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# The Proalgebraic Completion of Rigid Groups

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**Abstract.** A finitely generated group $\Gamma$ is called *representation rigid* (briefly, rigid) if for every $n$, $\Gamma$ has only finitely many classes of simple $\mathbb{C}$ representations in dimension $n$. Examples include higher rank $S$-arithmetic groups. By Margulis super rigidity, the latter have a stronger property: they are *representation super rigid*; i.e., their proalgebraic completion is finite dimensional. We construct examples of nonlinear rigid groups which are not super rigid, and which exhibit every possible type of infinite dimensionality. Whether linear representation rigid groups are super rigid remains an open question.

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## 1. Introduction

Let $\Gamma$ be a finitely generated group and let $k$ be an algebrically closed field of characteristic zero (usually $k = \mathbb{C}$). $\Gamma$ is said to be *representation rigid* (briefly, rigid) if, for each $n$, $\Gamma$ has only finitely many isomorphism classes of irreducible representations of degree $n$.

A useful way to study the representations of $\Gamma$ over $k$ is to consider the *proalgebraic completion* $A(\Gamma)$ of $\Gamma$, also called the Hochschild–Mostow group of $\Gamma$. $A(\Gamma)$ is the proalgebraic (more precisely, proaffine algebraic) group with a homomorphism $P: \Gamma \to A(\Gamma)$ such that for any representation $\rho$ of $\Gamma$ there is unique algebraic representation $\overline{\rho}$ of $A(\Gamma)$ such that $\overline{\rho} \circ P = \rho$.

This means that the representation theory of $\Gamma$ is equivalent to the algebraic representation theory of $A(\Gamma)$.

The prounipotent radical of $A(\Gamma)$ is denoted $U(\Gamma)$, and $Q(\Gamma) = A(\Gamma)/U(\Gamma)$ denotes the maximal proreductive quotient. In fact, $A(\Gamma)$ is the semidirect product of $U(\Gamma)$ and any maximal proreductive subgroup [13]. The identity component is denoted $A^0(\Gamma)$.

$\Gamma$ is called *representation super rigid* (briefly, super rigid), if $A(\Gamma)$ is finite-dimensional (i.e. the identity component $A^0(\Gamma)$ is an affine algebraic group). Super rigid groups are rigid (Corollary (9)).

Throughout this introduction, and usually also throughout this paper, we assume that $\Gamma$ is a finitely generated residually finite group. (Note that for a finitely generated group the finite-dimensional representations of $\Gamma$ separate the points of $\Gamma$ if and only if $\Gamma$ is residually finite.)

Examples of residually finite super rigid groups include the finitely generated torsion groups constructed by Golod (usually known as the groups of Golod–Shafarevich type). For these groups $\Gamma$, $A^0(\Gamma) = \{1\}$ and so $\dim(A(\Gamma)) = 0$. More interesting examples are the $S$-arithmetic subgroups of higher rank semi-simple groups, whose super rigidity was established by Margulis. Platonov conjectured that every finitely generated linear rigid group is of arithmetic type. A counter example to this conjecture was constructed in [5], where a rigid, even super rigid, linear nonarithmetic group is produced.

The main result of the current paper is the construction of rigid groups which are not super rigid. We produce examples of rigid groups where $A(\Gamma)$ is infinite dimensional in 'all possible ways'. For a rigid group $\Gamma$, the identity component $Q^0(\Gamma)$ is semi-simple, and is in fact a direct product of simple simply-connected algebraic groups $S_i$ (Corollary 3). We construct examples of rigid groups of each of the following types:

(1) $U(\Gamma)$ is infinite-dimensional and $Q(\Gamma)$ is finite-dimensional.
(2) $U(\Gamma)$ is finite-dimensional (in fact $U(\Gamma) = \{1\}$) and $Q(\Gamma)$ is infinite-dimensional, infinitely many different simple factors $S_i$ occur, and each appears with a finite multiplicity.
(3) $U(\Gamma) = \{1\}$ and $Q^0(\Gamma) = S_1^\infty \times S_2$ for some simple algebraic groups $S_1$ and $S_2$.

These examples are constructed in Sections 5 (of type (1)) and 6 (of types (2) and (3)).

In Section 3, we give general results on the structure of the proalgebraic completion of a rigid group and we give criteria for $\Gamma$ to be rigid in terms of properties of $A(\Gamma)$. For example, we define the degree $n$ proalgebraic completion $A_n(\Gamma)$ as $A(\Gamma)/K_n(\Gamma)$, where $K_n(\Gamma)$ is the intersection of the kernels of all the $n$ dimensional representations of $A(\Gamma)$. All $n$ dimensional representations of $\Gamma$ factor uniquely through $A_n(\Gamma)$.

One can easily see that $A(\Gamma) = \varprojlim A_n(\Gamma)$. We prove:

THEOREM A. *The following are equivalent:*

(1) $\Gamma$ *is a rigid group.*
(2) $\forall n$, $A_n(\Gamma)$ *is an affine algebraic group.*
(3) $\forall n$, $\dim(A(\Gamma)) < \infty$.

Thus rigidity is equivalent to $A_n(\Gamma)$ being finite-dimensional for all $n$, and super rigidity means that there is a common bound for the dimensions of $A_n(\Gamma)$ for all $n$.

Here, $A_n(\Gamma)$ can be viewed as an analogue, for groups, of the process, for algebras, of imposing the identities of $n \times n$ matrix algebras.

For more conditions equivalent to the rigidity of $\Gamma$, see Section 3. We also show in section (4) that if $\Gamma$ is super rigid, it has a finite index normal subgroup $\Gamma_0$ for which $A(\Gamma_0) = A^0(\Gamma_0) \times \widehat{\Gamma_0} = A^0(\Gamma) \times \widehat{\Gamma_0}$.

In Section 2 we prove two general results on the proalgebraic completion of any finitely generated group $\Gamma$ which are of independent interest:

THEOREM B. *$A(\Gamma)$ is simply connected (in the sense of Definition 6).*

THEOREM C. *$A(\Gamma)$ has profinite component lifting; i.e., a closed profinite subgroup which meets every connected component.*

In Section 7 we make some suggestions for further research on some sequences of numerical invariants associated with rigid groups.

### CONVENTIONS AND DEFINITIONS OF RIGIDITY

For the reader's convenience, we collect here the notations, conventions and definitions introduced in this introduction.

CONVENTION 1. $k$ denotes an algebraically closed field of characteristic 0; without loss of generality, $k$ can be assumed to be $\mathbb{C}$.

CONVENTION 2. $\Gamma$ denotes a discrete group, usually assumed to be finitely generated and residually finite. A *proalgebraic group $A$* is identified with its $k$ rational pojnts, and *homomorphisms* of these are assumed to be algebraic, and continuous for the pro-Zariski topology. This applies in particular to profinite groups. *Representations* are assumed to be finite $k$-dimensional linear representations. (*Prorepresentations* are projective limits of these.) We write $\Gamma^{\mathrm{ab}}$ for the Abelianization, $\Gamma/(\Gamma, \Gamma)$ of $\Gamma$. Similarly for $A^{\mathrm{ab}} = A/(A, A)$, except that in the proalgebraic category we always understand commutator subgroups to be closed, i.e. the closure of the algebraic commutator subgroup.

NOTATION 1. $R_n(\Gamma) = \mathrm{Hom}(\Gamma, \mathrm{GL}_n(k))$

DEFINITION 1. A discrete or proalgebraic group is *$n$ representation rigid* (briefly, *$n$ rigid*) if it has only finitely many isomorphism classes of simple representations in dimension $n$ or less. It is (*representation*) *rigid* if it is $n$ rigid for all $n$.

DEFINITION 2. A discrete or proalgebraic group is *representation reductive* (briefly, *reductive*) if every representation is semi-simple.

Let $G$ be an algebraic group, $U$ its unipotent radical and $Q^0 = G^0/U$ its connected reductive quotient. Then $G$ is rigid if and only it $Q^0$ is semisimple, i.e. has finite center, and $G$ is reductive if and only if $U = \{1\}$.

NOTATION 2. The profinite completion $\widehat{\rho}: \Gamma \to \widehat{\Gamma}$ is universal for maps from $\Gamma$ to finite groups. It is injective if and only if $\Gamma$ is residually finite.

NOTATION 3. $A(\Gamma)$ denotes the proalgebraic completion of $\Gamma$. (See Definition 4 below). $P: \Gamma \to A(\Gamma)$ is the canonical homomorphism, universal with respect to maps from $\Gamma$ to algebraic (or proalgebraic) groups; $\mathrm{Ker}(P) = \mathrm{Ker}(\widehat{\rho})$ for $\Gamma$ finitely generated. $A^0(\Gamma)$ is the identity component of $A(\Gamma)$, $U(\Gamma)$ is the prounipotnet radical of $A(\Gamma)$, and $Q(\Gamma)$ is the maximal proreductive quotient $A(\Gamma)/U(\Gamma)$. $Q^0(\Gamma)$ is the identity component of $Q(\Gamma)$.

DEFINITION 3. $\Gamma$ is *representation super rigid* (briefly, *super rigid*) if $A(\Gamma)$ is finite dimensional.

REFERENCES

We rely on and commend to the reader the following references on proalgebraic groups in general and proalgebraic completions in particular: ‘Representations and representative functions of Lie groups’ [7] and ‘Pro-affine algebraic groups’ [8] by G. Hochschild and G. D. Mostow; ‘Pro-affine algebraic groups’ [13] by F. Minbashian; and [10, Chapter 4].

## 2. Proalgebraic Completions

### 2.1. BASICS

We begin with an arbitrary group $\Gamma$ and define the proalgebraic completion of $\Gamma$ in terms of its universal property:

DEFINITION 4. A *proalgebraic completion* for $\Gamma$ relative to $k$ is a pair $(\rho_u, \mathcal{G})$ consisting of a proalgebraic $k$ group $\mathcal{G}$ and a homomorphism $\rho_u: \Gamma \to \mathcal{G}$ such that for any proalgebraic group $G$ and any homomorphism $\rho: \Gamma \to G$ there is a unique morphism $q_\rho: \mathcal{G} \to G$ such that $\rho = \rho_u \circ q_\rho$.

It is immediate from the definition that a proalgebraic completion for $\Gamma$ is unique up to unique isomorphism. Moreover, $\rho_u(\Gamma)$ is Zariski dense in $\mathcal{G}$. In fact, let $G$ denote the Zariski closure of $\rho_u(\Gamma)$ in $\mathcal{G}$. The universal property then furnishes a retraction $q: \mathcal{G} \to G \leqslant \mathcal{G}$. Since $q$ and $\mathrm{Id}_{\mathcal{G}}$ are endomorphisms of $\mathcal{G}$ that agree on $\rho_u(\Gamma)$, they are equal, hence $q = \mathrm{Id}_{\mathcal{G}}$, i.e. $G = \mathcal{G}$. We denote this group $A_k(\Gamma)$. Moreover, it is also easy to see that it is enough for a proalgebraic completion for $\Gamma$ only to satisfy the definition for the case that $G$ is an affine $k$ group, and hence for the case that $G = \mathrm{GL}_n(k)$, some $n$.

Our field $k$ is usually fixed, and we generally drop the subscript $k$ and write $A(\Gamma)$ for the proalgebraic completion. However, we point out that $A(\Gamma)$ depends on the field in a crucial way, and that for base change $k \subset K$ we may have $A_K(\Gamma) \neq K \otimes_k A_k(\Gamma)$.

There are two standard constructions for $A(\Gamma)$: the first considers the product

$$\mathcal{P} = \prod \{\mathrm{GL}_n(k) \mid \rho \in \mathrm{Hom}(\Gamma, \mathrm{GL}_n(k)),\ n \in \mathbb{N}\}$$

of all the ranges of all the finite-dimensional $k$ representations of $\Gamma$. $\mathcal{P}$ is a proalgebraic group, and there is an obvious diagonal homomorphism $P : \Gamma \to \mathcal{P}$. Then taking the Zariski closure of $P(\Gamma)$ in $\mathcal{P}$ produces a pair satisfying Definition (4).

The other construction begins by directly producing the ind-affine coordinate ring $\mathcal{O}$ of a proalgebraic completion: by Zariski density, functions in $\mathcal{O}$ are determined by their values on $\Gamma$, so $\mathcal{O}$ may be regarded as a ring of functions on $\Gamma$. Any function in the coordinate ring of the range of any representation of $\Gamma$ pulls back via $\rho_u$ to a function in $\mathcal{O}$. One checks that these are precisely the $k$-valued functions on $\Gamma$ whose translates by $\Gamma$ span a finite-dimensional vector space over $k$; these are termed *representative functions* on $\Gamma$. The set of representative functions on $\gamma$ is denoted $\mathcal{O}_k(\Gamma)$. It is seen to be a Hopf algebra whose associated proalgebraic group, of $k$ algebra homomorphisms to $k$, $\mathrm{Alg}_k(\mathcal{O}_k(\Gamma), k)$, is a proalgebraic for completion $\Gamma$ ($\Gamma \to \mathrm{Alg}_k(\mathcal{O}_k(\Gamma), k)$ is given by sending $\gamma \in \Gamma$ to evaluation at $\gamma$.)

A third common construction of the proalgebraic completion is as the group of tensor product preserving automorphisms of the forgetful functor from the category of finite-dimensional $\Gamma$ modules to the category of finite-dimensional $k$ vector spaces ('Tannaka Duality'); see [11].

As per Notation 3 above, we use $(P, A_k(\Gamma))$ to denote the (equivalent) proalgebraic completions resulting from either construction.

The case $\Gamma = \mathbb{Z}$ is instructive:

EXAMPLE 1. The Zariski closures of the representations of $\mathbb{Z}$ are the closures of the cyclic subgroups of $\mathrm{GL}_n(k)$. These are Abelian, can have an (at most) one-dimensional unipotent radical, a torus of aribtrary size, and a finite cyclic group on top. The divisibility of the first two types of subgroups shows that the group is a direct product of the three types. Hence, $A(\mathbb{Z}) = \mathbb{G}_a(k) \times T \times \widehat{\mathbb{Z}}$, where $T = T(\mathbb{Z})$ is an infinite-dimensional protorus whose character group is the divisible group $\mathrm{Hom}(\mathbb{Z}, k^\times) \cong k^\times$. Here $U(\mathbb{Z}) = \mathbb{G}_a(k) \cong k$ and $A^0(\mathbb{Z}) = \mathbb{G}_a(k) \times T$, and $Q^0(\mathbb{Z})$ is isomorphic to $T$. Note that the groups $A(\mathbb{Z})$, $A^0(\mathbb{Z})$, and $Q^0(\mathbb{Z}) = T(\mathbb{Z})$ are all infinite-dimensional.

EXAMPLE 2. More generally, if $\Gamma$ is Abelian, then $A(\Gamma) = U(\Gamma) \times T \times \widehat{\Gamma}$. Here $U(\Gamma)$ is $k$ dual to the $k$ vector space $\mathrm{Hom}(\Gamma, k)$, (finite-dimensional if $\Gamma$ is finitely generated, zero if and only if $\Gamma$ is torsion). The character group $X(\Gamma)$ of the protorus $T$ is isomorphic to the torsion free quotient of $\mathrm{Hom}(\Gamma, k^\times)$. The torsion subgroup of

$\mathrm{Hom}(\Gamma, k^\times)$ is Pontryagin dual to $\widehat{\Gamma}$. Now $\mathrm{Hom}(\Gamma, k^\times)$ is torsion if and only if $\Gamma$ is torsion of bounded exponent. In all other cases, $\mathrm{Hom}(\Gamma, k^\times)$ has infinite torsion-free rank, as can easily be checked. Thus, either $\Gamma$ is torsion of bounded exponent, and $T = \{1\}$, or else $T$ is infinite-dimensional. Thus we have the equivalence of the conditions:

(1) $\Gamma$ is torsion of bounded exponent.
(2) $X(\Gamma)$ is torsion.
(3) $T(\Gamma) = \{1\}$.
(4) $\mathrm{Dim}(T(\Gamma)) < \infty$.

Moreover, these conditions imply the following equivalent conditions:

(5) $\mathrm{Hom}(\Gamma, k) = \{0\}$, i.e. $\Gamma$ is torsion.
(6) $U(\Gamma) = \{1\}$.

EXAMPLE 3. If $\Gamma$ is no longer assumed to be abelian, then the above analysis describes $A(\Gamma)^{\mathrm{ab}} = A(\Gamma^{\mathrm{ab}})$ (see Remark 1 below). Namely, $A(\Gamma)^{\mathrm{ab}} = U \times T \times P$, where $P = \widehat{\Gamma^{\mathrm{ab}}}$, and is Pontryagin dual to the torsion subgroup of $X(G) := \mathrm{Hom}(\Gamma, k^\times)$; $T$ is the protorus whose character module is the torsion free quotient of $X(\Gamma)$; and $U$ is the $k$ module dual of the $k$ vector space $\mathrm{Hom}(\Gamma, k)$.

The following simple results on finite index subgroups, which we recall here with proofs, are basic for our analysis.

PROPOSITION 1. *Let $\Gamma$ be a group and let $\Gamma^0$ be a finite index subgroup. Then $A(\Gamma^0) \to A(\Gamma)$ is injective, and $q: \Gamma/\Gamma^0 \to A(\Gamma)/A(\Gamma^0)$ is bijective.*

*Proof.* Every finite-dimensional $\Gamma^0$ module $M$ is a $\Gamma^0$ submodule of a finite-dimensional $\Gamma$ module $k[\Gamma] \otimes_{k[\Gamma^0]} M$, so every representative function on $\Gamma^0$ is the restriction of a representative function on $\Gamma$. Thus the restriction $R(\Gamma) \to R(\Gamma^0)$ is surjective, which makes $A(\Gamma^0) \to A(\Gamma)$ is injective. If $\gamma_1, \ldots, \gamma_d$ are coset representatives for $\Gamma/\Gamma^0$, then $A(\Gamma^0)\gamma_1 \cup \cdots \cup A(\Gamma^0)\gamma_d$ is a closed subset of $A(\Gamma)$ which contains $\Gamma$, and hence $A(\Gamma)$, so $q$ is surjective. The permutation representation of $\Gamma$ on $k[\Gamma/\Gamma^0]$ extends to a representation $\rho$ of $A(\Gamma)$ such that $\rho(A(\Gamma)) = \rho(\Gamma)$, $\rho(A(\Gamma^0)) = \rho(\Gamma^0)$, and the $\rho(\gamma_i)$ are distinct modulo $\rho(\Gamma^0)$. Hence, $q$ is also injective. □

COROLLARY 1. *Let $\Gamma$ be a group and let $\Gamma^0$ be a finite index subgroup. Then $A^0(\Gamma^0) \to A^0(\Gamma)$ is an isomorphism.*

*Proof.* We consider $A(\Gamma^0) \to A(\Gamma)$ an inclusion. Without loss of generality, we may asume that $\Gamma^0$ is normal in $\Gamma$, which in turn implies that $A(\Gamma^0)$ is normal in $A(\Gamma)$ and, hence, so is the characteristic subgroup $A^0(\Gamma^0)$. $A(\Gamma^0)/A^0(\Gamma^0)$ is profinite and is of finite index in $A(\Gamma)/A^0(\Gamma^0)$, which implies that the latter is profinite as well. Thus $A^0(\Gamma^0)$ is a connected normal subgroup of $A(\Gamma)$ with profinite quotient, which implies that $A^0(\Gamma^0) \to A^0(\Gamma)$ is an isomorphism. □

*Remark* 1. Let $\Gamma_0 \to \Gamma \to \Gamma_1 \to 1$ be an exact sequence of discrete groups.

(a) The sequence $A(\Gamma_0) \to A(\Gamma) \to A(\Gamma_1) \to 1$ is exact.

(b) Call $\Gamma_0 \to \Gamma$ observable if every representation of $\Gamma_0$ is $\Gamma_0$ equivariantly embeddable in a representation of $\Gamma_1$. This is the case for example when $\Gamma_0$ is a finite index subgroup of $\Gamma$. From the point of view of representative functions (see the discussion following Definition 4) we see that this is necessary and sufficient for the injectivity of $A(\Gamma_0) \to A(\Gamma)$.

(c) Taking $\Gamma_0$ to be the commutator subgroup of $\Gamma$, so that $\Gamma_1$ is the Abelianization $\Gamma^{ab} = \Gamma/(\Gamma, \Gamma)$, we see that $A(\Gamma)^{ab} = A(\Gamma^{ab})$, and the latter group is as described in Example 2 above. Since $A(\Gamma) \cong U(\Gamma) \rtimes Q(\Gamma)$, we have

$$A(\Gamma)^{ab} = U(\Gamma^{ab}) \times T(\Gamma^{ab}) \times \widehat{\Gamma^{ab}},$$

where

$$U(\Gamma^{ab}) = U(\Gamma)/(A(\Gamma), U(\Gamma))$$

(recall that commutator groups are here always understood to be the closures of the algebraic commutator subgroups); $U(\Gamma^{ab})$ is $k$ dual to the $k$ vector space $\mathrm{Hom}(\Gamma, k)$. Moreover, $Q(\Gamma^{ab}) = T(\Gamma^{ab}) \times \widehat{\Gamma^{ab}}$ is a proreductive Abelian group with character group $X(\Gamma) = \mathrm{Hom}(\Gamma, k^{\times})$, and the character group $X(T(\Gamma^{ab}))$ is the torsion free quotient of $X(\Gamma)$.

(d) For any group $\Gamma$ we put $S(\Gamma) = (Q^0(\Gamma), Q^0(\Gamma))$, a prosemisimple group, and $T(\Gamma) = Z_{Q^0(\Gamma)}(S(\Gamma))^0$, the connected center of $Q^0(\Gamma)$, which is a protorus. (This notation is consistent with the notation $T(\Gamma^{ab})$ in (c) above.) We have $Q^0(\Gamma) = S(\Gamma) \cdot T(\Gamma)$.

Remark 1 gives a necessary condition for rigidity, which we will now name and give some equivalent formulations of:

DEFINITION 5. We introduce the following conditions on a group $\Gamma$.

(TAb) $\Gamma_0^{ab} = \Gamma_0/(\Gamma_0, \Gamma_0)$ is torsion for all $\Gamma_0$ of finite index in $\Gamma$,

(BTAb) $\Gamma_0^{ab} = \Gamma_0/(\Gamma_0, \Gamma_0)$ is torsion of bounded exponent for all $\Gamma_0$ of finite index in $\Gamma$,

(FAb) $\Gamma_0^{ab} = \Gamma_0/(\Gamma_0, \Gamma_0)$ is finite for all $\Gamma_0$ of finite index in $\Gamma$.

*Remark* 2. Clearly (FAb) implies (BTAb) implies (TAb), and they are all equivalent if $\Gamma$ is finitely generated. We have

(TAb) if and only if $\mathrm{Hom}(\Gamma_0, k) = \{0\}$ for all $\Gamma_0$ of finite index in $\Gamma$; and

(BTAb) if and only if $\mathrm{Hom}(\Gamma_0, k^{\times}) = \{1\}$ for all $\Gamma_0$ of finite index in $\Gamma$.

$\Gamma = \mathbb{Q}/\mathbb{Z}$ satisfies (TAb) but not (BTAb). $\Gamma = \mathbb{F}_p[t]$ satisfies (BTAb) but not (FAb). If $\Gamma$ is a weak direct product of infinitely many copies of s finite simple group, then $\Gamma$ satisfies (FAb) but $\widehat{\Gamma}$ is not rigid.

PROPOSITION 2. *Let* $\Gamma$ *be a discrete group.*

(1) *Condition* (a) *implies condition* (b)*:*

(a) $\widehat{\Gamma}$ *is rigid,*
(b) $\Gamma$ *has* (FAb).

(2) *The following conditions on* $\Gamma$ *are equivalent:*

(a) $\Gamma$ *has* (BTAb)*,*
(b) $T(\Gamma) = \{1\}$*,*
(c) $\mathrm{Dim}(T(\Gamma) < \infty$.

(3) *The following conditions on* $\Gamma$ *are equivalent:*

(a) (TAb)*,*
(b) $U(\Gamma) = (A^0(\Gamma), U(\Gamma))$.

*When* $\Gamma$ *is finitely generated, all of the above conditions are equivalent.*

*From* (2) *it follows that if there is one representation* $\rho: \Gamma \to \mathrm{GL}_n(k)$ *such that* $\rho(A^0(\Gamma))$ *has a nontrivial linear character, then* $T(\Gamma)$ *is infinite-dimensional.*

*Proof.* It will suffice to prove the following implications:

(1)(a) implies (1)(b):

Let $\Gamma_0$ be a finite index subgroup of $\Gamma$. If $\Gamma_0^{\mathrm{ab}}$ is infinite then $\Gamma_0$ has infinitely many one-dimensional representations with finite image, and these induce to representations with finite image of $\Gamma$ in dimension $[\Gamma : \Gamma_0]$ with infinitely many distinct characters, thus violating rigidity of $\widehat{\Gamma}$.

(2)(a) implies (2)(b):

If $T(\Gamma) \neq \{1\}$ there is an epimorphism $A^0(\Gamma) \to k^\times$. This appears in an algebraic quotient of $A(\Gamma)$, whose connected component pulls back to an open subgroup of $A(\Gamma)$ whose intersection, $\Gamma_0$, with $\Gamma$ is a finite index subgroup mapping to $k^\times$ with Zariski dense (i.e. infinite) image. Thus $\Gamma_0^{\mathrm{ab}}$ is not torsion of bounded exponent, contradicting (BTAb).

(2)(c) implies (2)(a):

Let $\Gamma_0$ be a finite index subgroup of $\Gamma$, and $X(\Gamma_0) = \mathrm{Hom}(\Gamma_0, k^\times)$. Then $T(\Gamma_0^{\mathrm{ab}})$ is a quotient of $T(\Gamma)$, and hence finite-dimensional, by hypothesis. By Example (2), this can happen only if $T(\Gamma_0^{\mathrm{ab}}) = \{1\}$, i.e. if $X(\Gamma_0)$ is torsion, and this happens only if $\Gamma_0^{\mathrm{ab}}$ is torsion of bounded exponent, whence (BTAb).

(3)(a) implies (3)(b):

Let $W = U(\Gamma)/(A^0(\Gamma), U(\Gamma))$. If $W \neq \{1\}$ then there is an algebraic quotient $G$ of $A(\Gamma)$ such that $(G^0)^{\mathrm{ab}}$ has a nontrivial unipotent radical, which is a direct factor. This produces, as usual, a finite index subgroup $\Gamma_0$ of G which maps to $k$ with Zariski dense (i.e. nonzero) image, whence $\Gamma_0^{\mathrm{ab}}$ is not torsion, contradicting (TAb).

(3)(b) implies (3)(a):

If (TAb) fails then $\mathrm{Hom}(\Gamma_0, k) \neq \{0\}$ for some finite index $\Gamma_0$ in $\Gamma$. This entails a non-trivial unipotent quotient of $A(\Gamma_0)$, and so $W \neq \{1\}$.

(1)(b) implies (1)(a) for $\Gamma$ finitely generated:

If $\widehat{\Gamma}$ is not rigid then, in some $\mathrm{GL}_n(k)$, $\Gamma$ has infinitely many conjugacy classes of representations with finite image, and therefore images of unbounded size. Let $\Gamma_n$ denote the intersection of all subgroups of $\Gamma$ of index at most $j = j(n)$ as in Jordan's Theorem (below). Since $\Gamma$ is finitely generated, the latter are finite in number, and so $\Gamma_n$ itself has finite index in $\Gamma$. Moreover, Jordan's Theorem implies that, for each representation $\rho : \Gamma \to \mathrm{GL}_n(k)$, $\rho(\Gamma_n)$ is Abelian. Since these images have unbounded size, it follows that $\Gamma_n^{\mathrm{ab}}$ is infinite, thus violating hypothesis (FAb). □

In the previous proof, and in other results below, we have used Jordan's Theorem. We recall its statement and some consequences:

JORDAN'S THEOREM *There is a number $j = j(n)$ such that each finite subgroup of* $\mathrm{GL}_n(k)$ *has an Abelian normal subgroup of index at most j.*

*Consequences*. For an integer $N > 0$, call a group $G$ *N-residual* if the quotients of $G$ of order at most $N$ separate the points of $G$. For any group $G$, let $G^N$ denote the intersection of the normal subgroups of index at most $N$. Then $G/G^N$ is the $N$-residual quotient of $G$. If $G$ is finitely generated (discrete or profinite) then it is clear that $G/G^N$ is finite. If the $n$-dimensional representations of $G$ separate points, then it follows from Jordan's Theorem that $G^{j(n)}$ is Abelian. Thus, if $G$ is finitely generated and its $n$-dimensional representations separate points then $G^{j(n)}$ is an Abelian normal subgroup of finite index (and $G/G^{j(n)}$ is $j(n)$-residual).

If $\Gamma$ is rigid, then Proposition (2) (1)(a) holds (it is a special case of rigidity). Hence:

COROLLARY 2. *A finitely generated rigid group has* (FAb).

## 2.2. SIMPLY CONNECTIVITY OF PROALGEBRAIC COMPLETION IDENTITY COMPONENT

In this section, we observe that, for a finitely generated group $\Gamma$, the identity component $A^0(\Gamma)$ of $A(\Gamma)$ is simply connected, in the sense which we now define.

DEFINITION 6. A connected proalgebraic group $G$ is said to be simply connected if every surjection $p: G_1 \to G$ where $G_1$ is connected proalgebraic and the kernel of $p$ is finite is an isomorphism. (It follows that the same property holds if we assume only that Ker(p) is profinite.)

In terms of structure, this signifies the following. Write $G = U \rtimes Q$, where $U$ is the prounipotent radical of $G$, and $Q$ is connected and proreductive. In turn, we can write $Q = S \cdot T$, where $S = (Q, Q)$ is connected prosemisimple and $T = Z_Q(S)^0$ is a protorus. Further we can write $S$ as an almost direct product of simple algebraic groups $S_I$, in the sense that the map $q: \prod S_I \to S$ is surjective with central kernel. Now, with this notation, $G$ is simply connected if and only if $Q = S \times T$, each $S_I$ is simply connected (as algebraic group), the map $q$ is an isomorphism, and the character group $X(T)$ is divisible.

The following proposition gives a more convenient version of the simply connected property:

PROPOSITION 3. *Let $G$ be a connected proalgebraic group. The following are equivalent:*

(1) *$G$ is simply connected*
(2) *If $H$ is a normal subgroup of $G$ such that $G/H$ is affine, and $q: \bar{G} \to G/H$ is an eprimorphism of affine groups with finite kernel, then there is a homomorphism $r: G \to \bar{G}$ such that $q \circ r$ is the canonical map $G \to G/H$.*

*Proof.* Suppose $G$ is simply connected and $q: \bar{G} \to G/H$ is a map as in (2). Let $G_2 = G \times_{G/H} \bar{G}$, and let $p_1: G_2 \to G$ be projection on the first factor. Then $p_1$ is surjective (since $q$ is) and $\mathrm{Ker}(p_1) \cong \mathrm{Ker}(q)$ is finite. Then, by hypothesis, $p: G_2^0 \to G$ is an isomorphism. Then $r = p_2 \circ p^{-1}$, where $p_2$: $G_2^0 \leqslant G_2 \to \bar{G}$ is projection on the second factor, is the desired homomorphism.

Now suppose that $G$ satisfies (2) and that there is a surjection $p: G_1 \to G$ where $G_1$ is connected proalgebraic and the kernel $K$ of $p$ is finite. Let $H_1$ be a connected normal subgroup of $G_1$ such that $G_1/H_1$ is affine and such that $K \cap H_1$ is the identity. Let $H = p(H_1)$. Then $q: G_1/H_1 \to G/H$ is a surjection of affine groups with finite kernel $K$, and so there is a map $f: G \to G_1/H_1$ such that $q(f(g)) = gH$ for $g \in G$. It follows that $f(H) \leqslant K$, and since $H$ is connected and $K$ finite, this implies that $f(H) = \{e\}$. But then $f$ factors through $G/H$ and provides a section $G/H \to G_1/H_1$ of $q$. Thus $G_1/H_1 \cong G/H \times K$, and since $G_1/H_1$ is connected this implies that $K = 1$ and $p$ is an isomorphism. So $G$ is simply connected. □

Now we show that identity components of proalgebraic completions of finitely generated groups have this property:

THEOREM 1. *Let $\Gamma$ be a finitely generated group. Then $A^0(\Gamma)$ is simply connected.*

*Proof.* We apply Proposition 3. Assume that $H$ is normal in $A^0(\Gamma)$ and such that $A^0(\Gamma)/H$ is affine, and $q: G \to A^0(\Gamma)/H$ is surjective with finite kernel with $G$ connected. Suppose $H_1 \leqslant H$ is also normal in $A^0(\Gamma)$ with affine quotient. Then

$$G_1 = (G \times_{A^0(\Gamma)/H} A^0(\Gamma)/H_1)^0 \to A^0(\Gamma)/H_1$$

by projection on the second factor is also surjective with finite kernel, and a map $A^0(\Gamma) \to G_1$ will give the desired map to $G$ when followed by projection on the first factor. So we can replace $H$ by smaller normal subgroups. In particular, we can replace $H$ by a subgroup normal in $A(\Gamma)$. (If $A(\Gamma) = \varprojlim A(\Gamma)/H_\alpha$ then $A^0(\Gamma) = \varprojlim A^0(\Gamma)/(A^0(\Gamma) \cap H_\alpha)$).

Let $\rho$ be the representation of $\Gamma$ corresponding to $A(\Gamma) \to A(\Gamma)/H$ and let $\Gamma^0 = \rho^{-1}(\rho(\Gamma) \cap A^0(\Gamma)/H)$. Then $\rho(\Gamma^0)$ is Zariski dense in $A^0(\Gamma)/H$. Let $\Lambda = q^{-1}(\rho(\Gamma^0))$. $\Lambda$ is an extension of $\rho(\Gamma^0)$ by the finite Abelian group $K$. $\Lambda$ is a finitely generated linear group and, hence, residually finite. Let $\Lambda^1$ be a finite index normal subgroup of $\Lambda$ with $\Lambda^1 \cap K$ the identity. Let $\Gamma^1 = q(\Lambda^1)$. The extension $\Lambda$ of $\Gamma^0$ by $K$ is thus split over $\Gamma^1$. This means that the map $\Gamma^1 \to A^0(\Gamma)/H$ lifts to a map $\Gamma^1 \to G$. This in turn gives rise to a map $A(\Gamma^1) \to G$ and, hence, a map $A^0(\Gamma^1) \to G$. By Corollary 1, $A^0(\Gamma^1) \to A^0(\Gamma)$ is an isomorphism, so we have a map $A^0(\Gamma) \to G$ as required by proposition 3. It follows that $A^0(\Gamma)$ is simply connected. □

As an important consequence of simple connectivity, we have the following:

COROLLARY 3. *Let $\Gamma$ be a finitely generated group. Then $Q^0(\Gamma)$ is the direct product of a protorus $T(\Gamma)$ with uniquely divisible character group and the closed commutator subgroup $S(\Gamma)$ of $Q^0(\Gamma)$; $S(\Gamma)$ is the (possibly infinite) direct product of simply connected simple algebraic groups.*

It is possible of course for $T(\Gamma)$ in Corollary 3 to be trivial. From Proposition 2(2) once it is nontrivial, then it is in fact infinite-dimensional.

## 2.3. LIFTING PROFINITE QUOTIENTS, AND PROJECTIVE PROALGEBRAIC GROUPS

DEFINITION 7. Let $G$ be a proalgebraic group. A component quotient lift (briefly, lifting) is a profinite subgroup of $G$ which maps onto $G/G^0$.

The main goal of this section will be to show that all proalgebraic groups admit component quotient lifts.

For the case of an affine algebraic group, such liftings are due to V. Platonov [14]. We obtain the existence of liftings in the proalgebraic case by a reduction to the case treated by Platonov: we introduce the notion of projective proalgebraic group, and show that projectivity can be tested on affine surjections. From this, we deduce that a projective profinite group is projective as a proalgebraic group. It follows that free profinite groups are projective. This provides a component lifting for the case of $A(F)$, $F$ free, from which the existence of component liftings in general then will follow,

We begin with the definition of projective proalgebraic:

DEFINITION 8. A proalgebraic group $P$ is projective if for every epimorphism $\alpha: A \to B$ of proalgebraic groups and for every homomorphism $f: P \to B$ there is a homomorphism $\phi: P \to A$ such that $f = \alpha \circ \phi$. We call $\phi$ a lifting of $f$ (through $\alpha$).

Symbolically, we want to complete the diagram

$$\begin{array}{ccc} & & A \\ & & \downarrow \\ P & \to & B \end{array}$$

with a diagonal map

$$\begin{array}{ccc} & & A \\ & \nearrow & \downarrow \\ P & \rightarrow & B \end{array}$$

For example, when $F$ is a free group (possibly on an infinite set), then $A(F)$ is clearly a projective proalgebraic group.

The following proposition is the key step in our reduction argument.

PROPOSITION 4. *Let $P$ be a proalgebraic group. The following are equivalent:*

(1) *$P$ is projective.*
(2) *For every $(\alpha, f)$ as in Definition* (8) *with* $\mathrm{Ker}(\alpha)$ *algebraic $f$ admits a lifting through $\alpha$.*
(3) *For every $(\alpha, f)$ as in Definition* (8) *with $A$ algebraic $f$ admits a lifting through $\alpha$.*

*Proof.* Clearly (1) implies (2) implies (3). We prove the converse of each.

Assume $P$ satisfies (3) and let $\alpha: A \to B, f: P \to A$, be such that $\alpha$ has algebraic kernel $K$. Since $K$ has the descending chain condition on closed subgroups, and the closed normal subgroups of $A$ with affine quotient have intersection the identity, we can find such a normal subgroup $N$ of $A$ with $N \cap K = \{e\}$. We can identify $B$ with $A/K$ and hence identify $B/\alpha(N)$ with $A/KN$. Then $A/N$ is algebraic and the induced maps $A/N \to A/KN$ and $P \to A/KN$ admit an extension $\phi: P \to A/N$. Then $\Phi = (f, \phi)$ maps $P$ to the fibre product $A/K \times_{A/KN} A/N$. On the other hand, it is easy to see that since the intersection $K \cap N$ is the identity, the map of $A$ to the fibre product induced by the canonical projections $A \to A/N$ and $A \to A/K$ is an isomorphism ('Chinese Remainder Theorem' for groups). Thus we can regard $\Phi$ as a map to $A$, and it is an extension of $\alpha$ and $f$ as needed for (2).

Now assume $P$ satisfies (2) and assume $\alpha: A \to B$ is any surjection and $f: P \to B$ is a morphism. It will be convenient to write $B$ as $A/L$. For any normal subgroup $N$ of $A$ contained in $L$, we will call a morphism $\phi: P \to A/N$ an $N$ partial extension for $f$ if the composition of $\phi$ and $A/N \to A/L$ is $f$. The set $\mathcal{N}$ of pairs $(N, \phi)$ where $\phi$ is an $N$ partial extension of $f$ is partially ordered: we say $(N, \phi) \leqslant (N', \phi')$ if $N' \leqslant N$ and $\phi'$ induces $\phi$ mod $N$. We claim that any chain $\mathcal{C} = \{(N_i, \phi_i) \mid i \in I\}$ in $\mathcal{N}$ has a maximal element. (Note: the indexing set $I$ in $\mathcal{C}$ is not necessary countable.) Let $N_0 = \cap_i N_i$. $A/N_0 = \varprojlim A/N_i$ and the maps $\phi_i$ induce a map $\phi_0: P \to A/N_0$. $(N_0, \phi_0)$ is a maximal element for $\mathcal{C}$. By Zorn's lemma, $\mathcal{N}$ has a maximal element $(N, \phi)$. If $N = \{e\}$, $\phi$ is an extension of $\alpha, f$ as desired. If not $N \neq \{e\}$, then there is a normal subgroup $M$ of $A$ with $A/M$ affine such that $M \cap N = L$ is a proper subgroup of $N$. The surjection $A/L \to A/N$ has algebraic kernel $K = MN/M$. By (2), there is an extension $\psi$ for $A/L \to A/N$ and $\phi: P \to A/N$, which implies that $(N, \phi) < (L, \psi)$, contrary to the maximality of $(N, \phi)$. Thus $N = \{e\}$ and (1) follows. □

As previously noted, Platonov proved component lifting for algebraic groups. Using Proposition (4), we use this to conclude that projective profinite groups are projective proalgebraic:

COROLLARY 4. *A profinite group which is projective in the category of profinite groups is projective in the category of proalgebraic groups.*

*Proof.* Let $P$ be a profinite group which is projective in the profinite group category. By Proposition 4, $P$ will be projective proalgebraic if morphisms $f: P \to B$ and $\alpha: A \to B$, $\alpha$ onto and $A$ and $B$ algebraic, have an extension. Since $B$ is algebraic, $f(P)$ is finite. Let $C = \alpha^{-1}(f(P))$. By [14] there is a finite subgroup $F \leqslant C$ mapping onto $f(P)$. Since $P$ is profinite projective, there is an extension $\phi: P \to F$ for the maps $P \to f(P)$ and $F \to f(P)$. Then regarding $\phi$ as a map to $A$ we have an extension for $f$ and $\alpha$. □

It is clear that the profinite completion $\widehat{F}$ of a free group $F$ is projective profinite and, hence, by Corollary 4 is projective proalgebraic.

COROLLARY 5. *Let $F$ be a free group. Then there is a profinite subgroup $\Delta$ of $A(F)$ which maps isomorphically onto $A(F)/A^0(F)$.*

We can regard any group $\Gamma$ as a homomorphic image of a free group $F$, which makes $A(\Gamma)$ a homomorphic image of $A(F)$ and, hence, $A(\Gamma)/A^0(\Gamma)$ an image of $A(F)/A^0(F)$. If $\Delta$ is a profinite subgroup of $A(F)$ as in Corollary 5, then its image in $A(\Gamma)$ maps onto $A(\Gamma)/A^0(\Gamma)$. Hence,

COROLLARY 6. *Let $\Gamma$ be a group. Then there is a profinite subgroup of $A(\Gamma)$ which meets every coset of $A^0(\Gamma)$.*

The argument used to establish Corollary 6 applies to any proalgebraic group which is a homomorphic image of $A(F)$:

COROLLARY 7. *Let $G$ be a proalgebraic group. Then there is a profinite subgroup of $G$ which meets every coset of $G^0$.*

*Proof.* Let $g_a$, $a \in \mathcal{A}$ be elements of $G$ that generate $G/G^0$ as a proalgebraic group.Let $F$ be a free group on $x_a$, $a \in \mathcal{A}$ and define a morphism $F \to G$ by $x_a \mapsto g_a$. By the universal property of proalgebraic completions, this extends to a morphism $p: A(F) \to G$ which is surjective and, by construction, gives rise to a surjection $A(F)/A^0(F) \to G/G^0$. If $\Delta$ is a profinite subgroup of $A(F)$ as in Corollary 5, then $p(\Delta)$ is a profinite subgroup of $G$ which maps onto $G/G^0$, as asserted. □

## 3. The Proalgebraic *n* Completion and Rigidity Criteria

There is an analogue of $A(\Gamma)$ which has a corresponding universal property for representations of dimension $n$ or less. In this section we define it and discuss its connection with rigidity. We begin with a definition:

DEFINITION 9. Let $K_n(\Gamma)$ be the intersection of the kernels of all the algebraic representations of $A(\Gamma)$ of dimension at most $n$. The proalgebraic $n$ completion of $\Gamma$, denoted $A_n(\Gamma)$, is the quotient $A(\Gamma)/K_n(\Gamma)$. We let $U_n(\Gamma)$ and $Q_n(\Gamma)$ denote the prounipotent radical and maximal reductive quotient of $A_n(\Gamma)$. We use $V_n(\Gamma)$ for the quotient of $U_n(\Gamma)$ by its closed commutator subgroup.

It is clear from the definition that representations of $\Gamma$ of dimension $n$ or less extend uniquely to representations of $A_n(\Gamma)$ of the same dimension. Since the representations of $\Gamma$ are assumed to separate points (of $\Gamma$ and hence of $A(\Gamma)$), the representations of dimension $n$ or less separate the points of $A_n(\Gamma)$.

As an application of Corollary 7 and Jordan's Theorem, we have the following property for $A_n(\Gamma)$ when $\Gamma$ satisfies (FAb):

COROLLARY 8. *Let $\Gamma$ be a finitely generated group and n a positive integer.*

(1) *There exists a finite index normal subgroup $\Gamma^n$ of $\Gamma$ such that*

$$A_n(\Gamma^n)/A_n^0(\Gamma) = (Q_n(\Gamma^n)/QA_n^0(\Gamma))$$

*is Abelian.*

(2) *If $\Gamma$ satisfies* (FAb) *then $A_n(\Gamma^n)/A_n^0(\Gamma)$ is finite, $Q_n^0(\Gamma)$ is prosemisimple and we can choose $\Gamma^n$ in* (1) *so that $A_n(\Gamma^n) = A_n^0(\Gamma)$.*

(3) *If $\Gamma$ satisfies* (FAb) *and* $\mathrm{Dim}(Q_n(\Gamma)) < \infty$*, then $Q_n(\Gamma)$ is algebraic and rigid.*

*Proof.* Let $D$ be a profinite subgroup of $A_n(\Gamma)$ such that $A_n(\Gamma) = A_n^0(\Gamma)D$. Since $\Gamma$ is finitely generated we can select $D$ to be finitely generated as a profinite group. The consequences of Jordan's Theorem furnish an Abelian normal subgroup $D^{j(n)}$ of finite index in $D$. Then $A_n^0(\Gamma) \cdot D^{j(n)}$ is an open normal subgroup of $A_n(\Gamma)$ whose intersection $\Gamma^n$ with $\Gamma$ satisfies the condition of (1).

If $\Gamma$ satisfies (FAb) then the Abelian image of $\Gamma^n$ modulo $A_n^0(\Gamma)$ must be finite, so, by making $\Gamma^n$ smaller by finite index we can put $\Gamma^n$ inside $A_n^0(\Gamma)$. From this and Proposition 2, (2) follows. Clearly (3) follows now from (2). □

We are going to see the connection between rigidity and the finite dimensionality of the $A_n(\Gamma)$. In this connection, we note that a proalgebraic group is algebraic if (and only if) it is finite-dimensional and has finitely many connected components.

The main result of this section is the following theorem:

THEOREM 2. *The following are equivalent for the finitely generated group $\Gamma$:*

(1) *$\Gamma$ is rigid.*
(2) *$\forall n$, $A_n(\Gamma)$ is rigid.*
(3) *$\forall n$, $A_n(\Gamma)$ is an algebraic group.*
(4) *$\forall n$, $A_n(\Gamma)$ is finite-dimensional.*
(5) *$\forall n$, $Q_n(\Gamma)$ is rigid.*

(6) *$\forall n$, $Q_n(\Gamma)$ is an algebraic group.*
(7) *$\forall n$, $Q_n(\Gamma)$ is finite-dimensional.*

We will prove Theorem 2 by means of Theorem 3 below, which makes more precise the connections between the various properties.

We begin by enumerating the various rigidity and finiteness conditions to be considered:

NOTATION 4. $n$ denotes a positive integer

$(R)_n$ $\Gamma$ is $n$-rigid.
$(AR)_n$ $A_n(\Gamma)$ is rigid.
$(AA)_n$ $A_n(\Gamma)$ is an algebraic group.
$(AD)_n$ $\mathrm{Dim}(A_n(\Gamma)) < \infty$.
$(QR)_n$ $Q_n(\Gamma)$ is rigid.
$(QA)_n$ $Q_n(\Gamma)$ is an algebraic group.
$(QD)_n$ $\mathrm{Dim}(Q_n(\Gamma)) < \infty$.

For each of the properties $\mathcal{P} = R, AR, AA, AD, QR, QA, QD$ we will write $(\mathcal{P})_\infty$ to mean that $(\mathcal{P})_n$ holds for all $n$. Note that $(R)_\infty$ is equivalent to rigid.

We have some obvious implications:

(1) $(QR)_n$ is equivalent to $(AR)_n$ and both imply $(R)_n$.
(2) $(AA)_n$ holds if and only if $(AD)_n$ holds and $A_n(\Gamma)/A_n^0(\Gamma)$ is finite.
(3) $(QA)_n$ holds if and only if $(QD)_n$ holds and $Q_n(\Gamma)/Q_n^0(\Gamma)$ is finite.
(4) Thus $(AD)_n$ and $(QA)_n$ implies $(AA)_n$.
(5) By Corollary 8 (2) $(AD)_n$ and (FAb) implies $(AA)_n$; and
(6) $(QD)_n$ and (FAb) implies $(QA)_n$.
(7) Finally, we note that $(R)_\infty$ implies (FAb).

The following theorem records some of the main relationships among the properties of Notation 4.

THEOREM 3. *For all integers n the following implications hold:*

$$
\text{(I)}\qquad
\begin{array}{ccccccc}
 & & & & (\mathrm{R})_{n^2} & & \\
 & & & & \Downarrow & & \\
 & & (\mathrm{AA})_n & \Rightarrow & (\mathrm{AR})_n & \Rightarrow & (\mathrm{AD})_n \\
 & & \Updownarrow & & \Downarrow & & \Downarrow \\
[(\mathrm{QD})_n + (\mathrm{FAb})] & \Rightarrow & (\mathrm{QA})_n & \Rightarrow & (\mathrm{QR})_n & \Rightarrow & (\mathrm{QD})_n
\end{array}
$$

(II) $(\mathrm{AA})_n \Rightarrow (\mathrm{R})_n \Rightarrow (\mathrm{QR})_n$
(III) $(\mathrm{R})_\infty, (\mathrm{AA})_\infty, (\mathrm{AR})_\infty, (\mathrm{AD})_\infty, (\mathrm{QA})_\infty, (\mathrm{QR})_\infty, (\mathrm{QD})_\infty$ *are all equivalent.*

*Proof.* Assertion (III) follows from (I) and (II). In the proof of (I) and (II), the only implications that are nonobvious or are not covered by the discussion above, are the following:

$(\mathrm{R})_n \Rightarrow (\mathrm{QA})_n$, whence $(\mathrm{QR})_n \Rightarrow (\mathrm{QA})_n$. Assume $(\mathrm{R})_n$ (that $\Gamma$ is $n$-rigid). For a representation $\rho$ of $A(\Gamma)$, we will denote by $\rho_{ss}$ the associated semisimple representation. It is easy to see that $\rho_{ss}(A(\Gamma))$ is the quotient of $\rho(A(\Gamma))$ by its unipotent radical. It follows that the representations $\{\rho_{ss} \mid \rho \in R_n(\Gamma)\}$ separate the points of $Q_n(\Gamma)$; i.e., their kernels have trivial intersection. Since $\Gamma$ is $n$-rigid, there are only finitely many such kernels, which implies that $Q_n(\Gamma)$ is embedded in a finite product of algebraic groups, and hence is algebraic.

$[(\mathrm{QD})_n + (\mathrm{Fab})] \Rightarrow [(\mathrm{QR})_n + (\mathrm{QA})_n]$ This follows from Corollary 8(3) above.

$(\mathrm{R})_{n^2} \Rightarrow (\mathrm{AA})_n$. Since $n^2$-rigidity implies $n$-rigidity, we already have $(\mathrm{QA})_n$, proved above, so it remains to show the finite-dimensionality of $U_n(\Gamma)$. This follows from Proposition 5(2) below. □

PROPOSITION 5.

(1) *Let $G$ be a group, let $S$ be a simple $G$-module of dimension $d$, and let $V$ be an $S$-isotypic $G$-module generated by $r$ elements. Then* $\mathrm{Dim}(V) \leqslant rd^2$.
(2) *If $G = \Gamma$ is finitely generated and $n^2$-rigid then* $\mathrm{Dim}(U_n(\Gamma)) < \infty$.

*Proof.* We are grateful to R. Guralnick for the proof of part (1).

Proof of (1): An easy induction argument shows that it suffices to treat the case $r = 1$, and in this case it suffices to show that $S^{d+1}$ cannot be $G$ generated by a single element $v = (v_0, \ldots, v_d) \in S^{d+1}$. Since $V$ has dimension $d$, the components of $v$ are linearly dependent: there are $a_i \in k$, $0 \leqslant i \leqslant d$ not all 0 such that $\sum a_i v_i = 0$. Then $v$ belongs to the kernel $K$ of the nonzero $G$ linear map $\lambda: S^{d+1} \to S$ by $(w_0, \ldots, w_d) \mapsto \sum a_i w_i$ contrary to the assumption that $v$ generates.

Proof of (2): Since the $n$-dimensional representations of the prounipotent group $U_n(\Gamma)$ separate points, it is nilpotent. Thus, it suffices to show the finite dimensionality of the semisimple $G$-module $V = U_n(\Gamma)^{\mathrm{ab}}$. Since $\Gamma$ is finitely generated, $U_n(\Gamma)$, being a normal semidirect factor of $A_n(\Gamma)$, is finitely generated as a normal subgroup, which implies that $V$ is finitely generated as a $G$-module. The simple submodules of $V$ appear as subquotients of $n$-dimensional representations of $U_n(\Gamma)$ and so they have dimension $< n^2$. Hence, by $n^2$-rigidity of $\Gamma$, there are only finitely many classes of them. Now the finite-dimensionality of $V$ follows from part (1). □

Note that Theorem 3 III is simply a restatement of Theorem 2, and hence the latter is now proved.

COROLLARY 9. *A representation super rigid group is representation rigid.*

*Proof.* If $\Gamma$ is super rigid, $A(\Gamma)$ is finite-dimensional by definition. Hence $A_n(\Gamma)$ is finite-dimensional for all $n$, and so by Theorem 2 $\Gamma$ is rigid. □

The finite-dimensionality of $U(\Gamma)$, in fact the finite-dimensionality of $V(\Gamma)$, also implies rigidity in the presence of (FAb), as we now show.

THEOREM 4. *Let $\Gamma$ be a finitely generated group with* (FAb)*, and suppose that $U(\Gamma)$ is finitely generated as a prounipotent group. Then $\Gamma$ is rigid.*

*Proof.* We assume that $\Gamma$ has (FAb), that $U(\Gamma)$ is finitely generated (which means $V(\Gamma)$ is finite-dimensional) but that $\Gamma$ is not rigid. It follows from Theorem 3 that for some $n$, the prosemisimple group $Q_n^0(\Gamma)$ has infinitely many simple factors isomorphic to some simple algebraic subgroup $S \leqslant \mathrm{GL}_n(k)$. It follows that we have an epimorphism $A_n^0(\Gamma) \to \Pi = S^{\mathbb{N}} = \prod_{i \geqslant 0} S_i$ with each $S_i$ is isomorphic to $S$. Let $q_i: \Pi \to S$ denote projection on the $i$th factor. Choose the finite index subgroup $\Gamma^n \leqslant \Gamma$ as in Corollary 8 (2) so that $\Gamma^n$ projects onto (a Zariski dense subgroup of) $A_n^0(\Gamma)$. Let $p_i: \Gamma^n \to S$ be the composition of this projection with $q_i$. We claim that, for $i \neq j$, we cannot have $p_i = \alpha \circ p_j$ for any $\alpha \in \mathrm{Aut}(S)$, in particular any inner automorphism. For otherwise the image by $(p_i, p_j)$ of $\Gamma^n$ in $S \times S$ would lie in the graph of $\alpha = \{(s, \alpha(s) \mid s \in S\}$, a proper algebraic subgroup of $S \times S$, contradicting Zariski density of the image of $\Gamma^n$. Let $[p_i]$ denote the class of $p_i$ in the (categorical) quotient variety $X$ of $\mathrm{Hom}(\Gamma^n, S)$ by the conjugation of $S$. By choosing a simple $S$ module $V$, and noting that $p_i$ makes $V$ a simple $\Gamma^n$ module as well, by Zariski density, it follows that the $p_i$ have closed $S$ orbits in $\mathrm{Hom}(\Gamma^n, S)$, and so the points $[p_i]$ of $X$ are all distinct.

Let $L = \mathrm{Lie}(S)$ and $\mathrm{Ad} = \mathrm{Ad}_S: S \to \mathrm{GL}(L)$ the adjoint representation, a simple $S$ representation since $S$ is a simple algebraic group. By Zariski density of $p_i(\Gamma^n)$, and the observation above, the representations $\rho_i = \mathrm{Ad} \circ p_i$ are pairwise non isomorphic simple $\Gamma^n$ representations.

It follows from a theorem of Weil [15] that the tangent space $T_{[p_i]}(X)$ embedds in the cohomology space $H^1(\Gamma^n, \rho_i)$. From [10] it follows that for any simple $\Gamma^n$ representation $\rho$ we have

$$H^1(\Gamma^n, \rho) \cong \mathrm{Hom}_{\Gamma^n}(V(\Gamma^n), V_\rho),$$

and so $H^1(\Gamma^n, \rho) \neq 0$ implies that $\rho$ occurs in $V(\Gamma^n) = V(\Gamma)$. Thus the infinitely many $\rho_i$ occur in $V(\Gamma)$. Our hypothesis implies that $V(\Gamma)$ is finite-dimensional, so this is a contradiction. □

The proof of Theorem 4 actually shows that under the condition (FAb), if $\Gamma$ is not rigid then for some $n$ both $U_n(\Gamma)$ and $V_n(\Gamma)$ are infinite-dimensional. We deduce:

COROLLARY 10. *The following are equivalent for the group $\Gamma$:*

(1) $\Gamma$ *is rigid*
(2) $\Gamma$ *has* (FAb) *and for every $n$, $U_n(\Gamma)$ is finite-dimensional*
(3) $\Gamma$ *has* (FAb) *and for every $n$, $V_n(\Gamma)$ is finite-dimensional.*

We further mention that if $\Gamma$ has (FAb) and is not rigid, then we can deduce that $U(\Gamma)$ is not nilpotent. As in the proof of Theorem 4, there is a finite index subgroup $\Delta$ of $\Gamma$ which has infinitely many nonconjugate Zariski dense homomorphisms into a simple algebraic group $S$. One can show that this implies that there is a curve of such, and therefore that $\Delta$ has a Zariski dense representation into the pro-affine group $S(k[[t]])$. (See [1] for the analogous case where $\mathrm{GL}_n$ replaces $S$.) $S(k[[t]])$ is isomorphic to $U \rtimes S(k)$, where $U$ is an infinite-dimensional prounipotent group whose associated graded group is isomorphic to $\mathfrak{S} \otimes k[[t]]$, where $\mathfrak{S} = \mathrm{Lie}(S)$. One sees, using the density of the image of $\Delta$ in $S(k)$ and from the simplicity of $\mathfrak{S}$ as an $S$ module that $U(\Delta)$ maps onto $U$. Hence we deuce:

COROLLARY 11. *Suppose* $\Gamma$ *has* (FAb) *and that* $U(\Gamma)$ *is nilpotent. Then* $\Gamma$ *is rigid.*

To summarize: we have shown that various finiteness assertions on $A(\Gamma)$ imply, or are even equivalent to, rigidity. (We are including the observation that (FAb) is equivalent to $\dim(T_n(\Gamma)) < \infty$.)

In particular, we have shown:

COROLLARY 12. *If either the solvable radical or the maximal reductive quotient of* $A(\Gamma)$ *are finite-dimensional, then* $\Gamma$ *is rigid.*

In Sections 5 and 6 we exhibit examples of rigid groups with infinite dimensional unipotent or reductive parts. These show that the converse of Corollary 12 is not true. On the other hand, if both the unipotent and reductive parts of $A(\Gamma)$ are finite-dimensional, then $A(\Gamma)$ is finite-dimensional. We will see in Section 4 that in this case $\Gamma$ is super rigid.

## 4. Finite-Dimensional Proalgebraic Completions and Super Rigid Groups

Let $\Gamma$ be a finitely generated group. Its proalgebraic completion $A(\Gamma)$ is finite-dimensional when the latter's identity component $A^0(\Gamma)$ is finite dimensional (that is, is an affine proalgebraic group). By Corollary 6, there is a profinite subgroup $\Delta$ of $A(\Gamma)$ such that $A(\Gamma) = A^0(\Gamma) \cdot \Delta$ (not necessarily semidirect, of course.)

Consider the homomorphism $\Delta \to \mathrm{Aut}(A^0(\Gamma))$ given by conjugation. Since the automorphism group of an affine algebraic group is a discrete group extended by an affine algebraic group, the image of $\Delta$ in $\mathrm{Aut}(A^0(\Gamma))$ is finite and, hence, the kernel $\Delta_0$ is of finite index in $\Delta$. $\Delta_0$ commutes with $A^0(\Gamma)$. As it is normal in $\Delta$, $A^0(\Gamma) \cdot \Delta_0$ is a finite index normal subgroup of $A(\Gamma)$.

Consider the intersection $F = A^0(\Gamma) \cap \Delta$. This is a closed profinite subgroup in the affine algebraic group $A^0(\Gamma)$ and, hence, finite. It follows that there is a finite index normal subgroup $\Delta_1$ of $\Delta$ such that $F \cap \Delta_1$ is the identity. It follows that $A^0(\Gamma) \cdot \Delta_1$ is a semidirect product and is a finite index normal subgroup of $A(\Gamma)$.

Let $\Delta^0 = \Delta_0 \cap \Delta_1$. Then $A^0(\Gamma) \cdot \Delta^0 = A^0(\Gamma) \times \Delta^0$ is a finite index normal subgroup of $A(\Gamma)$. Let $\Gamma^0 = \Gamma \cap (A^0(\Gamma) \times \Delta^0)$ (we identify $\Gamma$ with its image in $A(\Gamma)$). $\Gamma_0$ is of finite index in $\Gamma$. The injective map $A(\Gamma^0) \to A(\Gamma)$ has image in $A^0(\Gamma) \times \Delta^0$ and induces an isomorphism $A^0(\Gamma^0) \to A^0(\Gamma)$. It follows that $A(\Gamma^0)/A^0(\Gamma^0) = \widehat{\Gamma^0}$ maps injectively to $\Delta^0 = (A^0(\Gamma) \times \Delta^0)/A^0(\Gamma)$. We replace $\Delta^0$ by this image, and sum up the result in the first assertion of the following theorem:

THEOREM 5. *let $\Gamma$ be a finitely generated linear group and suppose that its proalgebraic completion $A(\Gamma)$ is finite-dimensional. Then there is a normal subgroup $\Gamma^0$ of finite index in $\Gamma$ such that $A(\Gamma^0) \cong A^0(\Gamma^0) \times \widehat{\Gamma^0}$.*

(1) *Let $p$ be the composite $\Gamma^0 \to A(\Gamma^0) \to A^0(\Gamma^0)$, the second map being projection. The kernel of $p$ is finite. $\Gamma^0$ may be chosen so that $p$ is injective.*
(2) *Let $\rho\colon \Gamma^0 \to \mathrm{GL}_n(k)$ be any representation of $\Gamma^0$. Then there is a representation $\rho_a\colon A^0(\Gamma^0) \to \mathrm{GL}_n(k)$ and a finite index subgroup $\Gamma_1^0$ of $\Gamma^0$ such that $\rho = \rho_a \circ p$ on $\Gamma_1^0$.*
(3) *Let $\rho\colon \Gamma \to \mathrm{GL}_n(k)$ be any representation of $\Gamma$. Then there is a representation $\rho_a\colon A^0(\Gamma^0) \to \mathrm{GL}_n(k)$ and a finite index subgroup $\Gamma_1$ of $\Gamma$ such that $\rho = \rho_a \circ p$ on $\Gamma_1$.*

*Proof.* As noted, the isomorphism $A(\Gamma^0) \cong A^0(\Gamma^0) \times \widehat{\Gamma^0}$ is a consequence of the analysis preceding the theorem.

The kernel of $p$ is $N = \Gamma^0 \cap \Delta_0$ (we identify $\Gamma^0$ with its image in $A(\Gamma^0)$). The profinite subgroup $\Delta_0$ of $A(\Gamma^0)$ has finite image in every affine quotinet of $A(\Gamma^0)$, which means that $N$ has finite image in every representation of $\Gamma^0$, including a faithful one. So $N$ is finite and there is a finite index normal subgroup of $\Gamma$ contained in $\Gamma^0$ and meeting $N$ in the identity. Replacing $\Gamma^0$ by this subgroup makes $p$ injective. This proves (2).

Now suppose $\rho$ is a representation of $\Gamma_0$, and let $G$ denote the Zariski closure of the image of $\rho$. Let $\rho_a$ denote the map $A(\Gamma^0) \to G$ induced from $\rho$. $\rho_a(\Delta_0)$ is finite. $\rho$, and $\rho_a$, factor as

$$\Gamma^0 \to A^0(\Gamma^0) \times \rho_a(\Delta_0) \to G.$$

We take $\Gamma_1^0$ to be the inverse image of $A^0(\Gamma^0)$ under the first map; it has the properties claimed in (2).

Finally, (3) is an obvious consequence of (2). □

## 5. Rigid Groups with Large Prounipotent Radical

5.1. CONSTRUCTION

Let

$$\Gamma_i \leqslant G_i \quad (i = 0, 1) \tag{1}$$

be super-rigid embeddings of finitely generated infinite groups. Let

$$M = \text{a} \quad \mathbb{Z}[\Gamma_0]\text{-module, free of finite rank over } \mathbb{Z}. \tag{2}$$

and satisfying

$$M/(\Gamma_0', M) \quad \text{is finite for all } \Gamma_0' \quad \text{of finite index in } \Gamma_0. \tag{3}$$

Here $M/(\Gamma_0', M) = H_0(\Gamma_0') = M/JM$, where $J$ is the augmentation ideal in $\mathbb{Z}[\Gamma_0']$. The commutator notation applies inside $M \rtimes \Gamma_0$.

Now put

$$\begin{aligned} &\Gamma_+ = \Gamma_0 \times \Gamma_1 \\ &U = M \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma_1], \quad \text{a finitely generated } \mathbb{Z}[\Gamma_+]\text{-module, and} \\ &\Gamma = U \rtimes \Gamma_+, \quad \text{a finitely generated group.} \end{aligned} \tag{4}$$

We will show that $\Gamma$ is rigid, that $Q^0(\Gamma) = G_0 \times G_1$ (so finite dimensional) and that $U(\Gamma)$ is infinite dimensional. (See Theorem 6 below for a precise statement.)

LEMMA 1. (FAb) *If* $\Gamma' <_F \Gamma$, *then* $(\Gamma')^{\text{ab}} = 1$.

*Proof*. We are at liberty to replace $\Gamma'$ by a smaller finite index subgroup. First replace $\Gamma'$ by $U' \rtimes \Gamma_+'$, where $\Gamma_+' = \Gamma' \cap \Gamma_+ <_F \Gamma_+$ and $U' = \Gamma' \cap U <_F U$, a $\Gamma_+'$-invariant subgroup. We can then further reduce to the case $\Gamma_+' = \Gamma_0' \times \Gamma_1'$ where $\Gamma_i' = \Gamma_+' \cap \Gamma_i$ for $i = 0, 1$. Then we have

$$(\Gamma')^{\text{ab}} = (U' \rtimes \Gamma_+')^{\text{ab}} = (U'/(\Gamma_+', U')) \times (\Gamma_+')^{\text{ab}}.$$

Since the $\Gamma_i$ are rigid, so is $\Gamma_+ = \Gamma_0 \times \Gamma_1$. Thus it satisfies (FAb) and so $(\Gamma_+')^{\text{ab}}$ is finite.

It remains to show that $(U'/(\Gamma_+', U'))$ is finite. The finite group $U/U'$ is a $\Gamma_+'$ module and so is annihilated by a finite index two sided ideal $K$ of $\mathbb{Z}[\Gamma_+']$, whence

$$K \cdot U \leqslant U' \leqslant U.$$

Since $U = M \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma_1]$, and $\mathbb{Z}[\Gamma_1]$ is a finitely generated (free) $\mathbb{Z}[\Gamma_1']$ module, $U/KU = M \otimes_{\mathbb{Z}} (\mathbb{Z}[\Gamma_1]/K\mathbb{Z}[\Gamma_1])$ is the tensor product over $\mathbb{Z}$ of a finitely generated and a finite Abelian group and, hence, is itself finite. Thus $K \cdot U <_F U$, so it suffices to show that $KU/(\Gamma_+', KU)$ is finite.

Let $J_i'$ denote the augmentation ideal of $\mathbb{Z}[\Gamma_i']$. $\mathbb{Z}[\Gamma_+'] = \mathbb{Z}[\Gamma_0'] \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma_1']$ has augementation ideal $J' = J_0' \otimes \mathbb{Z}[\Gamma_1'] + \mathbb{Z}[\Gamma_0'] \otimes J_1'$. Since $KU = M \otimes_{\mathbb{Z}} K\mathbb{Z}[\Gamma_1]$, we have

$$\frac{KU}{(\Gamma_+', KU)} = \frac{KU}{JKU} = \frac{M \otimes K\mathbb{Z}[\Gamma_1]}{J_0'M \otimes K\mathbb{Z}[\Gamma_1] + M \otimes J_1'K\mathbb{Z}[\Gamma_1]}$$

$$= (M/J_0') \otimes_{\mathbb{Z}} (K\mathbb{Z}[\Gamma_1]/J_1'K\mathbb{Z}[\Gamma_1]).$$

By assumption, $M/J_0'M = M/(\Gamma_0', M)$ is finite. Since $K/\mathbb{Z}[\Gamma_1']$ is finite and $\Gamma_1' <_F \Gamma_1$, $K\mathbb{Z}[\Gamma_1]$ is a finitely generated $\Gamma_1'$ module. It follows that $K\mathbb{Z}[\Gamma_1]/J_1'K\mathbb{Z}[\Gamma_1]$ is a finitely generated Abelian group, and therefore its tensor product with the finite group $M/J_0'$ is finite as well. This proves that $KU/(\Gamma_+', KU)$ is finite, as required. □

5.2. REPRESENTATIONS

Let $\rho\colon \Gamma \to \mathrm{GL}_n(\mathbb{C})$ be a finite $\mathbb{C}$-dimensional representation of $\Gamma$ and let $G$, $H$, and $W$ denote the Zariski closures of the images under $\rho$ of $\Gamma$, $\Gamma_+$, and $W$, respectively. Then $W$ is an Abelian normal subgroup of $G$ and $G = WH$. $W$, being Abelian, is a product $W = V \times T \times S$ where $V$ is a vector group (the unipotent radical of $W$), $T$ is a torus (the unique maximal torus of $W \times T$ so that $V \times T$ is the identity component $W^0$ of $W$, and $S$ is a finite group. Both $W^0$ and $T$ are characteristic in $W$ and, hence, normalized by $G$. Since the automorphism group of $T$ is discrete, $T$ is centralized by the identity component $G^0$. Thus $T$ is a central torus in $G^0$. Suppose that $T \neq 1$. Then $G^0$ has a nontrivial character $\chi\colon G^0 \to \mathbb{G}_m$. Let $\Gamma^0 = \rho^{-1}(G^0)$. Then $\chi(\rho(\Gamma^0))$ is Zariski dense in $\mathbb{G}_m$, and in particular infinite Abelian. Since $\Gamma^0$ is of finite index in $\Gamma$, which is (FAb) by Lemma 5.1, this is impossible, so in fact $T = 1$ and

$$W = V \times S \tag{5}$$

where $V$ is a vector group and $S$ is finite. (Note that $S$, being the torsion subgroup of $W$, is also characteristic in $W$ and hence normalized by $G$ and centralized by $G^0$.)

$H$, and hence $\Gamma_+$, is represented on the vector space $V$. In the proof of Lemma 5.1, it was shown that for $\Gamma'_+ <_F \Gamma_+$ and $U' <_F U$, $U'/(\Gamma'_+, U')$ is finite. It follows that

$$(H^0, V) = V. \tag{6}$$

The action of $\mathbb{C}[\Gamma_1]$ on $V$ factors through some ideal $K$ of finite codimension so that there is a $\mathbb{C}[\Gamma_+]$ module surjection

$$(\mathbb{C} \otimes_{\mathbb{Z}} M) \otimes_{\mathbb{C}} (\mathbb{C}[\Gamma_1]/K) \to V. \tag{7}$$

Let

$$M_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} M \quad \text{and} \quad U_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} U = M_{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{C}[\Gamma_1]. \tag{8}$$

Finally, let

$$\mathbb{C}[\Gamma_1] = \varprojlim_K (\mathbb{C}[\Gamma_1]/K) \tag{9}$$

where $K$ varies over two-sided ideals of finite codimension in $\mathbb{C}[\Gamma_1]$.

Since the representation $\rho$ here is arbitrary, we draw the following conclusions about proalgebraic completions:

$$\begin{aligned} &A(\Gamma) = A_*(U) \rtimes A(\Gamma_+),\\ &A_*(\Gamma) = \mathrm{Image}(A(U) \to A(\Gamma)),\\ &A(\Gamma_+) = A^0(\Gamma_+) \times \widehat{\Gamma_+},\\ &A^0(\Gamma_+) = G_0 \times G_1,\\ &\widehat{\Gamma_+} = \widehat{\Gamma}_0 \times \widehat{\Gamma}_1. \end{aligned} \tag{10}$$

Further,

$$A_*(U) = A^0_*(U) \times U^{\hat{*}}, \text{ where}$$
$$A^0(U)* = M_{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{C}[[\Gamma_1]] \leqslant U(\Gamma) \quad (11)$$
$$U^{\hat{*}} = \varprojlim U/U', U' \text{ ranging over finite index sub } \Gamma_+\text{-modules of U.}$$

If $U/U'$ is a finite $\Gamma_+$ module as above, then as in the proof of Lemma 5.1 above, the $\Gamma_1$ action factors through a finite quotient $\Gamma_1/\Gamma_1'$, and hence $U/U'$ is a quotient of $M \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma_1/\Gamma_1']$ and, hence, for some integer $N > 0$, of the finite $\Gamma_+$ module $(M/NM) \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma_1/\Gamma_1']$. Taking the inverse limit of these gives

$$U^{\hat{*}} = \hat{M} \otimes_{\hat{\mathbb{Z}}} \hat{\mathbb{Z}}[[\widehat{\Gamma_1}]], \quad (12)$$

where

$$\hat{\mathbb{Z}}[[\widehat{\Gamma_1}]] = \varprojlim(\mathbb{Z}/N\mathbb{Z})[\Gamma_1/\Gamma_1'] \text{ as } N \to \infty \quad \text{and} \quad \Gamma_1' \to 1.$$

Summarizing this section's discussion, then, we have the following class of examples of finitely generated rigid groups with infinite-dimensional prounipotent radical.

THEOREM 6. *Let* $\Gamma_i \leqslant G_i$, $(i = 0, 1)$, *be super rigid embeddings, and let* $\Gamma_+ = \Gamma_0 \times \Gamma_1 \leqslant G_0 \times G_1 = G_+$. *Let* $M$ *be a* $\mathbb{Z}[\Gamma_0]$ *module,* $\mathbb{Z}$ *free of finite rank such that for all* $\Gamma_0' <_F \Gamma_0$, $M/(\Gamma_0', M)$ *is finite. Finally, let* $U = M \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma_1]$ *regarded as a* $\Gamma_+$ *module, and define* $\Gamma = U \rtimes \Gamma_+$. *Then* $\Gamma$ *is a finitely generated rigid group, whose prounipotent radical* $U(\Gamma)$ *is infinite-dimensional, and whose maximal connected reductive quotient* $Q^0(\Gamma)$ *is finite-dimensional. If, moreover,* $G_+$ *is reductive, then* $U(\Gamma)$ *is Abelian.*

## 5.3. ADDITIONAL EXAMPLES

We conclude this section with a short discussion without details of another type of example of a rigid group with an infinite dimensional prounipotent radical.

Let $L \cong \mathbb{Z}^r$ be the free Abelian group on $t_1, \ldots, t_r$, and let $A = \mathbb{Z}[L] = \mathbb{Z}[t_1, t_1^{-1}, \ldots, t_r, t_r^{-1}]$ be its integral group algebra. $\Gamma_0 = \mathrm{SL}_r(\mathbb{Z})$ acts as a group of automorphisms of $L$ and, hence, of $A$, and $L$ acts on $A$ by multiplication. These actions are compatible and lead to an action of $\Gamma_1 = L \rtimes \Gamma_0$ on $A$. This action preserves the augmentation ideal $A'$ on $A$, and we can form the semidirect product

$$\Gamma = A' \rtimes \Gamma_1 = A' \rtimes (\mathbb{Z}^r \rtimes \mathrm{SL}_r(\mathbb{Z})).$$

For $r \geqslant 3$, $\Gamma_1$ is a rigid group [2], and it follows from [2] as well that $\Gamma$ will be rigid as long as its finite index subgroups have finite Abelianization. We omit that calculation.

The prounipotent radical of $A(\Gamma)$ maps onto a prounipotent group containing $A' \rtimes \mathbb{Z}^r$ Zariski densely.

## 6. Representation Reductive Groups with Finite or Infinite Multiplicity of Simple Factors in their Proalgebraic Completions

### 6.1. WEAK PRODUCTS

Let $\{\Gamma_n\}$, $n \geqslant 1$, be a sequence of groups. In the product $\Pi = \prod_{n \geqslant 1} \Gamma_n$ we identify each $\Gamma_n$ as a subgroup, in the $n$th component as usual. Then the weak direct product

$$\Pi' = \prod_{n \geqslant 1}^{\text{weak}} \Gamma_n \leqslant \Pi$$
$$= \left\langle \bigcup_{n \geqslant 1} \Gamma_n \right\rangle$$

is the subgroup generated by these component subgroups.

We also consider the condition

$$\Gamma_n^{\text{ab}} = 1 \quad \text{for all} \quad n \gg 1. \tag{Ab1}$$

PROPOSITION 6. *Assume* (Ab1). *Then we have natural isomorphisms*

$$A(\Pi') \to \prod_{n \geqslant 1} A(\Gamma_n), \tag{A}$$
$$A^0(\Pi') \to \prod_{n \geqslant 1} A^0(\Gamma_n), \quad \text{and} \tag{$A^0$}$$
$$\widehat{\Pi}' \to \prod_{n \geqslant 1} \widehat{\Gamma}_n. \tag{(ˆ)}$$

*If each* $\Gamma_n$ *is connected split, then so is* $\Pi'$.

*Proof.* (A) easily implies the other two assertions. To prove (A), we introduce the notation

$$\Pi'_{n+1} = \prod_{m > n}^{\text{weak}} \Gamma_n = \left\langle \bigcup_{m > n} \Gamma_n \right\rangle$$

and make the observation that (A) will follow if, for $\rho \colon \Pi' \to L = \mathrm{GL}(V)$ any finite-dimensional $\mathbb{C}$ representation, $\rho$ must vanish on $\Pi'_{n+1}$ for some $n$, or in other words, that $\rho$ must factor through a projection

$$\Pi' \to \Gamma_1 \times \cdots \times \Gamma_n \tag{13}$$

with kernel $\Pi'_{n+1}$.

For $U \leqslant L$, let $\overline{U}$ denote Zariski closure, and let $H = \overline{\rho(\Pi')}$ and $H_n = \overline{\rho(\Gamma_1 \times \cdots \times \Gamma_n)}$ for $n \geqslant 1$, so $H = \overline{\cup_n H_n}$. By (Ab1), we can choose $n_0$ so that

$$\Gamma_n^{\text{ab}} = 1 \quad \text{for } n > n_0. \tag{14}$$

On centralizers, we have

$$Z(H) = \bigcap_n Z_H(H_n) = Z_H(H_{n_1})$$

for some $n_1$. Let $N = \max(n_0, n_1)$.

We claim that

$$\rho(\Pi'_{N+1}) = 1. \tag{15}$$

and, hence, (13) and (A). For (15), it suffices to show

$$\rho(\Gamma_m) = 1 \quad \text{for } m > N. \tag{16}$$

But for $m > N$, $\Gamma_m$ centralizies $\Gamma_1 \times \cdots \times \Gamma_N$, so

$$\rho(\Gamma_m) \leqslant Z_H(\rho(\Gamma_1 \times \cdots \times \Gamma_N)) = Z_H(H_N). \tag{17}$$

Since $N \geqslant n_1$, $Z_H(H_N) = Z(H)$, so $\rho(\Gamma_m)$ is Abelian. Since $m > n_0$, by (14) $\Gamma_m^{\mathrm{ab}} = 1$. Thus $\rho(\Gamma_m) = 1$. Thus (16) and, hence, (A), follows. □

## 6.2. GROUP TOWERS

In this section we consider an ascending chain ('tower') of groups

$$\Gamma_1 < \Gamma_2 < \Gamma_3 \ldots \tag{18}$$

and put $\Gamma_\infty = \bigcup_{n \geqslant 1} \Gamma_n$.

As in Section 6.1 above, we have the weak products

$$\Pi' = \prod_{n \geqslant 1}^{\text{weak}} \Gamma_n \leqslant \Pi = \prod_{n \geqslant 1} \Gamma_n. \tag{19}$$

Let $x = (x_n)_{n \geqslant 1} \in \prod$; $x_n \in \Gamma_n$. We call $x$ *eventually constant* if, for some $n_0 = n_0(x)$, $x_n = x_{n_0}$ for all $n \geqslant n_0$. In this case we put

$$p_\infty(x) = x_n \quad (\forall n \geqslant n_0). \tag{20}$$

Put

$$\Delta = \{x \in \Pi \mid x \text{ is eventually constant}\}. \tag{21}$$

Then $\Delta$ is a group and we have an exact sequence

$$1 \to \Pi' \to \Delta \to \Gamma_\infty \to 1 \tag{22}$$

where the second map is inclusion and the third is given by $p_\infty$.

For $n \geqslant 1$, we can write $\Pi = (\Gamma_1 \times \cdots \times \Gamma_n) \times \Pi_{n+1}$ with $\Pi_{n+1} = \prod_{m>n} \Gamma_m$. We put $\Pi'_{n+1} = \Pi' \cap \Pi_{n+1}$ and $\Delta_{n+1} = \Delta \cap \prod_{n+1}$. Then we have

$$\Delta = (\Gamma_1 \times \cdots \times \Gamma_n) \times \Delta_{n+1}; \tag{23}$$

and

$$p_\infty \colon \Delta_{n+1}/\Pi'_{n+1} \to \Gamma_\infty \text{ is an isomorphism.}$$

### 6.3. $\Gamma_0$ ACTIONS

We retain the notion of the preceding sections.

Let $\Gamma_0$ be a group an suppose there is an action

$$\sigma_n\colon\ \Gamma_0 \to \mathrm{Aut}(\Gamma_n) \tag{24}$$

for each $n \geqslant 1$. We denote the actions

$$s \cdot_n x = \sigma_n(s)(x) \quad \text{for } s \in \Gamma_0, x \in \Gamma_n.$$

We do *not* assume that the inclusions $\Gamma_n < \Gamma_{n+1}$ are $\Gamma_0$ equivariant. However, we will need the following 'stable equivariance condition':

For $s \in \Gamma_0$ and $x \in \Gamma_n$ there is an $N = N(s, x) \geqslant n$ such that

$$s \cdot_m x = s \cdot_N x \quad \text{for all } m \geqslant N. \tag{25}$$

We then write

$$s \cdot_\infty x = s \cdot_m x \quad \text{for all } m \geqslant N(s, x). \tag{26}$$

It is then easily seen that (26) defines the unique action of $\Gamma_0$ on $\Gamma_\infty$ so that

(22) is an exact sequence of $\Gamma_0$ groups. (27)

We can thus form the semi-direct direct product sequence

$$1 \to \Pi' \to \Delta \rtimes \Gamma_0 \to \Gamma_\infty \rtimes \Gamma_0 \to 1 \tag{28}$$

In our next result, we will use the following hypotheses:

For all finite-dimensional $\mathbb{C}$ representations $\rho$ of $\Delta_\infty \rtimes \Gamma_0$,

we have $\rho(\Gamma_\infty) = 1$. $(\rho\Gamma_\infty = 1)$

For all $n \geqslant 1$, the action of $\Gamma_0$ on $\Gamma_n$ factors through a finite quotient of $\Gamma_0$. $(|\sigma_n \Gamma_0| < \infty)$

PROPOSITION 7. *Assume* (Ab1) *and* $(\rho\Gamma_\infty = 1)$. *Then any finite dimensional* $\mathbb{C}$ *representation of* $\Gamma = \Delta \times \Gamma_0$ *factors through some quotient* $\Gamma/\Delta_{n+1} = (\Gamma_1 \times \cdots \times \Gamma_n) \rtimes \Gamma_0$. *Hence we have a natural isomorphism*

$$A(\Gamma) \to \varprojlim_n A((\Gamma_1 \times \cdots \times \Gamma_n) \rtimes \Gamma_0). \tag{A}$$

*Further assume* $(|\sigma_n \Gamma_0| < \infty)$. *Then we have a natural isomorphism*

$$A^0(\Gamma) \to \prod_{n \geqslant 0} A^0(\Gamma_n) \tag{A$^0$}$$

*and a natural isomorphism*

$$\hat{\Gamma} \to \left( \prod_{n \geqslant 1} \widehat{\Gamma_n} \right) \times \widehat{\Gamma_0}, \tag{$\hat{\ }$}$$

*where the action of* $\widehat{\Gamma_0}$ *on each* $\widehat{\Gamma_n}$ *is defined because of* $(|\sigma_n \Gamma_0| < \infty)$.

*Proof*. Let $\rho\colon \Gamma \to \mathrm{GL}(V)$ be a finite-dimensional $\mathbb{C}$ representation. Because of (Ab1) and (.2) (A), on $\Pi' \leqslant \Delta$ the restriction of $\rho$ factors through a projection $\Pi' \to \Gamma_1 \times \cdots \times \Gamma_n$, with kernel $\Pi'_{n+1}$. So $\rho$ factors through $\Gamma/\Pi'_{n+1} = (\Delta/\Pi'_{n+1}) \rtimes \Gamma_0$. Because of (.3)(7),$(\Delta/\Pi'_{n+1}) = (\Gamma_1 \times \cdots \times \Gamma_n) \times \Gamma_\infty$. The hypothesis $(\rho\Gamma_\infty)$ applied to $(\Gamma_\infty) \rtimes \Gamma_0) \leqslant (\Delta/\Pi'_{n+1})$ implies that $\rho$ vanishes on $\Gamma_\infty$, so that $\rho$ factors through $\Gamma/\Delta_{n+1} = (\Gamma_1 \times \cdots \times \Gamma_n) \rtimes \Gamma_0$. This establishes assertion $(A)$.

Now assume $(|\sigma_n \Gamma_0| < \infty)$. Then the $\Gamma_0$ action on $\Gamma_1 \times \cdots \times \Gamma_n$ factors through some finite quotient $\Gamma_0/\Gamma_0^{(n)}$. So the (direct) product $(\Gamma_1 \times \cdots \times \Gamma_n) \times \Gamma_0^{(n)}$ has finite index in the semidirect product $(\Gamma_1 \times \cdots \times \Gamma_n) \rtimes \Gamma_0$, which implies that they have the same $A^0$:

$$A^0((\Gamma_1 \times \cdots \times \Gamma_n) \rtimes \Gamma_0) = A^0(\Gamma_1) \times \cdots \times A^0(\Gamma_n) \times A^0(\Gamma_0). \tag{29}$$

Passing to the inverse limit over $n$ in (29) we obtain assertion $(A^0)$.

Next, we observe that the action of the finite group $\Gamma_0/\Gamma_0^{(n)}$ on $\Gamma_n$ extends to an action on $\widehat{\Gamma_n}$ (every finite index subgroup of $\Gamma_n$ contains a $\Gamma_0$ invariant finite index subgroup). It follows easily from this that there is a natural isomorphism

$$[(\Gamma_1 \times \cdots \times \Gamma_n) \rtimes \Gamma_0]\widehat{\ } = (\widehat{\Gamma_1} \times \cdots \times \widehat{\Gamma_n}) \rtimes \widehat{\Gamma_0}. \tag{30}$$

Passing to the inverse limit over $n$ in (30) we obtain assertion $(\widehat{\cdot})$. □

## 6.4. FILTERING $\Gamma_0$

Let $\Gamma_0$ be a finitely generated residually finite infinite group with a finite generating set $S'_1$ such that

$$\Gamma_0 = \langle S'_1 \rangle, \quad 1 \in S'_1, \quad S'_1 = (S'_1)^{-1}, \quad |S'_1| \geqslant 3. \tag{31}$$

We inductively construct finite sets

$$S'_1 \subset S_1 \subset S_2 \subset \cdots \subset \Gamma_0 \tag{32}$$

and normal subgroups of $\Gamma_0$

$$\Gamma_0 > \Gamma_0^{(1)} > \Gamma_0^{(2)} > \cdots \tag{33}$$

such that for all $n \geqslant 1$ we have

$$S_n \to \Gamma_0/\Gamma_0^{(n)} \quad \text{is bijective} \tag{34}$$

and

$$S'_{n+1} \subset S_{n+1}, \quad \text{where } S'_{n+1} = S_1 \cdot S_n. \tag{35}$$

It will follow from (35) that $(S'_1)^n \subset S_n$, and so, in view of (31), we have

$$\Gamma_0 = \bigcup_{n \geqslant 1} S_n. \tag{36}$$

It then further follows from (36) and (34) that

$$\bigcap_{n \geqslant 1} \Gamma_0^{(n)} = 1. \tag{37}$$

The construction proceeds as follows: for $n \geqslant 1$, we first choose $\Gamma_0^{(1)} < \Gamma_0$ normal and of finite index, and so that $S'_1 \to \Gamma_0/\Gamma_0^{(1)}$ is injective (the later is possible because $\Gamma_0$ is residually finite). Then we enlarge $S'_1$ to a set of coset representatives of $\Gamma_0/\Gamma_0^{(1)}$.

Now assume, inductively, that $S_1, \ldots, S_n$ and $\Gamma_0^{(1)}, \ldots, \Gamma_0^{(n)}$ have been chosen as above. Then let $S'_{n+1} = S'_1 \cdot S_n$ and, using residual finiteness again, choose $\Gamma_0^{(n+1)} < \Gamma_0^{(n)}$ normal of finite index in $\Gamma_0$ such that $S'_{n+1} \to \Gamma_0/\Gamma_0^{(n+1)}$ is injective. Finally, enlarge $s'_{n+1}$ to a set of representatives of $\Gamma_0/\Gamma_0^{(n+1)}$.

We will use the following notation:

$$\text{For } s \in \Gamma_0,\ \text{define } s(n) \in S_n \text{ by } s^{-1}s(n) \in \Gamma_0^{(n)}. \tag{38}$$

### 6.5. THE CASE $\Gamma_n = \Lambda^{S_n}$

We fix a group $\Lambda$ and consider the set of $\Lambda$ valued functions on $\Gamma$:

$$\Lambda^{\Gamma_0} = \{x\colon\ \Gamma_0 \to \Lambda\}. \tag{39}$$

For $x \in \Lambda^{\Gamma_0}$, we define *support* by

$$\operatorname{supp}(x) = \{s \in \Gamma_0 \mid s(x) \neq 1\}. \tag{40}$$

For a subset $S \subset \Gamma_0$, we define

$$\Lambda^S = \{x\colon\ \Lambda^{\Gamma_0} \mid \operatorname{supp}(x) \in S\}. \tag{41}$$

From the sequence (32)

$$S_1 \subset S_2 \subset \cdots \subset \Gamma_0, \tag{42}$$

we obtain the tower of groups

$$\Gamma_1 < \Gamma_2 < \cdots < \Lambda^{\Gamma_0} \quad \text{where } \Gamma_n = \Lambda^{S_n} \tag{43}$$

and the group

$$\Gamma_\infty = \bigcup_{n \geqslant 1} \Gamma_n = \Lambda^{(\Gamma_0)}, \tag{44}$$

where

$$\Lambda^{(\Gamma_0)} = \{x \in \Lambda^{\Gamma_0} \mid\ \operatorname{supp}(x) \text{ is finite}\}.$$

$\Gamma_0$ acts on $\Lambda^{\Gamma_0}$ via left translations on $\Gamma_0$. We denote this action as follows:

$$\text{For } s \in \Gamma_0, x \in \Lambda^{\Gamma_0} \text{ and } t \in \Gamma_0, \quad (s \cdot_\infty x)(t) = x(s^{-1}t). \tag{45}$$

Under this action $\Gamma_\infty$ is $\Gamma_0$ invariant (and the groups $\Gamma_n$ are not). However, the bijection $S_n \to \Gamma_0/\Gamma_0^{(n)}$ defines an isomorphism $\Lambda^{(\Gamma_0/\Gamma_0^{(n)})} \to \Lambda^{S_n} = \Gamma_n$ so by transport of structure a $\Gamma_0$ action on $\Gamma_n$. To describe the action, we make the convention that for $u \in \Gamma_0$, $u(n) \in S_n$ denotes its representative modulo $\Gamma_0^{(0)}$. Then $s \in \Gamma_0$ acts on $x \in \Gamma_n$ by

$$(s \cdot_n x)(t) = \begin{cases} x((s^{-1}t)(n)), & \text{for } t \in S_n; \\ 1, & \text{for } t \notin S_n. \end{cases} \tag{46}$$

Now let $s \in \Gamma_0$ and $x \in \Gamma_n$. Choose $N = N(s, n)$ large enough so that $sS_n \subset S_N$. Then we claim that

$$s \cdot_m x = s \cdot_\infty x, \quad \text{for } m \geqslant N. \tag{47}$$

By definition, for $t \in \Gamma_0$, $(s \cdot_\infty x)(t) = x(s^{-1}t)$, and this is $\neq 1$ only for $t \in sS_n \subset S_m$. On the other hand, $(s \cdot_m x)(t) = 1$ for $t \notin S_m$ and, for $t \in S_m$, $(s \cdot_m x)(t) = x((s^{-1}t)(m))$. If $t \in sS_n$, $(s^{-1}t)(m) = s^{-1}t$. It remains to consider $t \in S_m - sS_n$. Then $(s^{-1}t)(m) \equiv s^{-1}t \pmod{\Gamma_0^{(m)}}$ so $s((s^{-1}t)(m)) \equiv t \pmod{\Gamma_0^{(m)}}$. If $(s^{-1}t)(m) \in S_n$, then $s((s^{-1}t)(m)) \in sS_n \subset S_m$. Since $t \in S_m$, the congruence implies that $t = s((s^{-1}t)(m)) \in sS_m$, contrary to assumption. Thus $(s^{-1}t)(m) \notin S_n$, so $(s \cdot_m x)(t) = 1 = (s \cdot_\infty x)(t)$, and (47) is proven.

### 6.6. THE GROUP $\Gamma(\Lambda, \Gamma_0)$

We retain the notation of 6.4 and 6.5. For $n \geqslant 1$, $\lambda \in \Lambda$, and $s \in \Gamma_0$, we define $\lambda_{n,s} \in \Gamma_n = \Lambda^{S_n}$ by

$$\lambda_{n,s}(t) = \begin{cases} \lambda, & \text{if } t = s(n); \\ 1, & \text{otherwise.} \end{cases} \tag{48}$$

Similarly, we let $\Lambda_{n,s}$ denote the $s(n)$ factor $\Lambda$ in $\Lambda^{S_n}$. Define

$$\begin{aligned} &\delta_{(n),s}\colon \Lambda \to \Pi_n = \prod_{m \geqslant n} \Gamma_n \qquad (\leqslant \Pi = \Pi_1) \\ &\delta_{(n),s}(\lambda) = (\lambda_{m,s})_{m \geqslant n}. \end{aligned} \tag{49}$$

Note that, with respect to the inclusion

$$\Gamma_n = \Lambda^{S_n} < \Gamma_\infty = \Lambda^{(\Gamma_0)},$$

the formula in (48) is valid for all $t \in \Gamma_0$. Moreover, in case that $s \in S_n$, then $s(n) = s$, so that the formula no longer involves $n$. Thus relative to $\Gamma_n < \Gamma_{n+1} < \cdots < \Gamma_\infty$

$$\text{If } s \in S_n, \text{ then} \quad \lambda_{n,s} = \lambda_{m,s} \quad \text{for all } m \geqslant n. \tag{50}$$

For any $s \in \Gamma_0$ we have $s \in S_m$ for $m$ sufficiently large. Hence

$$\delta_{(n),s}(\Lambda) \quad \text{consists of eventually constant sequences in } \Pi_n. \tag{51}$$

Let $u \in \Gamma_0$ then $(u \cdot_n \lambda_{n,s})(t) = 1$ if $t \notin S_n$. If $t \in S_n$, then $(u \cdot_n \lambda_{n,s})(t) = \lambda_{n,s}(u^{-1}t)$, and this equals 1 unless $u^{-1}t = s(n)$; that is, unless $t = u \cdot s(n)$. Since $t \in S_n$, this latter means that $t = (u \cdot s(n))(n) = (us)(n)$. Thus we have:

$$u \cdot_n \lambda_{n,s} = \lambda_{n,us}, \quad \text{and} \quad u \cdot \delta_{(n),s}(\lambda) = \delta_{(n),us}(\lambda) \tag{52}$$

where, in the latter, $u$ acts on $\Pi_n$ by the product action on factors.

Now let

$$\delta = \delta_{(1),1}\colon \Lambda \to \Pi = \Pi_1 \tag{53}$$

and put

$$\Gamma = \langle \delta(\Lambda), \Gamma_0 \rangle \leqslant \Pi \rtimes \Gamma_0. \tag{54}$$

Clearly

$$\Gamma = \Delta' \rtimes \Gamma_0,$$

where

$$\begin{aligned} \Delta' &= \Gamma \cap \Pi \\ &= \text{the } \Gamma_0 \text{ subgroup of } \Pi \text{ generated by } \delta(\Lambda). \end{aligned} \tag{55}$$

In 6.2 (21) we defined the group

$$\Delta = \{\text{eventually constant sequences in } \Pi\}. \tag{56}$$

In view of the $\Gamma_0$ invariance of $\Delta$ and (51), we have

$$\Delta' \leqslant \Delta. \tag{57}$$

Now assume that $\Lambda$ satisfies $(AB1)$:

$$\Lambda^{\mathrm{ab}} = 1. \tag{58}$$

Then we claim:

$$\Delta' = \Delta. \tag{59}$$

We begin the proof of (59) by showing that

$$\Pi'\left( = \prod^{\text{weak}} \Gamma_n \right) \leqslant \Delta'. \tag{60}$$

For $\lambda \in \Lambda$ and $s \in \Gamma_0$ we have $s \cdot \delta(\lambda) = (s \cdot_n \lambda_{n,1})_{n \geqslant 1} = (\lambda_{n,s})_{n \geqslant 1}$. Choose $s \in \Gamma_0^{(1)} - \Gamma_0^{(2)}$, so $s(1) = 1$ and $s(n) \neq 1$ for $n \geqslant 1$. Then, for $n > 1$ $\lambda' \in \Lambda$, $\lambda'_{n,1}$ and $\lambda_{n,s}$ belong to different factors of $\Gamma_n = \Lambda^{S_n}$, and thus commute. Thus

$$(\lambda_{n,s}, \lambda'_{n,1}) = \begin{cases} (\lambda, \lambda')_{1,1}, & \text{for } n = 1, \\ 1, & \text{for } n > 1. \end{cases}$$

Since $\Lambda = (\Lambda, \Lambda)$ by (58), it follows that $\Delta'$ contains $\Lambda_{1,1} \leqslant \Gamma_1 \leqslant \Lambda^{S_1}$. Since $\Lambda_{1,1}$ generates $\Gamma_1$ as a $\Gamma_0$ group, this implies that $\Gamma_1 \leqslant \Delta'$.

Now suppose that we have shown that $\Gamma_1, \ldots, \Gamma_n \leqslant \Delta'$. Modulo $\Gamma_1 \times \cdots \times \Gamma_{n-1}$, we can modify $\delta(\Lambda) = \delta_{(1),1}(\Lambda)$ to obtain $\delta_{(n),1}(\Lambda) \leqslant \Delta'$. Choose $s \in \Gamma_0^{(n)} - \Gamma_0^{(n+1)}$. Then, arguing as above, for $\lambda, \lambda' \in \Lambda$ and $m \geqslant n$ we have

$$(\lambda_{m,s}, \lambda'_{m,1}) = \begin{cases} (\lambda, \lambda')_{n,1,} & \text{for } m = n, \\ 1, & \text{for } m > n. \end{cases}$$

As before, since $\Lambda = (\Lambda, \Lambda)$ we then have that $\Delta'$ contains $\Lambda_{n,1}$, and it follows by $\Gamma_0$ invariance that $\Gamma_n \leqslant \Delta'$. Thus (60) follows by induction.

To complete the proof of the claim (59), we recall the exact sequence of $\Gamma_0$ groups 6.2 (22):

$$1 \to \Pi' \to \Delta \xrightarrow{p_\infty} \Gamma_\infty \to 1.$$

In view of (60), it suffices to show that $p_\infty(\Delta') = \Gamma_\infty$. We have

$$p_\infty(\delta(\Lambda)) \leqslant \Gamma_\infty = \Lambda^{\Gamma_0}$$

where

$$p_\infty(\delta(\lambda))(t) = \begin{cases} \lambda, & \text{for } t = 1, \\ 1, & \text{for } t \neq 1. \end{cases}$$

Thus $p_\infty(\delta(\Lambda))$ is the copy of $\Lambda$ in the 1-coordinate of $\Lambda^{(\Gamma_0)}$. Since this clearly generates $\Gamma_\infty$ as a $\Gamma_0$ group, $p_\infty(\Delta') = \Gamma_\infty$, as required, and (59) follows.

From (54) and (55), combined with (59), we have

$$\Gamma = \langle \delta(\Lambda), \Gamma_0 \rangle = \Delta \rtimes \Gamma_0. \tag{61}$$

We will sometimes write

$$\Gamma = \Gamma(\Lambda, \Gamma_0).$$

Note that its construction depends on the group $\Lambda$ (which is required to satisfy ($Ab1$) (58) and on the residually finite group $\Gamma_0$, as well as on the filtrations $(S_n)_{n \geqslant 1}$ and $(\Gamma_0^{(n)})_{n \geqslant 1}$ of 6.4.

From (61), we note that

If $\Lambda$ is finitely generated, so is $\Gamma$. (62)

### 6.7. $A(\Gamma(\Lambda, \Gamma_0))$

We are going to describe the proalgebraic completion $A(\Gamma)$ for the group $\Gamma = \Gamma(\Lambda, \Gamma_0)$ defined in 6.6 (61), using the Proposition 7. We begin by verifying the hypotheses (Ab1), ($\rho\Gamma_\infty = 1$), and ($|\sigma_n \Gamma_0| < \infty$) of that proposition.

Condition (Ab1) requires that $\Gamma_n^{\mathrm{ab}} = 1$ for all $n$ sufficiently large. This follows here because $\Gamma_n = \Lambda^{S_n}$ and because $\Lambda^{\mathrm{ab}} = 1$ by assumption.

Condition ($|\sigma_n \Gamma_0| < \infty$) means that $\Gamma_0$ acts on $\Gamma_n$ through a finite quotient for each $n$. This holds here since $\Gamma_0$ acts on $\Gamma_n = \Lambda^{S_n}$ through the permutation action on $S_n \cong \Gamma_0/\Gamma_0^{(n)}$.

Finally, the condition ($\rho\Gamma_\infty = 1$) requires that for any finite dimensional $\mathbb{C}$ representation $\rho$: $\Gamma_\infty \rtimes \Gamma_0 \to L = \mathrm{GL}(V)$ we have $\rho(\Gamma_\infty) = 1$. Since $\Gamma_\infty = \Lambda^{(\Gamma_0)}$, it follows from the fact that $\Lambda^{\mathrm{ab}} = 1$ and from Proposition 6 that $\rho(\Lambda^{(\Gamma_0 - S_n)}) = 1$ for some $n$.

Since $\mathrm{Ker}(\rho) \cap \Gamma_\infty$ is a $\Gamma_0$ invariant subgroup, and since $\rho(\Lambda^{(\Gamma_0 - S_n)})$ clearly generates $\Gamma_\infty = \Lambda^{(\Gamma_0)}$ as a $\Gamma_0$ group, we conclude that $\rho\Gamma_\infty = 1$ as required.

We now state the conclusions of Proposition 7 as a theorem:

THEOREM 7. *Let $\Gamma_0$ be a finitely generated residually finite group, filtered as in* 6.4, *and let $\Lambda$ be a group satisfying $\Lambda^{\mathrm{ab}} = 1$. Let $\Gamma = \Gamma(\Lambda, \Gamma_0) = \langle \delta(\Lambda), \Gamma_0 \rangle$.*

*Then there are natural isomorphisms*

$$A(\Gamma) \cong \varprojlim_n A((\Gamma_1 \times \cdots \times \Gamma_n) \rtimes \Gamma_0), \tag{A}$$

$$A^0(\Gamma) \cong \prod_{n \geqslant 0} A^0(\Gamma_n), \tag{$\mathrm{A}^0$}$$

$$\hat{\Gamma} \cong \left( \prod_{n \geqslant 1} \widehat{\Gamma_n} \right) \times \widehat{\Gamma_0}. \tag{$\hat{\cdot}$}$$

*If $\Lambda$ is finitely generated, then so is $\Gamma$.*

6.8. REMARKS

(1) For $\Gamma_n = \Lambda^{S_n}$, we have

$$A(\Gamma_n) = A(\Lambda)^{S_n}, \quad A^0(\Gamma_n) = A^0(\Lambda)^{S_n}, \quad \text{and } \widehat{\Gamma_n} = \widehat{\Lambda}^{S_n}. \tag{63}$$

Thus, putting

$$S = S_1 \amalg S_2 \amalg S_3 \amalg \ldots \tag{64}$$

we have, from Theorem 7 ($\mathrm{A}^0$) and ($\hat{\cdot}$),

$$A^0(\Gamma) \cong A^0(\Lambda)^S \times A^0(\Gamma_0) \tag{65}$$

and

$$\widehat{\Gamma} \cong \widehat{\Lambda}^S \rtimes \widehat{\Gamma_0}. \tag{66}$$

From (65) we see that, for suitable choice of $\Lambda$, simple groups can occur with infinite multiplicity in $Q(\Gamma)$. It follows that, if $\Gamma$ is rigid, it is not connected split. (Otherwise, it would have infinitely many irreducible representations in a single dimension.)

(2) To illustrate this last point, we could take $\Lambda = \Gamma_0 = \mathrm{SL}_d(\mathbb{Z})$ with $d \geqslant 3$. Then

$$A(\Lambda) = A^0(\Lambda) \times \widehat{\Lambda},$$
$$A^0(\Lambda) = \mathrm{SL}_d(\mathbb{C}),$$
$$\widehat{\Lambda} = \mathrm{SL}_d(\widehat{\mathbb{Z}}).$$

Hence, $A^0(\Gamma)$ is an infinite product of copies of $\mathrm{SL}_d(\mathbb{C})$. It follows that $\Gamma$ is not rigid.

(3) If $\Lambda$ and $\Gamma_0$ are rigid, then so is $\Gamma$, since representations of $\Gamma$ factor through a quotient $(\Gamma_1 \times \cdots \times \Gamma_n) \rtimes \Gamma_0$, and $(\Gamma_1 \times \cdots \times \Gamma_n) \times \Gamma_0^{(n)}$ is a rigid finite index subgroup of this quotient.

### 6.9. THE GROUPS $L_n = \mathbb{Z}^{S_n}$

We now make a construction like that of 6.5, but now with $\Lambda = \mathbb{Z}$ (in contrast with 6.6 where we assumed from (59) on that $\Lambda^{\mathrm{ab}} = 1$). We recall the notation and results of 6.5 in this context. We have the additive group

$$\mathbb{Z}^{\Gamma_0} = \{x\colon\ \Gamma_0 \to \mathbb{Z}\}. \tag{67}$$

For $x \in \mathbb{Z}^{\Gamma_0}$,

$$\mathrm{supp}(x) = \{s \in \Gamma_0 \mid x(s) \neq 0\}. \tag{68}$$

For $S \subset \Gamma_0$ we identify

$$\mathbb{Z}^S = \{x \in \mathbb{Z}^{\Gamma_0} \mid \mathrm{supp}(x) \subset S\}, \quad \text{and} \quad Z^{(S)} = \{x \in \mathbb{Z}^S \mid \mathrm{supp}(x) \text{ is finite}\} \tag{69}$$

From the sequence 6.4 (32) of finite sets

$$S_1 \subset S_2 \subset S_3 \subset \cdots \tag{70}$$

we obtain the tower of finitely generated free modules $\mathbb{Z}$ modules

$$L_1 \subset L_2 \subset L_3 \subset \ldots, L_n = \mathbb{Z}^{S_n} \tag{71}$$

and

$$L_\infty = \bigcup_{n \geqslant 1} L_n = \mathbb{Z}^{(\Gamma)} = \{x \in \mathbb{Z}^{\Gamma_0} \mid \mathrm{supp}(x) \text{ is finite}\}. \tag{72}$$

$\Gamma_0$ acts on $\mathbb{Z}^{\Gamma_0}$ by

$$(s \cdot_\infty x)(t) = x(s^{-1}t) \quad \text{for } s, t \in \Gamma_0, x \in \mathbb{Z}^{\Gamma_0} \tag{73}$$

and $L_\infty$ is $\Gamma_0$ invariant in this action.

$\Gamma_0$ acts on $L_n$ by

$$(s \cdot_n x)(t) = \begin{cases} x((s^{-1}t)(n)), & \text{if } t \in S_n, \\ 0, & \text{if } t \notin S_n. \end{cases} \tag{74}$$

As before, the inclusions $L_n \subset L_m \subset L_\infty$ are not $\Gamma_0$ invariant. Nonetheless, if $s \in \Gamma_0$, $x \in L_n$, and $N = N(s, n)$ is large enough so that $sS_n \subset S_N$, then

$$s \cdot_m x = s \cdot_\infty, \quad \text{for } m \geqslant N. \tag{75}$$

6.10. THE GROUPS $\Gamma_n = \mathrm{SL}(L_n)$

The free Abelian group $L_\infty = \mathbb{Z}^{(\Gamma_0)}$ has an evident basis indexed by $\Gamma_0$. We write

$$\mathrm{GL}(L_\infty) = \mathrm{Aut}(L_\infty),$$

and

$$\mathrm{GL}^f(L_\infty) = \{g \in \mathrm{GL}(L_\infty) \mid g \text{ fixes all but finitely many basis elements}\}. \tag{76}$$

The determinant is defined on $\mathrm{GL}^f(L_\infty)$, and we have the exact sequence

$$1 \to \mathrm{SL}^f(L_\infty) \to \mathrm{GL}^f(L_\infty) \to \{\pm 1\} \to 1. \tag{77}$$

For $S \subset \Gamma_0$, we can write

$$L_\infty = \mathbb{Z}^{(S)} \oplus \mathbb{Z}^{(\Gamma_0 - S)} \tag{78}$$

and we identify $g \in \mathrm{GL}^f(\mathbb{Z}^{(S)})$ with

$$g \oplus \mathrm{Id}_{\mathbb{Z}^{(\Gamma_0 - S)}} \in \mathrm{GL}^f(L_\infty). \tag{79}$$

When $S$ is finite,

$$\mathbb{Z}^{(S)} = \mathbb{Z}^S, \qquad \mathrm{GL}^f(\mathbb{Z}^{(S)}) = \mathrm{GL}(\mathbb{Z}^S), \quad \text{and} \quad \mathrm{SL}^f(\mathbb{Z}^{(S)}) = \mathrm{SL}(\mathbb{Z}^S).$$

We have the tower of groups

$$\begin{aligned} &\Gamma_1 < \Gamma_2 < \Gamma_3 \ldots, \\ &\Gamma_n = \mathrm{SL}(L_n) = \mathrm{SL}(\mathbb{Z}^{S_n}), \quad \text{and} \quad \Gamma_\infty := \bigcup_{n \geqslant 1} \Gamma_n = \mathrm{SL}^f(L_\infty). \end{aligned} \tag{80}$$

The actions 6.9 (73) and (74) of $\Gamma_0$ on $L_n$ and $L_\infty$ (permuting bases) correspond to homomorphisms $\sigma_0 \colon \Gamma_0 \to \mathrm{GL}(L_\infty)$, $1 \leqslant n \leqslant \infty$. These define actions of $\Gamma_0$ on $\Gamma_n$ as follows: for $s \in \Gamma_0$ and $g \in \Gamma_n$, define

$$s \cdot_n g = \sigma_n(s) g \sigma_n(s)^{-1} \quad (1 \leqslant n \leqslant \infty). \tag{81}$$

For $n < \infty$, this action factors through $\Gamma_0/\Gamma_0^{(n)}$.

Here, $g$ operates only on $L_n = \mathbb{Z}^{S_n}$ and is the identity on $\mathbb{Z}^{(\Gamma_0 - s)}$. In $L_\infty = \mathbb{Z}^{(\Gamma)}$, $s \cdot_\infty g$ is like $g$, but transferred from the basis $S_n$ of $L_n$ to the basis $sS_n$ of $\mathbb{Z}^{sS_n}$. Choose $N = N(s, n)$ large enough so that $sS_n \cup S_n \subset S_N$. Then:

$$s \cdot_m g = s \cdot_\infty g \quad \text{for all } m \geqslant N. \tag{82}$$

LEMMA 2 ($\Gamma_0$ generation of $\Gamma_n$). *For $1 \leqslant n \leqslant \infty$, $\Gamma_n$ is generated by $\Gamma_1$ as a $\Gamma_0$ group.*

*Proof.* We have $\Gamma_n = \mathrm{SL}^f(\mathbb{Z}^{(S_n)})$, where we can identify $S_n$ with $\Gamma_0/\Gamma_0^{(n)}$, taking $\Gamma_0^{(\infty)} = 1$ when $n = \infty$, and the action of $\Gamma_0$ on $\Gamma_n$ is via the translation action of $S_n$

on itself. Now $\Gamma_1 = \mathrm{SL}(\mathbb{Z}^{S_1})$, so the $\Gamma_0$ group generated by $\Gamma_1$ is the group generated by all $\mathrm{SL}(\mathbb{Z}^{sS_1})$ $(s \in S_n)$.

Consider the graph with vertex set $S_n$ and edges the pairs $\{s, t\}$ such that $sS_1 \cap tS_1 \neq \emptyset$. It is easily seen that the connected component of $1 \in S_n$ in this graph is $\langle S_1 \rangle S_n$; that is, the graph is connected. The lemma now follows from the following lemma: □

LEMMA 3. *Let $S = U \cup V$ be finite sets with $U \cap V \neq \emptyset$. Then $\mathrm{SL}(\mathbb{Z}^S)$ is generated by its subgroups $\mathrm{SL}(\mathbb{Z}^U)$ and $\mathrm{SL}(\mathbb{Z}^V)$.*

*Proof.* $\mathrm{SL}(\mathbb{Z}^S)$ is generated by the matrices $x_{s,t} = I + e_{s,t}$ $(s \neq t)$ where $e_{s,t}$ has a single nonzero entry, 1 in the $(s, t)$ position. Moreover, we have the commutator formula:

$$x_{s,u} = (x_{s,t}, x_{t,u}) \quad \text{for } s, t, u \text{ distinct.}$$

If $s, t \in U$ then $x_{s,t} \in \mathrm{SL}(\mathbb{Z}^U)$. If $s, t \in V$ then $x_{s,t} \in \mathrm{SL}(\mathbb{Z}^V)$. If neither is the case, say $s \in U$ and $t \in V$, we can choose $u \in U \cap V$ ($\neq \emptyset$, by assumption). Then $s, t, u$ are distinct, so

$$x_{s,t} = (x_{s,u}, x_{u,v}) \in (\mathrm{SL}(\mathbb{Z}^U), \mathrm{SL}(\mathbb{Z}^V)),$$

and the lemma follows. □

The next proposition affirms the hypotheses of Preposition 7, in preparation for its application, and also prepares for the proof of the finite generation of $\Delta \rtimes \Gamma_0$. We also find it convenient to introduce some additional terminology.

DEFINITION 10. We call two groups $A$ and $B$ *estranged* if they have no nontrivial isomorphic quotient groups. In other words, if $A \twoheadrightarrow Q \twoheadleftarrow B$ are epimorphisms, then $Q = 1$.

PROPOSITION 8. *The groups $\Gamma_n$, $1 \leqslant n \leqslant \infty$ satisfy.*

(Ab1) $\Gamma_n^{\mathrm{ab}} = 1$.
$(|\sigma_n \Gamma_0| < \infty)$ *the action of $\Gamma_0$ on $\Gamma_n$ factors through the finite quotient $\Gamma_0/\Gamma_0^{(n)}$,* $(n < \infty)$.
$(\rho\Gamma_\infty = 1)$ *For any finite-dimensional $\mathbb{C}$ representation $\rho$ of $\Gamma_\infty$,* $\rho(\Gamma_\infty) = 1$.

(Est)

*For $1 \leqslant n < m \leqslant \infty$, $\Gamma_n$ and $\Gamma_m$ are estranged.*

*Proof.* (Ab1) follows since $\mathrm{SL}(\mathbb{Z}^S)^{\mathrm{ab}} = 1$ whenever $|S| > 2$. The condition $(|\sigma_n \Gamma_0| < \infty)$ follows from our construction.

Since $|S_n| \geqslant 3$ for all $n$, it follows from the Congruence Subgroup Theorem that the quotients of $\Gamma_n = \mathrm{SL}^f(\mathbb{Z}^{(S_n)})$ are all of the form $\mathrm{SL}_{|S_n|}(\mathbb{Z}/q\mathbb{Z})/Z$, for some integer

$q \geqslant 0$, and where $Z$ is a central subgroup. It is immediate that succh groups cannot be isomorphic for $n < m$ (since $|S_n| < |S_m|$), and so condition (Est) follows.

Finally, we note that, for $m = \infty$, $\mathrm{SL}_\infty(\mathbb{Z}/q\mathbb{Z})$ has trivial center, and cannot be embedded in any $\mathrm{GL}_N(\mathbb{C})$ (which implies condition ($\rho\Gamma_\infty = 1$)). When $q = 0$, this follows since $\mathrm{SL}_d(\mathbb{Z})$ ($d < \infty$) has no faithful representations of dimension less than $d$. When $q > 0$, $\mathrm{SL}_\infty(\mathbb{Z}/q\mathbb{Z})$ is an infinite, locally finite group. If it were linear, by Jordan's Theorem it would have a normal Abelian subgroup of finite index. This is obviously not the case for $\mathrm{SL}_\infty(\mathbb{Z}/p\mathbb{Z})$ for $p$ prime, and the case for $\mathrm{SL}_\infty(\mathbb{Z}/q\mathbb{Z})$, $q$, reduces to the prime case by passage to a quotient. □

### 6.11. FINITE GENERATION OF $\Gamma = \Delta \rtimes \Gamma_0$

Recall from 6.2 that $\Delta$ is the group of eventually constant sequences in $\Pi = \prod_{n \geqslant 1} \Gamma_n$, and it is generated by the groups $\delta_n(\Gamma_n)$, where $\delta_n$ is the diagonal embedding of $\Gamma_n$ into $\Pi_n = \prod_{m \geqslant n} \Gamma_n$.

Put

$$\Gamma' = \langle \delta_1(\Gamma_1), \Gamma_0 \rangle \leqslant \Pi \rtimes \Gamma_0. \tag{83}$$

We show that $\Gamma = \Delta \rtimes \Gamma_0$ is finitely generated by showing that $\Gamma' = \Gamma$. Clearly

$$\Gamma' = D \rtimes \Gamma_0, \quad \text{where } D = \text{the } \Gamma_0 \text{ group generated by } \delta_1(\Gamma_1). \tag{84}$$

We must show that

the inclusion $D \leqslant \Delta$ is an equality. (Claim)

For $1 \leqslant n \leqslant \infty$, the projection $p_n : \Delta \to \Gamma_n$ maps $D$ to the $\Gamma_0$-group generated by $p_n(\delta_1(\Gamma_1)) = \Gamma_1$ in $\Gamma_n$. From Lemma 2 we can conclude that

$$p_n(D) = \Gamma_n, \quad \text{for } 1 \leqslant n \leqslant \infty. \tag{85}$$

In view of the above discussion and Proposition 8, the next proposition will imply that (Claim) obtains, and hence that

$$\Gamma := \Delta \rtimes \Gamma_0 = \langle \delta_1(\Gamma_1), \Gamma_0 \rangle, \quad \text{a finitely generated group.} \tag{86}$$

PROPOSITION 9. *Let $D \leqslant \Delta$ be a subgroup such that $p_n(D) = \Gamma_n$ for $1 \leqslant n \leqslant \infty$. Assume that*

$(Ab1)\Gamma_n^{\mathrm{ab}} = 1$

*and*

(*Est*) *For $1 \leqslant n < m \leqslant \infty$, $\Gamma_n$ and $\Gamma_m$ are estranged.*

*Then $D = \Delta$.*

Before starting the proof, we first establish a lemma:

LEMMA 4. *Assume* (*Est*).

(1) *For* $1 < n < \infty$, $\Gamma_1 \times \cdots \times \Gamma_{n-1}$ *and* $\Gamma_n$ *are estranged.*
(2) *If* $E \leqslant \Gamma_1 \times \cdots \times \Gamma_n$ *and* $p_i(E) = \Gamma_i$ *for each projection* $p_i$, $1 \leqslant i \leqslant n$ *then* $E \leqslant \Gamma_1 \times \cdots \times \Gamma_n$.

*Proof*. Let $p\colon \Gamma_1 \times \cdots \times \Gamma_{n-1} \to Q \leftarrow \Gamma_n\colon q$ be epimorphisms. For (1), we must show that $Q = 1$. If $p(\Gamma_1) = Q$, this follows because $\Gamma_1$ and $\Gamma_n$ are estranged. In general, since $\Gamma_1$ is normal in $\Gamma_1 \times \cdots \times \Gamma_{n-1}$, $p(\Gamma_1)$ is normal in $Q$, so that $Q/p(\Gamma_1)$ is a common quotient of $\Gamma_2 \times \cdots \times \Gamma_{n-1}$ and $\Gamma_n$, so $Q/p(\Gamma_1) = 1$ by induction on $n$.

To prove (2), we also argue by induction on $n$, the case $n = 1$ being trivial. For $n = 2$, put $E_i = E \cap \Gamma_i = \mathrm{Ker}(p_{2-i}|E)$, which is a normal subgroup of $E$ for $i = 1, 2$. We have $E/E_i \cong \Gamma_{2-i}$ since $p_i(E) = \Gamma_i$. Thus $E/(E_1 \cdot E_2)$ is a common quotient of $\Gamma_1$ and $\Gamma_2$ and, hence, trivial by (Est), so $E = E_1 \cdot E_2$. Clearly $E_1 \cap E_2 = 1$, so $E = E_1 \times E_2$. Since $E_i = p_i(E) = \Gamma_i$ $(i = 1, 2)$, we have $E = \Gamma_1 \times \Gamma_2$.

For $n > 2$, $\Gamma_1 \times \cdots \times \Gamma_{n-1}$ and $\Gamma_n$ are estranged by (1), and $E$ projects onto $\Gamma_1 \times \cdots \times \Gamma_{n-1}$ by induction. Hence, $E = (\Gamma_1 \times \cdots \times \Gamma_{n-1}) \times \Gamma_n$ by the case $n = 2$. □

Now we prove Proposition 9.

*Proof*. We have a commutative diagram with exact rows (see 6.2)

$$\begin{array}{ccccccccc} 1 & \to & \Pi' & \to & \Delta & \to & \Gamma_\infty & \to & 1 \\ & & \cup & & \cup & & \| & & \\ 1 & \to & D' & \to & D & \to & \Gamma_\infty & \to & 1 \end{array} \tag{1}$$

where $D' = D \cap \Pi'$, and $D \to \Gamma_\infty$ is onto since $p_\infty(D) = \Gamma_\infty$ by hypothesis. Moreover, by hypothesis $p_n(D) = \Gamma_n$ for $1 \leqslant n < \infty$, so it follows from Lemma 4 that

$$D \text{ projects onto } \Gamma_1 \times \cdots \times \Gamma_n \text{ for } 1 \leqslant n < \infty. \tag{2}$$

Since $D'$ is normal in $D$, it follows from (2) that

$$D' \lhd \Pi' = \prod^{\text{weak}} \Gamma_n,$$

and hence that $D'$ is normal in $\Pi$, since all normal subgroups of $\Pi'$ are. In particular,

$$D' \lhd \Delta \tag{3}$$

Next put

$$\begin{aligned} &D_n = D \cap \Gamma_n \qquad (= D' \cap \Gamma_n), \quad 1 \leqslant n < \infty \\ &D^* = \prod_{n \geqslant 1}^{\text{weak}} \leqslant D'. \end{aligned} \tag{4}$$

In view of (3),

$$D_n,\ D^* \lhd \Delta. \tag{5}$$

Put $\bar{\Delta} = \Delta/D^*$; for $H \leqslant \Delta$, let $\bar{H}$ denote the image of $H$ in $\bar{\Delta}$. Thus $\bar{\Gamma_n} = \Gamma_n/D_n$ and $\overline{\Pi'} = \prod_{n \geqslant 1}^{\text{weak}} \overline{\Gamma_n}$. We claim that

$$\bar{D} \cap \overline{\Gamma_n} = 1, \quad \text{for } 1 \leqslant n < \infty. \tag{6}$$

For if $g \in \Gamma_n$ and $\bar{g} \in \bar{D} \cap \bar{\Gamma}_n$, then $gd \in D$ for some $d = (d_m)_{m \geqslant 1} \in D^*$. It follows from the definition of $D^*$ that $d_n \in D_n$ and $gd_n \in D$, whence $g \in D \cap \Gamma_n = D_n$, so $\bar{g} = 1$.

Now from (6) and the fact that $\bar{D}' \lhd \Pi'$, we have

$$(\overline{\Gamma_n}, \bar{D}') \leqslant \bar{D}' \cap \Gamma_n = 1,$$

hence $\bar{D}'$ centralizes $\Gamma_n$ for all $n$, and so

$$\bar{D}' \leqslant \bar{Z}' := \prod^{\text{weak}} \bar{Z}_n, \quad \bar{Z}_n = Z(\overline{\Gamma_n})$$
$$= Z(\overline{\Pi'} \lhd \bar{\Delta}). \tag{7}$$

The inverse image $Z'$ of $\bar{Z}'$ modulo $D^*$ has the form

$$Z' := \prod^{\text{weak}} Z_n, \quad D_n \leqslant Z_n \leqslant \Gamma_n, \; D' \leqslant Z' \leqslant \Pi'. \tag{8}$$

Put

$$\tilde{\Delta} = \Delta/Z' = \bar{\Delta}/\bar{Z}' \quad \text{and} \quad H = \text{the image of } H \text{ in } \tilde{\Delta} \text{ for } H \leqslant \Delta. \tag{9}$$

From (1), (8), and (9), we have

$$\tilde{D} \cong D/D \cap Z' = D/D' \cong \Gamma_\infty \tag{10}$$

For $1 \leqslant n < \infty$, the projection $p_n: \Delta \to \Gamma_n$ induces a projection $\widetilde{p_n} : \widetilde{\Delta} \to \Gamma_n/p_n(Z') = \Gamma_n/Z_n$. Restricting to $\widetilde{D} \cong \Gamma_\infty$ (see (10)) and recalling that $p_n(D) = \Gamma_n$ by hypothesis, we obtain an epimorphism $\Gamma_\infty \to \Gamma_n/Z_n$. Since $\Gamma_\infty$ and $\Gamma_n$ are estranged by assumption, we have $\Gamma_n = Z_n$. Now $\overline{\Gamma_n} = Z_n/D_n = \bar{Z}_n = Z(\overline{\Gamma_n})$, so $\overline{\Gamma_n}$ is Abelian. By assumption, $\Gamma_n^{\text{ab}} = 1$. Thus $\overline{\Gamma_n = 1}$; that is, $\Gamma_n = D_n$. Hence

$$D^* = \prod^{\text{weak}} D_n = \prod^{\text{weak}} \Gamma_n = \Pi' \leqslant D.$$

In view of (1), this implies that $D = \Delta$, as claimed. This completes the proof of Proposition 9. □

Now we combine (86), Proposition 8, and Proposition 7 to obtain the following theorem:

THEOREM 8. *Let $\Gamma_0$ be finitely generated, residually finite, and filtered as in* 6.4. *Let*

$$\Gamma_1 < \Gamma_2 < \Gamma_3 \ldots$$
$$\Gamma_n = \mathrm{SL}(\mathbb{Z}^{S_n}), \quad S < S_2 < S_3 < \dot{<}\Gamma_0,$$
$$S_n \to \Gamma_0/\Gamma,^{(0)}$$

*be as in* 6.10*, with* $\Gamma_0$ *acting on* $\Gamma_n$ *via translation on* $S_n$ (*identified with* $\Gamma_0/\Gamma_0^{(n)}$)*, hence also on* $\Pi = \prod_{n \geqslant 1} \Gamma_n$*. Let*

$$\delta\colon \Gamma_1 \to \Pi, \quad \delta(q) = (q, q, q, \ldots)$$

*be the diagonally embedding, and put* $\Gamma = \langle \delta(\Gamma_1), \Gamma_0 \rangle \leqslant \Pi \rtimes \Gamma_0$*. Then* $\Gamma$ *is finitely generated, and* $\Gamma = \Delta \rtimes \Gamma_0$*, where* $\Delta$ *is the group of eventually constant sequences in* $\Pi$.

*Every finite-dimensional* $\mathbb{C}$ *representation* $\rho$ *of* $\Gamma$ *factors through some quotient* $\Gamma/\Delta_{n+1} = (\Gamma_1 \times \cdots \times \Gamma_n) \rtimes \Gamma_0$*. We have*

$$A(\Gamma) = \varprojlim_{n} A((\Gamma_1 \times \cdots \times \Gamma_n) \rtimes \Gamma_0). \tag{$A$}$$

$$A^0(\Gamma) = \prod_{n \geqslant 0} A^0(\Gamma_n) \tag{$A^0$}$$

*and*

$$\hat{\Gamma} \to \left( \prod_{n \geqslant 1} \widehat{\Gamma_n} \right) \times \widehat{\Gamma_0}, \tag{$\hat{\cdot}$}$$

*Put* $d(n) = |S_n|$ *for* $n \geqslant 3$*. Then for* $n \geqslant 1$*, we have*

$$A(\Gamma_n) = A^0(\Gamma_n) \times \widehat{\Gamma_n},$$
$$A^0(\Gamma_n) = \mathrm{SL}_{d(n)}(\mathbb{C}),$$
$$\widehat{\Gamma_n} = \mathrm{SL}_{d(n)}(\widehat{\mathbb{Z}})$$

(The final assertions, about $A(\Gamma)$, for $n \geqslant 1$, follow from the strict congruence subgroup theorem and rigidity properties of $\mathrm{SL}_d(\mathbb{Z})$ for $d \geqslant 3$; see [4].)

## 7. Concluding Remarks

The results of this paper show that a number of sets associated to a finitely generated residually finite representation rigid group $\Gamma$ are finite or finite dimensional, for example $S_n(\Gamma)$ (the isomorphism classes of simple $n$-dimensional representations of $\Gamma$), or $A_n(\Gamma)$. Thus we have a number of numeric sequences associated to $\Gamma$. We list them in this section. It should be of interest to relate these numer–theoretic functions to each other, and to consider their growth and/or other structural properties.

In addition to the objects previously defined, we also refer in the definition of the sequences to $SS_n(\Gamma)$, the isomorphism classes of semi–simple $n$ dimensional representations of $\Gamma$; and to $V_n(\Gamma)$, the commutator quotient of $U_n(\Gamma)$.

DEFINITION 11. Let $\Gamma$ be a finitely generated residually finite rigid group. Associated to $\Gamma$ are the following numeric sequences:

(1) $s_n(\Gamma) = |S_n(\Gamma)|$
(2) $ss_n(\Gamma) = |SS_n(\Gamma)|$
(3) $a_n(\Gamma) = \dim(A_n(\Gamma))$
(4) $p_n(\Gamma) = |A_n(\Gamma)/A_n(\Gamma)^0|$
(5) $q_n(\Gamma) = \dim(Q_n(\Gamma))$
(6) $u_n(\Gamma) = \dim(U_n(\Gamma))$
(7) $v_n(\Gamma) = \dim(V_n(\Gamma))$

One may also speculate about the relation of these sequences to the sequence $\alpha_n(\Gamma)$ which counts the number of subgroups of $\Gamma$ of index $n$. If $\Gamma$ has (FAb), $\alpha_n(\Gamma)$ grows strictly slower than $n^{\log n}$, for example, one can conclude that $\Gamma$ is super rigid: for this condition implies that for every prime $p$ the pro-$p$ completion of every finite index subgroup of $\Gamma$ is $p$-adic analytic, and this latter condition implies $\dim(Q(\Gamma)) < \infty$.

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