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On the Canonical Decomposition of Quiver Representations

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Abstract. Kac introduced the notion of the canonical decomposition for a dimension vector of a quiver. Here we will give an efficient algorithm to compute the canonical decomposition. Our study of the canonical decomposition for quivers with three vertices gives us fractal-like pictures.

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1. Introduction

Let Q be a quiver without oriented cycles. Let α be a dimension vector for Q. For a fixed field K we denote by $\mathrm{Rep}_K(Q, \alpha)$ the space of representations of Q of dimension α. An expression $\alpha = \beta_1 \oplus \beta_2 \oplus \cdots \oplus \beta_k$ is called the canonical decomposition of the dimension vector α if there is a Zariski open subset $\mathcal{U}$ of $\mathrm{Rep}_K(Q, \alpha)$ such that for $V \in \mathcal{U}$ the represention V decomposes $V = \bigoplus_{i=1}^k V_i$ with V_i indecomposable of dimension β_i for $i = 1, \ldots, k$. The notion of canonical decomposition was introduced by Kac who also proved its first properties and gave several conjectures (see [2] and [3]).

In the paper [5] Schofield proved many fundamental properties of the canonical decomposition. He also gave an inductive procedure for calculating the canonical decomposition of any dimension vector. His procedure, however, assumes that before we calculate the decomposition of a dimension vector α we know all decompositions of the coordinatewise smaller dimension vectors. The reason is that we need the inductive calculation of the generic dimensions of certain Hom and Ext spaces. This makes this procedure not so easy to use in practice.

In the recent preprint [6], Schofield suggests a more efficient algorithm. This algorithm presented there is fast at least for some examples, but still uses a recursion which seems inefficient for large dimension vectors or large quivers.

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In this paper we give a very efficient algorithm for calculating canonical decomposition. It is based on the notion of a compartment which is a modification of the notion of an exceptional sequence. The important feature of the algorithm is that it is expressed exclusively in terms of the Euler form. The calculations of dimensions of Hom and Ext spaces are needed to justify the outcome of the algorithm, but each step in the algorithm depends just on the Euler form. This allows to see many general features of canonical decomposition, which seem to be new.

2. Preliminaries

A quiver Q is a pair $Q = (Q_0, Q_1)$ consisting of the set of vertices Q_0 and the set of arrows Q_1. Each arrow a has its head ha and tail ta, both in Q_0; $ta \xrightarrow{a} ha$. We fix an algebraically closed field K. A representation V of Q is a family of finite-dimensional K-vector spaces $V(x)$ $(x \in Q_0)$ and of linear maps $V(a)\colon V(ta) \to V(ha)$ $(a \in Q_1)$. The dimension vector of a representation V is the function $\underline{d}_V$ defined by $\underline{d}_V(x) := \dim V(x)$. The dimension vectors lie in the space Γ of integer-valued functions on Q_0. A morphism $f\colon V \to V'$ of two representations is a collection of linear maps $f(x)\colon V(x) \to V'(x)$ such that for each $a \in Q_1$ we have $V'(a)f(ta) = f(ha)V(a)$. We denote the linear space of morphisms from V to V' by $\mathrm{Hom}_Q(V, V')$.

A path p in Q is a sequence of arrows $p = a_1, \dots, a_n$ such that $ha_i = ta_{i+1}$ $(1 \leqslant i \leqslant n-1)$. We define $tp = ta_1, hp = ha_n$. We also have trivial path $e(x)$ from x to x. We define $[x, y]$ to be the vector space on the basis of paths from x to y. We assume throughout the paper that Q has no oriented cycles, i.e., there are no paths $p = a_1 \dots a_n$ such that $ta_1 = ha_n$. Under this assumption the spaces $[x, y]$ are finite dimensional.

The category $\mathrm{Rep}(Q)$ is an Abelian category. Its basic property is that it is hereditary, i.e. every representation has projective dimension $\leqslant 1$. More specifically, for two representations V, W we have a canonical map

$$\bigoplus_{x \in Q_0} \mathrm{Hom}_K(V(x), W(x)) \xrightarrow{d^V_W} \bigoplus_{a \in Q_1} \mathrm{Hom}_K(V(ta), W(ha))$$

defined by the formula

$$d^V_W(\{f_x\}_{x \in Q_0}) = \{W(a)f_{ta} - f_{ha}V(a)\}_{a \in Q_1}.$$

The kernel of d^V_W is isomorphic to $\mathrm{Hom}_Q(V, W)$, and the cokernel to $\mathrm{Ext}_Q(V, W)$.

Let α, β be two elements of Γ. We define the Euler inner product

$$\langle \alpha, \beta \rangle = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha).$$

It follows that

$$\langle \underline{d}_V, \underline{d}_W \rangle = \dim_K \mathrm{Hom}_Q(V, W) - \dim_K \mathrm{Ext}_Q(V, W). \tag{1}$$

We define the Euler matrix E as the matrix corresponding to $\langle\ ,\ \rangle$ in the standard basis, i.e., the matrix satisfying

$$\langle \alpha, \beta \rangle = {}^{t}\alpha E \beta$$

(we will treat dimension vectors as column vectors). For a dimension vector α we denote by $\mathrm{Rep}(Q, \alpha)$ the vector space of representations of Q of dimension vector α.

We define the Coxeter transform to be the linear map $\tau \colon \Gamma \to \Gamma$ given by the formula $\tau = -({}^{t}E)^{-1}E$ where E is the Euler matrix. The linear map τ preserves the Euler form. We also have Auslander–Reiten duality

$$\langle \alpha, \beta \rangle = -\langle \beta, \tau\alpha \rangle.$$

Let α, β be two dimension vectors. The functions

$$(V, W) \mapsto \dim \mathrm{Hom}_Q(V, W),$$
$$(V, W) \mapsto \dim \mathrm{Ext}_Q(V, W)$$

are upper-semicontinuous on $\mathrm{Rep}(Q, \alpha) \times \mathrm{Rep}(Q, \beta)$. We denote their generic values by $\hom_Q(\alpha, \beta)$, $\mathrm{ext}_Q(\alpha, \beta)$ respectively.

By forgetting the orientation of the arrows of the quiver Q we obtain an undirected graph to which we can associate a Kac–Moody Lie algebra. Recall that by Kac's Theorem ([2]) a dimension vector α is a root for the associated Kac–Moody algebra if and only if there are indecomposable representations of dimension α. A root α is real when $\langle \alpha, \alpha \rangle = 1$, imaginary if $\langle \alpha, \alpha \rangle \leqslant 0$ and isotropic if $\langle \alpha, \alpha \rangle = 0$. A dimension vector α is called a Schur root if a general representation of dimension α is indecomposable. Equivalently (see [3]), the endomorphism ring $\mathrm{End}_Q(V)$ consists of scalars for a general representation V of dimension α.

It is clear that if $\alpha = \bigoplus_{i=1}^{k} \beta_i$ is the canonical decomposition then each β_i is a Schur root. Moreover, the following is proven in [3].

THEOREM 1 (Kac). *The sum* $\alpha = \bigoplus_{i=1}^{k} \beta_i$ *is the canonical decomposition if and only if all* β_i *are Schur roots and* $\mathrm{ext}(\beta_i, \beta_j) = 0$ *for all* $i \neq j$.

Remark 2. In [5, Theorem 3.8] Schofield proved that the canonical decomposition is homogeneous in the following sense. For a Schur root β we define $(n\beta)$ as follows

$$(n\beta) = \begin{cases} n\beta, & \text{if } \beta \text{ is imaginary, nonisotropic;} \\ \beta \oplus \cdots \oplus \beta (n \text{ copies}), & \text{if } \beta \text{ is real or isotropic.} \end{cases}$$

THEOREM 3 (Schofield). *Let* $\alpha = \bigoplus_{i=1}^{k} \beta_i$ *be a canonical decomposition. Then the canonical decomposition of* $n\alpha$ *is*

$$n\alpha = \bigoplus_{i=1}^{k} (n\beta_i).$$

3. The Canonical Decomposition for Quivers with Two Vertices

Let us assume that $Q = \theta(r)$ is a quiver with two vertices x and y and with r arrows $a_1, \ldots, a_r$ with $ta_i = x, ha_i = y$ for $i = 1, \ldots, r$. If we identify a dimension vector α with the column vector $\binom{\alpha(x)}{\alpha(y)}$ then the Euler matrix and Coxeter transform are given by the formulae

$$E = \begin{pmatrix} 1 & -r \\ 0 & 1 \end{pmatrix}, \qquad \tau = \begin{pmatrix} -1 & r \\ -r & r^2 - 1 \end{pmatrix}.$$

The real Schur roots for $\theta(r)$ are

$$\pi_{2k} = \tau^k \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \pi_{2k+1} = \tau^k \begin{pmatrix} 1 \\ r \end{pmatrix} \quad (k = 0, 1, \ldots)$$

and the dimensions of preinjective modules

$$\rho_{2k} = \tau^{-k} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \rho_{2k+1} = \tau^{-k} \begin{pmatrix} r \\ 1 \end{pmatrix} \quad (k = 0, 1, \ldots).$$

Let $p_m = \pi_m(x)/\pi_m(y)$ and similarly $q_m = \rho_m(x)/\rho_m(y)$. Then one sees easily that $p_m < p_{m+1}$ and

$$\lim_{m \to \infty} p_m = \frac{r - \sqrt{r^2 - 4}}{2}.$$

Similarly, $q_m = 1/p_m$, so $q_m > q_{m+1}$ and

$$\lim_{m \to \infty} q_m = \frac{r + \sqrt{r^2 - 4}}{2}.$$

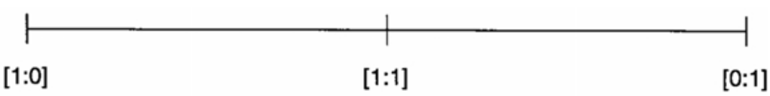
For a dimension vector α denote $a := \alpha(x)/\alpha(y)$. The canonical decomposition is expressed in the easiest way in terms of a. If we find m such that $p_m < a < p_{m+1}$ then α decomposes into a multiple of π_m and a multiple of π_{m+1}. If for some m $q_m > a > q_{m+1}$ then α decomposes into a multiple of ρ_m and a multiple of ρ_{m+1}. If

$$\frac{r - \sqrt{r^2 - 4}}{2} \leqslant a \leqslant \frac{r + \sqrt{r^2 - 4}}{2},$$

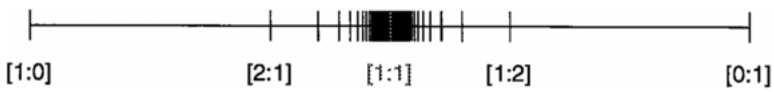
then α is imaginary (and α can be isotropic only when $r = 2$, $a = 1$). All of this is proven in [3, page 159].

For $r = 1, 2, 3$ we will draw $p_0, p_1, p_2, \ldots$ and $q_0, q_1, q_2, \ldots$ on the projective line. We get the following pictures:

r=1: $p_0 = q_2 = 0 = [0:1], p_1 = q_1 = 1 = [1:1], p_2 = q_0 = \infty = [1:0]$.



r=2: $p_0 = [0:1], p_1 = [1:2], p_2 = [2:3]$ and $q_0 = [1:0], q_1 = [2:1], q_2 = [3:2]$.



r=3: $p_0 = [0:1], p_1 = [1:3], p_2 = [3:8]$ and $q_0 = [1:0], q_1 = [3:1], q_2 = [8:3]$.

If $r = 2$ then there is one limit point, namely $[1:1]$ which corresponds to the imaginary Schur root. For $r = 3$ there is a red hole between $[1 : (3 - \sqrt{5})/2]$ and $[(3 - \sqrt{5})/2 : 1]$ corresponding to the imaginary Schur roots. For $r \geqslant 4$ the picture looks similar to that for $r = 3$.

4. The Algorithm for Canonical Decomposition

In this section we discuss the main result of this paper—the algorithm to calculate the canonical decomposition. It is based on the notion of a compartment which was inspired by the notion of exceptional sequence. Let us start with some definitions.

We fix a quiver Q without oriented cycles. Assume that Q has n vertices.

DEFINITION 4. Let $\mathcal{A}$ be an Abelian hereditary category. An object X from $\mathcal{A}$ is exceptional if $\mathrm{Hom}_{\mathcal{A}}(X, X) = K$, $\mathrm{Ext}_{\mathcal{A}}(X, X) = 0$. A sequence $\mathcal{E} = (X_1, \dots, X_r)$ of length r of objects of an Abelian hereditary category $\mathcal{A}$ is exceptional if or each i X_i is an exceptional object and for each $i < j$ we have $\mathrm{Hom}_{\mathcal{A}}(X_i, X_j) = \mathrm{Ext}_{\mathcal{A}}(X_i, X_j) = 0$.

If $\mathcal{A}$ is the category of representations of the quiver Q, the exceptional sequence of length $n = \#Q_0$ is called a complete sequence. Crawley-Boevey [1] defined the action of braid group B_n on n strings on the set of complete exceptional sequences and proved that this action is transitive.

We modify the notion of exceptional sequence by relaxing the condition on vertices allowing arbitrary Schur roots, but adding nonnegativity condition on the values of the Euler form. A dimension vector α is called left orthogonal to β if $\hom_Q(\alpha, \beta) = \mathrm{ext}_Q(\alpha, \beta) = 0$.

DEFINITION 5. A sequence $C = (\alpha^1, \dots, \alpha^s)$ of length s of dimension vectors from Γ is a compartment if α^i is a Schur root for each i, and

(1) for each $i < j$ α^i is left orthogonal to α^j,
(2) for each $i < j$ $\langle \alpha^j, \alpha^i \rangle \geqslant 0$.

The sequence C is called an exceptional compartment if all α^i are real Schur roots.

Remark 6. If α, β are Schur roots, and $\mathrm{ext}(\alpha, \beta) = 0$ then we have $\mathrm{ext}(\beta, \alpha) = 0$ or $\hom(\beta, \alpha) = 0$ (see [5, Theorem 4.1]). From (1) we see that $\mathrm{ext}(\beta, \alpha) = 0$ if and only if $\langle \beta, \alpha \rangle \geqslant 0$. In Definition 5 we could have replaced (2) by: for each $i < j$ $\mathrm{ext}(\alpha^j, \alpha^i) = 0$. □

PROPOSITION 7. *Suppose that $C = (\alpha^1, \dots, \alpha^s)$ is a compartment and α is a dimension vector such that $\alpha = \sum_{i=1}^s r_i \alpha^i$ for some nonnegative integers $r_1, \dots, r_s$. We also assume that $r_i = 1$ whenever α_i is imaginary and nonisotropic. Then the canonical decomposition of α is*

$$\alpha = (\alpha^1)^{\oplus r_1} \oplus (\alpha^2)^{\oplus r_2} \oplus \cdots \oplus (\alpha^s)^{\oplus r_s}.$$

Proof. This follows at once from Theorem 1, Definition 5 and Remark 6 and Remark 2. □

Thus to find a canonical decomposition of a dimension vector α it is enough to find a compartment $C = (\alpha^1, \dots, \alpha^s)$ such that α lies in the monoid spanned by $\alpha^1, \dots, \alpha^s$. Let α be a dimension vector. The algorithm to find a compartment containing α is as follows.

We try to find the expression $\alpha = \sum_{i=1}^s r_i \alpha^i$ such that

(1) $r_i \geqslant 0$ for $i = 1, \dots, s$,
(2) each α^i is a Schur root,
(3) for each $i < j$, α^i is left orthogonal to α^j,
(4) $r_i = 1$ whenever α^i is imaginary and not isotropic,
(5) for each $i < j$ $\langle \alpha^j, \alpha^i \rangle \geqslant 0$.

ALGORITHM 8.

(1) `input` : A quiver Q with n vertices, a dimension vector α.
(2) Write $\alpha = \sum_{i=1}^s r_i \alpha^i$ satisfying (1)–(4) (for example, take $s = n$ and $\alpha^i = e_i$ for all i).
(3) Omit all summands $r_i \alpha^i$ with $r_i = 0$. We may assume $r_i > 0$ for all i.
(4) If $\alpha = \sum_{i=1}^s r_i \alpha^i$ satisfies (5), terminate with `output` $: \alpha = \sum_{i=1}^s r_i \alpha^i$.
(5) There exist i and j with $i < j$ such that $\langle \alpha^j, \alpha^i \rangle < 0$. We take i and j such that $j - i$ is minimal. Let $\alpha^{k_1}, \alpha^{k_2}, \dots, \alpha^{k_a}$ be the subsequence of $\alpha^{i+1}, \dots, \alpha^{j-1}$ of all α^m with $\langle \alpha^j, \alpha^m \rangle > 0$. Let $\alpha^{l_1}, \dots, \alpha^{l_c}$ be the subsequence obtained from $\alpha^{i+1}, \dots, \alpha^{j-1}$ by omitting $\alpha^{k_1}, \alpha^{k_2}, \dots, \alpha^{k_a}$ (exactly those α^m such that $\langle \alpha^j, \alpha^m \rangle = 0$). We rearrange $(\alpha^1, \dots, \alpha^s)$ and $(r^1, \dots, r^s)$ as follows:

$$(\alpha^1, \dots, \alpha^s) := (\alpha^1, \dots, \alpha^{i-1}, \alpha^{k_1}, \dots, \alpha^{k_b}, \alpha^i, \alpha^j, \alpha^{l_1}, \dots, \alpha^{l_c}, \alpha^{j+1}, \dots, \alpha^s),$$

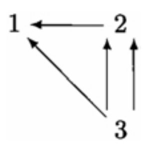
$$(r^1, \dots, r^s) := (r^1, \dots, r^{i-1}, r^{k_1}, \dots, r^{k_b}, r^i, r^j, r^{l_1}, \dots, r^{l_c}, r^{j+1}, \dots, r^s).$$

(6) There exists an i such that $\langle \alpha^{i+1}, \alpha^i \rangle < 0$. We modify the pair (α^i, α^{i+1}) (and (r_i, r_{i+1})) so that the Euler form $\langle \alpha^{i+1}, \alpha^i \rangle$ after modification becomes nonnegative. The modification is performed as follows. Write $\xi = \alpha^i$ $\eta = \alpha^{i+1}$. Let us write $p = r_i, q = r_{i+1}$. Let us also write $\zeta = p\xi + q\eta$. We will replace (ξ, η) by (ξ', η') where ξ', η' are positive linear combinations of ξ, η. Sometimes we will replace (ξ, η) by just one positive linear combination of ξ, η, thus reducing the number s. The replacement is performed according to the following scheme:
   (a) ξ, η are real Schur roots. The category spanned by ξ and η is the category of representations of a quiver with two vertices, no cycles and $m = -\langle \eta, \xi \rangle$ arrows. The canonical decomposition of ζ in this category is as described in Section 3.
      (i) $\langle \zeta, \zeta \rangle > 0$: There exists nonnegative combinations $\xi', \eta' \in \Gamma$ of ξ and η such that ξ' is left orthogonal to η', $\langle \eta', \xi' \rangle = m \geqslant 0$ and $\zeta = p'\xi' + q'\eta'$ for certain nonnegative integers p', q'. Replace (ξ, η) by (ξ', η').
      (ii) $\langle \zeta, \zeta \rangle = 0$: Replace (ξ, η) by ζ', where $\zeta = k\zeta'$ with k a positive integer and $\zeta' \in \Gamma$ is indivisible.
      (iii) $\langle \zeta, \zeta \rangle < 0$: Replace (ξ, η) by ζ.
   (b) ξ is real, η is imaginary:
      (i) $p + q\langle \eta, \xi \rangle \geqslant 0$: Replace (ξ, η) by (ξ', η'), where $\xi' = \eta - \langle \eta, \xi \rangle \xi$ and $\eta' = \xi$. Now we have $\zeta = q\xi' + (p + q\langle \eta, \xi \rangle)\eta'$.
      (ii) $p + q\langle \eta, \xi \rangle < 0$: Replace (ξ, η) by ζ.
   (c) ξ is imaginary, η is real:
      (i) $q + p\langle \eta, \xi \rangle \geqslant 0$: Replace (ξ, η) by (ξ', η'), where $\xi' = \eta$ and $\eta' = \xi - \langle \eta, \xi \rangle \eta$. Now we have $\zeta = (q + p\langle \eta, \xi \rangle)\xi' + p\eta'$.
      (ii) $q + p\langle \eta, \xi \rangle < 0$: Replace (ξ, η) by ζ.
   (d) ξ, η are imaginary: Replace (ξ, η) by ζ.

(7) Keep repeating step 3, 4, 5, 6 (until the loop gets broken in step 4).

Before proving the algorithm, we give an example to illustrate the algorithm.

EXAMPLE 9. Consider the following quiver with labeled vertices:



Because of the labeling, there is a natural way of identifying Γ with three-dimensional column vectors. Let us find the decomposition of the vector

$$d = \begin{pmatrix} 6 \\ 33 \\ 17 \end{pmatrix}.$$

At each step, we can put the column vectors $\alpha^1, \dots, \alpha^s$ in a matrix A, and we put the integers $r_1, \dots, r_s$ in a column vector r. So at each step we have $d = Ar$. We start with

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad r = \begin{pmatrix} 6 \\ 33 \\ 17 \end{pmatrix}.$$

Now α^1, α^2 are real Schur roots and $\langle \alpha^2, \alpha^1 \rangle = -1$. So $\zeta = 6\alpha^1 + 33\alpha^2$ and $\langle \zeta, \zeta \rangle = 927 > 0$. We are in case (a(i)). We replace (α^1, α^2) by $(\alpha^2 + \alpha^1, \alpha^1)$, and we obtain

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad r = \begin{pmatrix} 27 \\ 6 \\ 17 \end{pmatrix}.$$

Now $\langle \alpha^2, \alpha^3 \rangle = -3$ and α^2, α^3 are real Schur roots. We set $\zeta = 6\alpha^2 + 17\alpha^3$. We have $\langle \zeta, \zeta \rangle = 19$, so again we are in case (a(i)). We are dealing with a quiver with two vertices $-\langle \alpha^3, \alpha^2 \rangle = 3$ arrows. As we have seen in Section 3 we get a decomposition

$$\begin{pmatrix} 6 \\ 17 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}^{\oplus 3} \oplus \begin{pmatrix} 3 \\ 8 \end{pmatrix}.$$

So we replace (α^2, α^3) by $(\alpha^2 + 3\alpha^3, 3\alpha^2 + 8\alpha^3)$ and we get

$$A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 1 & 3 \\ 0 & 3 & 8 \end{pmatrix}, \qquad r = \begin{pmatrix} 27 \\ 3 \\ 1 \end{pmatrix}.$$

Now $\langle \alpha^2, \alpha^1 \rangle = -5$ and α^1, α^2 are real. We are in case (a(i)), and we replace (α^1, α^2) by $(\alpha^2 + 5\alpha^1, \alpha^1)$. We have

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 6 & 1 & 3 \\ 3 & 0 & 8 \end{pmatrix}, \qquad r = \begin{pmatrix} 3 \\ 12 \\ 1 \end{pmatrix}.$$

Now we have $\langle \alpha^3, \alpha^2 \rangle = -16$, α^2 and α^3 are real, $\zeta = 12\alpha^2 + \alpha^3$ and $\langle \zeta, \zeta \rangle = -47$. We are in case (a(iii)). We replace (α^2, α^3) by ζ and we obtain

$$A = \begin{pmatrix} 1 & 3 \\ 6 & 15 \\ 3 & 8 \end{pmatrix}, \qquad r = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Now α^1 is real and α^2 is imaginary and $\langle \alpha^2, \alpha^1 \rangle = -2$. The value of $3 + 1 \cdot \langle \alpha^2, \alpha^1 \rangle = 1$ is positive, so we are in case (b(i)). We replace (α^1, α^2) by $(\alpha^2 + 2\alpha^1, \alpha^1)$. We now have

$$A = \begin{pmatrix} 5 & 1 \\ 27 & 6 \\ 14 & 3 \end{pmatrix}, \qquad r = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We get $\langle \alpha^2, \alpha^1 \rangle = 2$. So we have found the canonical decomposition

$$\begin{pmatrix} 6 \\ 33 \\ 17 \end{pmatrix} = \begin{pmatrix} 5 \\ 27 \\ 14 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}.$$

LEMMA 10 (Schofield). *Let α, β, γ be three Schur roots. Assume that α is left orthogonal to β and γ and that β is left orthogonal to γ. Assume that $\langle \gamma, \beta \rangle > 0$, $\langle \beta, \alpha \rangle > 0$. Then $\langle \gamma, \alpha \rangle > 0$.*

Proof. This follows at once from Theorem 2.4 and Theorem 4.1 in [5]. □

LEMMA 11. *Suppose X and Y are representations of the quiver Q, and we have an exact sequence*

$$0 \to X \to Z \to Y \to 0.$$

If $\mathrm{Hom}(X, Y) = \mathrm{Hom}(Y, X) = 0$, $\mathrm{Hom}(Y, Y) = K$ *and in the long exact sequence*

$$\begin{aligned} 0 &\to \mathrm{Hom}(Y, X) \to \mathrm{Hom}(Z, X) \to \mathrm{Hom}(X, X) \\ &\to \mathrm{Ext}(Y, X) \to \mathrm{Ext}(Z, X) \to \mathrm{Ext}(X, X) \to 0, \end{aligned} \tag{2}$$

the connecting homomorphism $\mathrm{Hom}(X, X) \to \mathrm{Ext}(Y, X)$ *is injective, then* $\mathrm{Hom}(Z, X) = 0$ *and* $\mathrm{Hom}(Z, Z) = K$.

Proof. Clearly $\mathrm{Hom}(Z, X) \cong \mathrm{Hom}(Y, X) = 0$ from (2). From the exact sequence

$$0 \to \mathrm{Hom}(Y, Y) \to \mathrm{Hom}(Z, Y) \to \mathrm{Hom}(X, Y) \tag{3}$$

and $\mathrm{Hom}(X, Y) = 0$ follows that $\mathrm{Hom}(Z, Y) = \mathrm{Hom}(Y, Y) = K$. From the exact sequence

$$0 \to \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, Z) \to \mathrm{Hom}(Z, Y) \tag{4}$$

and $\mathrm{Hom}(Z, X) = 0$ now follows that $\mathrm{Hom}(Z, Z) = K$. □

COROLLARY 12. *If α, β are Schur roots with* $\hom(\alpha, \beta) = \hom(\beta, \alpha) = 0$ *and* $\mathrm{ext}(\beta, \alpha) \neq 0$*, then $\alpha + \beta$ is a Schur root.*

Proof. Let X be a general representation of dimension α, Y be a general representation of dimension β and let Z be a nontrivial extension of X and Y. Clearly

$$\mathrm{Hom}(Y, Y) = K, \qquad \mathrm{Hom}(X, Y) = \mathrm{Hom}(Y, X) = 0$$

and the homomorphism $\mathrm{Hom}(X, X) \to \mathrm{Ext}(Y, X)$ is injective because $\mathrm{Hom}(X, X) = K$ and the identity is mapped to the nontrivial element $Z \in \mathrm{Ext}(Y, X)$. Now the corollary follows from Lemma 11. □

COROLLARY 13. *If α, β are imaginary Schur roots with* $\hom(\alpha, \beta) = \hom(\beta, \alpha) = 0$ *and* $\mathrm{ext}(\beta, \alpha) \neq 0$*, then $p\alpha + q\beta$ is a Schur root for all $p, q > 0$.*

Proof. We will prove by induction on p and q that $p\alpha + q\beta$ is a Schur root. If $p\alpha$ and $q\beta$ both are Schur roots then so is $p\alpha + q\beta$ by Corollary 12. If $p\alpha$ is not a Schur root, then α is isotropic and $p > 1$. By induction hypothesis we may assume that $(p-1)\xi + q\eta$ is a Schur root. Now $\hom(\alpha,\alpha) = 0$, $\hom(\alpha,\beta) = 0$, therefore $\hom(\alpha, (p-1)\alpha + q\beta) = \hom((p-1)\alpha + q\beta, \alpha) = 0$ and $\mathrm{ext}((p-1)\alpha + q\beta, \alpha) = -q\langle\beta,\alpha\rangle > 0$. We can apply Corollary 12 and conclude that $p\alpha + q\beta = \alpha + ((p-1)\alpha + q\beta)$ is a Schur root. In a similar way $p\alpha + q\beta$ is a Schur root if $q\eta$ is not a Schur root. □

COROLLARY 14. *If α is a real Schur root and β is a Schur root such that* $\hom(\alpha,\beta) = \hom(\beta,\alpha) = 0$. *then for $0 \leqslant t \leqslant -\langle\beta,\alpha\rangle$ $\beta + t\alpha$ is a Schur root and* $\hom(\beta + t\alpha, \alpha) = 0$.

Proof. Let W be a general representation of dimension α, Y be a general representation of dimension β. We have

$$\mathrm{Hom}(W, Y) = \mathrm{Hom}(Y, W) = 0 \quad \text{and} \quad \mathrm{Hom}(Y, Y) = K. \quad \text{Put } t = -\langle\beta,\alpha\rangle.$$

A general representation X of dimension $t\alpha$ is isomorphic to the direct sum of t copies of W. We have $\mathrm{Hom}(X, Y) = \mathrm{Hom}(Y, X) = 0$. Let $Z \in \mathrm{Ext}(Y, X)$ be a general extension. In the long exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}(Y, W) \to \mathrm{Hom}(Z, W) \to \mathrm{Hom}(X, W) \to \mathrm{Ext}(Y, W) \\ &\to \mathrm{Ext}(Z, W) \to \mathrm{Ext}(X, W) \to 0, \end{aligned} \tag{5}$$

the connecting homomorphism $\mathrm{Hom}(X, W) \to \mathrm{Ext}(Y, W)$ is injective because Z is a generic extension and $\dim(\mathrm{Hom}(X, W)) = t \leqslant -\langle\beta,\alpha\rangle = \dim(\mathrm{Ext}(Y, W))$. In the long exact sequence (2), the connecting homomorphism $\mathrm{Hom}(X, X) \to \mathrm{Ext}(Y, X)$ is a direct sum of homomorphisms $\mathrm{Hom}(X, W) \to \mathrm{Ext}(Y, W)$ coming from (5). It follows that $\mathrm{Hom}(X, X) \to \mathrm{Ext}(Y, X)$ is injective and by Lemma 11 we conclude that $\mathrm{Hom}(Z, Z) = K$ and $\mathrm{Hom}(Z, X) = 0$. So $\beta + t\alpha$ is a Schur root, and $\hom(\beta + t\alpha, \alpha) = 0$. □

COROLLARY 15. *If α is a Schur root and β is a real Schur root such that* $\hom(\alpha,\beta) = \hom(\beta,\alpha) = 0$ *then for $0 \leqslant t \leqslant -\langle\beta,\alpha\rangle$, $\alpha + t\beta$ is a Schur root and* $\hom(\beta, \alpha + t\beta) = 0$.

Proof. Let Q^{op} be the quiver obtained from Q by reversing all arrows. We can identify dimension vectors for Q and Q^{op}. Notice that $\hom_Q(\alpha,\beta) = \hom_{Q^{\mathrm{op}}}(\beta,\alpha)$ and $\mathrm{ext}_Q(\alpha,\beta) = \mathrm{ext}_{Q^{\mathrm{op}}}(\beta,\alpha)$. Also α is a real or imaginary Schur root for Q if and only if it is a real or imaginary Schur root for Q^{op}. From the assumtions $\hom_Q(\alpha,\beta) = \hom_Q(\beta,\alpha) = 0$ follows that $\hom_{Q^{\mathrm{op}}}(\alpha,\beta) = \hom_{Q^{\mathrm{op}}}(\beta,\alpha) = 0$. We can apply Corollary 14 (with α and β interchanged) to Q^{op}, and we obtain that $\alpha + t\beta$ is a Schur root when $0 \leqslant -\langle\alpha,\beta\rangle_{Q^{\mathrm{op}}} = -\langle\beta,\alpha\rangle_Q$. Also we have that $\hom_{Q^{\mathrm{op}}}(\alpha + t\beta, \beta) = \hom_Q(\beta, \alpha + t\beta) = 0$. □

THEOREM 16. *Algorithm 8 terminates after finitely many steps with output* $\alpha = \sum_{i=1}^{s} r_i \alpha^i$ *satisfying* (1)–(5). *The canonical decomposition of* α *is*

$$\alpha = (\alpha^1)^{\oplus r_1} \oplus (\alpha^2)^{\oplus r_2} \oplus \cdots \oplus (\alpha^s)^{\oplus r_s}.$$

Proof. First, we will prove that throughout the algorithm, $\alpha = \sum_{i=1}^{s} r_i \alpha_i$ satisfies (1)–(4). When we are in step 2, $\alpha = \sum_{i=1}^{s} r_i \alpha_i$ is unchanged. Clearly, after step 3, $\alpha = \sum_{i=1}^{s} r_i \alpha_i$ still satisfies (1)–(4). It is clear that after step 5, $\alpha = \sum_{i=1}^{s} r_i \alpha_i$ satisfies (1), (2) and (4). Let us show that it also satisfies (3).

Now $\langle \alpha^{k_p}, \alpha^{l_q} \rangle \geqslant 0$ for all p, q by the assumptions on minimality of $j - i$. From Lemma 10 below, $\langle \alpha^j, \alpha^{l_q} \rangle = 0$ and $\langle \alpha^j, \alpha^{k_p} \rangle > 0$ it follows that also $\langle \alpha^{k_p}, \alpha^{l_q} \rangle \leqslant 0$. We conclude that α^{k_p} is left orthogonal to α^{l_q} for all p, q. Similarly one shows that α^{k_p} is left orthogonal to α^i for all p. We also have that α^{l_q} is right orthogonal to α^j for all q.

After step 6, it is clear that $\alpha = \sum_{i=1}^{s} r_i \alpha^i$ satisfies (1). We show that (4) is also satisfied. In case a(i), (ξ, η) is replaced by (ξ', η') with ξ', η' both real. There is nothing to prove. In (a)(ii) ζ' is isotropic, so again, there is nothing to prove. In cases (a)(iii) (b)(ii) (c)(ii) and (d) (ξ, η) is being replaced by ζ. Now ζ will have coefficient 1 in the decomposition of α. In case (b)(i), (ξ, η) is being replaced by $(\xi'.\eta')$ with η' real and ξ' imaginary. If ξ' is nonisotropic, then $\langle \xi', \xi' \rangle = \langle \eta, \eta \rangle < 0$, so η is also nonisotropic. Assuming that (4) was satisfied before step 6, we get that $q = 1$. Now q is exactly the coefficient of ξ' in the decomposition of α after step 6. In a similar way, we can prove that if we are in case (c)(i), (4) will be satisfied after step 6.

Now we will prove that after step 6, properties (2) and (3) are still satisfied. Notice that a positive linear combination of $\xi = \alpha^i$ and $\eta = \alpha^{i+1}$ is right orthogonal to α^j for $j < i$ and left orthogonal to α^j if $j > i + 1$. So we'll have to prove that whenever (ξ, η) is replaced by a pair (ξ', η'), then ξ', η' are Schur roots and ξ' is left orthogonal to η'. Whenever (ξ, η) is replaced by a single dimension vector ζ, then we must prove that ζ is a Schur root.

In case (a), the problem can be reduced to the category spanned by ξ and η. We reduce to the case of an quiver without cycles with two vertices and properties (2) and (3) follow from Section 3.

In case (b)(i) and (c)(i), properties (2) and (3) follow immediately from Corollary 14 and 15 respectively.

Suppose that we are in case (b)(ii). First we assume that η is not isotropic. Then $q = 1$ and $p < -\langle \eta, \xi \rangle$. From Corollary 14 follows that $\zeta = p\eta + \xi$ is a Schur root. If η is isotropic, then we notice that for $t = -\langle \eta, \xi \rangle$, $\eta + t\xi$ is a Schur root by 14. We have $\hom(\eta, \eta + t\xi) = \hom(\eta + t\xi, \eta) = 0$ because $\hom(\eta, \xi) = \hom(\xi, \eta) = 0$ and $\hom(\eta, \eta) = 0$. Also we have $\operatorname{ext}(\eta + t\xi, \eta) \neq -\langle \eta + t\xi, \eta \rangle = -t\langle \xi, \eta \rangle > 0$. Now we can write $t\zeta = p(\eta + t\xi) + (tq - p)\eta$ and by Corollary 12 we we get that $t\zeta$ is a Schur root. Therefore ζ is a Schur root. In a similar way, in case (c)(ii), property (2) follows from Corollaries 15 and 13.

In case (d), property (2) follows Corollary 13. This completes the proof that after step 6, properties (1)–(4) always are satisfied.

Finally we will prove that the algorithm terminates. After step 3, we have an expression $\alpha = \sum_{i=1}^{s} r_i\alpha^i$, with $r_i > 0$ for all i. It is clear that $\sum_{i=1}^{k} r_i$ gets smaller after each loop. Therefore the algorithm must terminate. $\square$

For the remaining of this section we allow the quiver Q to have oriented cycles. We will show how to compute the canonical decomposition of a dimension vector in this more general setting. Define a new quiver $\hat{Q} = (\hat{Q}_0, \hat{Q}_1)$ where $\hat{Q}_0 = Q_0 \times \{0, 1\}$ and

$$\hat{Q}_1 = \{c_x : x_0 \to x_1 \mid x \in Q_0\} \cup \{\hat{a} : (ta)_0 \to (ha)_1 \mid a \in Q_1\}.$$

Notice that $\hat{Q}$ does not have oriented cycles, because all arrows go from $Q_0 \times \{0\}$ to $Q_0 \times \{1\}$. If α is a dimension vector for Q then we define a dimension vector $\hat{\alpha}$ for $\hat{Q}$ by $\hat{\alpha}(x_i) = \alpha(x)$ for all $x \in Q_0$ and $i \in \{0, 1\}$.

LEMMA 17. *If the canonical decomposition of α is $\alpha = \alpha^1 \oplus \alpha^2 \oplus \cdots \oplus \alpha^s$ then the canonical decomposition of $\hat{\alpha}$ is $\hat{\alpha} = \hat{\alpha}^1 \oplus \hat{\alpha}^2 \oplus \cdots \oplus \hat{\alpha}^s$.*

Proof. For any representation of V of Q we define a representation $\hat{V}$ of $\hat{Q}$ by $\hat{V}(x_i) = V(x)$ for all $x \in Q_0$ and $i \in \{0, 1\}$, $\hat{V}(\hat{a}) = V(a)$ for all $a \in Q_1$ and $\hat{V}(c_x) = \mathrm{id}$ for all $x \in Q_0$. If W is a representation of Q such that $W(c_x)$ is invertible for all $x \in Q_0$, then we define a representation $\tilde{W}$ by $\tilde{W}(x) = W(x_0)$ for all $x \in Q_0$ and $\tilde{W}(a) = W(c_x)^{-1}W(\hat{a})$ for all $a \in Q_1$. It is easy to check that for a representation V of Q we have $\tilde{\hat{V}} \cong V$.

Notice that if V is an indecomposable representation of Q, then $\hat{V}$ is indecomposable. Indeed, suppose that $\hat{V} \cong W_1 \oplus W_2$ with W_1 and W_2 representations of $\hat{Q}$. Since $\hat{V}(c_x)$ is an isomorphism, $W_1(c_x)$ and $W_2(c_x)$ are both isomorphisms for all $x \in Q_0$. Now we have $V \cong \tilde{\hat{V}} \cong \tilde{W}_1 \oplus \tilde{W}_2$ which contradicts that V is indecomposable.

Suppose that W is a general representation of dimension $\hat{\alpha}$. We may assume that $W(c_x)$ is invertible for all x and that the decomposition of $\tilde{W}$ is the canonical decomposition of α. We can choose bases of $W(x_0)$ and $W(x_1)$ in such a way that $W(c_x)$ will be the identity. This shows that $W \cong \hat{V}$ for some representation V of Q. In fact, V is isomorphic to $\tilde{\hat{V}} \cong \tilde{W}$. Let

$$V \cong \tilde{W} \cong V_1 \oplus V_2 \oplus \cdots \oplus V_s$$

be the canonical decomposition of V, such that $\underline{d}_{V_i} = \alpha^i$. Then the canonical decomposition of W is $W \cong \hat{V} \cong \hat{V}_1 \oplus \hat{V}_2 \oplus \cdots \hat{V}_s$ and $\underline{d}_{\hat{V}_i} = \hat{\alpha}_i$ for all i. $\square$

If β is a dimension vector for $\hat{Q}$, then we define a dimension vector $\tilde{\beta}$ by $\tilde{\beta}(x) = \beta(x_0)$ for all $x \in Q_0$. Notice that $\hat{\tilde{\beta}} = \beta$.

COROLLARY 18. *If $\hat{\alpha} = \beta^1 \oplus \beta^2 \oplus \cdots \beta^s$ is the canonical decomposition of $\hat{\alpha}$, then the canonical decomposition of α is $\alpha = \tilde{\beta}^1 \oplus \tilde{\beta}^2 \oplus \cdots \tilde{\beta}^s$.*

Proof. Let $\alpha = \alpha^1 \oplus \alpha^2 \oplus \cdots \alpha^t$ be the canonical decomposition of α. By Lemma 17 the canonical decomposition of $\tilde{\alpha}$ is $\tilde{\alpha} = \tilde{\alpha}^1 \oplus \tilde{\alpha}^2 \oplus \cdots \tilde{\alpha}^t$. It follows that $\beta^1, \dots, \beta^s$ is a permutation of $\tilde{\alpha}^1, \dots, \tilde{\alpha}^t$, and $\tilde{\beta}^1, \dots, \tilde{\beta}^s$ is a permutation of $\alpha^1, \dots, \alpha^t$. □

Lemma 18 gives us a method for computing the canonical decomposition for arbitrary quivers. If α is a dimension vector for Q, then we can compute the canonical decomposition of $\hat{\alpha}$ using Algorithm 8, say

$$\hat{\alpha} = (\beta^1)^{r_1} \oplus (\beta^2)^{r_2} \oplus \cdots \oplus (\beta^s)^{r_s}.$$

Then we have

$$\alpha = (\tilde{\beta}^1)^{r_1} \oplus (\tilde{\beta}^2)^{r_2} \oplus \cdots \oplus (\tilde{\beta}^s)^{r_s}.$$

5. Consequences

The algorithm presented in the previous section allows to draw several conclusions about the nature of canonical decomposition.

COROLLARY 19. *Suppose that*

$$\alpha = (\beta^1)^{\oplus t_1} \oplus (\beta^2)^{\oplus t_2} \oplus \cdots \oplus (\beta^u)^{\oplus t_u}.$$

is the canonical decomposition of α ($t_i > 0$ for all i), and we can write $\alpha = \sum_{i=1}^s r_i \alpha^i$ satisfying properties (1)–(3), then any β_i is a nonnegative integral combination of $\alpha^1, \dots, \alpha^s$.

Proof. Whenever α_i is an imaginary Schur root, we can replace α_i by $r_i\alpha_i$ and r_i by 1. We may assume that $\alpha = \sum_{i=1}^s r_i \alpha^i$ satisfies (1)–(4). In step 2 of Algorithm 8, we could start with these $\alpha^1, \dots, \alpha^s$. In step 6 of the algorithm, some α^i and α^{i+1} are being replaced by one or two Schur roots which are positive combinations of $\alpha^1, \dots, \alpha^s$. The corollary follows. □

COROLLARY 20. *Suppose that we can write $\alpha = \sum_{i=1}^s r_i \alpha^i$ satisfying (1)–(3) and $r_i \neq 0$ for all i. If one of the α^i is imaginary, then α does not have a dense orbit.*

Proof. Whenever α_i is an imaginary Schur root, we can replace α_i by $r_i\alpha_i$ and r_i by 1. We may assume that $\alpha = \sum_{i=1}^s r_i \alpha^i$ satisfies (1)–(4). In step 2 of Algorithm 8, we could start with these $\alpha^1, \dots, \alpha^s$. It is clear from step 6 that in each loop there always will be an i such that α^i is imaginary. When the algorithm terminates, one of the summands of the canonical decomposition of α will be imaginary, so α does not have a dense orbit. □

COROLLARY 21. *For a dimension vector α, let $r(\alpha)$ be the number of distinct real Schur roots and $i(\alpha)$ be the number of distinct imaginary Schur roots in the canonical decomposition of α. Then we have $r(\alpha) + 2i(\alpha) \leqslant n$ where n is the number of vertices of the quiver Q.*

Proof. In Algorithm 8 we can start with n simple Schur roots $\alpha^1, \dots, \alpha^n$. In step 6 of the algorithm, the number of roots always decreases or stays the same. The only way for the number of imaginary roots to increase is, is when two real Schur roots are being replaced by one imaginary Schur root. □

We will now try to understand the canonical decomposition in more geometric terms. Suppose that Q is a quiver with vertices $x_1, \dots, x_n$ without oriented cycles. For a dimension vector α, we define a point

$$[\alpha] := [\alpha(x_1) : \alpha(x_2) : \cdots : \alpha(x_n)]$$

in projective space $\mathbb{P}^{n-1}$. If $C = (\alpha^1, \dots, \alpha^s)$ is a compartment we define $[C]$ as the simplex in $\mathbb{P}^{n-1}$ spanned by $[\alpha^1], \dots, [\alpha^s]$.

Notice that it is possible that $[C] = [C']$, without the compartments C and C' being equal. For example, if $\langle \alpha^i, \alpha^{i+1} \rangle = 0$, then we can make such a compartment C' by interchanging α^i and α^{i+1} in C. If α^i is a Schur root, then we can make another compartment C' by replacing α^i in C by a multiple of itself.

COROLLARY 22. *If $[\alpha]$ lies in $[C]$ for some compartment $C = (\alpha^1, \dots, \alpha^s)$, then we can write $\alpha = \sum_{i=1}^{s} r_i \alpha^i$ and the canonical decomposition of α is*

$$\alpha = (r_1\alpha^1) \oplus (r_2\alpha^2) \oplus \cdots \oplus (r_s\alpha^s)$$

(*see Remark 2*).

Proof. This follows from Proposition [7]. □

It is clear from the uniqueness of the canonical decomposition, that for every dimension vector α there is a unique open simplex $[C]^\circ$ containing $[\alpha]$.

COROLLARY 23. *Let S be the set of all $[C]$ with C an exceptional compartment. Let $D \subset \mathbb{P}^{n-1}$ be the union of S. Then S is a triangulation of the topological space D. A dimension vector α has a dense orbit if and only if $[\alpha] \in D$.*

6. Quivers with Three Vertices

We assume that Q is a quiver with three vertices, without oriented cycles. We label the vertices with 1, 2 and 3. Let $b_{i,j}$ be the number of arrows $i \to j$. We assume that $b_{i,j} = 0$ for $i \leqslant j$. In view of Corollary 21 there are the following possibilities for a dimension vector α:

(1) α has a dense orbit, and the canonical decomposition only involves real Schur roots;
(2) α decomposes $\beta^{\oplus r} \oplus \gamma^{\oplus s}$ with β a real and γ an imaginary Schur root, $r, s > 0$;
(3) α is an imaginary Schur root.

The first case we already studied. The set $D \subset \mathbb{P}^2$ of dimension vectors with dense orbit has a nice triangulation (Corollary 23). Using the braid group action as defined in [1] we can obtain all exceptional sequences $C = (\alpha^1, \alpha^2, \alpha^3)$. Of those exceptional sequences, we can select the exceptional compartments. A compartment C corresponds to a triangle $[C]$. The canonical decomposition in this triangle is given by Corollary 22.

We will study the second case. Suppose that $\alpha = \beta^{\oplus r} \oplus \gamma^{\oplus s}$ is the canonical decomposition with $r, s > 0$ and β is real, γ is imaginary. Let us fix a real Schur root β. We will investigate what the possibilities are for α. There are two possibilities, β is *left* orthogonal to γ, or β is *right* orthogonal to γ. Let us assume β is left orthogonal to γ.

The right orthogonal category of β is again a quiver category, because β is real. This category is generated by two dimension vectors, β_1, β_2, with β_1 left orthogonal to β_2 and $p = \langle \beta_2, \beta_1 \rangle \leqslant 0$. We can write $\gamma = r_1\beta_1 + r_2\beta_2$ with $r_1, r_2 \geqslant 0$. The root γ is imaginary, so $\langle \gamma, \gamma \rangle \leqslant 0$. This means that

$$\frac{1}{\lambda} \leqslant \frac{r_1}{r_2} \leqslant \lambda$$

where $\lambda = (p + \sqrt{p^2 - 4})/2$. The line segment between $[\beta_1 + \lambda\beta_2]$ and $[\lambda\beta_1 + \beta_2]$ corresponds to all imaginary roots right orthogonal to β.

An other condition is that $\langle \gamma, \beta \rangle \geqslant 0$. Choose a point $I([\beta])$ on the intersection of the lines $\langle \cdot, \beta \rangle = 0$ intersects $\langle \beta, \cdot \rangle = 0$ (usually this point is unique). There are four cases

(1) $\langle \beta_1 + \lambda\beta_2, \beta \rangle \geqslant 0$, $\langle \lambda\beta_1 + \beta_2, \beta \rangle \geqslant 0$: The point $[\gamma]$ can be anywhere in the interval between $[\beta_1 + \lambda\beta_2]$ and $[\lambda\beta_1 + \beta_2]$.
(2) $\langle \beta_1 + \lambda\beta_2, \beta \rangle < 0$, $\langle \lambda\beta_1 + \beta_2, \beta \rangle \geqslant 0$: Now $I([\beta])$ lies in the line segment between $[\beta_1 + \lambda\beta_2]$ and $[\lambda\beta_1 + \beta_2]$. The point $[\gamma]$ can be anywhere in the interval between $I([\beta])$ and $[\lambda\beta_1 + \beta_2]$.
(3) $\langle \beta_1 + \lambda\beta_2, \beta \rangle \geqslant 0$, $\langle \lambda\beta_1 + \beta_2, \beta \rangle < 0$: Now $I([\beta])$ lies in the line segment between $[\beta_1 + \lambda\beta_2]$ and $[\lambda\beta_1 + \beta_2]$. The point $[\gamma]$ can be anywhere in the interval between $I([\beta])$ and $[\beta_1 + \lambda\beta_2]$.
(4) $\langle \beta_1 + \lambda\beta_2, \beta \rangle < 0$, $\langle \lambda\beta_1 + \beta_2, \beta \rangle < 0$: This gives a contradiction with the existance of γ.

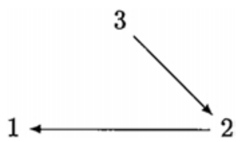
COROLLARY 24. *If we fix a real Schur root β, then α has a canonical decomposition $\beta^{\oplus r} \oplus \gamma^{\oplus s}$ with $r, s > 0$ and γ imaginary and right orthogonal to β if and only if $[\alpha]$ lies in the relative interior of the triangle T_β defined by*

(1) $\langle \beta_1 + \lambda\beta_2, \beta \rangle \geqslant 0$, $\langle \lambda\beta_1 + \beta_2, \beta \rangle \geqslant 0$: T_β is spanned by $[\beta]$, $[\beta_1 + \lambda\beta_2]$ and $[\lambda\beta_1 + \beta_2]$.
(2) $\langle \beta_1 + \lambda\beta_2, \beta \rangle < 0$, $\langle \lambda\beta_1 + \beta_2, \beta \rangle \geqslant 0$: T_β is spanned by $[\beta]$, $I([\beta])$ and $[\lambda\beta_1 + \beta_2]$.
(3) $\langle \beta_1 + \lambda\beta_2, \beta \rangle \geqslant 0$, $\langle \lambda\beta_1 + \beta_2, \beta \rangle < 0$: T_β is spanned by $[\beta]$, $I([\beta])$ and $[\beta_1 + \lambda\beta_2]$.
(4) $\langle \beta_1 + \lambda\beta_2, \beta \rangle < 0$, $\langle \lambda\beta_1 + \beta_2, \beta \rangle < 0$: $T_\beta = \emptyset$.

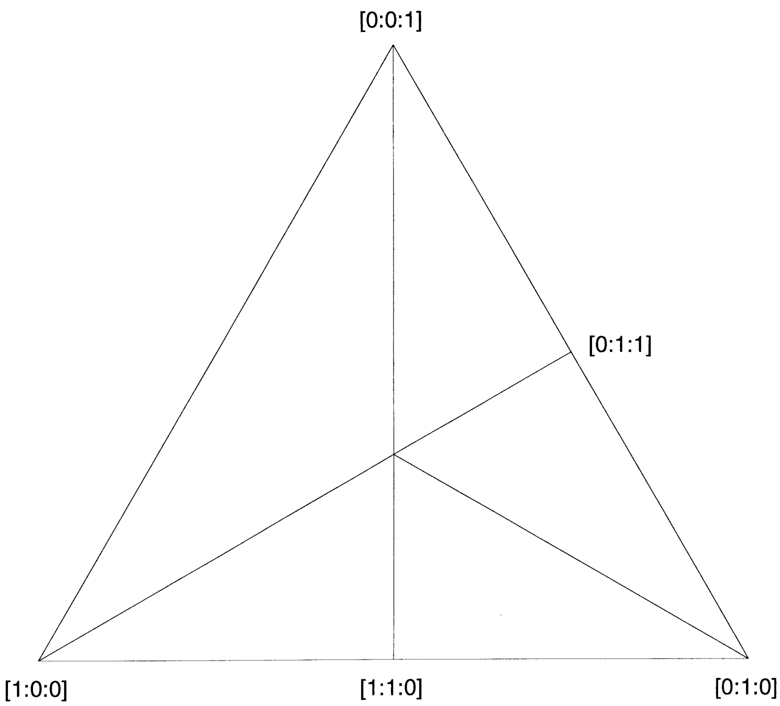
If $p = 2$ (and $\lambda = 1$) we are in a degenerate case: T_β will be a line segment.

In a similar way we can define a T'_β such that α has a canonical decomposition $\beta^{\oplus r} \oplus \gamma^{\oplus s}$ with $r, s > 0$ and γ imaginary and *left* orthogonal to β if and only if $[\alpha]$ lies in the relative interior of the triangle T'_β. So for each real Schur root β we have defined two triangles T_β and T'_β (which may be empty). In many cases, the two triangles are adjacent and have the vertices $[\beta]$ and $I([\beta])$ in common.

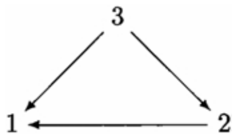
EXAMPLE 25. We consider the quiver



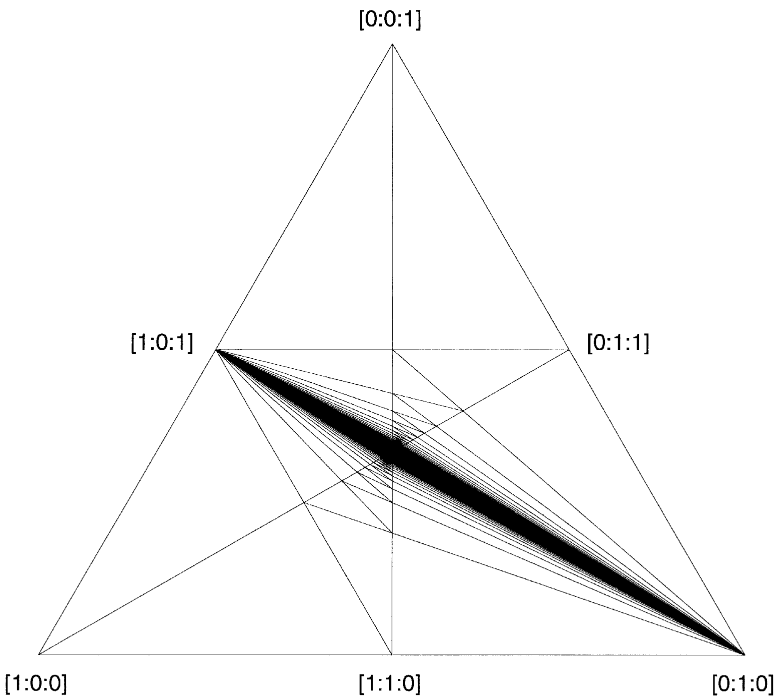
This quiver is of finite type (Dynkin type A_3). Every dimension vector has a dense orbit. The set of dimension vectors is divided up in 5 triangles, corresponding to exceptional compartments of length 3.



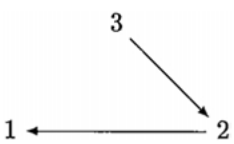
EXAMPLE 26. We consider the quiver



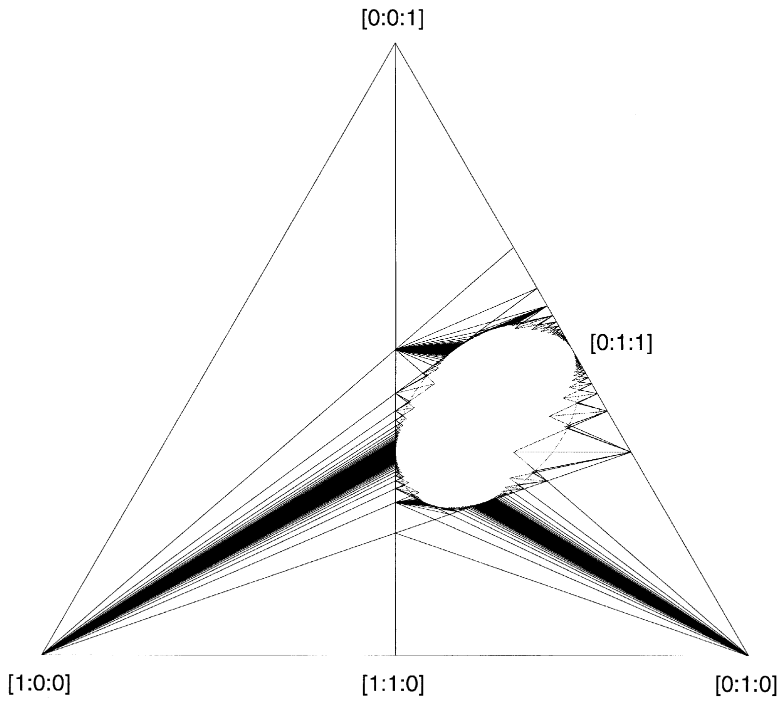
This quiver is of tame type (type $\tilde{A}_2$). The dimension vectors with no dense orbit correspond to the open line segment between [0 : 1 : 0] and [1 : 0 : 1]. The set $D \subset \mathbb{P}^2$ of vectors with dense orbit, is triangulated with infinitely many triangles. The only imaginary Schur root is [1 : 1 : 1].



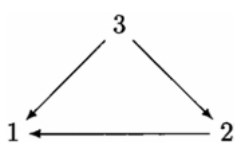
EXAMPLE 27. We consider the quiver



This quiver is of wild type. The black triangles correspond to exceptional compartments. The red triangles correspond to dimension vectors who decompose into a multiple of an imaginary Schur root and a multiple of a real Schur root. The star shaped region in the interior corresponds to the imaginary Schur roots. Notice that this region is not convex. The picture is like a *fractal*. Also, notice how the red triangles come in pairs. The quadric $\langle \alpha, \alpha \rangle = 0$ is graphed in green.



EXAMPLE 28. We consider the quiver

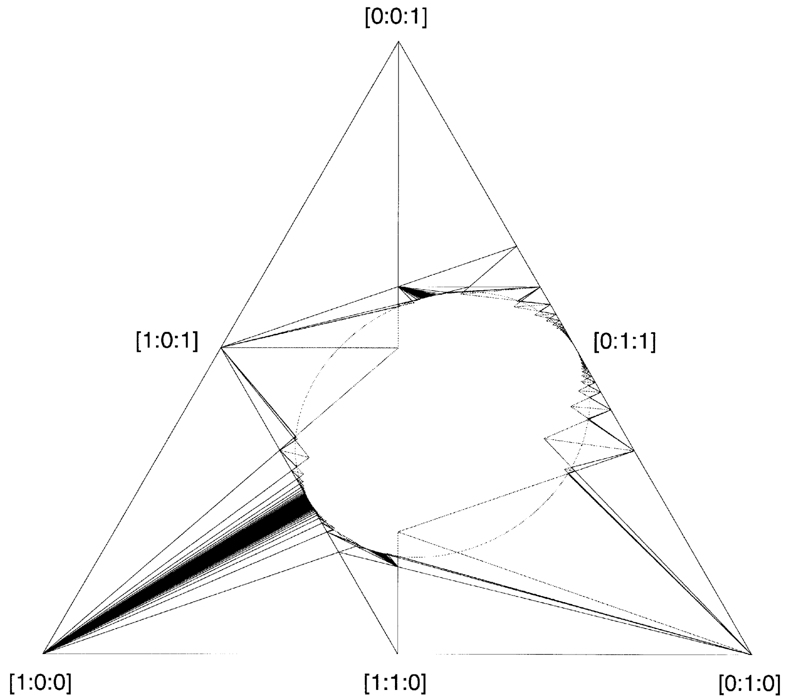


This is another example of a wild quiver. For real roots $[\beta]$, $I([\beta])$ seems to be a mirror image of $[\beta]$ inside the quadric $\langle \alpha, \alpha \rangle = 0$. We will study the map I.

Let us consider the transformation $I : \mathbb{P}^2 \to \mathbb{P}^2$ of the projective plane which we have defined as

$$\alpha \mapsto \{\langle \alpha, - \rangle = \langle -, \alpha \rangle = 0\}$$

sending a point α to the (generically unique up to scalar) solution of linear equations $\langle \alpha, - \rangle = \langle -, \alpha \rangle = 0$.



PROPOSITION 29. *The map I is a birational involution of the projective plane. In fact, it is a quadratic transformation corresponding to three points which are the eigenvectors of the Coxeter transform τ related to the Euler form $\langle \cdot, \cdot \rangle$.*

Proof. Let us start with the Euler matrix

$$E = \begin{pmatrix} 1 & 0 & 0 \\ b_{2,1} & 1 & 0 \\ -b_{3,1} & -b_{2,1} & 1 \end{pmatrix},$$

where $b_{i,j}$ is the number of arrows from the ith to jth vertex.

By direct calculation one can see that the characteristic polynomial of $\tau - ({}^{t}E)^{-1}E$ is given by the formula

$$\chi(\tau, \lambda) = 1 + (3 - J)\lambda + (3 - J)\lambda^2 + \lambda^3,$$

where

$$J = b_{2,1}^2 + b_{3,1}^2 + b_{3,2}^2 + b_{2,1}b_{3,1}b_{3,2}$$

is the basic invariant of the triple $(b_{1,2}, b_{1,3}, b_{2,3})$.

The symmetry of $\chi(\tau, \lambda)$ means that -1 is an eigenvalue. In fact the vector

$$\beta = \begin{pmatrix} b_{2,3} \\ -b_{1,3} \\ b_{1,2} \end{pmatrix}$$

is the corresponding eigenvector. The other two eigenvalues λ_1, λ_2 have to satisfy $\lambda_1 \lambda_2 = 1$.

PROPOSITION 30. *The quiver Q is of finite type if $J = 0, 1, 2$, is of tame type if $J = 4$ and is of wilde type if $J > 4$. If Q is wild, then the Coxeter transform has three distinct real eigenvalues. The eigenvectors corresponding to eigenvalues λ_1, λ_2 lie on the ellipse of isotropic dimension vectors, given by the equation $\langle \alpha, \alpha \rangle = 0$.*

Proof. The first statement is an easy check. After dividing the characteristic polynomial $\chi(\tau, \lambda)$ by $\lambda + 1$ we get the quadratic polynomial $1 + (2 - J)\lambda + \lambda^2$. Its discriminant equals $J^2 - 4J$, so we have two distinct positive real roots λ_1, λ_2 for $J > 4$. Let v_1 and v_2 be the eigenvectors corresponding to λ_1 and λ_2 respectively. Then we have

$$\lambda_1 \langle v_1, v_1 \rangle = \langle v_1, \tau v_1 \rangle = -\langle v_1, v_1 \rangle$$

so $\langle v_1, v_1 \rangle = 0$ because $\lambda_1 \neq -1$. In a similar way, v_2 is isotropic. □

The rational map I is given by the formulas

$$(\alpha_1, \alpha_2, \alpha_3) \mapsto (M_{3,2}, -M_{3,1}, M_{2,1}),$$

where

$$M_{3,2} = b_{3,2} \langle \alpha, \alpha \rangle + \alpha_1 u,$$

$$-M_{3,1} = -b_{3,1} \langle \alpha, \alpha \rangle + \alpha_2 u,$$

$$M_{2,1} = b_{2,1} \langle \alpha, \alpha \rangle + \alpha_3 u,$$

where

$$u = -b_{3,2}\alpha_1 + (b_{3,1} + b_{2,1} b_{3,2})\alpha_2 - b_{2,1}\alpha_3.$$

The rational map I is an involution by definition, therefore a birational map of the projective plane. The three quadrics $M_{1,2}, M_{1,3}, M_{2,3}$ that define it have three common zeros: the point β and two intersections of the line $u = 0$ with the ellipse $\langle \alpha, \alpha \rangle = 0$. It follows that I is a quadratic transformation blowing up these three points and collapsing the three lines joining them. The involution I fixes every point on the ellipse $\langle \alpha, \alpha \rangle = 0$. □

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