is surjective, but this is obvious from the previous lemma. □

Consider the diagram:

$$\begin{array}{ccccc} \bigoplus_{\Phi\in I} X^*(T^{\Phi}) & \longrightarrow & X^*(T^K) & \longrightarrow & X^*(S^K) \\ \downarrow{\scriptstyle \alpha''} & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} \\ \bigoplus_{\Pi\in I'} X^*(L^{\Pi}) & \longrightarrow & X^*(L^K) & \longrightarrow & X^*(P^K). \end{array}$$

The last lemma shows that the composite of the maps $\mathrm{Ker}(\alpha'') \to \mathrm{Ker}(\alpha') \to \mathrm{Ker}(\alpha)$ is surjective, which implies that $\mathrm{Ker}\,(\alpha') \to \mathrm{Ker}(\alpha)$ is surjective. Therefore the right-hand square is almost Cartesian, which completes the proof of Theorem 6.1.

7. The Hodge Conjecture Implies the Tate Conjecture

Let X be a smooth projective variety over $\mathbb{C}$. We say that the Hodge conjecture holds for X if, for all r, the $\mathbb{Q}$-vector space $H^{2r}(X(\mathbb{C}),\mathbb{Q}) \cap H^{r,r}$ is spanned by the classes of algebraic cycles. This section will be occupied with proving the following theorem.

THEOREM 7.1. *If the Hodge conjecture holds for all Abelian varieties of CM-type over $\mathbb{C}$, then the Tate conjecture* (0.1) *holds for all Abelian varieties over the algebraic closure $\mathbb{F}$ of a finite field.*

We shall derive Theorem 7.1 from two further propositions. Before stating them, it will be useful to review some of the theory of characteristic polynomials.

Let $\mathbf{T}$ be a pseudo-Abelian rigid tensor category over a field k (in particular, this means that $k = \mathrm{End}(\mathbb{1})$). Then, for any X in $\mathbf{T}$, $\mathrm{End}\,(X) = \mathrm{Hom}(\mathbb{1}, X^\vee \otimes X)$, and the trace $\mathrm{Tr}(\alpha|X)$ of an endomorphism α of X is its composite with ev: $X^\vee \otimes X \to \mathbb{1}$ (regarded as an element of k). For any integer r,

$$a_r \overset{\mathrm{df}}{=} \frac{1}{r!}\sum \mathrm{sgn}(\sigma)\cdot\sigma\colon X^{\otimes r} \to X^{\otimes r}$$

(sum over the elements of the symmetric group on r letters) is an idempotent in $\mathrm{End}\,(X^{\otimes r})$, and we define $\bigwedge^r X$ to be its image. Assume that $d =_{\mathrm{df}} \mathrm{Tr}(1|X) \in \mathbb{N}$. The characteristic polynomial $f_\alpha(t)$ of an endomorphism α of X is defined to be

$$c_0 + c_1 t + c_2 t^2 + \cdots + c_d t^d, \qquad c_{d-i} = (-1)^i\,\mathrm{Tr}(\alpha|\Lambda^i X).$$

When this definition is applied to an endomorphism of a vector space, it leads to the usual characteristic polynomial. Clearly, for any k-linear tensor functor $F\colon \mathbf{T} \to \mathbf{T}'$ from $\mathbf{T}$ to a similar category $\mathbf{T}'$, $f_\alpha(t) = f_{F(\alpha)}(t)$.

For a field k and an adequate equivalence relation $\sim$, let $\mathbf{Mot}_\sim(k)$ be the category of motives generated by the Abelian varieties over k with the algebraic cycles modulo $\sim$ as the correspondences. When $\sim$ is taken to be numerical equivalence, we obtain a semisimple $\mathbb{Q}$-linear Tannakian category $\mathbf{Mot}_{\mathrm{num}}(k)$ (Jannsen, 1992).

Let M be the fundamental group of $\mathbf{Mot}_{\mathrm{num}}(\mathbb{F})$. Since every Lefschetz class is algebraic, there is a canonical $\mathbb{Q}$-linear tensor functor $w\colon \mathbf{LMot}(\mathbb{F}) \to \mathbf{Mot}_{\mathrm{num}}(\mathbb{F})$ which is faithful (because of 1.6) and exact. The homomorphism $M \to w(L)$ of fundamental groups defined by w is injective because $\mathbf{Mot}_{\mathrm{num}}(\mathbb{F})$ is generated by the image of w. Therefore, M is commutative, and so can be regarded as an affine group scheme over $\mathbb{Q}$ in the usual sense. Because $\mathbf{Mot}_{\mathrm{num}}(\mathbb{F})$ is semisimple,

M is an affine group scheme of multiplicative type, and the functor w defines an inclusion $M \hookrightarrow L$.

It is known that the Frobenius maps on projective smooth varieties over a finite field commute with algebraic correspondences (e.g., Soulé (1984), Proposition 2). Therefore, any X_0 in $\mathbf{Mot}_{\text{num}}(\mathbb{F}_q)$ admits a Frobenius automorphism π_{X_0}. The family $\{\pi_{X_0}\}$ is an automorphism of the identity functor $\mathbf{Mot}_{\text{num}}(\mathbb{F}_q) \to \mathbf{Mot}_{\text{num}}(\mathbb{F}_q)$. The characteristic polynomial $f_{\pi_{X_0}}$ of π_{X_0} has coefficients in $\mathbb{Q}$, and its roots in $\mathbb{Q}^{\text{al}}$ are Weil q-numbers. To see the second statement, choose a fibre ω: $\mathbf{Mot}_{\text{num}}(\mathbb{F}_q) \to \mathbf{Vec}_{\mathbb{Q}^{\text{al}}}$, and note that $f_{\pi_{X_0}}$ is also the characteristic polynomial of $\omega(\pi_{X_0})$ acting on $\omega(X_0)$ and that $\omega(X_0)$ occurs a factor of $\omega(A_0)$ (possibly twisted) for some Abelian variety A_0 over $\mathbb{F}_q$. Let M^{X_0} be the fundamental group of the Tannakian subcategory $\langle X_0\rangle^{\otimes}$ generated by X_0 and the Tate object. A comparison with the Lefschetz group again shows that M^{X_0} is commutative, and therefore equals $\underline{\text{Aut}}^{\otimes}(\text{id}_{\langle X_0\rangle^{\otimes}})$. Hence, $\pi_{X_0} \in \text{Aut}^{\otimes}(\text{id}_{\langle X_0\rangle^{\otimes}}) = M^{X_0}(\mathbb{Q})$.

Now let X be an object on $\mathbf{Mot}_{\text{num}}(\mathbb{F})$, and let M^X (quotient of M) be the fundamental group of $\langle X\rangle^{\otimes}$. Let X_0 be a model of X over some finite subfield $\mathbb{F}_{p^n}$ of $\mathbb{F}$. As we enlarge $\mathbb{F}_{p^n}$, M^{X_0} may be replaced by a smaller algebraic group, but after a certain finite extension it will become constant, and equal to M^X. Therefore, for some $N > 1$, $\pi_X =_{\text{df}} \pi_{X_0}^N \in M^X(\mathbb{Q})$. For any character χ of M^X, $\chi(\pi_X)$ is a Weil p^{nN}-number (for any fibre functor ω: $\langle X\rangle^{\otimes} \to \mathbf{Vec}_{\mathbb{Q}^{\text{al}}}$, it occurs as an eigenvalue of π_X acting on $\omega(Y)$ for some Y in $\langle X\rangle^{\otimes}$). Hence, we can apply Proposition 3.3 of Milne (1994) to obtain a well-defined homomorphism $P \to M^X$. These homomorphisms are compatible for varying X, and so define a homomorphism $P \to M$. The composite of this with the homomorphism $M \to L$ defined above is the homomorphism β of Section 4 (apply ib. 3.3 again). It follows that $P \to M$ is injective, and we identify P with a subgroup scheme of M.

Remark. 7.2. The pro-torus P is generated by a certain 'germ of an element' (ib. p. 435), which (by definition) the homomorphism $P \to M^X$ 'sends to' π_X. We can use this observation to characterize the image of P in M^X. Let π_{X_0} be as in the above discussion. As we observed, for N sufficiently divisible, $\pi_{X_0}^N$ lies in $M(\mathbb{Q})$. The smallest algebraic subgroup of M^X containing $\pi_{X_0}^N$ will be independent of N if N is sufficiently divisible – this smallest algebraic subgroup will then be the image P in M^X. In the following, we shall always use π_X to denote an element $\pi_{X_0}^N$ of $M^X(\mathbb{Q})$ with N chosen to be sufficiently divisible that π_X generates the image of P in M^X.

Theorem 7.1 will follow from the next two propositions.

PROPOSITION 7.3. *If the Hodge conjecture holds for all Abelian varieties of CM-type over $\mathbb{C}$, then $P = M$.*

PROPOSITION 7.4. *The Tate conjecture* (0.1) *holds for all Abelian varieties over* $\mathbb{F}$ *if and only if* $M = P$.

Proof of Proposition 7.3. For an Abelian variety A over $\mathbb{Q}^{\mathrm{al}}$, the map from the space of absolute Hodge classes on A to that of $A_{/\mathbb{C}}$ is bijective (Deligne, 1982, 2.9). Since a similar statement is true for the spaces of algebraic classes*, our assumption implies that every absolute Hodge class on an Abelian variety of CM-type over $\mathbb{Q}^{\mathrm{al}}$ is algebraic. Therefore, there is a reduction functor R: $\mathbf{CM}(\mathbb{Q}^{\mathrm{al}}) \to \mathbf{Mot}_{\mathrm{num}}(\mathbb{F})$, and hence a commutative diagram of Tannakian categories and exact $\mathbb{Q}$-linear tensor functors:

$$\begin{tikzcd}
\mathbf{LCM}(\mathbb{Q})^{\mathrm{al}}) \arrow[r] \arrow[d, "R"'] & \mathbf{CM}(\mathbb{Q}^{\mathrm{al}}) \arrow[d, "R"'] \\
\mathbf{LMot}(\mathbb{F}) \arrow[r] & \mathbf{Mot}_{\mathrm{num}}(\mathbb{F}).
\end{tikzcd}$$

From this diagram, we obtain a commutative diagram of fundamental groups:

$$\begin{tikzcd}
T & S \arrow[l, "\gamma"'] \\
L \arrow[u, "\alpha'"] & M. \arrow[l] \arrow[u]
\end{tikzcd}$$

Hence

$$M \subset S \cap L, \quad (\text{in } T).$$

Because $P \subset M$, Theorem 6.1 forces $M = P$.

Proof of Proposition 7.4. That the Tate conjecture implies $M = P$ is shown in Milne 1994, Proposition 2.38.

For the converse, suppose initially that numerical equivalence equals ℓ-adic homological equivalence on $\mathcal{Z}^r(A) \otimes \mathbb{Q}$ for all Abelian varieties A over $\mathbb{F}$ and all r. Then M acts on $H^{2r}(A, \mathbb{Q}_\ell(r))$, and the classes it fixes are precisely those in the $\mathbb{Q}_\ell$-subspace generated by the algebraic classes. On the other hand, the classes fixed by P are precisely those in $\mathcal{T}_\ell^r(A)$ (to be fixed by P is to be fixed by some power of the Frobenius element). Hence, $P = M$ implies that $\mathcal{T}_\ell^r(A)$ is spanned by algebraic classes.

It remains to prove that $P = M$ implies that numerical equivalence equals ℓ-adic homological equivalence. The following elementary statement will be used (Tate, 1966, p. 138).

> Let $f(t) \in \mathbb{Q}[t]$, and let $f(t) = \prod P(t)^{m(P)}$ be the unique factorization of P into a product of distinct irreducible polynomials over a field $k \supset \mathbb{Q}$.

* To show surjectivity, consider a specialization of an algebraic cycle over $\mathbb{C}$.

The integer $r(f) = \sum m(P)^2 \deg(P)$ is independent of k. If a semisimple endomorphism γ of a k-vector space V has characteristic polynomial $f(t)$, then $\dim_k \mathrm{End}_{k[\gamma]}(V) = r(f)$.

For an adequate equivalence relation $\sim$ we define $\mathbf{Mot}_{\sim}(\mathbb{F})_\ell$ to be the category of motives (with the corrected commutativity constraint) generated by Abelian varieties over $\mathbb{F}$ and using as correspondences the spaces $(\mathcal{Z}(A) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)/\sim$. We shall show that the natural functor $X \mapsto \bar{X}$: $\mathbf{Mot}_{\text{hom}}(\mathbb{F})_\ell \to \mathbf{Mot}_{\text{num}}(\mathbb{F})_\ell$ is faithful (hom $=$ ℓ-adic homological equivalence on $\mathcal{Z}(A) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$). For this it suffices to show that the natural map End $(X) \to \mathrm{End}(\bar{X})$ is injective for all X. Note that it is automatically surjective.

Let X be in $\mathbf{Mot}_{\text{hom}}(\mathbb{F})_\ell$. The fibre functor ω_ℓ on $\mathbf{Mot}_{\text{hom}}(\mathbb{F})_\ell$ defined by ℓ-adic étale cohomology is faithful, and so $\dim_{\mathbb{Q}_\ell}\mathrm{End}(X) \leqslant \dim_{\mathbb{Q}_\ell}\mathrm{End}_{\mathbb{Q}_\ell[\pi_X]}$ $(\omega_\ell(X))$. Because $f_{\omega_\ell(\pi_X)}(t) = f_{\pi_X}(t)$ and π_X acts semisimply on $\omega_\ell(X)$, $\dim_{\mathbb{Q}_\ell}\mathrm{End}_{\mathbb{Q}_\ell[\pi_X]}(\omega_\ell(X)) = r(f_{\pi_X})$.

Let Y be in $\mathbf{Mot}_{\text{num}}(\mathbb{F})_\ell$. For any field $k \supset \mathbb{Q}_\ell$ and fibre functor ω: $\mathbf{Mot}_{\text{num}}(\mathbb{F})_\ell \to \mathbf{Vec}_k$, $\mathrm{End}(Y) \otimes_{\mathbb{Q}_\ell} k \cong \mathrm{End}_k(\omega(Y))^M$. If $P = M$, so that M^Y is generated as an algebraic group by $\pi_Y \in M^Y(\mathbb{Q})$ (see 7.2), then the dimension of the second space is $r(f_{\pi_Y})$.

On taking $Y = \bar{X}$, we find that

$$r(f_{\pi_{\bar{X}}}) = \dim_{\mathbb{Q}_\ell}\mathrm{End}(\bar{X}) \leqslant \dim_{\mathbb{Q}_\ell}\mathrm{End}(X) \leqslant r(f_{\pi_X}). \tag{$*$}$$

Since $f_{\pi_X}(t) = f_{\pi_{\bar{X}}}(t)$, both inequalities must be equalities, and so End $(X) \to \mathrm{End}(\bar{X})$ is an isomorphism.

This completes the proof that the functor $\mathbf{Mot}_{\text{hom}}(\mathbb{F})_\ell \to \mathbf{Mot}_{\text{num}}(\mathbb{F})_\ell$ is faithful. Since Hom $(\mathbb{1}, h^{2r}_{\sim}(A)(r)) = \mathcal{C}^r_{\sim}(A)$, we now know (for all A and r) that the map

$$\mathcal{Z}^r(A) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell/\text{hom} \to \mathcal{Z}^r(A) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell/\text{num}$$

is injective, i.e., that if $z \in \mathcal{Z}^r(A) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ has nonzero cohomology class, then there exists a $z' \in \mathcal{Z}^{\dim A - r} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ such that $z \cdot z' \neq 0$. By elementary linear algebra, this implies the same statement with the '$\otimes_{\mathbb{Q}}\mathbb{Q}_\ell$' removed, i.e., that numerical equivalence coincides with ℓ-adic homological equivalence on $\mathcal{Z}^r(A)$. This completes the proof of Proposition 7.4. □

Remark 7.5. Without the assumption $P = M$, there seems to be no reason why both inequalities in $(*)$ should not be strict. For example, we might (perhaps) have an X in $\mathbf{Mot}_{\text{hom}}(X)_\ell$ of rank 2 with $\mathrm{End}_{\mathbb{Q}_\ell[\pi_X]}(\omega_\ell(X)) = M_2(\mathbb{Q}_\ell)$,

$$\mathrm{End}(X) = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \quad \text{and} \quad \mathrm{End}(\bar{X}) = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}.$$

Remark 7.6. (a) Let K be a CM-field as in the final part of the proof of Theorem 6.1. Then the above argument shows that if the Hodge conjecture holds for all Abelian varieties of CM-type over $\mathbb{C}$ with reflex field contained in K, then the Tate conjecture holds for all Abelian varieties over $\mathbb{F}$ with endomorphism algebra split by K.

(b) Once one knows the Tate conjecture for all Abelian varieties over $\mathbb{F}$, then one obtains it for all smooth projective varieties over $\mathbb{F}$ whose motive, defined using algebraic cycles modulo homological equivalence, lies in the Tannakian subcategory generated by Abelian varieties, for example, for products of curves.

(c) A similar argument to the above shows that the Hodge conjecture for Abelian varieties of CM-type over $\mathbb{C}$ implies the crystalline analogue of the Tate conjecture for Abelian varieties over $\mathbb{F}$.

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