Geometriae Dedicata **91:** 175–196, 2002.

Collapsing and Dirac-Type Operators

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(Received: 20 March 2001)

Abstract. We analyze the limit of the spectrum of a geometric Dirac-type operator under a collapse with bounded diameter and bounded sectional curvature. In the case of a smooth limit space B, we show that the limit of the spectrum is given by the spectrum of a certain first-order differential operator on B, which can be constructed using superconnections. In the case of a general limit space X, we express the limit operator in terms of a transversally elliptic operator on a G-manifold $\check{X}$ with $X = \check{X}/G$. As an application, we give a characterization of manifolds which do not admit uniform upper bounds, in terms of diameter and sectional curvature, on the k-th eigenvalue of the square of a Dirac-type operator. We also give a formula for the essential spectrum of a Dirac-type operator on a finite-volume manifold with pinched negative sectional curvature.

Mathematics Subject Classifications (2000). 58C40, 58J50, 35P15.

Key words. collapsing, Dirac operator, eigenvalues.

1. Introduction

In previous papers we analyzed the limit of the spectrum of the differential form Laplacian on a manifold, under a collapse with bounded diameter and bounded sectional curvature [17, 22]. In the present paper, we extend the analysis of [17, 22] to geometric Dirac-type operators. As the present paper is a sequel to [17, 22], we refer to the introduction of [17] for background information about collapsing with bounded curvature and its relation to analytic questions.

Let M be a connected closed oriented Riemannian manifold of dimension $n > 0$. If M is spin then we put $G = \mathrm{Spin}(n)$ and if M is not spin then we put $G = \mathrm{SO}(n)$. The spinor-type fields that we consider are sections of a vector bundle E^M associated to a G-Clifford module V, the latter being in the sense of Definition 2 of Section 2. The ensuing Dirac-type operator D^M acts on sections of E^M. We will think of the spectrum $\sigma(D^M)$ of D^M as a set of real numbers with multiplicities, corresponding to possible multiple eigenvalues. For simplicity, in this introduction we will sometimes refer to the Dirac-type operators as acting on spinors, even though the results are more general.

* Research supported by NSF grant DMS-9704633

We first consider a collapse in which the limit space is a smooth Riemannian manifold. The model case is that of a Riemannian affine fiber bundle.

DEFINITION 1. An affine fiber bundle is a smooth fiber bundle π: $M \to B$ whose fiber Z is an infranilmanifold and whose structure group can be reduced from $\mathrm{Diff}(Z)$ to $\mathrm{Aff}(Z)$. A Riemannian affine fiber bundle is an affine fiber bundle along with

- A horizontal distribution $T^H M$ whose holonomy lies in $\mathrm{Aff}(Z)$,
- A family of vertical Riemannian metrics g^{TZ} which are parallel with respect to the flat affine connections on the fibers Z_b and
- A Riemannian metric g^{TB} on B.

Given a Riemannian affine fiber bundle π: $M \to B$, there is a Riemannian metric g^{TM} on M constructed from $T^H M$, g^{TZ} and g^{TB}. Let R^M denote the Riemann curvature tensor of (M, g^{TM}), let Π denote the second fundamental forms of the fibers $\{Z_b\}_{b \in B}$ and let $T \in \Omega^2(M; TZ)$ be the curvature of $T^H M$. Given $b \in B$, there is a natural flat connection on $E^M\big|_{Z_b}$ which is constructed using the affine structure of Z_b. We define a Clifford bundle E^B on B whose fiber over $b \in B$ consists of the parallel sections of $E^M\big|_{Z_b}$. The operator D^M restricts to a first-order differential operator D^B on $C^\infty(B; E^B)$. If V happens to be the spinor module then we show that D^B is the 'quantization' of a certain superconnection on B. For general V, there is an additional zeroth-order term in D^B which depends on Π and T.

We show that the spectrum of D^M coincides with that of D^B up to a high level, which depends on the maximum diameter $\mathrm{diam}(Z)$ of the fibers $\{Z_b\}_{b \in B}$.

THEOREM 1. *There are positive constants A, A' and C which only depend on n and V such that if $\| R^Z \|_\infty \,\mathrm{diam}\,(Z)^2 \leqslant A'$ then the intersection of $\sigma(D^M)$ with the interval*

$$[-(A\,\mathrm{diam}\,(Z)^{-2} - C(\| R^M \|_\infty + \| \Pi \|_\infty^2 + \| T \|_\infty^2))^{1/2},\quad (A\,\mathrm{diam}\,(Z)^{-2} - C(\| R^M \|_\infty + \| \Pi \|_\infty^2 + \| T \|_\infty^2))^{1/2}] \tag{1.1}$$

equals the intersection of $\sigma(D^B)$ with (1.1).

If $Z = S^1$, $\Pi = 0$ and V is the spinor module then we recover some results of [1, Section 4]; see also [12, Theorem 1.5]. The proof of Theorem 1 follows the same strategy as the proof of the analogous [17, Theorem 1]. Consequently, in the proof of Theorem 1, we only indicate the changes that need to be made in the proof of [17, Theorem 1] and refer to [17] for details.

Given B, Cheeger, Fukaya and Gromov showed that under some curvature bounds, any Riemannian manifold M which is sufficiently Gromov–Hausdorff close to B can be well approximated by a Riemannian affine fiber bundle [11]. Using this fact, we show that the spectrum of D^M can be uniformly approximated by that of a certain first-order differential operator D^B on B, at least up to a high level which depends on the Gromov-Hausdorff distance between M and B.

Given $\varepsilon > 0$ and two collections of real numbers $\{a_i\}_{i \in I}$ and $\{b_j\}_{j \in J}$, we say that $\{a_i\}_{i \in I}$ and $\{b_j\}_{j \in J}$ are ε-close if there is a bijection $\alpha\colon I \to J$ such that for all $i \in I$, $|b_{\alpha(i)} - a_i| \leqslant \varepsilon$.

THEOREM 2. *Let B be a fixed smooth connected closed Riemannian manifold. Given $n \in \mathbb{Z}^+$, take $G \in \{\mathrm{SO}(n), \mathrm{Spin}(n)\}$ and let V be a G-Clifford module. Then for any $\varepsilon > 0$ and $K > 0$, there are positive constants $A(B, n, V, \varepsilon, K)$, $A'(B, n, V, \varepsilon, K)$, and $C(B, n, V, \varepsilon, K)$ so that the following holds. Let M be an n-dimensional connected closed oriented Riemannian manifold with a G-structure such that $\| R^M \|_\infty \leqslant K$ and $d_{GH}(M, B) \leqslant A'$. Then there are a Clifford module E^B on B and a certain first-order differential operator D^B on $C^\infty(B; E^B)$ such that*

(1) $\{\sinh^{-1}(\lambda/\sqrt{2K})\colon \lambda \in \sigma(D^M)$ *and* $\lambda^2 \leqslant \mathrm{A}\, d_{GH}(M, B)^{-2} - C\}$ *is ε-close to a subset of* $\{\sinh^{-1}(\lambda/\sqrt{2K})\colon \lambda \in \sigma(D^B)\}$, *and*

(2) $\{\sinh^{-1}(\lambda/\sqrt{2K})\colon \lambda \in \sigma(D^B)$ *and* $\lambda^2 \leqslant \mathrm{A}\, d_{GH}(M, B)^{-2} - C\}$ *is ε-close to a subset of* $\{\sinh^{-1}(\lambda/\sqrt{2K})\colon \lambda \in \sigma(D^M)\}$.

The other results in this paper concern collapsing to a possibly-singular space. Let X be a limit space of a sequence $\{M_i\}_{i=1}^\infty$ of n-dimensional connected closed oriented Riemannian manifolds with uniformly bounded diameter and uniformly bounded sectional curvature. In general, X is not homeomorphic to a manifold. However, Fukaya showed that X is homeomorphic to $\check{X}/G$, where $\check{X}$ is a manifold and G is a compact Lie group which acts on $\check{X}$ [15]. This comes from writing $M_i = P_i/G$, where $G = \mathrm{SO}(n)$ and P_i is the oriented orthonormal frame bundle of M_i. There is a canonical Riemannian metric on P_i. Then $\{P_i\}_{i=1}^\infty$ has a subsequence which Gromov–Hausdorff converges to a manifold $\check{X}$. As the convergence argument can be done G-equivariantly, the corresponding subsequence of $\{M_i\}_{i=1}^\infty$ converges to $X = \check{X}/G$. In general, $\check{X}$ is a smooth manifold with a metric which is $C^{1,\alpha}$ regular for all $\alpha \in (0, 1)$.

In [22] we dealt with the limit of the spectra of the differential form Laplacians $\{\triangle^{M_i}\}_{i=1}^\infty$ on the manifolds $\{M_i\}_{i=1}^\infty$. We defined a limit operator $\triangle^X$ which acts on the 'differential forms' on X, coupled to a superconnection. In order to make this precise, we defined the 'differential forms' on X to be the G-basic differential forms on $\check{X}$. We constructed the corresponding differential form Laplacian $\triangle^X$ and showed that its spectrum described the limit of the spectra of $\{\triangle^{M_i}\}_{i=1}^\infty$. We refer to [22] for the precise statements.

In the case of geometric Dirac-type operators D^{M_i}, there is a fundamental problem in extending this approach. Namely, if $\check{X}$ is a spin manifold on which a compact Lie group G acts isometrically and preserving the spin structure then there does not seem to be a notion of G-basic spinors on $\check{X}$. In order to get around this problem, we take a different approach. For a given n-dimensional Riemannian spin manifold

M, put $G = \mathrm{Spin}(n)$, let P be the principal $\mathrm{Spin}(n)$-bundle of M and let V be the spinor module. One can identify the spinor fields on M with $(C^\infty(P) \otimes V)^G$, the G-invariant subspace of $C^\infty(P) \otimes V$. There are canonical horizontal vector fields $\{\mathfrak{Y}_j\}_{j=1}^n$ on P and the Dirac operator takes the form $D^M = -i\sum_{j=1}^n \gamma^j \mathfrak{Y}_j$. Furthermore, $(D^M)^2$ can be written in a particularly simple form. As in equation (4.2) below, when acting on $(C^\infty(P) \otimes V)^G$, $(D^M)^2$ becomes the scalar Laplacian on P (acting on V-valued functions) plus a zeroth-order term.

Following this viewpoint, it makes sense to define the limiting 'spinor fields' on X to be the elements of $(C^\infty(\check{X}) \otimes V)^G$. We can then extend Theorem 1 to the setting of G-equivariant Riemannian affine fiber bundles. Namely, the limit operator D^X turns out to be a G-invariant first-order differential operator on $C^\infty(\check{X}) \otimes V$, transversally elliptic in the sense of Atiyah [2], which one then restricts to the G-invariant subspace $(C^\infty(\check{X}) \otimes V)^G$. In Theorem 6 below, we show that the analog of Theorem 1 holds, in which D^B is replaced by D^X.

Theorem 6 refers to a given G-equivariant Riemannian affine fiber bundle. In order to deal with arbitrary collapsing sequences, we use the aforementioned representation of $(D^M)^2$ as a Laplace-type operator on P. If $\{M_i\}_{i=1}^\infty$ is a sequence of n-dimensional Riemannian manifolds with uniformly bounded diameter and uniformly bounded sectional curvature then we show that after taking a subsequence, the spectra of $\{(D^{M_i})^2\}_{i=1}^\infty$ converge to the spectrum of a Laplace-type operator on a limit space. Let $\{\lambda_k(|D^M|)\}_{k=1}^\infty$ denote the eigenvalues of $|D^M|$, counted with multiplicity.

THEOREM 3. *Given $n \in \mathbb{Z}^+$ and $G \in \{\mathrm{SO}(n), \mathrm{Spin(n)}\}$, let $\{M_i\}_{i=1}^\infty$ be a sequence of connected closed oriented n-dimensional Riemannian manifolds with a G-structure. Let V be a G-Clifford module. Suppose that for some $D, K > 0$ and for each $i \in \mathbb{Z}^+$, we have $\mathrm{diam}\,(M_i) \leqslant D$ and $\| R^{M_i} \|_\infty \leqslant K$. Then there are*

(1) *A subsequence of $\{M_i\}_{i=1}^\infty$, which we relabel as $\{M_i\}_{i=1}^\infty$,*
(2) *A smooth closed G-manifold $\check{X}$ with a G-invariant Riemannian metric $g^{T\check{X}}$ which is $C^{1,\alpha}$-regular for all $\alpha \in (0, 1)$,*
(3) *A positive G-invariant function $\chi \in C(\check{X})$ with $\int_{\check{X}} \chi dvol = 1$ and*
(4) *A G-invariant function $\mathcal{V} \in L^\infty(\check{X}) \otimes \mathrm{End(V)}$ such that if $\triangle^{\check{X}}$ denotes the Laplacian on $L^2(\check{X}, \chi d\mathrm{vol}) \otimes \mathrm{V}$ [14, (0.8)] and $|D^X|$ denotes the operator $\sqrt{\triangle^{\check{X}} + \mathcal{V}}$ acting on $(L^2(\check{X}, \chi d\mathrm{vol}) \otimes \mathrm{V})^{\mathrm{G}}$ then for all $k \in \mathbb{Z}^+$,*

$$\lim_{i\to\infty} \lambda_k\big(|D^{M_i}|\big) = \lambda_k\big(|D^X|\big). \tag{1.2}$$

In the special case of the signature operator, the proof of Theorem 3 is somewhat simpler than that of the analogous [22, Proposition 3], in that we essentially only have to deal with scalar Laplacians. However, [22, Proposition 3] gives more detailed information. In particular, it expresses the limit operator in terms of a basic flat degree-1 superconnection on $\check{X}$. This seems to be necessary in order to prove the results of [22] concerning small eigenvalues. Of course, one does not expect to have analogous results concerning the small eigenvalues of general geometric Dirac-type operators, as their zero-eigenvalues have no topological meaning.

As an application of Theorem 3, we give a characterization of manifolds which do not have a uniform upper bound on the k-th eigenvalue of $|D^M|$, in terms of diameter and sectional curvature.

THEOREM 4. *Let M be a connected closed oriented manifold with a G-structure. Let V be a G-Clifford module. Suppose that for some $K > 0$ and $k \in \mathbb{Z}^+$, there is no uniform upper bound on $\lambda_k(|D^M|)$ among Riemannian metrics on M with* $\operatorname{diam}(M) = 1$ *and* $\| R^M \|_\infty \leqslant K$. *Then M is the total space of a possibly-singular affine fiber bundle $M \to X$ whose generic fiber is an infranilmanifold Z such that the restriction of E^M to Z does not have any nonzero affine-parallel sections.*

As a partial converse, let M be the total space of a smooth affine fiber bundle whose fiber is Z and whose base B has positive dimension. If the restriction of E^M to Z does not have any nonzero affine-parallel sections then there is some $K > 0$ such that for any $k \in \mathbb{Z}^+$, there is no uniform upper bound on $\lambda_k(|D^M|)$ among Riemannian metrics on M with $\operatorname{diam}(M) = 1$ *and* $\| R^M \|_\infty \leqslant K$.

More precisely, the possibly-singular affine fiber bundle $M \to X$ of Theorem 4 is the G-quotient of a G-equivariant affine fiber bundle $P \to \check{X}$. Theorem 4 is an analog of [22, Theorem 1.2]. A simple example of Theorem 4 comes from considering spinors on $M = S^1 \times N$, where N is a spin manifold and the spin structure on S^1 is the one that does not admit a harmonic spinor. Upon shrinking the S^1-fiber, the eigenvalues of D_M go off to $\pm\infty$.

Finally, we give a result about the essential spectrum of a geometric Dirac-type operator on a finite-volume manifold of pinched negative curvature, which is an analog of [19, Theorem 2]. Let M be a complete connected oriented n-dimensional Riemannian manifold with a G-structure. Suppose that M has finite volume and its sectional curvatures satisfy $-b^2 \leqslant K \leqslant -a^2$, with $0 < a \leqslant b$. Let V be a G-Clifford module. Label the ends of M by $I \in \{1, \ldots, N\}$. An end of M has a neighborhood U_I whose closure is homeomorphic to $[0, \infty) \times Z_I$, where the first coordinate is the Busemann function corresponding to a ray exiting the end, and Z_I is an infranilmanifold. Let E^M be the vector bundle on M associated to the pair (G, V) and let D^M be the corresponding Dirac-type operator. If U_I lies far enough out the end then for each $s \in [0, \infty)$, $C^\infty(\{s\} \times Z_I; E^M|_{\{s\}\times Z_I})$ decomposes as the direct sum of a finite-dimensional space $E^B_{I,s}$, consisting of 'bounded energy' sections, and its orthogonal complement, consisting of 'high energy' sections. The vector spaces $\{E^B_{I,s}\}_{s\in[0,\infty)}$ fit together to form a vector bundle E^B_I on $[0, \infty)$. Let P_0 be orthogonal projection from $\bigoplus_{I=1}^N C^\infty(\overline{U_I}; E^M|_{\overline{U_I}})$ to $\bigoplus_{I=1}^N C^\infty([0, \infty); E^B_I)$. Let D^M_{end} be the restriction of D^M to $\bigoplus_{I=1}^N C^\infty(\overline{U_I}; E^M|_{\overline{U_I}})$, say with Atiyah-Patodi-Singer boundary conditions. Then $P_0 D^M_{\mathrm{end}} P_0$ is a first-order ordinary differential operator on $\bigoplus_{I=1}^N C^\infty\big([0, \infty); E^B_I\big)$.

THEOREM 5. *The essential spectrum of* D^M *is the same as that of* $P_0 D^M_{\mathrm{end}} P_0$.

There is some intersection between Theorem 5 and the results of [4, Theorem 0.1], concerning the essential spectrum of D^M when $n = 2$ and under an additional curvature assumption, and [5, Theorem 1], concerning the essential spectrum of D^M when M is hyperbolic and V is the spinor module.

2. Dirac-type Operators and Infranilmanifolds

Given $n \in \mathbb{Z}^+$, let G be either $\mathrm{SO}(n)$ or $\mathrm{Spin}(n)$.

DEFINITION 2. A G-Clifford module consists of a finite-dimensional Hermitian G-vector space V and a G-equivariant linear map $\gamma : \mathbb{R}^n \to \mathrm{End}(V)$ such that $\gamma(v)^2 = |v|^2\mathrm{Id}.$ and $\gamma(v)^* = \gamma(v)$.

Let M be a connected closed oriented smooth n-dimensional Riemannian manifold. Put $G = \mathrm{Spin}(n)$ or $G = \mathrm{SO}(n)$, according as to whether or not M is spin. If M is spin, fix a spin structure. Let P be the corresponding principal G-bundle, covering the oriented orthonormal frame bundle. Its topological isomorphism class is independent of the choice of Riemannian metric. Given the Riemannian metric, there is a canonical $\mathbb{R}^n$-valued 1-form θ on P, the soldering form.

With respect to the standard basis $\{e_j\}_{j=1}^n$ of $\mathbb{R}^n$, we write $\gamma^j = \gamma(e_j)$. We also take generators $\{\sigma^{ab}\}_{a,b=1}^n$ for the representation of the Lie algebra $\mathfrak{g}$ on V, so that $\sigma^{ba} = -\sigma^{ab}$, $\left(\sigma^{ab}\right)^* = -\sigma^{ab}$ and

$$[\sigma^{ab}, \sigma^{cd}] = \delta^{ad}\sigma^{bc} - \delta^{ac}\sigma^{bd} + \delta^{bc}\sigma^{ad} - \delta^{bd}\sigma^{ac}. \tag{2.1}$$

The G-equivariance of γ implies

$$[\gamma^a, \sigma^{bc}] = \delta^{ab}\gamma^c - \delta^{ac}\gamma^b. \tag{2.2}$$

EXAMPLES. (1) If $G = \mathrm{Spin}(n)$ and V is the spinor representation of G then $\sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$.

(2) If $G = \mathrm{SO}(n)$ and $V = \Lambda^*(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C}$, let E^j and I^j denote exterior and interior multiplication by e^j, respectively. Put $\gamma^j = i\left(E^j - I^j\right)$ and $\widehat{\gamma}^j = E^j + I^j$. Then $\sigma^{ab} = \frac{1}{4}\left([\gamma^a, \gamma^b] + [\widehat{\gamma}^a, \widehat{\gamma}^b]\right)$.

Put $E^M = P \times_G V$. The Dirac-type operator D^M acts on the space $C^\infty(M; E^M)$. As the topological vector space $C^\infty(M; E^M)$ is independent of any choice of Riemannian metric on M, it makes sense to compare Dirac-type operators for different Riemannian metrics on M; see [18, Section 2] for further discussion.

Let g^{TM} be the Riemannian metric on M. Let ω be the Levi-Civita connection on P. Let $\{e_j\}_{j=1}^n$ be a local oriented orthonormal basis of TM, with dual basis

$\{\tau^j\}_{j=1}^n$. Then we can write ω locally as a matrix-valued 1-form $\omega_b^a = \sum_{j=1}^n \omega_{bj}^a \tau^j$, and

$$D^M = -i \sum_{j=1}^n \gamma^j \nabla_{e_j} = -i \sum_{j=1}^n \gamma^j \left(e_j + \frac{1}{2} \sum_{a,b=1}^n \omega_{abj} \sigma^{ab} \right). \tag{2.3}$$

We have the Bochner-type equation

$$(D^M)^2 = \nabla^* \nabla - \frac{1}{8} \sum_{a,b,i,j=1}^n R_{abij}^M (\gamma^i \gamma^j - \gamma^j \gamma^i) \sigma^{ab}. \tag{2.4}$$

As the set of Riemannian metrics on M is an open convex subset of a Fréchet space, it makes sense to talk about an analytic 1-parameter family $\{c(t)\}_{t \in [0,1]}$ of metrics. Then for $t \in [0, 1]$, $\dot{c}(t)$ is a symmetric 2-tensor on M. Let $\| \dot{c}(t) \|_{c(t)}$ denote the norm of $\dot{c}(t)$ with respect to $c(t)$, i.e.

$$\| \dot{c}(t) \|_{c(t)} = \sup_{v \in TM - 0} \frac{|\dot{c}(t)(v, v)|}{c(t)(v, v)}. \tag{2.5}$$

Put $l(c) = \int_0^1 \| \dot{c}(t) \|_{c(t)} \, dt$. We extend the definition of $l(c)$ to piecewise-analytic families of metrics in the obvious way. Given $K > 0$, let $\mathcal{M}(M, K)$ be the set of Riemannian metrics on M with $\| R^M \|_\infty \leqslant K$. Let d be the corresponding length metric on $\mathcal{M}(M, K)$, computed using piecewise-analytic paths in $\mathcal{M}(M, K)$. Let $\sigma(D^M, g^{TM})$ denote the spectrum of D^M as computed with g^{TM}, a discrete subset of $\mathbb{R}$ which is counted with multiplicity.

PROPOSITION 1. *There is a constant $C = C(n, V) > 0$ such that for all $K > 0$ and $g_1^{TM}, g_2^{TM} \in \mathcal{M}(M, K)$,*

$$\left\{ \sinh^{-1}\left(\frac{\lambda}{\sqrt{K}} \right) : \lambda \in \sigma(D^M, g_1^{TM}) \right\} \tag{2.6}$$

and

$$\left\{ \sinh^{-1}\left(\frac{\lambda}{\sqrt{K}} \right) : \lambda \in \sigma(D^M, g_2^{TM}) \right\} \tag{2.7}$$

are $Cd(g_1^{TM}, g_2^{TM})$-close.

Proof. It is enough to show that there is a number C such that if $\{c(t)\}_{t \in [0,1]}$ is an analytic 1-parameter family of metrics contained in $\mathcal{M}(M, K)$ then

$$\left\{ \sinh^{-1}\left(\frac{\lambda}{\sqrt{K}} \right) : \lambda \in \sigma(D^M, c(0)) \right\}$$

and

$$\left\{ \sinh^{-1}\left(\frac{\lambda}{\sqrt{K}} \right) : \lambda \in \sigma(D^M, c(1)) \right\}$$

are $Cd(c(0), c(1))$-close. By eigenvalue perturbation theory [20, Chapter XII], the subset $\bigcup_{t\in[0,1]}\{t\} \times \sigma(D^M, c(t))$ of $\mathbb{R}^2$ is the union of the graphs of functions $\{\lambda_j(t)\}_{j\in\mathbb{Z}}$ which are analytic in t. Thus it is enough to show that for each $j \in \mathbb{Z}$,

$$\left|\sinh^{-1}\left(\frac{\lambda_j(1)}{\sqrt{K}}\right) - \sinh^{-1}\left(\frac{\lambda_j(0)}{\sqrt{K}}\right)\right| \leqslant Cl(c). \tag{2.8}$$

Let $D(t)$ denote the Dirac-type operator constructed with the metric $c(t)$. It is self-adjoint when acting on $L^2(E^M, d\mathrm{vol}(t))$. In order to have all of the operators $\{D(t)\}_{t\in[0,1]}$ acting on the same Hilbert space, define $f(t) \in C^\infty(M)$ by $f(t) = d\mathrm{vol}(t)/d\mathrm{vol}(0)$. Then the spectrum of $D(t)$, acting on $L^2(E^M, d\mathrm{vol}(t))$, is the same as the spectrum of the self-adjoint operator $f(t)^{1/2}D(t)f(t)^{-1/2}$ acting on $L^2(E^M, d\mathrm{vol}(0))$. One can now compute $d\lambda_j/dt$ using eigenvalue perturbation theory, as in [20, Chapter XII]. Let $\psi_j(t)$ be a smoothly-varying unit eigenvector whose eigenvalue is $\lambda_j(t)$. Define a quadratic form $T(t)$ on TM by

$$\begin{aligned} T(t)(X, Y) = {}& \langle\psi_j, -i(\gamma(X)\nabla_Y\psi_j + \gamma(Y)\nabla_X\psi_j)\rangle + \\ & + \langle -i(\gamma(X)\nabla_Y\psi_j + \gamma(Y)\nabla_X\psi_j), \psi_j\rangle. \end{aligned} \tag{2.9}$$

Using the metric $c(t)$ to convert the symmetric tensors $\dot{c}(t)$ and $T(t)$ to self-adjoint sections of $End(TM)$, one finds

$$\frac{\mathrm{d}\lambda_\mathrm{j}}{\mathrm{d}t} = -\frac{1}{8}\int_M \mathrm{Tr}(\dot{c}(t)T(t))d\mathrm{vol}(t). \tag{2.10}$$

(This equation was shown for the pure Dirac operator, by different means, in [10].) Then

$$\left|\frac{\mathrm{d}\lambda_j}{\mathrm{d}t}\right| \leqslant \text{const.} \parallel \dot{c}(t) \parallel_{c(t)} \int_M \mathrm{Tr}(|T(t)|)d\mathrm{vol}(t). \tag{2.11}$$

Letting $\{x_i\}_{i=1}^n$ be an orthonormal basis of eigenvectors of $T(t)$ at a point $m \in M$, we have $\mathrm{Tr}(|T(t)|) = \sum_{i=1}^n |T(t)(x_i, x_i)|$. Then from (2.9), we obtain

$$\int_M \mathrm{Tr}(|T(t)|)d\mathrm{vol}(t) \leqslant \text{const.}\left(\int_M |\nabla\psi_j|^2 d\mathrm{vol}(t)\right)^{1/2}. \tag{2.12}$$

From (2.4),

$$\int_M |\nabla\psi_j|^2 d\mathrm{vol}(t) \leqslant \lambda_j^2 + \text{const. } K. \tag{2.13}$$

In summary, from (2.11), (2.12) and (2.13), there is a positive constant C such that

$$\left|\frac{\mathrm{d}\lambda_j}{\mathrm{d}t}\right| \leqslant C \parallel \dot{c}(t) \parallel_{c(t)} \left(\lambda_j^2 + K\right)^{1/2}. \tag{2.14}$$

Integration gives Equation (2.8). The proposition follows. □

For some basic facts about infranilmanifolds, we refer to [17, Section 3]. Let N be a simply-connected connected nilpotent Lie group. Let Γ be a discrete subgroup of $\mathrm{Aff}(N)$ which acts freely and cocompactly on N, with $\Gamma \cap N$ of finite index in Γ. Put $Z = \Gamma\backslash N$, an infranilmanifold. There is a canonical flat linear connection ∇^{aff} on TZ. Put $\widehat{\Gamma} = \Gamma \cap N$, a cocompact subgroup of N. There is a short exact sequence

$$1 \longrightarrow \widehat{\Gamma} \longrightarrow \Gamma \longrightarrow F \longrightarrow 1, \tag{2.15}$$

with F a finite group. Put $\widehat{Z} = \widehat{\Gamma}\backslash N$, a nilmanifold which finitely covers Z with covering group F.

Let g^{TZ} be a Riemannian metric on Z which is parallel with respect to ∇^{aff}. Let us discuss the condition for Z to be spin. Suppose first that Z is spin. Choose a spin structure on Z. Fix the basepoint $z_0 = \Gamma e \in Z$. As ∇^{aff} preserves g^{TZ}, its holonomy lies in $\mathrm{SO}(n)$. Hence ∇^{aff} lifts to a flat connection on the principal $\mathrm{Spin}(n)$-bundle, which we also denote by ∇^{aff}. There is a corresponding holonomy representation $\Gamma \to \mathrm{Spin}(n)$.

Conversely, suppose that we do not know *a priori* if Z is spin. Suppose that the affine holonomy $\Gamma \to F \to \mathrm{SO}(n)$ lifts to a homomorphism $\Gamma \to \mathrm{Spin}(n)$. Naturally, the existence of this lifting is independent of the particular choice of g^{TZ}. Then there is a corresponding spin structure on Z with principal bundle $\Gamma\backslash(N \times \mathrm{Spin}(n))$. The different spin structures on Z correspond to different lifts of $\Gamma \to \mathrm{SO}(n)$ to $\Gamma \to \mathrm{Spin}(n)$. These are labelled by $H^1(\Gamma; \mathbb{Z}_2) \cong H^1(Z; \mathbb{Z}_2)$. Note that there are examples of nonspin flat manifolds [3]. Also, even if Z is spin and has a fixed spin structure, the action of $\mathrm{Aff}(Z)$ on Z generally does not lift to the principal $\mathrm{Spin}(n)$-bundle, as can be seen for the $SL(n, \mathbb{Z})$-action on $Z = T^n$.

Now let G be either $\mathrm{SO}(n)$ or $\mathrm{Spin}(n)$. Let V be a G-Clifford module. Suppose that Z has a G-structure. If $G = \mathrm{SO}(n)$ then we have the affine holonomy homomorphism $\rho\colon \Gamma \to \mathrm{SO}(n)$. If $G = \mathrm{Spin}(n)$ then we have a given lift of it to $\rho\colon \Gamma \to \mathrm{Spin}(n)$. In either case, there is an action of Γ on V coming from $\Gamma \xrightarrow{\rho} G \to Aut(V)$. The vector bundle E^Z can now be written as $E^Z = \Gamma\backslash(N \times V)$. We see that the vector space of sections of E^Z which are parallel with respect to ∇^{aff} is isomorphic to V^Γ, the subspace of V which is fixed by the action of Γ.

If V is the spinor representation of $G = \mathrm{Spin}(n)$ then let us consider the conditions for V^Γ to be nonzero. First, as the restriction of $\rho\colon \Gamma \to \mathrm{Spin}(n)$ to $\widehat{\Gamma}$ maps $\widehat{\Gamma}$ to ± 1, we must have $\rho\big|_{\widehat{\Gamma}} = 1$. Given this, the homomorphism ρ factors through a homomorphism $F \to \mathrm{Spin}(n)$. Then we have $V^\Gamma = V^F$. This may be nonzero even if the homomorphism $F \to \mathrm{Spin}(n)$ is nontrivial.

Returning to the case of general V, as g^{TZ} is parallel with respect to ∇^{aff}, the operator D^Z preserves the space V^Γ of affine-parallel sections of E^Z. Let D^{inv} be the restriction of D^Z to V^Γ.

PROPOSITION 2. *There are positive constants A and A' depending only on* $\dim(Z)$ *and V such that if* $\| R^Z \|_\infty \operatorname{diam}(Z)^2 \leqslant A'$ *then the spectrum $\sigma(D^Z)$ of D^Z satisfies*

$$\sigma(D^Z) \cap [-A \operatorname{diam}(Z)^{-1}, A \operatorname{diam}(Z)^{-1}] \\ = \sigma(D^{inv}) \cap [-A \operatorname{diam}(Z)^{-1}, A \operatorname{diam}(Z)^{-1}]. \tag{2.16}$$

Proof. As D^Z is diagonal with respect to the orthogonal decomposition

$$C^\infty(Z; E^Z) = V^\Gamma \oplus \left(V^\Gamma\right)^\perp, \tag{2.17}$$

it is enough to show that there are constants A and A' as in the statement of the proposition such that the eigenvalues of $(D^Z)^2\big|_{(V^\Gamma)^\perp}$ are greater than $A^2 \operatorname{diam}(Z)^{-2}$. As in the proof of [17, Proposition 2], we can reduce to the case when $F = \{e\}$, i.e. Z is a nilmanifold $\Gamma \backslash N$. Then

$$C^\infty(Z; E^Z) \cong (C^\infty(N) \otimes V)^\Gamma. \tag{2.18}$$

Using an orthonormal frame $\{e_i\}_{i=1}^{\dim(Z)}$ for the Lie algebra $\mathfrak{n}$ as in the proof of [17, Proposition 2], we can write

$$\nabla^{aff}_{e_i} = e_i \otimes \mathrm{Id}. \tag{2.19}$$

and

$$\nabla^Z_{e_i} = (e_i \otimes \mathrm{Id.}) + \left(\mathrm{Id.} \otimes \frac{1}{2} \sum_{a,b=1}^{\dim(Z)} \omega_{abi} \sigma^{ab} \right). \tag{2.20}$$

The rest of the proof now proceeds as in that of [17, Proposition 2], to which we refer for details. □

3. Collapsing to a Smooth Base

For background information about superconnections and their applications, we refer to [7]. Let M be a connected closed oriented Riemannian manifold which is the total space of a Riemannian submersion $\pi: M \to B$. Suppose that M has a G^M-structure and that V^M is a G^M-Clifford module, as in Section 2. If $G^M = \mathrm{SO}(n)$, put $G^Z = \mathrm{SO}(\dim(Z))$ and $G^B = \mathrm{SO}(\dim(B))$. If $G^M = \mathrm{Spin}(n)$, put $G^Z = \mathrm{Spin}(\dim(Z))$ and $G^B = \mathrm{Spin}(\dim(B))$. As a fiber Z_b has a trivial normal bundle in M, it admits a G^Z-structure. Fixing an orientation of T_bB fixes the G^Z-structure of Z_b. Note, however, that B does not necessarily have a G^B-structure. For example, if M is oriented then B is not necessarily oriented, as is shown in the example of $S^1 \times_{\mathbb{Z}_2} S^2 \to \mathbb{R}P^2$, where the generator of $\mathbb{Z}_2$ acts on S^1 by complex conjugation and on S^2 by the antipodal map. And if M is spin then B is not necessarily spin, as is shown in the example of $S^5 \to \mathbb{C}P^2$. What is true is that if the vertical tangent bundle TZ, a vector bundle on M, has a G^Z-structure then B has a G^B-structure.

Put $E^M = P \times_{G^M} V^M$. There is a Clifford bundle C on B with the property that $C^\infty(B; C) \cong C^\infty(M; E^M)$ [7, Section 9.2]. If $\dim(Z) > 0$ then $\dim(C) = \infty$. To describe C more explicitly, let $V^M = \bigoplus_{l \in L} V_l^B \otimes V_l^Z$ be the decomposition of V_M into irreducible representations of $G^B \times G^Z \subset G^M$.

EXAMPLES. (1) If $G^M = \mathrm{Spin}(n)$ and V^M is the spinor representation then V^B and V^Z are spinor representations.

(2) If $G^M = \mathrm{SO}(n)$ and $V^M = \Lambda^*(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C}$, then $V^B = \Lambda^*(\mathbb{R}^{\dim(B)}) \otimes_{\mathbb{R}} \mathbb{C}$ and $V^Z = \Lambda^*(\mathbb{R}^{\dim(Z)}) \otimes_{\mathbb{R}} \mathbb{C}$.

Let U be a contractible open subset of B. Choose an orientation on U. For $b \in U$, let $E_{b,l}^Z$ be the vector bundle on Z_b associated to the pair (G^Z, V_l^Z). Then $E^M\big|_{Z_b} \cong \bigoplus_{l \in L} V_l^B \otimes E_{b,l}^Z$. The vector bundles $\{E_{b,l}^Z\}_{b \in U}$ are the fiberwise restrictions of a vector bundle E_l^Z on $\pi^{-1}(U)$, a vertical 'spinor' bundle. There is a pushforward vector bundle W_l on U whose fiber $W_{l,b}$ over $b \in U$ is $C^\infty(Z_b; E_{b,l}^Z)$. If $\dim(Z) > 0$ then $\dim(W_l) = \infty$. There are Hermitian inner products $\{h^{W_l}\}_{l \in L}$ on $\{W_l\}_{l \in L}$ induced from the vertical Riemannian metric g^{TZ}. Furthermore, there are Clifford bundles $\{C_l\}_{l \in L}$ on U for which the fiber $C_{l,b}$ of C_l over $b \in U$ is isomorphic to $V_l^B \otimes W_{l,b}$. By construction, $C^\infty(Z_b; E^M\big|_{Z_b}) \cong \bigoplus_{l \in L} C_{l,b}$. The Clifford bundles $\{C_l\}_{l \in L}$ exist globally on B and $C = \bigoplus_{l \in L} C_l$. The Dirac-type operator D^M decomposes as $D^M = \bigoplus_{l \in L} D_l^M$, where D_l^M acts on $C^\infty(B; C_l)$.

In order to write D_l^M explicitly, let us recall the Bismut superconnection on W_l. We will deal with each $l \in L$ separately and so we drop the subscript l for the moment. We use the notation of [9, Section III(c)] to describe the local geometry of the fiber bundle $M \to B$, and the Einstein summation convention. Let ∇^{TZ} denote the Bismut connection on TZ [7, Proposition 10.2], which we extend to a connection on E_l^Z. The Bismut superconnection on W [7, Proposition 10.15] is of the form

$$A = D^W + \nabla^W - \tfrac{1}{4} c(T). \tag{3.1}$$

Here D^W is the fiberwise Dirac-type operator and has the form

$$D^W = -i\gamma^j \nabla_{e_j}^{TZ} = -i\gamma^j \big(e_j + \tfrac{1}{2}\omega_{pqj}\sigma^{pq}\big). \tag{3.2}$$

Next, ∇^W is a Hermitian connection on W given by

$$\nabla^W = \tau^\alpha \Big(\nabla_{e_\alpha}^{TZ} - \tfrac{1}{2}\omega_{\alpha jj}\Big) = \tau^\alpha \big(e_\alpha + \tfrac{1}{2}\omega_{jk\alpha}\sigma^{jk} - \tfrac{1}{2}\omega_{\alpha jj}\big). \tag{3.3}$$

Finally,

$$c(T) = i\omega_{\alpha\beta j}\gamma^j \tau^\alpha \tau^\beta. \tag{3.4}$$

The superconnection A can be 'quantized' into an operator D^A on $C^\infty(B; V^B \otimes W)$.

Explicitly,

$$D^A = -i\gamma^j\big(e_j + \tfrac{1}{2}\omega_{pqj}\sigma^{pq}\big) - \\ - i\gamma^\alpha\big(e_\alpha + \tfrac{1}{2}\omega_{\beta\gamma\alpha}\sigma^{\beta\gamma} + \tfrac{1}{2}\omega_{jk\alpha}\sigma^{jk} - \tfrac{1}{2}\omega_{\alpha jj}\big) + \\ + i\tfrac{1}{2}\omega_{\alpha\beta j}\gamma^j\sigma^{\alpha\beta.} \tag{3.5}$$

Let $\mathcal{V} \in \mathrm{End}(C_l)$ be the self-adjoint operator given by

$$\mathcal{V} = -i\big(\omega_{\alpha jk}\gamma^k\sigma^{\alpha j} + \tfrac{1}{2}\omega_{\alpha jj}\gamma^\alpha + \omega_{\alpha\beta j}(\gamma^j\sigma^{\alpha\beta} + \gamma^\alpha\sigma^{j\beta})\big). \tag{3.6}$$

Then restoring the index l everywhere,

$$D_l^M = D^{A_l} + \mathcal{V}_l. \tag{3.7}$$

EXAMPLES. (1) If $G^M = \mathrm{Spin}(n)$ and V^M is the spinor representation then $\mathcal{V} = 0$.
(2) If $G^M = \mathrm{SO}(n)$ and $V^M = \Lambda^*(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C}$, then

$$\mathcal{V} = -\tfrac{1}{4}i\big(\omega_{\alpha jk}\gamma^k[\widehat{\gamma}^\alpha, \widehat{\gamma}^j] + \omega_{\alpha\beta j}(\gamma^j[\widehat{\gamma}^\alpha, \widehat{\gamma}^\beta] + \gamma^\alpha[\widehat{\gamma}^j, \widehat{\gamma}^\beta])\big). \tag{3.8}$$

Now suppose that $\pi\colon M \to B$ is a Riemannian affine fiber bundle. Then $E^M\big|_{Z_b}$ inherits a flat connection from the flat affine connections on $\{E^Z_{b,l}\}_{l\in L}$. Let E^B be the Clifford bundle on B whose fiber over $b \in B$ is the space of parallel sections of $E^M\big|_{Z_b}$. Then D^M restricts to a first-order differential operator D^B on $C^\infty(B; E^B)$.

Given $b \in U$ and $l \in L$, let $W^{\mathrm{inv}}_{l,b}$ be the finite-dimensional subspace of $W_{l,b}$ consisting of affine-parallel elements of $C^\infty(Z_b; E^Z_{b,l})$. From the discussion in Section 2, $W^{\mathrm{inv}}_{l,b}$ is isomorphic to $\big(V_l^Z\big)^\Gamma$. The vector spaces $W^{\mathrm{inv}}_{l,b}$ fit together to form a finite-dimensional subbundle W_l^{inv} of W_l. There is a corresponding finite-dimensional Clifford subbundle C_l^{inv} of C_l whose fiber over $b \in U$ is isomorphic to $V_l^B \otimes W^{\mathrm{inv}}_{l,b}$. Again, C_l^{inv} exists globally on B. Then $E^B = \bigoplus_{l\in L} C_l^{\mathrm{inv}}$. Let D_l^B be the restriction of D_l^M to $C^\infty(B; C_l^{\mathrm{inv}})$. Then

$$D^B = \bigoplus_{l\in L} D_l^B. \tag{3.9}$$

The superconnection A_l restricts to an superconnection A_l^{inv} on W_l^{inv}, the endomorphism $\mathcal{V}_l$ restricts to an endomorphism of C_l^{inv} and D_l^M restricts to the first-order differential operator

$$D_l^B = D^{A_l^{\mathrm{inv}}} + \mathcal{V}_l^{\mathrm{inv}} \tag{3.10}$$

on $C^\infty(B; C_l^{\mathrm{inv}})$.

Proof of Theorem 1. The operator D_l^M is diagonal with respect to the orthogonal decomposition

$$C_l = C_l^{\mathrm{inv}} \oplus \big(C_l^{\mathrm{inv}}\big)^\perp. \tag{3.11}$$

Thus it suffices to show that there are constants A, A' and C such that the spectrum of $\sigma(D_l^M)$, when restricted to $\left(C_l^{\mathrm{inv}}\right)^{\perp}$, is disjoint from (1.1).

For simplicity, we drop the subscript l. Given $\eta \in C^{\infty}(B; (C^{\mathrm{inv}})^{\perp}) \subset C^{\infty}(M; E^M)$, it is enough to show that for suitable constants,

$$\langle D^M\eta, D^M\eta\rangle \geqslant \left(\text{const. diam }(Z)^{-2} - \text{const.}\left(\| R^M \|_{\infty} + \| \Pi \|_{\infty}^2 + \| T \|_{\infty}^2\right)\right) \langle \eta, \eta\rangle. \tag{3.12}$$

Using (2.4), it is enough to show that

$$\langle \nabla^M\eta, \nabla^M\eta\rangle \geqslant \left(\text{const. diam }(Z)^{-2} - \text{const.}\left(\| R^M \|_{\infty} + \| \Pi \|_{\infty}^2 + \| T \|_{\infty}^2\right)\right) \langle \eta, \eta\rangle. \tag{3.13}$$

We can write $\nabla^M = \nabla^V + \nabla^H$, where

$$\nabla^V\colon C^{\infty}(M; E^M) \to C^{\infty}(M; T^*Z \otimes E^M) \tag{3.14}$$

denotes covariant differentiation in the vertical direction and

$$\nabla^H\colon C^{\infty}(M; E^M) \to C^{\infty}(M; \pi^*T^*B \otimes E^M) \tag{3.15}$$

denotes covariant differentiation in the horizontal direction. Then

$$\begin{aligned}\langle \nabla^M\eta, \nabla^M\eta\rangle &= \langle \nabla^V\eta, \nabla^V\eta\rangle + \langle \nabla^H\eta, \nabla^H\eta\rangle \\ &\geqslant \langle \nabla^V\eta, \nabla^V\eta\rangle \\ &= \int_B \int_{Z_b} \left|\nabla^V\eta\right|^2(z)\,d\mathrm{vol}_{Z_b}\,d\mathrm{vol}_B.\end{aligned} \tag{3.16}$$

On a given fiber Z_b, we have

$$E^M\big|_{Z_b} \cong V^B \otimes E_b^Z. \tag{3.17}$$

Hence we can also use the Bismut connection ∇^{TZ} to vertically differentiate sections of E^M. That is, we can define

$$\nabla^{TZ}\colon C^{\infty}(M; E^M) \to C^{\infty}(M; T^*Z \otimes E^M). \tag{3.18}$$

Explicitly, with respect to a local framing,

$$\nabla_{e_j}^{TZ} = e_j\eta + \tfrac{1}{2}\omega_{pqj}\sigma^{pq}\eta \tag{3.19}$$

and

$$\nabla_{e_j}^{V} = e_j\eta + \tfrac{1}{2}\omega_{pqj}\sigma^{pq}\eta + \omega_{\alpha kj}\sigma^{\alpha k}\eta + \tfrac{1}{2}\omega_{\alpha\beta j}\sigma^{\alpha\beta}\eta. \tag{3.20}$$

Then from (3.16), (3.19) and (3.20),

$$\langle \nabla^M \eta, \nabla^M \eta \rangle \geqslant \int_B \left[\int_{Z_b} \left| \nabla^{TZ} \eta \right|^2 (z) - \text{const.} \left(\| T_b \|^2 + \| \Pi_b \|^2 \right) \left| \eta(z) \right|^2 \right] d\text{vol}_{Z_b} d\text{vol}_B. \tag{3.21}$$

Thus it suffices to bound $\int_{Z_b} \left| \nabla^{TZ} \eta \right|^2 (z) d\text{vol}_{Z_b}$ from below on a given fiber Z_b in terms of $\langle \eta, \eta \rangle_{Z_b}$, under the assumption that $\eta \in (W_b^{\text{inv}})^\perp$. Using the Gauss–Codazzi equation, we can estimate $\| R^{Z_b} \|_\infty$ in terms of $\| R^M \|_\infty$ and $\| \Pi \|_\infty^2$. Then the desired bound on $\int_{Z_b} \left| \nabla^{TZ} \eta \right|^2 (z) d\text{vol}_{Z_b}$ follows from Proposition 2. □

Proof of Theorem 2. Let g_0^{TM} denote the Riemannian metric on M. From Proposition 1, if a Riemannian metric g_1^{TM} on M is close to g_0^{TM} in $(\mathcal{M}(M, 2K), d)$ then applying the function $x \to \sinh^{-1}(x/\sqrt{2K})$ to $\sigma(D^M, g_0^{TM})$ gives a collection of numbers which is close to that obtained by applying $x \to \sinh^{-1}(x/\sqrt{2K})$ to $\sigma(D^M, g_1^{TM})$. We will use the geometric results of [11] to find a metric g_2^{TM} on M which is close to g_0^{TM} and to which we can apply Theorem 1.

First, as in [11, (2.4.1)], by the smoothing results of Abresch and others [11 , Theorem 1.12], for any $\varepsilon > 0$ we can find metrics on M and B which are ε-close in the C^1-topology to the original metrics such that the new metrics satisfy $\| \nabla^i R \|_\infty \leqslant A_i(n, \varepsilon)$ for some appropriate sequence $\{A_i(n, \varepsilon)\}_{i=0}^\infty$. Let g_1^{TM} denote the new metric on M. In the proof of the smoothing result, such as using the Ricci flow [21, Proposition 2.5], one obtains an explicit smooth 1-parameter family of metrics on M in $\mathcal{M}(M, K')$, for some $K' > K$, going from g_0^{TM} to g_1^{TM}. We can approximate this family by a piecewise-analytic family. Hence one obtains an upper bound on $d\left(g_0^{TM}, g_1^{TM}\right)$ in $\mathcal{M}(M, K')$, for some $K' > K$, which depends on K and is proportionate to ε. (Note that d is essentially the same as the C^0-metric on $\mathcal{M}(M, K')$.) By rescaling, we may assume that $\| R^M \|_\infty \leqslant 1$, $\| R^B \|_\infty \leqslant 1$ and $inj(B) \geqslant 1$. We now apply [11, Theorem 2.6], with B fixed. It implies that there are positive constants $\lambda(n)$ and $c(n, \varepsilon)$ so that if $d_{GH}(M, B) \leqslant \lambda(n)$ then there is a fibration $f : M \to B$ such that

(1) diam $\left(f^{-1}(b) \right) \leqslant c(n, \varepsilon) d_{GH}(M, B)$.
(2) f is a $c(n, \varepsilon)$-almost Riemannian submersion.
(3) $\| \Pi_{f^{-1}(b)} \|_\infty \leqslant c(n, \varepsilon)$.

As in [16], the Gauss–Codazzi equation, the curvature bound on M and the second fundamental form bound on $f^{-1}(b)$ imply a uniform bound on $\{\| R^{f^{-1}(b)} \|_\infty\}_{b \in B}$. Along with the diameter bound on $f^{-1}(b)$, this implies that if $d_{GH}(M, B)$ is sufficiently small then $f^{-1}(b)$ is almost flat.

From [11, Propositions 3.6 and 4.9], we can find another metric g_2^{TM} on M which is ε-close to g_1^{TM} in the C^1-topology so that the fibration $f : M \to B$ gives M the structure of a Riemannian affine fiber bundle. Furthermore, by [11, Proposition 4.9], there is a sequence $\{A_i'(n, \varepsilon)\}_{i=0}^\infty$ so that we may assume that g_1^{TM} and g_2^{TM} are close in the sense that

$$\| \nabla^i\big(g_1^{TM} - g_2^{TM}\big) \|_\infty \leqslant A_i'(n, \varepsilon) d_{GH}(M, B), \tag{3.22}$$

where the covariant derivative in (41) is that of the Levi-Civita connection of g_2^{TM}. Then we can interpolate linearly between g_1^{TM} and g_2^{TM} within $\mathcal{M}(M, K'')$ for some $K'' > K'$, and obtain an upper bound on $d\big(g_1^{TM}, g_2^{TM}\big)$ in $\mathcal{M}(M, K'')$ which is proportionate to ε. From [21, Theorem 2.1], we can take $K'' = 2K$ (or any number greater than K).

We now apply Theorem 1 to the Riemannian affine fiber bundle with metric g_2^{TM}. It remains to estimate the geometric terms appearing in (1.1). We have an estimate on $\| \Pi \|_\infty$ as above. Applying O'Neill's formula [8, (9.29)] to the Riemannian affine fiber bundle, we can estimate $\| T \|_\infty^2$ in terms of $\| R^M \|_\infty$ and $\| R^B \|_\infty$. Putting this together, the theorem follows. □

4. Collapsing to a Singular Base

Let $\mathfrak{p}$: $P \to M$ be the principal G-bundle of Section 2. Let $\{\mathfrak{Y}_j\}_{j=1}^n$ be the horizontal vector fields on P such that $\theta(\mathfrak{Y}_j) = e_j$. Put $D^P = -i \sum_{j=1}^n \gamma^j \mathfrak{Y}_j$, acting on $C^\infty(P) \otimes V$. There is an isomorphism $C^\infty(M; E^M) \cong (C^\infty(P) \otimes V)^G$. Under this isomorphism, $D^M \cong D^P\big|_{(C^\infty(P)\otimes V)^G}$. The Bochner-type equation (2.4) becomes

$$(D^M)^2 \cong -\sum_{j=1}^n \mathfrak{Y}_j^2 + \sum_{i,j=1}^n \omega^i_{jj} \mathfrak{Y}_i - \frac{1}{8} \sum_{a,b,i,j=1}^n (\mathfrak{p}^* R^M)_{abij} (\gamma^i\gamma^j - \gamma^j\gamma^i)\sigma^{ab} \tag{4.1}$$

when acting on $(C^\infty(P) \otimes V)^G$.

Let $\{x_a\}_{a=1}^{\dim(G)}$ be a basis for the Lie algebra $\mathfrak{g}$ which is orthonormal with respect to the negative of the Killing form. Let $\{\mathfrak{Y}_a\}_{a=1}^{\dim(G)}$ be the corresponding vector fields on P. Then $-\sum_{a=1}^{\dim(G)} \mathfrak{Y}_a^2$ acts on $(C^\infty(P) \otimes V)^G$ as $c_V \in (\mathrm{End}(V))^G$, the Casimir of the G-module V. Give P the Riemannian metric g^{TP} with the property that $\{\mathfrak{Y}_j, \mathfrak{Y}_a\}$ forms an orthonormal basis of vector fields. Let $\triangle^P$ denote the corresponding (nonnegative) scalar Laplacian on P, extended to act on $C^\infty(P) \otimes V$. Then when acting on $(C^\infty(P) \otimes V)^G$, equation (4.1) is equivalent to

$$(D^M)^2 \cong \triangle^P - \frac{1}{8} \sum_{a,b,i,j=1}^n (\mathfrak{p}^* R^M)_{abij} (\gamma^i\gamma^j - \gamma^j\gamma^i)\sigma^{ab} - c_V. \tag{4.2}$$

DEFINITION 3. A G-equivariant Riemannian affine fiber bundle structure on P consists of a Riemannian affine fiber bundle structure $\check{\pi} : P \to \check{X}$ which is G-equivariant.

In [11, Proposition 7.21] it is shown that one can make a small G-equivariant perturbation of g^{TP} in the $C^{1,\alpha}$-topology so that the new Riemannian metric is the total space of a G-equivariant Riemannian affine fiber bundle. The quotient space $M = P/G$ acquires a new quotient Riemannian metric, which is called an invariant metric [11, Section 8]. In [21, Theorem 2.1] it is shown that one can assume that the sectional curvatures of the invariant metric on M are bounded in terms of the sectional curvatures of the original metric on M. As we can take the new canonical Riemannian metric g^{TP} on P, the upshot is that we assume that the Riemannian metric on the total space of the G-equivariant affine fiber bundle $P \to \check{X}$ is the canonical metric coming from a Riemannian metric on M.

Given such a G-equivariant Riemannian affine fiber bundle, let $\check{Z}$ be the fiber of $\check{\pi}$: $P \to \check{X}$, an infranilmanifold. For collapsing purposes it suffices to take $\check{Z}$ to be a nilmanifold $\Gamma \backslash N$ [11, (7.2)]. We assume hereafter that this is the case. Put $X = \check{X}/G$, a possibly singular space. As the Lie algebra $\mathfrak{n}$ of N is represented by vector fields in a neighborhood of a point of P, and the local flow preserves the horizontal subspaces of $P \to M$, it follows that the vector fields $\{\mathfrak{Y}_j\}_{j=1}^n$ are projectable with respect to $\check{\pi}$ and push forward to vector fields $\{\mathcal{X}_j\}_{j=1}^n$ on $\check{X}$. Put $D^{\check{X}} = -i \sum_{j=1}^n \gamma^j \mathcal{X}_j$, acting on $C^\infty(\check{X}) \otimes V$. Let $v \in C^\infty(\check{X})$ be given by $v(\check{x}) = vol(\check{Z}_{\check{x}})$. We give $C^\infty(\check{X}) \otimes V$ the weighted L^2-inner product with respect to the weight function v.

We recall that there is a notion of a pseudodifferential operator being transversally elliptic with respect to the action of a Lie group G [2, Definition 1.3].

LEMMA 1. *$D^{\check{X}}$ is transversally elliptic on $\check{X}$.*

Proof. Let $s(D^{\check{X}}) \in C^\infty(T^*\check{X}) \otimes \mathrm{End}(V)$ denote the symbol of $D^{\check{X}}$. Suppose that $\xi \in T^*_{\check{x}}\check{X}$ satisfies $\xi(\check{v}) = 0$ for all $\check{v} \in T_{\check{x}}\check{X}$ which lie in the image of the representation of $\mathfrak{g}$ by vector fields on $\check{X}$. Then if $p \in \check{\pi}^{-1}(\check{x})$, we have that $(\check{\pi}^*\xi)(r) = 0$ for all $r \in T_pP$ which lie in the image of the representation of $\mathfrak{g}$ by vector fields on P. In other words, $\check{\pi}^*\xi$ is horizontal. Now $((s(D^{\check{X}}))(\xi))^2 = \sum_{j=1}^n \langle \xi, \mathcal{X}_j \rangle^2 = \sum_{j=1}^n \langle \check{\pi}^*\xi, \mathfrak{Y}_j \rangle^2$. If $(s(D^{\check{X}}))(\xi)$ fails to be an isomorphism then $\langle \check{\pi}^*\xi, \mathfrak{Y}_j \rangle = 0$ for all j. Along with the fact that $\check{\pi}^*\xi$ is horizontal, this implies that $\check{\pi}^*\xi = 0$. Thus $\xi = 0$, which proves the lemma.

DEFINITION 4. For notation, write $C^\infty(X; E^X) = (C^\infty(\check{X}) \otimes V)^G$. Let D^X be the restriction of $D^{\check{X}}$ to $C^\infty(X; E^X)$.

It will follow from the proof of the next theorem that D^X is self-adjoint on the Hilbert space completion of $C^\infty(X; E^X)$ with respect to the (weighted) inner product. As $D^{\check{X}}$ is transversally elliptic, it follows that D^X has a discrete spectrum [2, Proof of Theorem 2.2].

Let $\check{\Pi}$ denote the second fundamental forms of the fibers $\{\check{Z}_{\check{x}}\}_{\check{x} \in \check{X}}$. Let $\check{T} \in \Omega^2(P; T\check{Z})$ be the curvature of the horizontal distribution on the affine fiber bundle $P \to \check{X}$.

THEOREM 6. *There are positive constants A, A′ and C which only depend on n and V such that if* $\| R^{\check{Z}} \|_\infty$ diam $(\check{Z})^2 \leqslant A'$ *then the intersection of* $\sigma(D^M)$ *with*

$$[-(A\mathrm{diam}(\check{Z})^{-2} - C(1+ \| R^M \|_\infty + \| \check{\Pi} \|_\infty^2 + \| \check{T} \|_\infty^2))^{1/2},$$
$$(A\mathrm{diam}(\check{Z})^{-2} - C(1+ \| R^M \|_\infty + \| \check{\Pi} \|_\infty^2 + \| \check{T} \|_\infty^2))^{1/2}] \quad (4.3)$$

equals the intersection of $\sigma(D^X)$ *with* (4.3).

Proof. Let us write

$$C^\infty(P) \otimes V = \left(C^\infty(\check{X}) \otimes V\right) \oplus \left(C^\infty(\check{X}) \otimes V\right)^\perp, \tag{4.4}$$

where we think of $C^\infty(\check{X}) \otimes V$ as the elements of $C^\infty(P) \otimes V$ which are constant along the fibers of the fiber bundle $\check{\pi}$: $P \to \check{X}$. Taking G-invariant subspaces, we have an orthogonal decomposition

$$C^\infty(M; E^M) = C^\infty(X; E^X) \oplus \left(C^\infty(X; E^X)\right)^\perp, \tag{4.5}$$

with respect to which D^M decomposes as

$$D^M = D^X \oplus D^M\big|_{(C^\infty(X;E^X))^\perp}. \tag{4.6}$$

As in the proof of Theorem 1, it suffices to obtain a lower bound on the spectrum of $(D^M)^2\big|_{(C^\infty(X;E^X))^\perp}$. As $(C^\infty(X; E^X))^\perp \subset (C^\infty(\check{X}) \otimes V)^\perp$, using (4.2) it suffices to obtain a lower bound on the spectrum of $\triangle^P\big|_{(C^\infty(\check{X})\otimes V)^\perp}$. This follows from the arguments of the proof of Theorem 1, using the fact that $\| R^P \|_\infty \leqslant \text{const.}(1+ \| R^M \|_\infty)$. We omit the details. In fact, it is somewhat easier than the proof of Theorem 1, since we are now only dealing with the scalar Laplacian and so can replace Proposition 2 by standard eigenvalue estimates (which just involve a lower Ricci curvature bound); see [6] and references therein.

Proof of Theorem 3. Everything in the proof will be done in a G-equivariant way, so we may omit to mention this explicitly. Let P_i be the principal G-bundle of M_i, equipped with a Riemannian metric as in the beginning of the section. From the G-equivariant version of Gromov's compactness theorem, we obtain a subsequence $\{P_i\}_{i=1}^\infty$ which converges in the equivariant Gromov–Hausdorff topology to a G-Riemannian manifold $(\check{X}, g^{T\check{X}})$ with a $C^{1,\alpha}$-regular metric. As in [14, Section 3], the measure $\chi d\mathrm{vol}_{\check{X}}$ is a weak-$*$ limit point of the pushforwards of the normalized Riemannian measures on $\{P_i\}_{i=1}^\infty$. As in [14, p. 535], after smoothing we may assume that we have G-equivariant Riemannian affine fiber bundles $\check{\pi}_i$: $P'_i \to \check{X}_i$, with G acting freely on P'_i, along with G-diffeomorphisms $\check{\phi}_i$: $P_i \to P'_i$ and Φ_i: $\check{X} \to \check{X}_i$. Put $M'_i = P'_i/G$. Then $\check{\phi}_i$ descends to a diffeomorphism ϕ_i: $M_i \to M'_i$ and we may also assume, as in the proof of Theorem 2, that

(1) $\phi_i^* g^{TM'_i} \in \mathcal{M}(M_i, \text{const.}K)$,
(2) $d(\phi_i^* g^{TM'_i}, g^{TM_i}) \leqslant 2^{-i}$ in $\mathcal{M}(M_i, \text{const.}K)$ and
(3) $\lim_{i\to\infty} \Phi_i^* g^{T\check{X}_i} = g^{T\check{X}}$ in the $C^{1,\alpha}$-topology.

Using Proposition 1, we can effectively replace M_i by M'_i for the purposes of the argument. For simplicity, we relabel M'_i as M_i and P'_i as P_i. For the purposes of the limiting argument, using Theorem 6 and (4.2), we may replace the spectrum of $|D^{M_i}|$ by the spectrum of the operator $|D^{X_i}| \equiv \sqrt{\triangle^{\check{X}_i} + \mathcal{V}_i}$ acting on $C^\infty(X_i, E^{X_i}) = (C^\infty(\check{X}_i) \otimes V)^G$, where $\mathcal{V}_i$ is the restriction of

$$-\frac{1}{8} \sum_{a,b,i,j=1}^{n} (\mathfrak{p}^* R^{M_i})_{abij} (\gamma^i \gamma^j - \gamma^j \gamma^i) \sigma^{ab} - c_V \tag{4.7}$$

to the elements of $(C^\infty(P_i) \otimes V)^G$ which are constant along the fibers of $\check{\pi}_i \colon P_i \to \check{X}_i$, i.e. to $C^\infty(X_i, E^{X_i})$.

From the curvature bound, we have a uniform bound on $\{\| \mathcal{V}_i \|_\infty\}_{i=1}^\infty$. Using the weak-$*$ compactness of the unit ball, let $\mathcal{V}$ be a weak-$*$ limit point of $\{\Phi_i^* \mathcal{V}_i\}_{i=1}^\infty$ in $L^\infty(\check{X}) \otimes \mathrm{End}(V) = (L^1(\check{X}) \otimes \mathrm{End}(V))^*$. We claim that with this choice of $\check{X}$, χ and $\mathcal{V}$, equation (1.2) holds.

To see this, we use the minimax characterization of eigenvalues as in [14, Section 5]. Using the diffeomorphisms $\{\Phi_i\}_{i=1}^\infty$, we identify each $\check{X}_i$ with $\check{X}$. We denote by $\langle \cdot, \cdot \rangle_{X_i}$ an L^2-inner product constructed using $\Phi_i^* g^{T\check{X}_i}$ and the weight function $(\check{\pi}_i)_*(d\mathrm{vol}_{P_i}) / \int_{\check{X}_i} (\check{\pi}_i)_*(d\mathrm{vol}_{P_i})$. We denote by $\langle \cdot, \cdot \rangle_X$ an L^2-inner product constructed using $g^{T\check{X}}$ and the weight function $\chi d\mathrm{vol}_{\check{X}}$. As $\triangle^{\check{X}}$ has a compact resolvent, it follows that $|D^X|^2$ has a compact resolvent. Then

$$\lambda_k(|D^X|)^2 = \inf_W \sup_{\psi \in W - 0} \frac{\langle \mathrm{d}\psi, \mathrm{d}\psi \rangle_X + \langle \psi, \mathcal{V}\psi \rangle_X}{\langle \psi, \psi \rangle_X}, \tag{4.8}$$

where W ranges over the k-dimensional subspaces of the Sobolev space $H^1(X; E^X)$. Given $\varepsilon > 0$, let W_∞ be a k-dimensional subspace such that

$$\sup_{\psi \in W_\infty - 0} \frac{\langle \mathrm{d}\psi, \mathrm{d}\psi \rangle_X + \langle \psi, \mathcal{V}\psi \rangle_X}{\langle \psi, \psi \rangle_X} \leqslant \lambda_k(|D^X|)^2 + \varepsilon. \tag{4.9}$$

As $\psi \otimes \psi^*$ lies in the finite-dimensional subspace $W_\infty \otimes W_\infty^*$ of $L^1(\check{X}) \otimes \mathrm{End}(V)$, it follows that

$$\lim_{i \to \infty} \langle \psi, \mathcal{V}_i \psi \rangle_X = \langle \psi, \mathcal{V}\psi \rangle_X \tag{4.10}$$

uniformly on $\{\psi \in W_\infty \colon \langle \psi, \psi \rangle_X = 1\}$. Then

$$\lim_{i \to \infty} \sup_{\psi \in W_\infty - 0} \frac{\langle \mathrm{d}\psi, \mathrm{d}\psi \rangle_{X_i} + \langle \psi, \mathcal{V}_i \psi \rangle_{X_i}}{\langle \psi, \psi \rangle_{X_i}} = \sup_{\psi \in W_\infty - 0} \frac{\langle \mathrm{d}\psi, \mathrm{d}\psi \rangle_X + \langle \psi, \mathcal{V}\psi \rangle_X}{\langle \psi, \psi \rangle_X}. \tag{4.11}$$

As

$$\lambda_k(|D^{X_i}|)^2 = \inf_W \sup_{\psi \in W - 0} \frac{\langle \mathrm{d}\psi, \mathrm{d}\psi \rangle_{X_i} + \langle \psi, \mathcal{V}_i \psi \rangle_{X_i}}{\langle \psi, \psi \rangle_{X_i}}, \tag{4.12}$$

it follows that

$$\limsup_{i \to \infty} \lambda_k(|D^{X_i}|) \leqslant \lambda_k(|D^X|). \tag{4.13}$$

We now show that

$$\liminf_{i\to\infty} \lambda_k(|D^{X_i}|) \geqslant \lambda_k(|D^X|). \tag{4.14}$$

Along with (4.13), this will prove the theorem. Suppose that (4.14) is not true. Then there is some $\varepsilon > 0$ and some infinite subsequence of $\{M_i\}_{i=1}^\infty$, which we relabel as $\{M_i\}_{i=1}^\infty$, such that for all $i \in \mathbb{Z}^+$,

$$\lambda_k(|D^{X_i}|)^2 \leqslant \lambda_k(|D^X|)^2 - 2\varepsilon. \tag{4.15}$$

For each $i \in \mathbb{Z}^+$, let W_i be a k-dimensional subspace of $H^1(X; E^X)$ such that

$$\sup_{\psi\in W_i - 0} \frac{\langle \mathrm{d}\psi, \mathrm{d}\psi\rangle_{X_i} + \langle \psi, \mathcal{V}_i\psi\rangle_{X_i}}{\langle \psi, \psi\rangle_{X_i}} \leqslant \lambda_k(|D^{X_i}|)^2 + \varepsilon. \tag{4.16}$$

Let $\{f_{i,j}\}_{j=1}^k$ be a basis for W_i which is orthonormal with respect to $\langle\cdot,\cdot\rangle_X$. Then for a given j, the sequence $\{f_{i,j}\}_{i=1}^\infty$ is bounded in $H^1(X; E^X)$. After taking a subsequence, which we relabel as $\{f_{i,j}\}_{i=1}^\infty$, we can assume that $\{f_{i,j}\}_{i=1}^\infty$ converges weakly in $H^1(X; E^X)$ to some $f_{\infty,j}$. Doing this successively for $j \in \{1, \dots, k\}$, we can assume that for each j, $\lim_{i\to\infty} f_{i,j} = f_{\infty,j}$ weakly in $H^1(X; E^X)$. Then from the compactness of the embedding $H^1(X; E^X) \to L^2(X; E^X)$, we have strong convergence in $L^2(X; E^X)$. In particular, $\{f_{\infty,j}\}_{j=1}^k$ are orthonormal. Put $W_\infty = \mathrm{span}(f_{\infty,1}, \dots, f_{\infty,k})$.

If $w_\infty = \sum_{j=1}^k c_j f_{\infty,j}$ is a nonzero element of W_∞, put $w_i = \sum_{j=1}^k c_j f_{i,j}$. Then $\{w_i\}_{i=1}^\infty$ converges weakly to w_∞ in $H^1(X; E^X)$ and hence converges strongly to w_∞ in $L^2(X; E^X)$. From a general result about weak limits, we have

$$\langle w_\infty, w_\infty\rangle_{H^1} \leqslant \limsup_{i\to\infty} \langle w_i, w_i\rangle_{H^1}. \tag{4.17}$$

Along with the L^2-convergence of $\{w_i\}_{i=1}^\infty$ to w_∞, this implies that

$$\langle dw_\infty, dw_\infty\rangle_X \leqslant \limsup_{i\to\infty} \langle dw_i, dw_i\rangle_{X_i}. \tag{4.18}$$

As $w_i \otimes w_i^*$ converges in $L^1(\check{X}) \otimes \mathrm{End}(E)$ to $w_\infty \otimes w_\infty^*$, we have

$$\begin{aligned}\lim_{i\to\infty} \langle w_i, \mathcal{V}_i w_i\rangle_X &= \lim_{i\to\infty} (\langle w_\infty, \mathcal{V}_i w_\infty\rangle_X + (\langle w_i, \mathcal{V}_i w_i\rangle_X - \langle w_\infty, \mathcal{V}_i w_\infty\rangle_X)) \\ &= \langle w_\infty, \mathcal{V} w_\infty\rangle_X.\end{aligned} \tag{4.19}$$

Then

$$\sup_{\psi\in W_\infty - 0} \frac{\langle \mathrm{d}\psi, \mathrm{d}\psi\rangle_X + \langle \psi, \mathcal{V}\psi\rangle_X}{\langle \psi, \psi\rangle_X} \leqslant \limsup_{i\to\infty} \sup_{\psi\in W_i - 0} \frac{\langle \mathrm{d}\psi, \mathrm{d}\psi\rangle_{X_i} + \langle \psi, \mathcal{V}_i\psi\rangle_{X_i}}{\langle \psi, \psi\rangle_{X_i}}. \tag{4.20}$$

Thus from (4.15), (4.16) and (4.20),

$$\inf_{W} \sup_{\psi \in W-0} \frac{\langle \mathrm{d}\psi, \mathrm{d}\psi \rangle_X + \langle \psi, \mathcal{V}\psi \rangle_X}{\langle \psi, \psi \rangle_X} \leqslant \lambda_k(|D^X|)^2 - \varepsilon, \tag{4.21}$$

which is a contradiction. This proves the theorem. □

Proof of Theorem 4. Let $\{g_i^{TM}\}_{i=1}^{\infty}$ be a sequence of Riemannian metrics on M as in the statement of the theorem, with respect to which $\lambda_k(|D^M|)$ goes to infinity. Let P be the principal G-bundle of M and let $\check{X}$ be the limit space of Theorem 3, a smooth manifold with a $C^{1,\alpha}$-regular metric. As the limit space $X = \check{X}/G$ has diameter 1, it has positive dimension. As in the proof of Theorem 3, after slightly smoothing the metric on $\check{X}$, there is a G-equivariant Riemannian affine fiber bundle $\check{\pi}$: $P \to \check{X}$ whose fiber is a nilmanifold $\check{Z}$. Let $\check{x}$ be a point in a principal orbit for the G-action on $\check{X}$, with isotropy group $H \subset G$. Then H acts affinely on the nilmanifold fiber $\check{Z}_{\check{x}}$. In particular, H is virtually abelian. The quotient $Z = \check{Z}_{\check{x}}/H$ is the generic fiber of the possibly-singular affine fiber bundle π: $M \to X$, the G-quotient of $\check{\pi}$: $P \to \check{X}$. Then $E^M\big|_Z = \check{Z}_{\check{x}} \times_H V$. In particular, the vector space of affine-parallel sections of $E^M\big|_Z$ is isomorphic to V^H. On the other hand, if $C^\infty(X; E^X) \neq 0$ then $|D^X|$ has an infinite discrete spectrum. Theorem 3 now implies that $C^\infty(X; E^X) \cong (C^\infty(\check{X}) \otimes V)^G$ must be the zero space. As the orbit $\check{x} \cdot G$ has a neighborhood consisting of principal orbits, the restriction map from $(C^\infty(\check{X}) \otimes V)^G$ to $(C^\infty(\check{x} \cdot G) \otimes V)^G$ is surjective. However, $(C^\infty(\check{x} \cdot G) \otimes V)^G$ is isomorphic to V^H. Thus $V^H = 0$.

Conversely, let π: $M \to B$ be an affine fiber bundle. Theorem 1 implies that if $E^M\big|_Z$ does not have any nonzero affine-parallel sections then upon collapsing M to B as in [16, Section 6], the eigenvalues of D_M go off to $\pm\infty$. This proves the theorem. □

5. Proof of Theorem 5

As the proof of Theorem 5 is similar to [19, Pf. of Theorem 2], we only indicate the structure of the proof and the necessary modifications to [19, Pf. of Theorem 2].

The closure $\overline{U_I}$ of an appropriate neighborhood of an end has the (affine) structure of an affine fiber bundle over $[0, \infty)$ with fiber Z_I. The vector bundle E_I^B is the trivial vector bundle over $[0, \infty)$ whose fiber over $s \in [0, \infty)$ consists of the affine-parallel sections of $E^M\big|_{\{s\} \times Z_I}$. As in [19, Section 4], if U_I is sufficiently far out the end then we can use Propositions 1 and 2 of the present paper to construct an embedding of $C^\infty([0, \infty); E_I^B)$ into $C^\infty(\overline{U_I}; E^M\big|_{\overline{U_I}})$ whose image consists of elements with 'bounded energy' fiberwise restrictions. Let P_0 be the Hilbert space extension of orthogonal projection from $\bigoplus_{I=1}^N C^\infty(\overline{U_I}; E^M\big|_{\overline{U_I}})$ to $\bigoplus_{I=1}^N C^\infty([0, \infty); E_I^B)$. By standard arguments as in [13, Pf. of Proposition 2.1], the essential spectrum of D^M equals that of D^M_{end}. With respect to the decomposition of the Hilbert space into

$\mathrm{Im}(P_0) \oplus \mathrm{Im}(I - P_0)$, we write

$$D^M_{\mathrm{end}} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}. \tag{5.1}$$

The operators $\mathcal{B}$ and $\mathcal{C}$ are bounded, as can be seen by the method of proof of [19, Proposition 2], replacing the operator $\widehat{d} + \widehat{d}^*$ of [19, Pf. of Proposition 2] by D^{Z_I}. As in [19, Proposition 3], the operator $\mathcal{D}$ has vanishing essential spectrum. Put $\mathcal{L} = \left(\begin{smallmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{D} \end{smallmatrix}\right)$. To prove the theorem, it suffices to show that D^M_{end} and $\mathcal{L}$ have the same essential spectrum. For this, it suffices to show that $\left(D^M_{\mathrm{end}} + ki\right)^{-1} - (\mathcal{L} + ki)^{-1}$ is compact for some $k > 0$ [20, Vol. IV, Chapter XIII.4, Corollary 1].

We use the general identity that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} = \begin{pmatrix} \alpha^{-1} + \alpha^{-1}\beta\left(\delta - \gamma\alpha^{-1}\beta\right)^{-1}\gamma\alpha^{-1} & -\alpha^{-1}\beta\left(\delta - \gamma\alpha^{-1}\beta\right)^{-1} \\ -\left(\delta - \gamma\alpha^{-1}\beta\right)^{-1}\gamma\alpha^{-1} & \left(\delta - \gamma\alpha^{-1}\beta\right)^{-1} \end{pmatrix} \tag{5.2}$$

provided that α and $\delta - \gamma\alpha^{-1}\beta$ are invertible. Put

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = D^M_{\mathrm{end}} + ki = \begin{pmatrix} \mathcal{A} + ki & \mathcal{B} \\ \mathcal{C} & \mathcal{D} + ki \end{pmatrix}. \tag{5.3}$$

If k is positive then α and δ are invertible, with δ^{-1} being compact. If k is large enough then $\| \delta^{-1/2}\gamma\alpha^{-1}\beta\delta^{-1/2} \| < 1$. Writing

$$\delta - \gamma\alpha^{-1}\beta = \delta^{1/2}\left(I - \delta^{-1/2}\gamma\alpha^{-1}\beta\delta^{-1/2}\right)\delta^{1/2}, \tag{5.4}$$

we now see that $\delta - \gamma\alpha^{-1}\beta$ is invertible. It also follows from (5.4) that $\left(\delta - \gamma\alpha^{-1}\beta\right)^{-1}$ is compact. Using (5.2), the theorem follows.

Acknowledgements

I thank the Max-Planck-Institut-Bonn for its hospitality while this research was performed.

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