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Actions of Superrigid Non-Kazhdan Lattices on Compact Manifolds

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Abstract. In this paper, we examine the actions of lattices in superrigid non-Kazhdan simple groups on compact manifolds. The geometric results are obtained by analyzing the properties of amenable ergodic groupoids.

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In this paper, we study the action of a discrete Lie group on a compact manifold, continuing the investigation begun by Zimmer in [6]–[8]. In particular, in [8], Zimmer was able to deduce a number of geometric consequences by using Kazhdan's property as a replacement for measurable superrigidity for cocycles. This was done, in part, by analyzing the algebraic hull of a cocycle from a Kazhdan group into an amenable group. Since many Kazhdan groups are superrigid, i.e. satisfy the hypotheses of measurable superrigidity [5, Ch. 5], one would hope that similar results hold true in the case of superrigid non-Kazhdan groups. We establish such results in this paper.

THEOREM 1. *Let Γ be an irreducible lattice in G a semisimple Lie group with finite center, $\mathbb{R}$-rank$(G) \geq 2$, and no compact factors. Suppose Γ acts ergodically on a compact manifold M of dimension 2 preserving a volume density and a connection. Then $M = S^2$ or $P_2(\mathbb{R})$.*

THEOREM 2. *Let Γ and G be as in Theorem 1. Suppose Γ acts ergodically on a compact manifold M preserving a Lorentz metric. Then either,*

(1) Γ *leaves a Riemannian metric invariant and hence the action factors to* $\Gamma \to K \to \mathrm{Diff}(M)$ *where K is a compact Lie group,* or
(2) Γ *factors to* $\Gamma \to \mathrm{SL}(2, \mathbb{R}) \times K \to \mathrm{Diff}(M)$ *where K is a compact Lie group. In this case, the simple components of G must all be locally isomorphic to* $\mathrm{SL}(2, \mathbb{R})$.

These results are generalizations of theorems by Zimmer in [6], [8], where Γ was assumed to be a Kazhdan group. Consequently, the proofs of the theorems here will assume Γ to be a non-Kazhdan group, i.e. a lattice in a product of rank 1 simple groups.

We assume the reader is familiar with the elements of the theory of superrigidity for cocycles as presented in [5] and [9].

1. Amenable Ergodic Groupoids

Before beginning the actual proof of these theorems, it is necessary to discuss some related topics. First, we wish to show that the algebraic hull of the cocycle of a class of lattices cannot be amenable. To do this we will consider the properties of ergodic groupoids.

Let S be an ergodic G space. Then $S \times G$ naturally defines an ergodic groupoid $\mathcal{G}$. Let $R(\mathcal{G})$ be the naturally associated principal ergodic groupoid, i.e.

$$R(\mathcal{G}) = \{(s,\, t) | \exists g \ni s.g = t\} \subset S \times S.$$

Next, define $S(\mathcal{G}) = \{(s,\, g) | g \in G_s\} \subset S \times G$. Note that there exists natural maps $i\colon S(\mathcal{G}) \to \mathcal{G}$ and $p\colon \mathcal{G} \to R(\mathcal{G})$ defined by $i(s,\, g) = (s,\, g)$ and $p(s,\, g) = (s,\, s.g)$. As in [4], we have an exact sequence of ergodic groupoids

$$0 \to S(\mathcal{G}) \to \mathcal{G} \to R(\mathcal{G}) \to 0.$$

Recall the definition for an ergodic groupoid to be amenable: $\mathcal{G}$, a measured groupoid, is *amenable* if for every separable Banach space E, and for every cocycle $\phi\colon \mathcal{G} \to \mathrm{Iso}(E)$, the set of isometric automorphisms of E, and for every measurable field $\mathcal{U}_{\mathcal{G}} \to E_1^*$ invariant under ϕ (E_1^* is the unit ball in E^*, $\mathcal{U}_{\mathcal{G}}$ is the units in $\mathcal{G}$, a measurable field is an assignment for every $x \in \mathcal{U}_{\mathcal{G}}$, $A_x \subset E_1^*$ a compact convex subset such that $\{(x,\, A_x)\} \subset S \times E_1^*$ is measurable, and invariant meaning $\phi^*(x)A_{d(x)} = A_{r(x)}$, $d(x)$ the right identity of x, and $r(x)$ the left identity of x), then there exists a fixed point in $F(\mathcal{U}_{\mathcal{G}},\, \{A_x\})$, i.e. there exists a function $a\colon \mathcal{U}_{\mathcal{G}} \to E^*$ such that $a(x) \in A_x$ and $\phi^*(x)a(d(x)) = a(r(x))$.

LEMMA 3. *If $S(\mathcal{G})$ and $R(\mathcal{G})$ are amenable, then so is $\mathcal{G}$.*

Proof. Let ϕ and $\{(x,\, A_x)\}$ be as above. Since $S(\mathcal{G})$ is amenable, the set of $S(\mathcal{G})$ fixed points in $F(\mathcal{U}_{\mathcal{G}},\, \{A_x\})$ is nonempty. Call this set A_0. Thus $a \in A_0$ means $\phi^*(s,\, g)a(d(s,\, g)) = a(r(s,\, g))$ where $g \in G_s$. But we have that $d(s,\, g) = (s.g,\, e)$ and $r(s,\, g) = (s,\, e)$. Since $g \in G_s$, this translates to $\phi^*(s,\, g)a(s) = a(s)$ (viewing $a\colon S \to A_x$, since $S \cong \mathcal{U}_{\mathcal{G}}$). For the moment, assume that A_0 is a $\mathcal{G}$ invariant set in $F(S,\, \{A_s\})$. Since $S(\mathcal{G})$ fixes A_0, we have a natural action by $R(\mathcal{G})$ on $A_0 \subset F(S,\, \{A_s\})$ (note that $\mathcal{U}_{R(\mathcal{G})} \cong S$). Since $R(\mathcal{G})$ is amenable, there exists an $R(\mathcal{G})$ fixed point in a_0, we must then be a $\mathcal{G}$ fixed point. Therefore $\mathcal{G}$ is amenable.

Thus, it remains to show that A_0 is a $\mathcal{G}$ invariant set. We need to show that $g.a \in A_0$ provided $a \in A_0$. Since $(g.a)(s) = \alpha^*(s, g)a(s, g)$, this amounts to showing that $(g_s.(g.a))(s) = (g.a)(s)$ where $g_s \in G_s$. But we have

$$\begin{aligned}(g_s(g.a))(s) &= \alpha^*(s, g_s)\alpha^*(s, g)\phi(sg_sg) \\ &= \alpha^*(s, g_sg)\phi(sg_sg) \quad (\text{since } g_s \in G_s) \\ &= \alpha^*(s, gg_{sg})\phi(sgg_{sg}) \quad (\text{where } g_{sg} \in G_{sg}) \\ &= \alpha^*(s, g)(\alpha^*(sg, g_{sg})\phi(sg)) \\ &= \alpha^*(s, g)\phi(sg) = (g.a)(s),\end{aligned}$$

thus completing the proof. □

We will now make use of Lemma 3 to prove the following.

THEOREM 4. *Suppose S is an ergodic G-space, and $\alpha: S \times G \to H$ is a cocycle. Suppose also that G acts tamely on $S \times_\alpha H$. If H is amenable and all stabilizers of G on S are amenable, then G acts amenably on S.*

Proof. Note that there exists a stable orbit equivalence between the G action on S and the H action on $(S \times_\alpha H)/G$. This follows simply because there is a 1–1 correspondence between the G orbits on S and the H orbits on $(S \times_\alpha H)/G$. If G acts tamely on $S \times_\alpha H$ then the Mackey range of the G action on S is the H action on $(S \times_\alpha H)/G$. This provides a stable orbit equivalence between the H action on the Mackey range and $R(\mathcal{G})$, which is just the groupoid formed from the G orbits on S. Since stable orbit equivalence preserves amenability, the amenability of H implies that $R(\mathcal{G})$ is amenable. Since all the stabilizers of the G action on S are amenable, we have that $S(\mathcal{G})$ is also amenable. The result then follows immediately from Lemma 3. □

We now provide one more result that will be useful in the proof of Theorem 1.

PROPOSITION 5. *Let $\Gamma \subset G^0_{\mathbb{R}}$ be an irreducible noncocompact lattice where $\mathbb{R}$-rank$(G) \geq 2$. If $\phi: \Gamma \to K$ is a homomorphism where K is a compact Lie group, then ϕ has finite image.*

Proof. We may assume that $\overline{\phi(\Gamma)} = K$. Since Γ is F-simple, it suffices to show that the kernel of ϕ is infinite. If G has a nontrivial center, we obtain a map

$$\Phi: \Gamma/(Z(G) \cap \Gamma) \to K/\phi(Z(G) \cap \Gamma).$$

Since $Z(G) \cap \Gamma$ is finite and normal, and as $\overline{\phi(\Gamma)} = K$, $\phi(Z(G) \cap \Gamma)$ is also finite and normal. Hence if Φ has finite image, so does ϕ. So we may reduce to the case where G has trivial center. By [5, Th. 6.1.10], we may assume that Γ is the $\mathbb{Z}$-points of a semisimple $\mathbb{Q}$-group H, and if $K = L_{\mathbb{R}}$ for some $\mathbb{R}$-group L, then $\phi: H \to L$ is an $\mathbb{R}$-map. Since Γ is a noncocompact lattice, by [5, 6.1.9], $H_{\mathbb{Q}}$ has nontrivial unipotent elements, and hence so must $H_{\mathbb{Z}}$. In fact, $H_{\mathbb{Z}}$ must then contain an infinite number of nontrivial unipotents. Since the image of a unipotent element under ϕ must again be a unipotent, and as K is compact, we have that all the unipotents in $H_{\mathbb{Z}}$ are in the kernel of ϕ. Thus the kernel is infinite. □

2. Proof of Theorem 1

Since Γ acts ergodically, and finite actions cannot be ergodic on manifolds of positive dimension, to demonstrate the action is trivial it suffices to see that it is finite. Let $\alpha: M \times \Gamma \to \mathrm{SL}(2, \mathbb{R})$ be the cocycle determined by the natural induced action of Γ on $P(M)$, the frame bundle over M. Since the Γ action preserves a volume density, we may safely assume that the image of α does lie inside SL(2, $\mathbb{R}$). Let H be the algebraic hull of the cocycle α. If H is compact (i.e. $H = \mathrm{O}(2, \mathbb{R})$), the Γ action on M must preserve a measurable Riemannian metric on M, [9, Prop. 2.2]. In addition, Γ preserves a connection, so by [9, Th. 2.5], Γ preserves a smooth Riemannian metric on M. Hence, there exists a homomorphism $\Theta: \Gamma \to \mathrm{Isom}(M)$. As M is compact and 2-dimensional, Isom(M) is compact and at most 3-dimensional. Thus $\overline{\Theta(\Gamma)}$ is either abelian or $\mathrm{O}(3, \mathbb{R})$ (or $\mathrm{SO}(3, \mathbb{R})$). In the former, since Γ is F-simple, $\Theta(\Gamma)$ must be finite. In the latter case, [2, Th. 2.3.1] implies that M must be either S^2 or $P_2(\mathbb{R})$, 2-dimensional projective space. Note that Proposition 5 assures us that this cannot happen in the noncocompact case.

So, we may assume that H is noncompact. Since Γ acts preserving a connection, we have a homomorphism $\phi: \Gamma \to A(M)$, the affine transformations on M. Since there is a natural inclusion of $A(M)$ in Isom $(P(M))$, we have a homomorphism $\Phi: \Gamma \to \mathrm{Isom}(P(M))$. Let $L = \overline{(\Phi(\Gamma))}^0$ and $\Lambda = \Phi(\Gamma) \cap \overline{(\Phi(\Gamma))}^0$. Since Λ is normal in $\Phi(\Gamma)$, using the F-simplicity of Γ, we have Λ is either finite or of finite index in $\Phi(\Gamma)$. So, we may assume that Λ is either trivial or $\Phi(\Gamma)$.

If Λ is trivial, then $\Phi(\Gamma)$ is closed in $\mathrm{Isom}(P(M))$, and therefore Γ acts properly on $P(M)$. Discreteness of Γ then implies that Γ acts tamely with finite stabilizers. Since $H \subset \mathrm{SL}(2, \mathbb{R})$ and is noncompact, H is either SL(2, $\mathbb{R}$) or is amenable. If H is SL(2), applying Superrigidity we have that α is equivalent to a cocycle corresponding to a homomorphism $\beta: \Gamma \to \mathrm{SL}(2)$, which extends to a homomorphism $\beta: G \to \mathrm{SL}(2)$. This is equivalent to the existence of a measurable trivialization of $P(M)$ to $M \times \mathrm{SL}(2)$ such that Γ acts by the product action (where Γ acts on SL(2) via β). By [1], the existence of such a β is impossible unless the Lie algebra of G consists of simple Lie algebras all of whose complexifications equal sl(2, $\mathbb{C}$). In this case, irreducibility of Γ implies that $\beta(\Gamma)$ is dense in SL(2). Hence, there exists a sequence of elements $\{\gamma_n\} \to \infty$ in Γ such that $\{\beta(\gamma_n)\} \to 1$ in SL(2). Select a compact set K containing 1 in SL(2) of nonzero Haar measure. Then for sufficiently large N, $n \geq N$ implies $\gamma_n.K \cap K \neq \emptyset$. Hence, $\gamma_n.(M \times K) \cap (M \times K) \neq \emptyset$ for sufficiently large n, contradicting the properness assumption.

If H is amenable, noting that the stabilizers are finite and therefore amenable, then applying Theorem 4, Γ acts amenably on M. However, M is compact and therefore of finite volume, so Γ itself must be amenable. Since Γ is a lattice in G, this is clearly impossible.

The final case to consider is that when $\Lambda = \Phi(\Gamma)$, i.e. $\Phi(\Gamma) \subset L$. By [3, Th. 9.6.15], L must be semisimple. Since H is noncompact, L must also be noncompact. (L compact implies the existence of a smooth invariant Riemannian metric on M, which, by [9, Prop. 2.4] implies α is equivalent to a cocycle into $O(2, \mathbb{R})$. Hence, H is compact.) Thus, we may apply Superrigidity to obtain a homomorphism $G \to L$. L must therefore contain a group locally isomorphic to a simple component G' of G, and so must $\overline{\Gamma} \subset A(M)$. By [10, Cor. 3.6], there is an open dense conull set of this subgroup for which the stabilizers are discrete. But this is impossible since any such G' is at least 3 dimensional and M is only 2 dimensional.

3. Proof of Theorem 2

Retaining the notation from the proof of Theorem 1 it will suffice to show that L must be compact. Again, we have two cases to consider: either Λ is trivial or equals $\Phi(\Gamma)$. In the former, we may rule out the possibility that H is amenable (using Theorem 4), hence, we once again obtain a homomorphism $\beta: G \to H \subset SO(1, n-1)$. Thus, H must contain a subgroup locally isomorphic to a simple factor of G. As in [8, Th. 4.1], this must be locally isomorphic to either $SO(1, m)$ or $SU(1, m)$. But, again, irreducibility of Γ implies that $\beta(\Gamma)$ is dense in this subgroup. As in Theorem 1, this is impossible.

If $\Lambda = \Phi(\Gamma)$, since L is semisimple and noncompact, we have by Superrigidity a homomorphism $G \to L$. By [8, Th. B], either $L \subset SL(2, \mathbb{R}) \times K$, where K is a compact Lie group, or L is amenable. F-simplicity of Γ rules out the latter possibility, so $L = SL(2, \mathbb{R}) \times K'$. However, this is impossible unless G is of the requisite form (using the main result from [1] and Superrigidity).

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