Geometriae Dedicata **96:** 211–240, 2003.

From the Boundary of the Convex Core to the Conformal Boundary

MARTIN BRIDGEMAN[1] and RICHARD D. CANARY[2★]
[1]*Department of Mathematics, Boston College, Chestnut Hill, MA 02167, U.S.A.*
e-mail: bridgem@bc.edu
[2]*Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, U.S.A.*
e-mail: canary@umich.edu

(Received: 21 September 2001)

Abstract. If N is a hyperbolic 3-manifold with finitely generated fundamental group, then the nearest point retraction is a proper homotopy equivalence from the conformal boundary of N to the boundary of the convex core of N. We show that the nearest point retraction is Lipschitz and has a Lipschitz homotopy inverse and that one may bound the Lipschitz constants in terms of the length of the shortest compressible curve on the conformal boundary.

Mathematics Subject Classifications (2000). 30F40, 57M50.

Key words. Kleinian group, hyperbolic manifold, convex core, limit set.

1. Introduction

If N is an orientable hyperbolic 3-manifold with finitely generated fundamental group, then the boundary $\partial C(N)$ of its convex core and its conformal boundary $\partial_c N$ are homeomorphic finite area hyperbolic surfaces. Sullivan showed that there exists some uniform constant K such that if $\partial C(N)$ is incompressible in the convex core $C(N)$, then there is a K-bi-Lipschitz homeomorphism between $\partial_c N$ and $\partial C(N)$, (see Epstein and Marden's paper [7]). In this paper, we investigate the relationship between the conformal boundary and the boundary of the convex core in the more general situation where one only assumes that N has finitely generated fundamental group.

If $N = \mathbf{H}^3/\Gamma$, then we may identify the sphere at infinity for $\mathbf{H}^3$ with the Riemann sphere $\hat{\mathbf{C}}$ and Γ acts as a group of conformal automorphisms of $\hat{\mathbf{C}}$. If we let $\Omega(\Gamma)$ be the domain of discontinuity for this action, i.e. the largest open subset of $\hat{\mathbf{C}}$ on which Γ acts properly discontinuously, then the conformal boundary $\partial_c N$ of N is the quotient $\Omega(\Gamma)/\Gamma$. If Γ is non-Abelian, then $\Omega(\Gamma)$ inherits a conformally invariant hyperbolic metric, called the *Poincaré metric*, and $\partial_c N$ is naturally a hyperbolic surface. The convex core $C(N)$ is the smallest convex submanifold of N. If the convex core

★Research supported in part by grants from the National Science Foundation.

is not two-dimensional, then $C(N)$ is homeomorphic to $\hat{N} = \partial_c N \cup N$ and $\partial C(N)$ is a hyperbolic surface (in its intrinsic metric). Epstein and Marden gave an extensive treatment of the convex core in [7].

One may produce sequences of examples of hyperbolic 3-manifolds where the minimal biLipschitz constant of a homeomorphism between the conformal boundary and the boundary of the convex core becomes arbitrarily large, see [7] or [6]. In these sequences, the length of the shortest compressible curve in the conformal boundary becomes arbitrarily small. It is thus natural to conjecture that there should be a biLipschitz homeomorphism between the conformal boundary and the boundary of the convex core, such that the biLipschitz constant is bounded above by a constant depending only on the length of the shortest compressible curve in the conformal boundary.

In this paper, we give a partial generalization of Sullivan's theorem to the setting of hyperbolic 3-manifolds with compressible conformal boundary. It is not difficult to combine Canary's estimates in [6] and Epstein and Marden's techniques used in [7] to show that the nearest point retraction is a Lipschitz map from the conformal boundary to the boundary of the convex core and that there is a bound on the Lipschitz constant depending only on a lower bound for the injectivity radius of the domain of discontinuity. We adapt Bridgeman's techniques from [4] to produce a homotopy inverse which is a Lipschitz map where again there is a bound on the Lipschitz constant depending only on a lower bound for the injectivity radius of the domain of discontinuity.

THEOREM 1. *There exist functions $J, L\colon (0,\infty) \to (0,\infty)$ such that if $N = \mathbf{H}^3/\Gamma$ is a hyperbolic 3-manifold with finitely generated, non-Abelian fundamental group and ρ_0 is a lower bound on the injectivity radius (in the Poincaré metric) of the domain of discontinuity $\Omega(\Gamma)$, then the nearest point retraction $r\colon \partial_c N \to \partial C(N)$ is $J(\rho_0)$-Lipschitz and has a $L(\rho_0)$-Lipschitz homotopy inverse.*

We will give explicit expressions for J and L later. For the moment, we simply note that as ρ_0 tends to 0, $J(\rho_0) = O(1/\rho_0)$ and $L(\rho_0) = O(e^{\frac{C}{\rho_0}})$ for some constant $C > 0$. Although these expressions may seem to grow quite fast we will also see that their basic forms cannot be substantially improved.

A lower bound on the injectivity radius of the domain of discontinuity (in the Poincaré metric) is equivalent to a lower bound on the length of the shortest compressible curve in the conformal boundary. If Γ is finitely generated, then Ahlfors' finiteness theorem [1] implies that there is a lower bound on the injectivity radius of the domain of discontinuity.

In the case that the conformal boundary is incompressible, our techniques improve on the bounds obtained by Bridgeman in [4]. We note that the conformal boundary is incompressible if and only if each component of the domain of discontinuity is simply connected.

THEOREM 2. *If $N = \mathbf{H}^3/\Gamma$ is a hyperbolic 3-manifold with finitely generated, non-Abelian fundamental group and each component of $\Omega(\Gamma)$ is simply connected, then the nearest point retraction $r\colon \partial_c N \to \partial C(N)$ is 4-Lipschitz and has a $(1 + \pi/\sinh^{-1}(1))$-Lipschitz homotopy inverse, where*

$$1 + \frac{\pi}{\sinh^{-1}(1)} \approx 4.56443.$$

The fact that r is 4-Lipschitz if the conformal boundary is incompressible is due to Epstein and Marden [7]. In a recent preprint [8], Epstein, Marden and Markovic establish that r is 2-Lipschitz in the same situation. Furthermore, they give a counter-example to Thurston's $K = 2$ Conjecture by exhibiting a hyperbolic 3-manifold with incompressible conformal boundary such that the nearest point retraction is not homotopic to a 2-quasiconformal map (see also [9, 10]).

One expects that the conclusions of Theorem 1 ought to guarantee the existence of a bi-Lipschitz homeomorphism between the conformal boundary and the boundary of the convex core and uniform bounds on the bi-Lipschitz constant. A realization of this expectation would produce a full generalization of Sullivan's theorem. In most cases, one uses Sullivan's theorem to assure that there is a bi-Lipschitz equivalence of lengths of corresponding closed geodesics. Theorem 1 does produce this bi-Lipschitz equivalence of lengths.

COROLLARY 1. *Let $N = \mathbf{H}^3/\Gamma$ be a hyperbolic 3-manifold with finitely generated, non-Abelian fundamental group and let ρ_0 be a lower bound for the injectivity radius of $\Omega(\Gamma)$. If α is a closed geodesic in $\partial_c N$ and $r(\alpha)^*$ denotes the closed geodesic in $\partial C(N)$ which is homotopic to $r(\alpha)$, then*

$$\frac{l_{\partial C(N)}(r(\alpha)^*)}{J(\rho_0)} \leqslant l_{\partial_c(N)}(\alpha) \leqslant L(\rho_0)\, l_{\partial C(N)}(r(\alpha)^*),$$

where $l_{\partial C(N)}(r(\alpha)^*)$ denotes the length of $r(\alpha)^*$ in $\partial C(N)$ and $l_{\partial_c N}(\alpha)$ denotes the length of α in $\partial_c N$.

We note that there is also a version of Theorem 1, where the bounds depend on the injectivity radius of the boundary of the convex hull $CH(L_\Gamma)$ of the limit set L_Γ of Γ, see Section 9. In fact, the bounds on the Lipschitz constant produced by generalizing the techniques of Bridgeman naturally give bounds which depend on the injectivity radius bounds on the boundary of the convex hull and it is necessary to prove that injectivity radius bounds on the boundary of the convex hull imply injectivity radius bounds on the domain of discontinuity (and vice versa.) We will also see that Theorem 1 holds more generally for analytically finite hyperbolic 3-manifolds and that Corollary 1 may be generalized to allow α to be any geodesic current on $\partial_c N$.

The key tool underlying the proofs of Theorems 1 and 2 is an estimate on the average bending of a curve in the boundary of the convex core. Suppose that N is a hyperbolic 3-manifold and α is a closed geodesic in $\partial C(N)$. We define the *average bending* $B(\alpha)$ of α to be

$$B(\alpha) = \frac{i(\alpha, \beta_N)}{l_{\partial C(N)}(\alpha)},$$

where $i(\alpha, \beta_N)$ is the total bending along α and $l_{\partial C(N)}(\alpha)$ is the hyperbolic length of α on $\partial C(N)$.

THEOREM 3. *There exists a function $K\colon (0,\infty) \to (0,\infty)$ such that if $N = \mathbf{H}^3/\Gamma$ is a hyperbolic 3-manifold with finitely generated, non-Abelian fundamental group and α is a closed geodesic on $\partial C(N)$, then*

(1) *If $\hat{\rho}_\alpha$ is a lower bound for the injectivity radius of $\partial CH(L_\Gamma)$ at any point in the support of a lift $\tilde{\alpha}$ of α, then $B(\alpha) \leqslant K(\hat{\rho}_\alpha)$*
(2) *If α is contained in an incompressible component of $\partial C(N)$, then $B(\alpha) \leqslant K_\infty$, where $K_\infty = \pi/\sinh^{-1}(1) \approx 3.56443$.*

2. Background

An orientable hyperbolic 3-manifold $\mathbf{H}^3/\Gamma$ is the quotient of hyperbolic 3-space $\mathbf{H}^3$ by a discrete torsion-free subgroup of the group $Isom_+(\mathbf{H}^3)$ of orientation preserving isometries of $\mathbf{H}^3$. We may identify $Isom_+(\mathbf{H}^3)$ with the group $\mathrm{PSL}_2(\mathbf{C})$ of Möbius transformations of $\hat{\mathbf{C}}$. The *domain of discontinuity* $\Omega(\Gamma)$ is the largest open set in $\hat{\mathbf{C}}$ on which Γ acts properly discontinuously, and the *limit set* L_Γ is its complement. The conformal boundary $\partial_c N$ of N is simply the quotient $\Omega(\Gamma)/\Gamma$. If Γ is non-Abelian, then L_Γ is infinite and $\Omega(\Gamma)$ admits a canonical hyperbolic metric $p(z)|\mathrm{d}z|$ called the Poincaré metric. We will assume throughout the paper that Γ is non-Abelian. The Kleinian group Γ acts as a group of isometries of the Poincaré metric, so $\partial_c N$ is a hyperbolic surface. The hyperbolic 3-manifold N is said to be *analytically finite* if $\partial_c N$ has finite area in this metric. Ahlfors' Finiteness Theorem [1] asserts that N is analytically finite if Γ is finitely generated. We note that if N is analytically finite then there is always a positive lower bound for the injectivity radius on $\Omega(\Gamma)$.

The *convex hull* $CH(L_\Gamma)$ of L_Γ is the smallest convex subset of $\mathbf{H}^3$ so that all geodesics with both endpoints in L_Γ are contained in $CH(L_\Gamma)$. The *convex core* $C(N)$ of $N = \mathbf{H}^3/\Gamma$ is the quotient of $CH(L_\Gamma)$ by Γ. The boundary $\partial C(N)$ of the convex core is a pleated surface, i.e. there is a path-wise isometry $f\colon S \to \partial C(N)$ from a hyperbolic surface S onto N which is totally geodesic in the complement of a disjoint collection β_N of geodesics which is called the *bending lamination*. The nearest point retraction $\tilde{r}\colon \mathbf{H}^3 \to CH(L_\Gamma)$ is the map which takes a point to the (unique) nearest point in $CH(L_\Gamma)$. It extends continuously to a map $\tilde{r}\colon \Omega(\Gamma) \cup \mathbf{H}^3 \to \partial CH(L_\Gamma)$, called the nearest point retraction, such that if $z \in \Omega(\Gamma)$, then $\tilde{r}(z)$ is the (unique) first point of contact of an expanding family of horospheres based at z with $\partial CH(L_\Gamma)$. This map descends to a map $r\colon \hat{N} \to \partial C(N)$. We will often consider the restriction of r to $\partial_c N$ (which we will simply call r) which gives a homotopy equivalence from $\partial_c N$ to $\partial C(N)$. Epstein and Marden [7] give a complete description of the geometry of the convex hull.

We have to modify the above description in the special case that L_Γ lies in a round circle. In this case, $CH(L_\Gamma)$ is a convex subset of a hyperbolic plane and $C(N)$ is a totally geodesic surface with boundary. In this case, we will consider $\partial C(N)$ to be the double of $C(N)$ (along its boundary considered as a hyperbolic surface) where we regard the two copies of $C(N)$ as having opposite normal vectors. One may still define $r\colon \partial_c N \to \partial C(N)$ in this setting and it remains a homotopy equivalence.

The bending lamination β_N inherits a measure on arcs transverse to β_N which records the total amount of bending along any transverse arc, so β_N is a measured lamination. A *measured lamination* on a finite area hyperbolic surface S consists of a closed subset λ of S which is the disjoint union of geodesics, together with countably additive invariant (with respect to projection along λ) measures on arcs transverse to λ. The simplest example of a measured lamination is a (real) multiple of a simple closed geodesic, where the measure on each transverse arc has an atom of fixed mass at each intersection point with the geodesic. Multiples of simple closed geodesics are dense in the space $ML(S)$ of all measured laminations on S (see [13]).

If we lift a measured lamination to the universal cover $\mathbf{H}^2$ of S, we obtain a $\pi_1(S)$-invariant subset of the space $G(\mathbf{H}^2)$ of geodesics on $\mathbf{H}^2$. The transverse measure on λ gives rise to a $\pi_1(S)$-invariant measure on $G(\mathbf{H}^2)$. More generally, a *geodesic current* is a $\pi_1(S)$-invariant measure on $G(\mathbf{H}^2)$. Bonahon [2, 3] has extensively studied the space $\mathcal{C}(S)$ of geodesic currents on S. The support of a geodesic current projects to a closed union of geodesics and multiples of closed geodesics are dense in $\mathcal{C}(S)$ (see also [12].) The function given by the length of a closed geodesic extends in a natural way to continuous functions on $ML(S)$ and $\mathcal{C}(S)$. Similarly, the geometric intersection number of two closed geodesics extends to a continuous functions on $\mathcal{C}(S) \times \mathcal{C}(S)$. Moreover, if $f\colon S \to T$ is a Lipschitz map between finite area hyperbolic surfaces it induces a homeomorphism $f_*\colon \mathcal{C}(S) \to \mathcal{C}(T)$.

3. Some Basic Facts from Hyperbolic Geometry

We begin by observing that among hyperbolic triangles with a side of fixed length and opposite angle of fixed value, the isosceles triangle maximizes perimeter. We will omit the proof which is an elementary calculation involving hyperbolic trigonometry.

LEMMA 3.1. *Consider the set of all hyperbolic triangles with one side of fixed length C and the opposite angle of fixed value θ, where $0 < \theta < \pi$. Then the unique triangle in this set with maximal length perimeter is the isosceles triangle having the fixed side as base. The other sides have length*

$$\sinh^{-1}\left(\frac{\sinh(C/2)}{\sin(\theta/2)}\right).$$

We will also need an elementary observation about configurations of planes in $\mathbf{H}^3$. We will later use such configurations to enclose the convex hull.

Let H_0, H_1, and H_2 be three closed half-spaces in $\mathbf{H}^3$. Let P_i denote the plane in $\mathbf{H}^3$ which bounds H_i and let D_i be the closed disk in S^2_∞ which is the intersection of the closure of H_i with S^2_∞. Suppose that $D_0 \cap D_1 = \{a\}$ and $D_1 \cap D_2 = \{b\}$. Let C be the closure of the complement of $H_1 \cup H_2 \cup H_3$.

Suppose that α is a parametrized curve $\alpha\colon [0, 2] \to C$ such that $\alpha(i) \in P_i$, for $i = 0, 1, 2$. We denote the length of α by l. Then α is a curve with one endpoint on P_0, the other on P_2, and an interior point on P_1. We show that if l is short enough, then D_0 and D_2 must intersect and that l determines an upper bound for their angle of intersection. Recall that the angle of intersection of two half-spaces equals the angle of intersection of the associated disks on the sphere at infinity.

LEMMA 3.2. *If $l \leqslant 2\sinh^{-1}(1)$, then D_0 and D_2 intersect and their angle of intersection θ satisfies $\theta \geqslant 2\cos^{-1}(\sinh(l/2))$.*

Proof. Let $\bar{\alpha}$ be the shortest curve in C with one endpoint on P_0, the other on P_2, and an interior point on P_1. Let H be the unique plane orthogonal to the three planes P_0, P_1 and P_2. We note that the circle on S^2_∞ which bounds H must pass through the two ideal points a and b described above. Thus, letting $L_i = P_i \cap H$, the line L_1 meets each of L_0 and L_2 in an ideal point. Furthermore, the disks D_0 and D_2 intersect if and only if the lines L_0 and L_2 intersect, and the angle of intersection of the lines is equal to the angle of intersection of the disks.

As orthogonal projection onto H decreases distance, $\bar{\alpha}$ must be contained in the plane H. Using planar hyperbolic geometry, one sees that the curve $\bar{\alpha}$ consists of two equal length geodesic segments with a common endpoint v on L_1 which are perpendicular to L_0 and L_2, respectively. If $l(\bar{\alpha})$ denotes the length of $\bar{\alpha}$, then $l \geqslant l(\bar{\alpha})$.

If L_0 and L_2 intersect in an angle θ, then we let T be the triangle given by the three lines L_0, L_1 and L_2. Applying elementary formulae from hyperbolic trigonometry one sees that $\sinh(l(\bar{\alpha})/2) = \cos(\theta/2)$. Thus,

$$l(\bar{\alpha}) = 2\sinh^{-1}(\cos(\theta/2)).$$

Since $l \geqslant l(\bar{\alpha})$,

$$l \geqslant 2\sinh^{-1}(\cos(\theta/2)).$$

The function $f(x) = \sinh^{-1}(\cos(x/2))$ is decreasing on $[0, \pi]$, so

$$\theta \geqslant 2\cos^{-1}(\sinh(l/2)). \tag{1}$$

If T is ideal then $\theta = 0$ and $l(\bar{\alpha}) = 2\sinh^{-1}(1)$.

If the closures of L_0 and L_2 do not intersect, then there is an ideal triangle T' with two ideal vertices equal to the ideal endpoints of L_1, whose other ideal vertex lies between the ideal endpoints of L_0 and L_2 which are not endpoints of L_1. Since T' is ideal, the intersection of $\bar{\alpha}$ with T' has length at least $2\sinh^{-1}(1)$, so

$$l \geqslant l(\bar{\alpha}) > 2\sinh^{-1}(1).$$

Therefore, if $l \leqslant 2\sinh^{-1}(1)$ the (closures of) L_0 and L_2 must intersect and inequality (1) must hold. □

4. Local Intersection Number Estimates

In this section we show that if a geodesic arc in the boundary of the convex hull is short enough then its 'total bending' is at most 2π. How short it is necessary to make the arc will be an explicit function of the injectivity radius of the convex hull at the starting point of the arc. This estimate, Lemma 4.3, underlies all the results in the paper.

We first need to recall some background material on convex hulls. For a full description of convex hulls, see [7]. We will assume throughout this section that Γ is analytically finite.

If Γ is a Kleinian group with convex hull $CH(L_\Gamma)$ then a *support plane* to $CH(L_\Gamma)$ is a hyperbolic plane P in $\mathbf{H}^3$ which bounds a closed half-space H_P whose intersection $H_P \cap CH(L_\Gamma)$ with the convex hull is nonempty and contained in P. We will consider P to be an oriented plane, with orientation chosen so that H_P lies above P. If P is a support plane and $P \cap \partial CH(L_\Gamma)$ is a single geodesic, then this geodesic is called a *bending line*, otherwise, the interior of $P \cap \partial CH(L_\Gamma)$ is called a *flat* and the geodesics in the frontier of the flat are also called bending lines. If P_1 and P_2 are distinct intersecting support planes, then $r = P_1 \cap P_2$ is called a *ridge line*.

If $x \in \partial CH(L_\Gamma)$ then either x lies in a flat or x is on some bending line. If x lies in a flat then there is a unique support plane P containing x. If $x \in b$, where b is a bending line, let $\Sigma(b)$ be the set of support planes to b. The set of oriented planes $S(b)$ containing b is a circle and $\Sigma(b) \subseteq S(b)$. As $\Sigma(b)$ is connected, it is either a closed arc or a point. If $\Sigma(b)$ is an arc, the endpoints are called *extreme* support planes and the *bending angle* $\beta(x)$ is defined to be the angle between the extreme support planes. Otherwise, we define $\beta(x) = 0$.

The union of the bending lines in $\partial CH(L_\Gamma)$ is denoted β_Γ and is called the *bending lamination*. Thurston defined a *transverse measure* on β_Γ called the *bending measure* which assigns to every arc α transverse to β_Γ a value $i(\alpha, \beta_\Gamma)$ corresponding to the amount of bending along α (see [7] or [13]). If the closed arc α is transverse to β_Γ and has endpoints x and y, then

$$i(\alpha, \beta_\Gamma) = \beta(x) + i(\alpha^0, \beta_\Gamma) + \beta(y),$$

where α^0 denotes the interior of α. The bending lamination β_Γ on $\partial CH(L_\Gamma)$ projects to the bending lamination β_N of $\partial C(N)$.

We now refine the analysis further to allow for arbitrary support planes at the endpoints. Since Γ is analytically finite, each bending line with positive angle covers one of finitely many closed geodesics in β_N. Therefore, any path $\alpha\colon [0, 1] \to \partial CH(L_\Gamma)$ which is transverse to β_Γ contains at most finitely many points where there is not a unique support plane to the image of α. If there is a unique support plane at

$\alpha(s)$, let Q_s be the unique support plane. We define the *initial* support plane at $\alpha(\bar{s})$ to be $Q_{\bar{s}}^- = \lim_{s \mapsto \bar{s}^-} Q_s$ and the *terminal* support plane at $\alpha(\bar{s})$ to be $Q_{\bar{s}}^+ = \lim_{s \mapsto \bar{s}^+} Q_s$. The initial support plane at $\alpha(0)$ is defined to be Q_0 if there is a unique support plane, and otherwise is the extreme support plane which is not terminal. The terminal support plane at $\alpha(1)$ is defined similarly. If $\beta(\alpha(\bar{s})) > 0$, then the initial and terminal support planes are the two extreme support planes.

Suppose that $\alpha\colon [0, 1] \to \partial CH(L_\Gamma)$ is a path transverse to β_Γ and that P and Q are support planes at $\alpha(0)$ and $\alpha(1)$. We define θ_P to be the exterior dihedral angle between P and the terminal support plane Q_0^+ and θ_Q to be the exterior dihedral angle between Q and the initial support plane Q_1^-. Then we define

$$i(\alpha, \beta_\Gamma)_P^Q = \theta_P + i(\alpha^0, \beta_\Gamma) + \theta_Q.$$

Notice that if $\bar{P}$ is the initial support plane at $\alpha(0)$ and $\bar{Q}$ is the terminal support plane at $\alpha(1)$, then $i(\alpha, \beta_\Gamma)_{\bar{P}}^{\bar{Q}} = i(\alpha, \beta_\Gamma)$.

If $\{0 = s_0 < s_1 < \cdots < s_n = 1\}$ is a subdivision of $[0, 1]$, then let α_i be the closed subarc obtained by restricting α to the interval $[s_{i-1}, s_i]$. Let Q_i be a support plane at $\alpha(s_i)$ with $Q_0 = P$ and $Q_n = Q$. Then it follows from the additivity of the standard intersection number that

$$i(\alpha, \beta_\Gamma)_P^Q = \sum_{i=1}^{n} i(\alpha_i, \beta_\Gamma)_{Q_{i-1}}^{Q_i}.$$

We now obtain an explicit description of a continuous path of support planes to α joining P to Q. Let $\{0 \leqslant s_1 < \cdots < s_{n-1} \leqslant 1\}$ be the points at which $\alpha(s)$ does not have a unique support plane. If s_i is not either 0 or 1, then let $\theta_i = \beta(\alpha(s_i)) > 0$ and let $\{Q_\theta^i | \theta \in [0, \theta_i]\}$ denote the one parameter family of all support planes to $\alpha(s_i)$ parameterized by the exterior angle the support plane makes with the initial support plane $Q_{s_i}^-$. If $s_1 = 0$, then we let θ_1 be the angle between P and the terminal support plane Q_0^+ at $\alpha(0)$ and we begin the parameterization $\{Q_\theta^i | \theta \in [0, \theta_1]\}$ at P. Similarly, if $s_{n-1} = 1$, then we let θ_{n-1} be the angle between Q and the initial support plane Q_1^- at $\alpha(1)$ and we end the parameterization $\{Q_\theta^i | \theta \in [\theta, \theta_{n-1}]\}$ at Q.

We obtain our continuous 1-parameter family of support planes along α by inserting the families $\{Q_\theta^i | \theta \in [0, \theta_i]\}$ between the intervals where the support planes are uniquely defined. Let $I_j = (s_{j-1}, s_j)$ for all $j = 1, \ldots, n$, where we define $s_0 = 0$ and $s_n = 1$. Let $\Theta_i = \sum_{j=1}^{i} \theta_j$ and let $k = 1 + \Theta_{n-1}$. We let

$$X_i = [s_i + \Theta_{i-1}, s_i + \Theta_i] \quad \text{and} \quad Y_i = (s_{i-1} + \Theta_{i-1}, s_i + \Theta_{i-1}).$$

We let $Y_1 = [0, s_1)$ and $Y_n = (s_{n-1} + \Theta_{n-1}, k]$. The intervals X_i and Y_i give a partition of $[0, k]$ and we define a piecewise linear continuous function $s\colon [0, k] \to [0, 1]$ by

$$s(t) = \begin{cases} t_i, & t \in X_i, \\ t - \Theta_{i-1}, & t \in Y_i. \end{cases}$$

The function s is a continuous monotonic function. We define the support planes P_t by letting $P_0 = P$, $P_k = Q$ and if $t \in (0, k)$ setting

$$P_t = \begin{cases} Q^i_{t-s_i-\Theta_{i-1}}, & t \in X_i, \\ Q_{s(t)}, & t \in Y_i. \end{cases}$$

The family $\{P_t | t \in [0, k]\}$ is called the *continuous* 1-*parameter family of support planes along* α *from* P *to* Q. Notice that P_t is a support plane to $\alpha(s(t))$ and that if $P_{t_1} = P_{t_2}$ and $s(t_1) = s(t_2)$, then $t_1 = t_2$.

The following lemma allows us to estimate the intersection number along a geodesic on $\partial CH(L_\Gamma)$ by using support planes. Its proof is given in the appendix.

Let $\{g_t\}$ be a continuous family of geodesics in a hyperbolic plane which is indexed by an interval J. We say that the family is *monotonic* on J if given $a, b \in J$ such that $a < b$ and $g_a \cap g_b \neq \emptyset$ then $g_t = g_a$ for all $t \in [a, b]$. Notice that if $\{g_t\}$ is monotonic over $[a, b)$ and continuous on $[a, b]$, then it is monotonic on $[a, b]$.

We say that (P, Q) is a *roof* over a path α if for all $t \in [0, k]$, $P \cap P_t \neq \emptyset$ and the interiors of the half spaces H_P and H_{P_t} also intersect.

LEMMA 4.1. *Let* $N = \mathbf{H}^3/\Gamma$ *be an analytically finite hyperbolic* 3-*manifold such that* L_Γ *is not contained in a round circle. Let* $\alpha\colon [0, 1] \to \partial CH(L_\Gamma)$ *be a geodesic path, in the intrinsic metric on* $\partial CH(L_\Gamma)$*, which is transverse to* β_Γ*. If* (P, Q) *is a roof over* α *and* $\{P_t |\ t \in [0, k]\}$ *is the continuous one-parameter family of support planes over* α *joining* P *to* Q*, then*

(1) $i(\alpha, \beta_\Gamma)^Q_P \leqslant \theta < \pi$, *where* θ *is the exterior dihedral angle between* P *and* Q*, and*
(2) *there is a* $\bar{t} \in [0, k]$ *such that* $P_t = P$ *if* $t \in [0, \bar{t}]$ *and the ridge lines* $\{r_t = P \cap P_t | t > \bar{t}\}$ *exist and form a monotonic family of geodesics on* P.

We say (P, Q) is a π-roof if (P, P_t) is a roof over $\alpha([0, s(t)])$ for all $0 \leqslant t < k$ but (P, Q) is not a roof over α. Notice that this implies that either $P = Q$, in which case the limit set L_Γ is contained in a round circle, or that the closures of P and Q intersect in a single point at infinity. The following corollary follows immediately from Lemma 4.1.

COROLLARY 4.2. *If* (P, Q) *is a* π-*roof over* α *then the interiors of the half spaces* H_P *and* H_Q *are disjoint and* $i(\alpha, \beta_\Gamma)^Q_P \leqslant \pi$.

The following functions arise naturally when we attempt to quantify how short we must make a geodesic in $\partial CH(L_\Gamma)$ in order to guarantee that its intersection with the bending measure is at most 2π. We define the functions F, G, K by

$$F(x) = \frac{x}{2} + \sinh^{-1}\left(\frac{\sinh\left(\frac{x}{2}\right)}{\sqrt{1 - \sinh^2\left(\frac{x}{2}\right)}}\right), \qquad G(x) = F^{-1}(x), \quad K(x) = \frac{2\pi}{G(x)}.$$

From the equation it is easy to see that F is monotonically increasing with domain $[0, 2\sinh^{-1}(1))$. The function $G(x)$ has asymptotic behavior $G(x) \asymp x$ as x tends to 0, and $G(x)$ approaches $2\sinh^{-1}(1)$ as x tends to ∞. We further define

$$G_\infty = 2\sinh^{-1}(1) \approx 1.76275 \quad \text{and} \quad K_\infty = \frac{\pi}{\sinh^{-1}(1)} \approx 3.56443.$$

The following lemma shows that if a short arc bends a lot, then it must begin at a point with small injectivity radius. In the next section, we will apply this local bound to obtain the global bound on average bending given in Theorem 3. If $x \in \partial CH(L_\Gamma)$, let $\hat{\rho}(x)$ denote the injectivity radius of $\partial CH(L_\Gamma)$ (in the intrinsic metric) at the point x.

LEMMA 4.3. *Let $N = \mathbf{H}^3/\Gamma$ be an analytically finite hyperbolic 3-manifold and let $\alpha\colon [0, 1] \to \partial CH(L_\Gamma)$ be a geodesic path of length $l(\alpha)$ which is transverse to β_Γ. If P is a support plane at $\alpha(0)$ and either*

(1) *$\alpha([0, 1])$ is contained in a simply connected component of $\partial CH(L_\Gamma)$ and $l(\alpha) \leqslant G_\infty$, or*
(2) *$l(\alpha) \leqslant G(\hat{\rho}(\alpha(0)))$,*

then there is a support plane Q at $\alpha(1)$ such that $i(\alpha, \beta_\Gamma)_P^Q \leqslant 2\pi$.

Proof. Let $\alpha\colon [0, 1] \to \partial CH(L_\Gamma)$ be a geodesic. We first deal with the special case that L_Γ is contained in a round circle. In this case, if α intersects more than one bending line, then the double of a subarc of α joining two bending lines is a homotopically nontrivial curve on $\partial CH(L_\Gamma)$, so $l(\alpha) \geqslant \hat{\rho}(\alpha(0))$. However, we have assumed that $l(\alpha) < G(\hat{\rho}(\alpha(0))) < \hat{\rho}(\alpha(0))$. Therefore, α can intersect at most one bending line, so $i(\alpha, \beta_\Gamma)_P^Q \leqslant \pi$. From now on we may assume that L_Γ is not contained in a round circle.

Let Q be the initial support plane at $\alpha(1)$ and let $\{P_t | t \in [0, k]\}$ be the continuous one parameter family of support planes to α joining P to Q. If (P, Q) is a roof over α, then, by Lemma 4.1, the exterior angle of intersection θ of P and Q is an upper bound for $i(\alpha, \beta_\Gamma)_P^Q$. Therefore, in this case, $i(\alpha, \beta_\Gamma)_P^Q \leqslant \theta < \pi$.

If (P, Q) is not a roof over α, let t_1 be the smallest value of $t > 0$ such that (P, P_t) is not a roof over $\alpha([0, s(t)])$. We let $s(t_1) = s_1$ and $\alpha_1 = \alpha|_{[0,s_1]}$. Then, (P_0, P_{t_1}) is a π-roof over α_1 and so, by Corollary 4.2, $i(\alpha_1, \beta_\Gamma)_P^{P_{t_1}} \leqslant \pi$.

If (P_{t_1}, Q) is a roof over $\alpha([s_1, 1])$, we let $\alpha_2 = \alpha|_{[s_1,1]}$. Then, the exterior angle of intersection θ_1 of P_{t_1} and Q is an upper bound for $i(\alpha_2, \beta_\Gamma)_{P_{t_1}}^Q$. Thus we have

$$i(\alpha, \beta_\Gamma)_P^Q = i(\alpha_1, \beta_\Gamma)_P^{P_{t_1}} + i(\alpha_2, \beta_\Gamma)_{P_{t_1}}^Q \leqslant \pi + \theta_1 < 2\pi.$$

In the final case we let t_2 be the smallest value of $t \in [t_1, k]$ such that (P_{t_1}, P_t) is not a roof over $\alpha([s_1, s(t)])$, and we let $s(t_2) = s_2$. If $s_2 = 1$, then (P_{t_1}, Q) is a π-roof over $\alpha_2 = \alpha([s_1, 1])$. Therefore $i(\alpha, \beta_\Gamma)_P^Q \leqslant 2\pi$ as above. Otherwise, let $\tilde{l} = l(\alpha([0, s_2]))$. Then $\tilde{l} < l(\alpha)$. As $G(\hat{\rho}(\alpha(0))) < G_\infty = 2\sinh^{-1}(1)$, we have $\tilde{l} < 2\sinh^{-1}(1)$.

Since (P, P_{t_1}) and (P_{t_1}, P_{t_2}) are π-roofs and L_Γ is not a round circle, the support planes P, P_{t_1}, and P_{t_2} have the configuration described in Lemma 3.2. Also the curve

$\alpha\colon [0, s_2] \to \partial CH(L_\Gamma)$ has one endpoint on P, the other on P_{t_2}, an interior point on P_{t_1} and length $\tilde{l} < 2\sinh^{-1}(1)$. Lemma 3.2 implies that P and P_{t_2} intersect and have angle of intersection θ satisfying

$$\theta \geqslant 2\cos^{-1}(\sinh(\tilde{l}/2)) > 0.$$

We join the endpoints $\alpha(0)$ and $\alpha(s_2)$ by the shortest curve v on $P \cup P_{t_2}$. This curve consists of two geodesic segments, v_1 and v_2. The segment v_1 lies on P and joins $\alpha(0)$ to a point $V \in P \cap P_{t_2}$, while v_2 lies on P_{t_2} and joins V to $\alpha(s_2)$. We consider the triangle T in $\mathbf{H}^3$ with vertices $\alpha(0)$, $\alpha(s_2)$, and V. The angle θ_V at V satisfies $\theta_V \geqslant \theta$. Also, the opposite side joining $\alpha(0)$ to $\alpha(s_2)$ has length $l_V \leqslant \tilde{l}$. Therefore, T has an angle bounded below by θ and opposite side bounded above by $\tilde{l}$. Lemma 3.1 implies that

$$l(v) \leqslant 2\sinh^{-1}\left(\frac{\sinh(\tilde{l}/2)}{\sin(\theta/2)}\right).$$

Applying the bound for θ we obtain

$$l(v) \leqslant 2\sinh^{-1}\left(\frac{\sinh(\tilde{l}/2)}{\sqrt{1-\sinh^2(\tilde{l}/2)}}\right).$$

We obtain a closed curve η by concatenating $\alpha([0, s_2])$ and v. Then,

$$l(\eta) \leqslant \tilde{l} + 2\sinh^{-1}\left(\frac{\sinh(\tilde{l}/2)}{\sqrt{1-\sinh^2(\tilde{l}/2)}}\right) = 2F(\tilde{l}).$$

Let $\gamma = \tilde{r}(\eta)$ where $\tilde{r}$ is the nearest point retraction. Therefore γ is the union of $\alpha([0, s_2])$ and $g = \tilde{r}(v)$. In particular, $l(\gamma) \leqslant l(\eta) \leqslant 2F(\tilde{l})$.

If α is in a simply connected component of $\partial CH(L_\Gamma)$, then γ is in a simply connected component, so γ must be homotopically trivial.

If α is in a nonsimply connected component, then $\tilde{l} < l(\alpha) \leqslant G(\hat{\rho}(\alpha(0)))$. Therefore, by monotonicity of F, $F(\tilde{l}) < \hat{\rho}(\alpha(0))$ and $l(\gamma) \leqslant 2F(\tilde{l}) < 2\hat{\rho}(\alpha(0))$. As γ contains the point $\alpha(0)$ and $l(\gamma) < 2\hat{\rho}(\alpha(0))$, γ is homotopically trivial in $\partial CH(L_\Gamma)$.

We now obtain a contradiction by showing that γ is not homotopically trivial in $\partial CH(L_\Gamma)$. Let b_1 be the first bending line on P_{t_1} that the curve $\alpha([0, s_2])$ intersects and let $\alpha(\bar{s})$ be this first point of intersection. We first show that $\alpha([0, s_2])$ intersects b_1 exactly once. We then show that g intersects b_1 in at most one point and that, if they do intersect, g does not cross b_1 at this point, i.e. near the point of intersection both component of $g - b_1$ lie on the same side of b_1. Since $\alpha([0, s_2])$ intersects b_1 transversely, it follows that γ may be perturbed slightly so that it is transverse to b_1 and intersects it exactly once. However, it is impossible for a properly embedded infinite geodesic on a hyperbolic surface to intersect a homotopically trivial transverse closed curve exactly once, so we will have achieved our contradiction.

Suppose that $\alpha([0, s_2])$ has a second intersection point with b_1 at a point x. Since $P_{t_1} \cap P_{t_2} = \emptyset$, $x = \alpha(\tilde{s})$ where $\bar{s} < \tilde{s} < s_2$. Let $\tilde{t}$ be such that $s(\tilde{t}) = \tilde{s}$. Then, by definition, $\bar{t} < \tilde{t} < t_2$ and $x \in P_{\tilde{t}}$, so $(P_{t_1}, P_{\tilde{t}})$ is a roof over $\alpha([\bar{s}, \tilde{s}])$. If $P_{t_1} = P_{\tilde{t}}$ then, by Lemma 4.1, $P_t = P_{t_1}$ for all $t \in [t_1, \tilde{t}]$. Therefore, $\alpha([\bar{s}, \tilde{s}])$ is a geodesic arc contained in P_{t_1} with two endpoints on the geodesic b_1. Thus, $\alpha([\bar{s}, \tilde{s}]) \subseteq b_1$ which contradicts the fact that $\alpha([0, s_2])$ intersects b_1 transversely.

If $P_{t_1} \neq P_{\tilde{t}}$, let $r_{\tilde{t}}$ be the ridge line $P_{t_1} \cap P_{\tilde{t}}$. By Lemma 1.9.2 in [7] (stated in the appendix as Lemma 10.2) if a ridge line intersects a bending line then they are equal. Since $x \in r_{\tilde{t}} \cap b_1$, we have $r_{\tilde{t}} = b_1$. By the monotonicity of the ridge lines, for each $t \in [t_1, \tilde{t}]$, either $P_t = P_{t_1}$ or $r_t = P_t \cap P_{t_1} = b_1$. Thus for all $t \in [t_1, \tilde{t}]$, P_t is a support plane to b_1. If b_1 has a unique support plane, then $P_t = P_{t_1}$ for all $t \in [t_1, \tilde{t}]$ and this reduces to the above case. If b_1 has more than one support plane then we let X and Y be the extreme support planes at b_1. If Z is another support plane for b_1, then $Z \cap \partial CH(L_\Gamma) = b_1$. As the only points of $\alpha([\bar{s}, \tilde{s}])$ in b_1 are the endpoints, the only possible support plane for any point in the open arc $\alpha((\bar{s}, \tilde{s}))$ is either X or Y. Since $\alpha((\bar{s}, \tilde{s}))$ is connected and $X \cap Y = b_1$, either $\alpha((\bar{s}, \tilde{s})) \subseteq X$ or $\alpha((\bar{s}, \tilde{s})) \subseteq Y$. We can assume $\alpha((\bar{s}, \tilde{s})) \subseteq X$. As the endpoints of $\alpha((\bar{s}, \tilde{s}))$ are in b_1, the geodesic arc $\alpha([\bar{s}, \tilde{s}])$ lies in X and intersects the geodesic b_1 at its endpoints. Therefore, $\alpha([\bar{s}, \tilde{s}]) \subseteq b_1$ which again contradicts the fact that $\alpha([0, s_2])$ intersects b_1 transversely. Thus, we have established that $\alpha([0, s_2])$ intersects b_1 exactly once.

We now consider $g = \tilde{r}(v)$. First suppose that b_1 has a unique support plane P_{t_1}. If g intersects b_1, then there is a point $x \in \tilde{r}^{-1}(b_1) \cap v$ which lies in the interior of $H_{P_{t_1}}$ and in either P or P_{t_2}. But, since P and P_{t_2} are support planes disjoint from P_{t_1}, this is impossible. Therefore, if b_1 has a unique support plane, then g does not intersect b_1.

Now suppose that b_1 does not have a unique support plane and let X and Y be the extreme support planes at b_1. Each support plane Q to b_1 determines a normal half-plane, i.e. the portion of the normal plane to Q which lies in H_Q. Then, $\tilde{r}^{-1}(b_1)$ is a wedge bounded by the normal half-planes to X and Y and is made up of the disjoint normal half-planes to all the support planes to b_1. Notice that the endpoints of v lie outside of $\tilde{r}^{-1}(b_1)$ and that v does not intersect b_1. If $\tilde{r}(v_1)$ intersects b_1 at an interior point, then the endpoints of v_1 lie outside this wedge and the geodesic segment v_1 must intersect every normal half-plane to a support plane for b_1. In particular, v_1 must intersect the normal half-plane to P_{t_1}. Therefore, there must exist a point $x \in \tilde{r}^{-1}(b_1) \cap v_1$ which lies in the interior of $H_{P_{t_1}}$ and in P, which is impossible. So, $\tilde{r}(v_1)$ cannot intersect b_1 at an interior point. Similarly, $\tilde{r}(v_2)$ cannot intersect b_1 at an interior point. Therefore, if g intersects b_1 it must do so at $\tilde{r}(V)$, in which case $V \in \tilde{r}^{-1}(b)$. If g crosses b_1 at $\tilde{r}(V)$, then v_1 and v_2 must intersect the normal half-planes to X and Y. By continuity, v must then intersect the normal half-plane to P_{t_1}. Again, we have found a point which lies both in the interior of $H_{P_{t_1}}$ and in either P or P_{t_2}, which is a contradiction. Therefore, as claimed, g can intersect b_1 at only one point, and it cannot cross b_1 at this point. This completes the proof. □

5. Global Intersection Number Estimates

The proofs of Theorems 1 and 2 rely heavily on the following global estimate on intersection numbers. Moreover, Theorem 3 is an immediate corollary.

PROPOSITION 5.1. *Suppose that* $N = \mathbf{H}^3/\Gamma$ *is an analytically finite hyperbolic* 3*-manifold and* α *is a closed geodesic on* $\partial C(N)$.

(1) *If* $\hat{\rho}_\alpha$ *is a lower bound for the injectivity radius of* $\partial CH(L_\Gamma)$ *at any point in the support of a lift* $\tilde{\alpha}$ *of* α, *then*

$$i(\alpha, \beta_N) \leqslant K(\hat{\rho}_\alpha) l_{\partial C(N)}(\alpha)$$

where $l_{\partial C(N)}$ *is the hyperbolic length of* α *on* $\partial C(N)$.

(2) *If* α *is contained in an incompressible component of* $\partial C(N)$, *then*

$$i(\alpha, \beta_N) \leqslant K_\infty l_{\partial C(N)}(\alpha).$$

We recall that

$$K_\infty = \frac{\pi}{\sinh^{-1}(1)} \approx 3.56443$$

and that $K(x) \asymp 2\pi/x$ as x tends to 0.

Proof. Let $\alpha\colon S^1 \to \partial C(N)$ be a closed geodesic on $\partial C(N)$. Either α lies in β_N or is transverse to β_N. If α lies in β_N, then $i(\alpha, \beta_N) = 0$, so we may assume that α is transverse to β_N. We identify S^1 with $\mathbf{R}/\mathbf{Z}$ and let $\tilde{\alpha}\colon \mathbf{R} \to \partial CH(L_\Gamma)$ be a lift of α to $\partial CH(L_\Gamma)$. We may assume, without loss of generality, that $\alpha(0)$ lies in a flat.

If we let $\tilde{\alpha}_n$ be the restriction of $\tilde{\alpha}$ to the interval $[0, n]$ then $i(\tilde{\alpha}_n, \beta_\Gamma) = n\, i(\alpha, \beta_N)$ and $l_{\partial CH(L_\Gamma)}(\tilde{\alpha}_n) = n\, l_{\partial C(N)}(\alpha)$. Let $\tilde{\alpha}$ be in the connected component C of $\partial CH(L_\Gamma)$. If C is simply connected we let $G = G_\infty$. Otherwise we let $G = G(\hat{\rho}_\alpha)$. We subdivide $\tilde{\alpha}_n$ into m subarcs of length less than or equal to G where m is given by

$$m = \left[\frac{l_{\partial CH(L_\Gamma)}(\tilde{\alpha}_n)}{G}\right]^+ \leqslant \frac{l_{\partial CH(L_\Gamma)}(\tilde{\alpha}_n)}{G} + 1$$

and $[x]^+$ is the least integer greater than or equal to x.

We let $\tilde{\alpha}_n^j$ be the subarcs, where $\tilde{\alpha}_n^j$ is restriction of $\tilde{\alpha}_n$ to the interval $[s_{j-1}, s_j]$ and $0 = s_0 < s_1 < \cdots < s_m = n$. We define support planes P_j at $\tilde{\alpha}_n(s_j)$ inductively. First, we let $P_0 = P$ where P is the unique support plane to $\tilde{\alpha}_n(0)$. If P_{j-1} is defined, then it is a support plane to $\tilde{\alpha}_n^j(s_{j-1}) = \tilde{\alpha}_n(s_{j-1})$. As the length of $\tilde{\alpha}_n^j$ is less than or equal to G, by Lemma 4.3, there is a support plane P_j at $\tilde{\alpha}_n^j(s_j) = \tilde{\alpha}_n(s_j)$ such that $i(\tilde{\alpha}_n^j, \beta_\Gamma)_{P_{j-1}}^{P_j} \leqslant 2\pi$. As $\tilde{\alpha}_n(n)$ is in a flat, P_m must be the unique support plane at $\tilde{\alpha}_n(n)$. Therefore, by additivity, we have

$$i(\tilde{\alpha}_n, \beta_\Gamma) = i(\tilde{\alpha}_n, \beta_\Gamma)_{P_0}^{P_m} = \sum_{j=1}^{m} i(\tilde{\alpha}_n^j, \beta_\Gamma)_{P_{j-1}}^{P_j} \leqslant 2\pi m.$$

Substituting the upper bound for m, we get

$$i(\tilde{\alpha}_n, \beta_\Gamma) \leqslant 2\pi\left(\frac{l_{\partial CH(L_\Gamma)}(\tilde{\alpha}_n)}{G} + 1\right).$$

Rewriting in terms of α, we get

$$n\, i(\alpha, \beta_N) \leqslant \frac{2\pi n l_{\partial C(N)}(\alpha)}{G} + 2\pi.$$

Dividing through by n, we get

$$i(\alpha, \beta_N) \leqslant \frac{2\pi l_{\partial C(N)}(\alpha)}{G} + \frac{2\pi}{n}.$$

As this holds for all n,

$$i(\alpha, \beta_N) \leqslant \frac{2\pi l_{\partial C(N)}(\alpha)}{G} = K\, l_{\partial C(N)}(\alpha),$$

where K equals either K_∞ or $K(\hat{\rho}_\alpha)$ depending on whether α is contained in an incompressible component of $\partial C(N)$ or not. □

The version of Theorem 3 stated in the introduction follows immediately from Proposition 5.1. We now give a version of Theorem 3 which applies to geodesic currents. If α is a geodesic current on $\partial C(N)$, then we may define its average bending to be

$$B(\alpha) = \frac{i(\alpha, \beta_N)}{l_{\partial C(N)}(\alpha)}.$$

Since multiples of closed geodesics are dense in $\mathcal{C}(\partial C(N))$ and the length and intersection functions are continuous, the following version of Theorem 3 also follows from Proposition 5.1.

THEOREM 3′. *Let $N = \mathbf{H}^3/\Gamma$ be an analytically finite hyperbolic 3-manifold and let $\alpha \in \mathcal{C}(\partial C(N))$ be a geodesic current in the boundary of the convex core of N.*

(1) *If $\partial C(N)$ is incompressible, then $B(\alpha) \leqslant K_\infty$.*
(2) *If $\partial C(N)$ is compressible and $\hat{\rho}_0$ is a lower bound for the injectivity radius of the boundary of the convex hull of the limit set, then $B(\alpha) \leqslant K(\hat{\rho}_0)$.*

6. A Homotopy Inverse for the Nearest Point Retraction

We now combine Proposition 5.1 with work of Thurston [14] to obtain a Lipschitz homotopy inverse to the nearest point retraction. One should note that the bounds on the Lipschitz constant of the homotopy inverse depend on the injectivity radius of the boundary of the convex hull of the limit set. We will see later how to obtain a lower bound on the injectivity radius of $\partial CH(L_\Gamma)$ from a lower bound on the injectivity radius of $\Omega(\Gamma)$.

PROPOSITION 6.1. *Let N be an analytically finite hyperbolic 3-manifold. If $\partial C(N)$ is compressible and $\hat{\rho}_0$ is a lower bound for the injectivity radius of $\partial CH(L_\Gamma)$, then the nearest point retraction r has a homotopy inverse that is $(1 + K(\hat{\rho}_0))$ Lipschitz. If $\partial C(N)$ is incompressible, then the homotopy inverse is $(1 + K_\infty)$-Lipschitz.*

Proof. Let $s\colon \partial C(N) \to \partial_c(N)$ be a homotopy inverse to the nearest point retraction r. Let K denote K_∞ if $\partial C(N)$ is incompressible and $K(\hat{\rho}_0)$ otherwise.

Let α be a simple closed geodesic in $\partial C(N)$ with length $l_{\partial C(N)}(\alpha)$ and let $l_{\partial_c N}(s(\alpha)^*)$ be the length of the geodesic representative of $s(\alpha)$ in $\partial_c N$. McMullen (Theorem 3.1 in [11]) showed that

$$l_{\partial_c N}(s(\alpha)^*) \leqslant l_{\partial C(N)}(\alpha) + i(\alpha, \beta_N).$$

Using Proposition 5.1, we get that

$$l_{\partial_c N}(s(\alpha)^*) \leqslant (1 + K) l_{\partial C(N)}(\alpha).$$

Thurston [14] proved that if $f\colon X \to Y$ is a homotopy equivalence between two finite area hyperbolic surfaces and

$$\frac{l_Y(f(\beta)^*)}{l_X(\beta)} \leqslant M$$

for any simple closed geodesic β on X, then f is homotopic to a M-Lipschitz map. Thus, we may conclude in our case that s is homotopic to a $(1 + K)$-Lipschitz map from $\partial C(N)$ to $\partial_c N$ as claimed. □

The following proposition indicates that one cannot improve much on the bounds obtained in Proposition 6.1. Recall that $K(\hat{\rho}_0) \asymp 2\pi/\hat{\rho}_0$ as $\hat{\rho}_0$ tends to 0. Notice that in Proposition 6.2 we do not need to assume that N is analytically finite.

PROPOSITION 6.2. *There exists $L > 0$ such that if N is a hyperbolic 3-manifold, there is a compressible closed geodesic γ on $\partial C(N)$ with length $l_0 < L$ and $s\colon \partial C(N) \to \partial_c N$ is a K-Lipschitz homotopy inverse to the nearest point retraction, then*

$$K \geqslant \frac{1}{l_0 \log\left(\frac{1}{l_0}\right)}.$$

Proof. Let l denote the length of $s(\gamma)^*$ in $\partial_c N$. Theorem 5.1 in [6] implies that if $l < 1$ then

$$l \geqslant \frac{\pi^2}{\sqrt{e}\log\left(\frac{4\pi e^{(.502)\pi}}{l_0}\right)}.$$

If $\log(l_0) \leqslant -2\log(4\pi e^{(.502)\pi})$ and $l < 1$, then

$$l \geqslant \frac{1}{\log\left(\frac{1}{l_0}\right)}.$$

Notice that the above inequality also holds if $l \geqslant 1$. Thus, if we choose

$$L = \frac{1}{(4\pi e^{(.502)\pi})^2} \quad \text{and} \quad l_0 \leqslant L,$$

then

$$K \geqslant \frac{l}{l_0} \geqslant \frac{1}{l_0 \log\left(\frac{1}{l_0}\right)}. \qquad \square$$

In particular, this shows that if $\partial C(N)$ contains arbitrarily short compressible curves, then there is no Lipschitz map from the convex core to the conformal boundary. This situation can occur only when N has infinitely generated fundamental group.

7. The Nearest Point Retraction is Lipschitz

In this section we show how to combine the techniques in Section 2.3 of [7] and the results of [6] to show that the nearest point retraction is itself Lipschitz (and to produce bounds on the Lipschitz constant.) We remark that Epstein and Marden showed that the nearest point retraction is 4-Lipschitz if $\partial C(N)$ is incompressible.

PROPOSITION 7.1. *If $N = \mathbf{H}^3/\Gamma$ is an analytically finite hyperbolic 3-manifold and ρ_0 is a lower bound for the injectivity radius of $\Omega(\Gamma)$, then the nearest point retraction $r\colon \partial_c N \to \partial C(N)$ is $J(\rho_0)$-Lipschitz where*

$$J(\rho_0) = 2\sqrt{2}\left(k + \frac{\pi^2}{2\rho_0}\right)$$

and $k = 4 + \log(3 + 2\sqrt{2}) \approx 5.763$.

Proof. Let $K = \sqrt{2}(k + (\pi^2/2\rho_0))$. We will show that given any point $z \in \Omega(\Gamma)$ and any $\delta \in (0, 1)$ there exists a neighborhood of z on which $\tilde{r}$ is $2K((1 + \delta)/(1 - \delta^2))$-Lipschitz. It follows that $\tilde{r}$ is itself $2K((1 + \delta)/(1 - \delta^2))$-Lipschitz. Since δ can be chosen to be arbitrarily close to 0, it follows that $\tilde{r}$ (and, hence, r) is $2K$-Lipschitz as claimed.

Let $z \in \Omega(\Gamma)$ and let P be the support plane to $\tilde{r}(z)$ which is orthogonal to $z\tilde{r}(z)$. We can always find a neighborhood U of z such that if $w \in U$ and Q is the support plane to $\tilde{r}(w)$ which is orthogonal to $w\tilde{r}(w)$, then P intersects Q. Given $\delta \in (0, 1)$, we may further restrict U so that it is contained in the ball of radius $\delta/4K$ about z in the Poincaré metric and that any point $w \in U$ may be joined to z by a unique geodesic in U of length $d_\Omega(z, w)$.

Let $w \in U$, let Q be the support plane to $\tilde{r}(w)$ which is orthogonal to $w\tilde{r}(w)$, and let g be the geodesic in U joining z to w. We normalize, in the upper half-space model for $\mathbf{H}^3$, so that $z = 0$, the unit circle is the boundary of the support plane P and $\infty \in L_\Gamma$. It is shown, in the proof of Proposition 4.1 of [6], that if $\rho_\Omega(z)|dz|$ denotes the Poincaré metric on $\Omega(\Gamma)$ then

$$p_\Omega(z) \geqslant \frac{1}{K\,\mathrm{d}(z, L_\Gamma)} \tag{2}$$

for all $z \in \Omega(\Gamma)$ where $\mathrm{d}(z, L_\Gamma)$ denotes the Euclidean distance from z to the limit set L_Γ.

Let D be the unit disk and let D_Q be the disk bounded by ∂Q. If we let $p_D(z)|\mathrm{d}z|$ denote the Poincaré metric on D, then

$$\frac{p_D(z)}{p_\Omega(z)} \leqslant \frac{2K\,\mathrm{d}(z, L_\Gamma)}{1 - |z|^2} \tag{3}$$

for all $z \in D$.

Since g has length at most $\delta/4K$, inequality (2) implies that g is contained in the ball of Euclidean radius δ about 0. In particular, if $z \in g$, then $p_D(z)/p_\Omega(z) \leqslant 2K((1+\delta)/(1-\delta^2))$. We divide g up into 3 segments:

$$g_1 = g \cap (D - D_Q), \qquad g_2 = g \cap (D \cap D_Q) \quad \text{and} \quad g_3 = g \cap (D_Q - D)$$

Inequality (3) then implies that $l_D(g_1) \leqslant 2K((1+\delta)/(1-\delta^2))l_\Omega(g_1)$ where $l_D(g_1)$ denotes the length of g_1 in the Poincaré metric on D and $l_\Omega(g_1)$ denotes the length of g_1 in the Poincaré metric on $\Omega(\Gamma)$. Similarly,

$$l_D(g_2) \leqslant 2K\left(\frac{1+\delta}{1-\delta^2}\right)l_\Omega(g_2) \quad \text{and} \quad l_{D_Q}(g_3) \leqslant 2K\left(\frac{1+\delta}{1-\delta^2}\right)l_\Omega(g_3).$$

Let $\Omega' = D \cup D_Q$ and let $r'\colon \Omega' \to CH(\partial\Omega')$ be the nearest point retraction. Notice that $r'(0) = r(0)$ and $r'(w) = r(w)$. Let

$$r_D\colon D \to P, \qquad r_Q\colon D_Q \to Q \quad \text{and} \quad r_L\colon D \cap D_Q \to L$$

be the nearest point retractions, where $L = P \cap Q$. Then

$$r'|_{D-D_Q} = r_D, \qquad r'|_{D_Q-D} = r_Q \quad \text{and} \quad r'|_{D\cap D_Q} = r_L.$$

Notice that r_D and r_Q are isometries with respect to the Poincaré metrics on P and Q and that r_L is 1-Lipschitz with respect to the Poincaré metric on either P or Q. It follows that

$$l_{\mathbf{H}^3}(r'(g)) \leqslant l_D(g_1) + l_D(g_2) + l_{D_Q}(g_3) \leqslant 2K\left(\frac{1+\delta}{1-\delta^2}\right)l_\Omega(g).$$

We recall that $\tilde{r}\colon \Omega(\Gamma) \to \partial CH(L_\Gamma)$ extends to $\tilde{r}\colon \mathbf{H}^3 \cup \Omega(\Gamma) \to CH(L_\Gamma)$. Then $\tilde{r}(r'(g))$ is a path joining $r(0)$ to $r(w)$ of length at most $2K((1+\delta)/(1-\delta^2))l_\Omega(g)$ (since $\tilde{r}$ is 1-Lipschitz on $\mathbf{H}^3$). It follows that

$$\mathrm{d}_{\partial CH(L_\Gamma)}(\tilde{r}(w), \tilde{r}(z)) \leqslant 2K\left(\frac{1+\delta}{1-\delta^2}\right)\mathrm{d}_\Omega(z, w).$$

Hence, $\tilde{r}$ is $2K((1+\delta)/(1-\delta^2))$-Lipschitz on U as required and we have completed the proof. $\square$

Remarks. (1) Epstein and Marden [7] showed that the nearest point retraction r is 4-Lipschitz if $\partial C(N)$ is incompressible. In [6] it is shown that r is homotopic to a $2\sqrt{2}$-Lipschitz map if $\partial C(N)$ is incompressible and to a $\sqrt{2}K$-Lipschitz map if not.

(2) In Section 6 of [6], Canary constructs an infinite sequence of hyperbolic manifolds $\{N_n\}$ such that, for all large enough n, the shortest geodesic in $\partial_c N$ has length $1/n$ and the shortest geodesic in $\partial C(N)$ has length at most $4\pi/e^{\pi(2n-1)}$ and the nearest point retraction is not even homotopic to a map which is $5n/(2\log(5n))$-Lipschitz. Hence, we cannot improve substantively on the form of the estimate obtained above.

8. The Proof of Theorem 1

The only issue remaining in the proof of Theorem 1 is that the bound on the Lipschitz constant in Proposition 6.1 depends on an injectivity radius bound in the boundary of the convex hull, while the assumptions of Theorem 1 only give us an injectivity radius bound on the domain of discontinuity. The following lemma guarantees that injectivity radius bounds on the domain of discontinuity give us injectivity radius bound in the boundary of the convex hull.

LEMMA 8.1. *Let $N = \mathbf{H}^3/\Gamma$ be a hyperbolic 3-manifold and let α be a geodesic on $\partial CH(L_\Gamma)$ with length $l(\alpha) < e^{-m} \approx .06798$, where $m = \cosh^{-1}(e^2) \approx 2.68854$, then*

$$l_{\partial_c N}(\tilde{s}(\alpha)^*) \leqslant \frac{\pi^2}{\log\left(\frac{1}{l(\alpha)}\right) - m}$$

where $\tilde{s}\colon \partial CH(L_\Gamma) \to \partial_c N$ is a lift of a homotopy inverse to $r\colon \partial_c N \to \partial C(N)$.

Proof. There is a collar neighborhood C of α on $\partial CH(L_\Gamma)$ which is isometric to $[-w, w] \times S^1$ with the metric

$$ds^2 = dr^2 + \left(\frac{l(\alpha)}{2\pi}\right)^2 \cosh^2 r \, dt^2,$$

where α is identified with $\{0\} \times S^1$ and

$$w = \sinh^{-1}\left(\frac{1}{\sinh(l(\alpha)/2)}\right)$$

(see Theorem 4.1.1 in [5].) Let α_1 and α_2 denote the boundary components of C. Then

$$\begin{aligned} l(\alpha_1) = l(\alpha_2) &= l(\alpha)\cosh\left(\sinh^{-1}\left(\frac{1}{\sinh(l(\alpha)/2)}\right)\right) \\ &= l(\alpha)\coth\left(\frac{l(\alpha)}{2}\right) \leqslant 4. \end{aligned}$$

(The last inequality follows since $l(\alpha)\coth(l(\alpha)/2)$ is an increasing function and $l(\alpha) < 1$.) Recall that every closed geodesic in $\partial CH(L_\Gamma)$ must intersect β_Γ, since otherwise there would be a closed geodesic contained entirely within a flat. We normalize the situation so that α passes through the origin, the origin lies on a bending line L for $\partial CH(L_\gamma)$ and that L is the z-axis in the Poincaré ball model for $\mathbf{H}^3$.

Let $\beta_1 = \tilde{r}^{-1}(\alpha_1)$ and $\beta_2 = \tilde{r}^{-1}(\alpha_2)$ be the set-theoretic pre-images of the curves α_1 and α_2 under $\tilde{r}$. Then, β_1 and β_2 are homotopic simple closed curves in $\Omega(\Gamma)$. Our goal is to prove that β_1 and β_2 bound a 'large' modulus annulus in $\Omega(\Gamma)$ and, hence, that the core curve of this annulus is 'short'. Since r is a homotopy inverse to s the core curve of the annulus is homotopic to $s(\alpha)$.

Notice that L must pass through C and intersects both α_1 and α_2 transversely at points, x_1 and x_2, and that

$$d(x_i, 0) \geqslant \sinh^{-1}\left(\frac{1}{\sinh(l(\alpha)/2))}\right) \geqslant \sinh^{-1}\left(\frac{1}{l(\alpha)}\right) \geqslant \log\left(\frac{1}{l(\alpha)}\right).$$

(The middle inequality follows from the facts that $\sinh^{-1}$ is an increasing function and that $\sinh(x) \leqslant 2x$ if $x \leqslant 1$.)

Let $r_L\colon \Omega(\Gamma) \to L$ denote the nearest point projection onto L. One may calculate that if $x \in L$, $y \in \mathbf{H}^3$, $\mathrm{d}(x, y) \leqslant 2$ and the family of horoballs about a point $z \in \Omega(\Gamma)$ hits y before it hits L, then $\mathrm{d}(x, r_L(z)) \leqslant \cosh^{-1}(\mathrm{e}^2)$. Let $m = \cosh^{-1}(\mathrm{e}^2)$. Let L_0 be the portion of L joining x_1 to x_2 and let L_m denote the portion of L_0 which is a distance more than m from both x_1 and x_2. Let $A_m = \pi_L^{-1}(L_m)$. (Notice that since $l(\alpha) < \mathrm{e}^{-m}$, L_m and A_m are nonempty.) Since $\beta_i = \tilde{r}^{-1}(\alpha_i)$ and $d(y, x_i) \leqslant 2$ for all $y \in \alpha_i$, β_1 and β_2 lie in opposite components of $\hat{\mathbf{C}} - A_m$. Therefore, since β_1 and β_2 are homotopic in $\Omega(\Gamma)$, $A_m \subset \Omega(\Gamma)$.

One may readily check that

$$\mathrm{mod}(A_m) \geqslant \frac{\log\left(\frac{1}{l(\alpha)}\right) - m}{\pi},$$

where $\mathrm{mod}(A_m)$ is the conformal modulus of A_m. If α' is the core curve of A_m, then, see for example Theorem 2.6 in [7], α' has length at most $\pi^2/(\log(1/l(\alpha)) - m)$ in the Poincaré metric on A_m and hence in the Poincaré metric on $\Omega(\Gamma)$. Since α' is homotopic to $s(\alpha)$, we see that

$$l_{\partial_c N}(s(\alpha)^*) \leqslant \frac{\pi^2}{\log\left(\frac{1}{l(\alpha)}\right) - m}. \qquad \square$$

In particular, Lemma 9.1 guarantees that if ρ_0 is a lower bound for the injectivity radius of $\Omega(\Gamma)$, then $g(\rho_0)$ is a lower bound for the injectivity radius of $\partial CH(L_\Gamma)$, where

$$g(\rho_0) = \frac{\mathrm{e}^{-m}\mathrm{e}^{\frac{-\pi^2}{2\rho_0}}}{2}.$$

If we define $L(\rho_0) = 1 + K(g(\rho_0))$, then we may combine Corollary 6.1 and Proposition 7.1 to obtain the following, slightly more general, version of Theorem 1:

THEOREM 1. *If $N = \mathbf{H}^3/\Gamma$ is an analytically finite hyperbolic 3-manifold and ρ_0 is a lower bound for the injectivity radius of $\Omega(\Gamma)$, then the nearest point retraction $r\colon \partial_c N \to \partial C(N)$ is $J(\rho_0)$-Lipschitz and has a $L(\rho_0)$-Lipschitz homotopy inverse.*

The following slightly more general version of Corollary 1 is an almost immediate corollary of Theorem 1.

COROLLARY 1. *Let $N = \mathbf{H}^3/\Gamma$ be an analytically finite hyperbolic 3-manifold and let ρ_0 be a lower bound for the injectivity radius of $\Omega(\Gamma)$. If α is a geodesic current in $\partial_c N$ and $r(\alpha)^*$ denotes the geodesic current in $\partial C(N)$ which is homotopic to $r(\alpha)$, then*

$$\frac{l_{\partial C(N)}(r(\alpha)^*)}{J(\rho_0)} \leqslant l_{\partial_c N}(\alpha) \leqslant L(\rho_0) l_{\partial C(N)}(r(\alpha)^*),$$

where $l_{\partial C(N)}(r(\alpha)^*)$ denotes the length of $r(\alpha)^*$ in $\partial C(N)$ and $l_{\partial_c(N)}(\alpha)$ denotes the length of α in $\partial_c(N)$.

Proof. We note that the bounds follow immediately from Theorem 1 when α is a closed geodesic. Recall that multiples of closed geodesics are dense in the space of geodesic currents, length is a continuous function on the space of geodesic currents on a surface, and $r_*\colon \mathcal{C}(\partial_c N) \to \mathcal{C}(\partial C(N))$ is continuous. The general result then follows. □

Theorem 2 follows immediately from Proposition 6.1 and Epstein and Marden's result that the nearest point retraction is 4-Lipschitz when each component of $\Omega(\Gamma)$ is incompressible. It has the following immediate corollary in the spirit of Corollary 1.

COROLLARY 2. *Let $N = \mathbf{H}^3/\Gamma$ be an analytically finite hyperbolic 3-manifold such that $\partial_c N$ is incompressible in $\hat{N} = N \cup \partial_c N$. If α is a geodesic current in $\partial_c N$ and $r(\alpha)^*$ denotes the geodesic current in $\partial C(N)$ which is homotopic to $r(\alpha)$, then*

$$\frac{l_{\partial C(N)}(r(\alpha)^*)}{4} \leqslant l_{\partial_c(N)}(\alpha) \leqslant \left(1 + \frac{\pi}{\sinh^{-1}(1)}\right) l_{\partial C(N)}(r(\alpha)^*).$$

Remark. Notice that

$$J(\rho_0) \asymp \frac{\sqrt{2}\pi^2}{\rho_0} \quad \text{and} \quad L(\rho_0) \asymp 4\pi e^m e^{\frac{\pi^2}{2\rho_0}}$$

as ρ_0 tends to 0. We observed in remark (2) in Section 7 that the form of $J(\rho_0)$ can not be substantially improved. It is an immediate consequence of Theorem 5.1 in [6] that if $\rho_0 < .5$ and s is a L-Lipschitz homotopy inverse to r, then

$$L \geqslant \frac{\rho_0 e^{\frac{\pi^2}{2\sqrt{e}\rho_0}}}{2\pi e^{(.502)\pi}}$$

so again the form of $L(\rho_0)$ cannot be substantially improved.

9. An Alternative Version of Theorem 1

The following lemma allows us to translate injectivity radius bounds on the boundary of the convex core to injectivity radius bounds on the conformal boundary.

LEMMA 9.1. *Let N be a hyperbolic 3-manifold and let $\hat{\rho}_0$ be a lower bound for the injectivity radius of $\partial CH(L_\Gamma)$. Then $f(\hat{\rho}_0)$ is a lower bound for the injectivity radius of $\Omega(\Gamma)$ where*

$$f(\hat{\rho}_0) = \min\left\{\frac{1}{2}, \frac{\pi^2}{2\sqrt{e}\log\left(\frac{4\pi e^{(.502)\pi}}{2\hat{\rho}_0}\right)}\right\}.$$

Notice that

$$f(\hat{\rho}_0) \asymp \frac{\pi^2}{2\sqrt{e}\log(1/\hat{\rho}_0)} \quad \text{as } \hat{\rho}_0 \text{ tends to } 0.$$

Proof. If not, there exists a compressible curve α on $\partial_c N$ with length L such that $L < 2f(\hat{\rho}_0)$. Theorem 5.1 in [6] then implies that $r(\alpha)^*$ is a compressible geodesic on $\partial C(N)$ with length less that $2\hat{\rho}_0$ which contradicts our assumptions. □

Therefore, if we set $J'(\hat{\rho}_0) = J(f(\hat{\rho}_0))$ and let $L'(\hat{\rho}_0) = 1 + K(\hat{\rho}_0)$, then we obtain the following alternative formulation of Theorem 1:

THEOREM 1′. *Let $N = \mathbf{H}^3/\Gamma$ be an analytically finite hyperbolic 3-manifold and let $\hat{\rho}_0$ be a lower bound for the injectivity radius of $\partial CH(L_\Gamma)$. Then the nearest point-retraction is a $J'(\hat{\rho}_0)$-Lipschitz map and has a homotopy inverse which is $L'(\hat{\rho}_0)$-Lipschitz map.*

We also get the following alternative formulation of Corollary 1.

COROLLARY 1′. *Let $N = \mathbf{H}^3/\Gamma$ be an analytically finite hyperbolic 3-manifold and let $s\colon \partial C(N) \to \partial_c N$ be a homotopy inverse to the nearest point retraction. If $\hat{\rho}_0$ is a* *lower bound for the injectivity radius of* $\partial CH(L_\Gamma)$ *and* α *is a geodesic current on* $\partial C(N)$, *then*

$$\frac{l_{\partial_c N}(s(\alpha)^*)}{L'(\hat\rho_0)} \leqslant l_{\partial C(N)}(\alpha) \leqslant J'(\hat\rho_0) l_{\partial_c N}(s(\alpha)^*).$$

Remark. Notice that

$$J'(\hat\rho_0) = \mathrm{O}\left(\log\left(\frac{1}{\hat\rho_0}\right)\right) \quad \text{and} \quad L'(\hat\rho_0) = \mathrm{O}\left(\frac{1}{\hat\rho_0}\right)$$

as $\hat\rho_0$ tends to 0. These asymptotics are much better than those in Theorem 1, since when $\Omega(\Gamma)$ has small injectivity radius, $\partial CH(L_\Gamma)$ has much smaller injectivity radius. Proposition 6.2 indicates that the form of L' can not be substantially improved, while the examples in Section 6 of [6] can be used to show that $J'(\hat\rho_0)$ must grow at least as fast as $D\log\left(\frac{1}{\hat\rho_0}\right)/\log\left(\log\left(\frac{1}{\hat\rho_0}\right)\right)$ as $\hat\rho_0$ tends to 0 (for some constant $D > 0$.)

10. Appendix: The Proof of Lemma 4.1

In this section we review some of the theory of convex hulls of limit sets, as developed by Epstein and Marden [7]. We then give a proof of Lemma 4.1 which asserts that ridge lines are monotonic for the support planes under a roof and that one can use the exterior dihedral angle of the roof to provide a bound on the bending measure. We will assume throughout the appendix that $N = \mathbf{H}^3/\Gamma$ is analytically finite and that L_Γ is not contained in a round circle.

We will say that a neighborhood U of x in $\partial CH(L_\Gamma)$ is *adapted to* x if it has the following two properties:

(1) U is a spherical shell adapted to x, see Definition 1.5.3 in [7]. In particular, U is simply connected and the intersection of any bending line or flat with U is connected and convex.
(2) If two bending lines b_1 and b_2 meet U, then any support plane to b_1 meets any support plane to b_2.

Lemma 1.8.3 in [7] guarantees that one can choose a set U having property (2) above and also guarantees that ridge lines to support planes in a small enough neighborhood must lie close to one another.

LEMMA 10.1 ([7]). *If* $x \in \partial CH(L_\Gamma)$ *then there is an open neighborhood* $U \subseteq \partial CH(L_\Gamma)$ *of* x *such that if two bending lines* b_1 *and* b_2 *meet* U *then any support plane to* b_1 *intersects any support plane to* b_2. *Furthermore, if* b *is a bending line containing* x *and* N *is a neighborhood of* b *in the space of geodesics, then, by taking* U *small enough, we may assume that any ridge line, which is formed by the intersection of two distinct support planes at points of* U *lies in* N.

Suppose that $x \in \partial CH(L_\Gamma)$ and U is a neighborhood adapted to x. If b_1 and b_2 are distinct bending lines which intersect U and lie in support planes P_1 and P_2, then l_1 and l_2 bound a strip in U. If $r = P_1 \cap P_2$, then we may define the corresponding *local roof* which is the union of the portion of P_1 between b_1 and r and the portion of P_2 between b_2 and r. We say the open strip between b_1 and b_2 in U is *under* this local roof.

We next recall the definition of the bending measure on β_Γ. Let $\alpha\colon [0,1] \to \partial CH(L_\Gamma)$ be a path which is transverse to β_Γ. We say that a partition

$$0 = s_0 < s_1 < \cdots < s_{n-1} < s_n = 1$$

of $[0, 1]$ is *allowable* if each sub-arc $\alpha([s_{i-1}, s_i])$ lies under a local roof. Let P_0 be the initial support plane at $\alpha(0)$ and let P_n be the terminal support plane at $\alpha(1)$. If $0 < i < n$, let P_i be a support plane at $\alpha(s_i)$. Let θ_i be the exterior dihedral angle between P_{i-1} and P_i. We define $i_{\mathcal{P}}(\alpha, \beta_\Gamma) = \sum_{i=1}^{n} \theta_i$ and let

$$i(\alpha, \beta_\Gamma) = \inf_{\mathcal{P}} i_{\mathcal{P}}(\alpha, \beta_\Gamma)$$

where we take the infimum over all allowable partitions.

Notice that, by the definition of $i(\alpha, \beta_\Gamma)$, if α is under a local roof then we have that $i(\alpha, \beta_\Gamma) \leqslant \theta$, where θ is the exterior dihedral angle between the support planes P and Q at the points $\alpha(0)$ and $\alpha(1)$.

We begin by showing that Lemma 4.1 is valid if the path remains under a local roof. We must first recall some basic facts about ridge lines and bending lines.

LEMMA 10.2 ([7], Lemma 1.9.2). *If any ridge line meets a bending line, then they are equal. If a bending line b lies under the local roof formed by the support planes P_1 and P_2 and the bending lines b_1 and b_2, then b is either equal to or disjoint from the ridge line $r = P_1 \cap P_2$. If P is a support plane to b then P is either disjoint from the ridge line or else contains it.*

LEMMA 10.3. *If three distinct support planes P_1, P_2, and P_3 intersect in a common line l, then l is a bending line with positive bending angle.*

Proof. As support planes are oriented, consider the three normals n_1, n_2, and n_3 to the planes P_1, P_2, and P_3 at a common point $p \in l$. The normals divide the circle of planes $S(l)$ containing l into three nonempty segments. At most one can be greater than or equal to π in length. Choose the normal n with segments of length less than π on either side of it. Then the corresponding support plane P is contained in the union of the half spaces of the other two. Therefore $P \cap \partial CH(L_\Gamma) \subseteq l$. As P is a support plane, l must be a bending line with positive bending angle. □

We now prove the local version of Lemma 4.1.

LEMMA 10.4. *Let* $\alpha\colon [0,1] \to \partial CH(L_\Gamma)$ *be a geodesic path which is transverse to* β_Γ *and such that* $\alpha([0,1])$ *is contained in a neighborhood* U *adapted to* $\alpha(0)$. *Let* P *be a support plane at* $\alpha(0)$, *and let* $\{P_t \mid t \in [0,k]\}$ *be the continuous one parameter family of support planes along* α *with* $P_0 = P$. *Then*

(1) *If* $t_1 < t_2$ *and* $P_{t_1} = P_{t_2}$, *then* $P_t = P_{t_1}$ *for all* $t \in [t_1, t_2]$.
(2) *There is a* $\bar{t} \in [0,k]$ *such that* $P_t = P$ *if* $t \in [0,\bar{t}]$ *and the ridge lines* $\{r_t = P \cap P_t \mid t > \bar{t}\}$ *exist and form a monotonic family of geodesics on* P.

Proof. Suppose that $t_1 < t_2$ and $P_{t_1} = P_{t_2}$ and let $s_1 = s(t_1)$ and $s_2 = s(t_2)$. Let $F = P_{t_1} \cap \partial CH(L_\Gamma)$. Since U is simply connected and $F \cap U$ is convex, $\alpha([s_1,s_2])$ is a geodesic arc in F. If $\alpha([s_1,s_2])$ is contained in a bending line b, then α intersects b at a single point, so $s_1 = s_2$. Since α intersects b transversely, the family $\{P_t \mid s(t) = s_1\}$ sweeps out an arc in $\Sigma(b)$. In this case, $P_{t_1} = P_{t_2}$ implies that $t_1 = t_2$. If $\alpha([s_1,s_2])$ is not contained in a bending line, then $\alpha(s)$ is contained in the interior of F for all $s \in (s_1,s_2)$. Thus, if $s(t) \in (s_1,s_2)$, then $P_t = P_{t_1}$. If $\alpha(s_1)$ lies in a boundary component b of the flat, then again $\{P_t \mid s(t) = s_1\}$ sweeps out an arc in $\Sigma(b)$. This arc ends at P_{t_1}, since P_{t_1} is the terminal support plane at $\alpha(s_1)$. So, if $t > t_1$, then $s(t) > s_1$. Similarly, if $t < t_2$, then $s(t) < s_2$. Therefore, if $t \in (t_1,t_2)$, then $s(t) \in (s_1,s_2)$, so $P_t = P_{t_1}$. This establishes claim (1).

Let $\bar{t} = \sup\{t \in [0,k] \mid P_t = P_0\}$. By continuity, $P_{\bar{t}} = P_0$ and, by claim (1), $P_t = P_0$ for all $t \in [0,\bar{t}]$. By definition, if $t > \bar{t}$, then $P_t \neq P_0$ and the ridge line $r_t = P_t \cap P_0$ exists.

In order to complete the proof of claim (2), it suffices to show that if $\bar{t} < t_1 < t_2$ and $r_{t_1} \cap r_{t_2} \neq \emptyset$, then $r_t = r_{t_1}$ for all $t \in [t_1,t_2]$. As P_0 and P_{t_2} form a local roof, Lemma 10.2 implies that P_{t_1} either contains r_{t_2} or is disjoint from it. If P_{t_1} is disjoint from r_{t_2} then $r_{t_1} \cap r_{t_2} = \emptyset$. If P_{t_1} contains r_{t_2}, then $r_{t_1} = r_{t_2}$, so the support planes P_0, P_{t_1}, and P_{t_2} all contain r_{t_2}. If $P_{t_1} = P_{t_2}$, then $P_t = P_{t_1}$ for all $t \in [t_1,t_2]$, which implies that $r_t = r_{t_1}$ for all $t \in [t_1,t_2]$. If $P_{t_1} \neq P_{t_2}$, then the three planes P_0, P_{t_1}, and P_{t_2} are distinct and Lemma 10.3 implies that r_{t_1} is a bending line with positive bending angle. If r_{t_1} is a bending line with positive bending angle, then, since U is simply connected and $\alpha(s(t_1))$ and $\alpha(s(t_2))$ lie in the closure of the two flats containing r_{t_1} in their boundary, $\alpha([s(t_1),s(t_2)]$ lies in the closure of the two flats. Moreover, since P_{t_1} contains r_{t_1}, either $\alpha(s(t_1)) \in r_{t_1}$ or P_{t_1} is the the terminal support plane at $\alpha(s(t_1))$. Similarly, either $\alpha(s(t_2)) \in r_{t_1}$ or P_{t_2} is the initial support plane at $\alpha(s(t_2))$. It follows that, for all $t_1 < t < t_2$, $\alpha(s(t))$ is either contained in r_{t_1} or is contained in a flat with r_{t_1} in its boundary. Thus, for all $t_1 < t < t_2$, P_t contains r_{t_1} and, since $r_{t_1} \subset P_0$, $r_t = r_{t_1}$ for all $t \in [t_1,t_2]$. We have completed the proof of claim (2). □

We next show that if the ridge lines are monotonic, then the exterior dihedral angle is monotonically increasing. We first recall some basic facts about angles of triples of planes in $\mathbf{H}^3$.

Suppose that P_1, P_2, and P_3 are three distinct planes bounding half spaces H_1, H_2, and H_3. We also suppose that, for all i and j, P_i and P_j intersect transversely with exterior dihedral angle θ_{ij}, and that there is no common point of intersection of the three planes. In this case, there is a plane or horoball P perpendicular to all three and the intersection of the planes P_1, P_2, and P_3 with P gives lines l_1, l_2, and l_3 that intersect to form a triangle T with vertices $v_{ij} = l_i \cap l_j$. The angle of T at v_{ij} is the (exterior or interior) dihedral angle between the planes P_i and P_j.

The following general fact is established in Section 1.10 of [7].

LEMMA 10.5. *Let P_1, P_2, and P_3 be support planes to a convex set in $\mathbf{H}^3$. If the interior of the triangle T is contained in the half space H_2 and is in the complement H_1^C of H_1, then T is also in the complement H_3^C of H_3 and $\theta_{12} + \theta_{23} \leqslant \theta_{13}$* (Figure 1).

We notice that if P_1 and P_3 form a local roof with P_2 under it, then the hypotheses of Lemma 10.5 are satisfied.

LEMMA 10.6. *Let P_1, P_2, and P_3 be support planes to $\partial CH(L_\Gamma)$ with b a bending line on P_1 and ridge lines $r_1 = P_1 \cap P_2$, $r_2 = P_1 \cap P_3$. If $r_1 \cap r_2 = \emptyset$ and r_1 separates b and r_2 on P_1, then P_1, P_2, and P_3 satisfy the assumptions of Lemma* 10.5.

Proof. For $i = 1, 2, 3$, let H_i be the half-space bounded by P_i whose interior does not intersect $CH(L_\Gamma)$. Since r_1 separates b and r_2, r_2 is in the interior of H_2. Since b and r_1 are on the same side of P_3, r_1 is in the interior of H_3^C. If $P_2 \cap P_3 = \emptyset$, then, since r_2 is in the interior of H_2, P_3 is in the interior of H_2, which contradicts the fact that P_3 is a support plane. Therefore, the ridge line $r_3 = P_2 \cap P_3$ exists and the planes P_1, P_2, and P_3 describe a triangle T as above. As r_2 is in the interior of H_2, T is contained in H_2. Also, since r_1 is in the interior of H_3^C, T is contained in the complement of H_3. Therefore, P_1, P_2, and P_3 satisfy the assumptions of Lemma 10.5. □

We are now ready to analyze the situation when the ridge lines are monotonic.

LEMMA 10.7. *Let $\alpha\colon [0, 1] \to \partial CH(L_\Gamma)$ be a geodesic path which is transverse to β_Γ and let $\{P_t \mid t \in [0, k]\}$ be a continuous one-parameter family of support planes to α.*

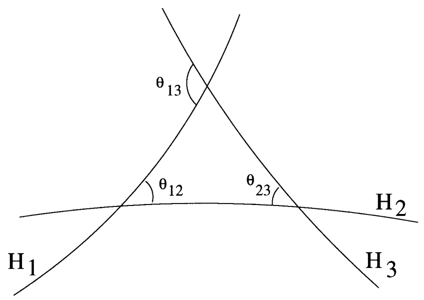

Figure 1. Configuration of planes in Lemma 10.5.

Suppose that the ridge lines $r_t = P_0 \cap P_t$ *exist for all* $t \in (0, k]$ *and form a monotonic family of geodesics. Let* $\lim_{t\to 0} r_t = b$ *where* b *is a bending line on* P_0. *Let* $t_1, t_2 \in (0, k]$ *with* $t_1 < t_2$.

(1) *If* r_{t_1} *is a bending line, then* $r_t = b$ *for* $t \in (0, t_1]$.
(2) *If* $r_{t_1} = r_{t_2}$, *then either*
$P_t = P_{t_1}$ *for all* $t \in [t_1, t_2]$ *or* $r_t = b$ *for all* $t \in (0, t_2]$.
(3) *If* $P_{t_1} = P_{t_2}$, *then* $P_t = P_{t_1}$ *for all* $t \in [t_1, t_2]$.
(4) *The exterior dihedral angle* θ_t *between* P_0 *and* P_t *is monotonically increasing.*

Proof. If r_{t_1} is a bending line b_0 and $b_0 = b$, then, by monotonicity, $r_t = b$ for all $t \in (0, t_1]$. If $b_0 \neq b$ then there must be a $t_3 \in (0, t_1)$ such that r_{t_3} separates b and b_0. Thus either b or b_0 is in the interior of H_{t_3}, the half space corresponding to P_{t_3}. This contradicts the fact that both b and b_0 are bending lines. Thus $b_0 = b$ and we have established claim (1).

Suppose that $r_{t_1} = r_{t_2}$. Then, by monotonicity, $r_t = r_{t_1}$ for all $t \in [t_1, t_2]$. Either $P_t = P_{t_1}$ for all $t \in [t_1, t_2]$ or there is some $t_3 \in (t_1, t_2]$ such that the support plane P_{t_3} is not equal to P_{t_1} or P_{t_2}. In this case, r_{t_1} is contained in the three distinct support planes P_0, P_{t_1}, and P_{t_3}. Therefore, by Lemma 10.3, r_{t_1} is a bending line b_0 on P_0. Thus, by claim (1), $b_0 = b$ and by monotonicity, $r_t = b$ for all $t \in (0, t_2]$. This establishes claim (2).

If $P_{t_1} = P_{t_2}$, then $r_{t_1} = r_{t_2}$. Therefore, by claim (2), either $P_t = P_{t_1}$ for all $t \in [t_1, t_2]$ or $r_t = b$ for all $t \in (0, t_2]$. If $r_t = b$ for all $t \in (0, t_2]$, then let X and Y be the extreme planes at b and let $s_2 = s(t_2)$. Since $\alpha([0, s_2]) \subset X \cup Y$, it intersects b only once. So, $\{P_t | t \in [t_1, t_2]\}$ sweeps out an arc in $\Sigma(b)$ joining P_{t_1} to P_{t_2}. Since $P_{t_1} = P_{t_2}$, P_t must equal P_{t_1} for all $t \in [t_1, t_2]$. Thus, in either case, $P_t = P_{t_1}$ for all $t \in [t_1, t_2]$, which is claim (3).

We now show monotonicity of θ_t. Let $t_1 \in (0, k]$, and let $s_1 = s(t_1)$. It suffices to show that there exists $\delta > 0$ such that $\theta_t \geqslant \theta_{t_1}$ for all $t \in [t_1, t_1 + \delta]$.

If $\alpha(s_1)$ is in a flat then there is some $\delta > 0$ so that $P_t = P_{t_1}$ for all $t \in [t_1, t_1 + \delta]$ and therefore $\theta_t = \theta_{t_1}$ for all $t \in [t_1, t_1 + \delta]$ which completes the proof in this case.

Now suppose that $\alpha(s_1)$ is contained in a bending line b_1 and $t_2 > t_1$. If $P_{t_1} = P_{t_2}$ then $\theta_{t_2} = \theta_{t_1}$. If $P_{t_1} \neq P_{t_2}$ and $r_{t_1} = r_{t_2}$ then, by claim (2), $r_t = b$ for all $t \in (0, t_2]$. Thus the support planes $\{P_t | t \in [0, t_2]\}$ sweep out an arc in $\Sigma(b)$ which begins at P_0, and again $\theta_{t_1} \leqslant \theta_{t_2}$. Therefore, $\theta_{t_1} \leqslant \theta_{t_2}$ if $P_{t_1} = P_{t_2}$ or $r_{t_1} = r_{t_2}$.

If $P_{t_1} \neq P_{t_2}$ and $r_{t_1} \neq r_{t_2}$, then, by monotonicity, either r_{t_1} separates b and r_{t_2}, or $r_{t_1} = b$. If r_{t_1} separates b and r_{t_2}, then we apply Lemma 10.6 to the support planes P_0, P_{t_1}, and P_{t_2} to see that $\theta_{t_1} \leqslant \theta_{t_2}$. By combining the above, we see that if $r_{t_1} \neq b$, then $\theta_{t_1} \leqslant \theta_{t_2}$ for all $t_2 \in [t_1, k]$.

If $r_{t_1} = b$, then, by monotonicity, $r_t = b$ for all $t \in [0, t_1]$. As $P_{t_1} \neq P_0$, b has positive bending angle. If b is the bending line b_1 which contains $\alpha(s_1)$, then we may choose $\delta > 0$ such that if $t \in [t_1, t_1 + \delta]$, then P_t is a support plane to b. This implies that if $t_2 \in [t_1, t_1 + \delta)$, then $r_{t_1} = r_{t_2} = b$. We saw above that this implies that $\theta_{t_1} \leqslant \theta_{t_2}$.

If $b \neq b_1$, we choose a neighborhood N of b_1 in the space of geodesics so that no geodesic in N intersects P_0. Lemma 10.1 assures us that we can choose $\delta > 0$ such that if $t_2 \in [t_1, t_1 + \delta)$ and $P_{t_2} \neq P_{t_1}$, then $r_{t_1,t_2} = P_{t_1} \cap P_{t_2} \subset N$. If $P_{t_1} = P_{t_2}$ or $r_{t_1} = r_{t_2}$, then we have previously shown that $\theta_{t_2} \geqslant \theta_{t_1}$. If $P_{t_1} \neq P_{t_2}$ and $r_{t_1} \neq r_{t_2}$, then $r_{t_1,t_2} \subseteq N$, so r_{t_1,t_2} is in the interior of H_0^C. Furthermore, b does not lie in P_{t_2}, so b is in the interior of $H_{t_2}^C$. In order to apply Lemma 10.5 to the half-spaces H_0, H_{t_1} and H_{t_2}, we need to show that r_{t_2} is in the interior of H_{t_1}. To do this we apply a simple continuity argument. Since $r_{t_2} \neq r_{t_1} = b$, there is a $t_3 \in [t_1, t_2]$ such that r_{t_3} separates b and r_{t_2}. Thus r_{t_2} is in the interior of H_{t_3}. Moreover, if $t \in [t_1, t_3]$, then $r_t \cap r_{t_2} = \emptyset$. So, for all $t \in [t_1, t_3]$, $P_t \cap r_{t_2} = \emptyset$. We consider the half spaces H_t for all $t \in [t_1, t_3]$. As r_{t_2} is in the interior of H_{t_3} and $P_t \cap r_{t_2} = \emptyset$ for all $t \in [t_1, t_3]$, then, by continuity, r_{t_2} is in the interior of H_t for all $t \in [t_1, t_3]$. In particular, r_{t_2} is in the interior of H_{t_1}. Lemma 10.5 then gives that $\theta_{t_1} \leqslant \theta_{t_2}$. So, if $r_{t_1} = b$, we have seen that there exists $\delta > 0$ such that $\theta_{t_1} \leqslant \theta_{t_2}$ for all $t_2 \in [t_1, t_1 + \delta]$. This completes the proof that θ_t is monotonic. □

We are now ready to establish Lemma 4.1, which we restate here for reference.

LEMMA 4.1. *Let $N = \mathbf{H}^3/\Gamma$ be an analytically finite hyperbolic 3-manifold such that L_Γ is not contained in a round circle. Let $\alpha\colon [0, 1] \to \partial CH(L_\Gamma)$ be a geodesic arc, in the intrinsic metric on $\partial CH(L_\Gamma)$, which is transverse to β_Γ. If (P, Q) is a roof over α, and $\{P_t |\ t \in [0, k]\}$ is the continuous one-parameter family of support planes over α joining P to Q, then*

(1) *$i(\alpha, \beta_\Gamma)_P^Q \leqslant \theta < \pi$, where θ is the exterior dihedral angle between P and Q, and*
(2) *there is a $\bar{t} \in [0, k]$ such that $P_t = P$ if $t \in [0, \bar{t}]$ and the ridge lines $\{r_t = P \cap P_t | t > \bar{t}\}$ exist and form a monotonic family of geodesics on P.*

Proof. We first prove claim (2), that the ridge lines are monotonic. Let $\{P_t |\ \in [0, k]\}$ be the continuous one-parameter family of support planes along α from P to Q. We let H_t be the half-space bounded by P_t and let D_t be the closed disk in $\hat{\mathbf{C}}$ associated to P_t.

Since (P, Q) is a roof over α, $P_0 \cap P_t \neq \emptyset$ for all $t \in [0, k]$. Let $\bar{t}$ be the maximum value such that $P_t = P_0$ for all $t \in [0, \bar{t}]$. If $\bar{t} = k$, then claim (2) is trivially true.

Consider the case when $\bar{t} < k$. Let $\bar{s} = s(\bar{t})$, then $\alpha([0, \bar{s}]) \subseteq P_0$. If $\alpha(\bar{s})$ is in a flat, we obtain a contradiction to the maximality of $\bar{t}$. So, $\alpha(\bar{s})$ is on a bending line b. Let U be adapted for $\alpha(\bar{s})$ and choose $k_1 > \bar{t}$ so that $\alpha([\bar{t}, k_1]) \subset U$. By Lemma 10.4, the ridge lines $\{r_t\}$ for $t \in (\bar{t}, k_1]$ are well-defined and monotonic. Also by continuity $\lim_{t \to \bar{t}^+} r_t = b$. Thus, if we define $r_{\bar{t}} = b$, we obtain a monotonic family of geodesics $\{r_t\}$ for $t \in [\bar{t}, k_1]$.

Since (P, Q) is a roof over α, if $P_0 \cap P_t$ is not a ridge line then $P_t = P_0$. Let T be the maximum value such that the ridge lines r_t exist and give a monotonic family of geodesics for $t \in (\bar{t}, T)$. Since $P_T \cap P_0 \neq \emptyset$, either $P_T = P_0$ or $r_T = P_T \cap P_0$ is a ridge line.

By Lemma 10.7, the angle θ_t is an increasing function on $(\bar{t}, T)$. Since $\theta_t \in (0, \pi)$ for all $t \in (\bar{t}, T)$, we see that if $P_T = P_0$ then $\theta_T = \pi$ and H_T has disjoint interior from H_0. This contradicts our assumption that (P, Q) is a roof for α.

Thus we can assume that the ridge line r_T exists. Then, by continuity, the family of geodesics $\{r_t | \, t \in (\bar{t}, T]\}$ is monotonic. If $T = k$, claim (2) holds. So assume that $T < k$.

Let $\bar{T}$ be the minimum value in $[\bar{t}, T]$ such that $P_{\bar{T}} = P_T$. Thus, since r_t is monotonic on $(\bar{t}, T)$, Lemma 10.7 implies that $P_t = P_T$ for all $t \in [\bar{T}, T]$.

We now consider the ridge lines $r_t^T = P_t \cap P_T$. By the choice of $\bar{T}$ there is some $\delta_1 > 0$ such that r_t^T is a ridge line for $t \in (\bar{T} - \delta_1, \bar{T})$. We define $b_T^- = \lim_{t \to \bar{T}^-} r_t^T$. Similarly, by our choice of T, there is some $\delta_2 > 0$ such that r_t^T is a ridge line for $t \in (T, T + \delta_2)$. We define $b_T^+ = \lim_{t \to T^+} r_t^T$. Then b_T^+ and b_T^- are both bending lines (possibly equal) on the support plane P_T. By definition of T, $\alpha(s(T)) \in b_T^+$.

As bending lines do not intersect, either $b_T^+ = b_T^-$ or they are disjoint geodesics on P_T. By Lemma 10.2, if a bending line intersects a ridge line they must be equal, so neither b_T^+ nor b_T^- transversely intersect r_T.

We will establish a contradiction by finding a $\delta > 0$ such that r_t is monotonic on $[\bar{t}, T + \delta)$. We first show that there is a $\delta > 0$ so that r_t is monotonic on $(T - \delta, T + \delta)$.

If b_T^+ intersects P_0 then, since ridge lines are equal or disjoint, by Lemma 10.2, $r_T = b_T^+$. Therefore, by Lemma 10.7, $r_t = b$ for all $t \in [\bar{t}, T]$. As $P_T \neq P_0$, b has a positive bending angle. Therefore, there exists $\delta > 0$ such that P_t is a support plane to $b_T^+ = b$ for all $t \in (T - \delta, T + \delta)$. Therefore, $r_t = b$ for all $t \in (T - \delta, T + \delta)$ and is thus trivially monotonic on this region (Figure 2).

If b_T^+ does not intersect P_0, choose a neighborhood N of b_T^+ so that every geodesic in N does not intersect P_0. Let U be adapted for $\alpha(s(T))$ so that the ridge line associated to any two support planes to U lies in N. Finally, we choose $\delta > 0$ so that $\alpha([s(T - \delta), s(T + \delta)]) \subset U$. If $t_1, t_2 \in (T - \delta, T + \delta)$ and $r_{t_1} \cap r_{t_2} \neq \emptyset$, then P_{t_1} must equal P_{t_2}, since otherwise $r_{t_1, t_2} \subset N$ contains a point of P_0. In this case, by Lemma 10.4,

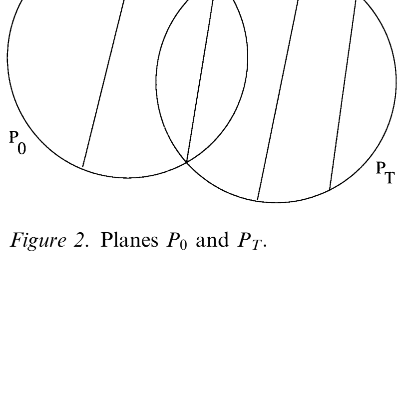

Figure 2. Planes P_0 and P_T.

we have that $P_t = P_{t_1}$ for all $t \in [t_1, t_2]$, so $r_t = r_{t_1}$ for all $t \in [t_1, t_2]$. Since $r_t = r_{t_1}$ for all $t \in [t_1, t_2]$ whenever $r_{t_1} \cap r_{t_2} \neq \emptyset$ and $t_1, t_2 \in (T - \delta, T + \delta)$, $\{r_t\}$ is monotonic on $(T - \delta, T + \delta)$.

We now know that there exists $\delta > 0$ such that $\{r_t\}$ is monotonic on $(\bar{t}, T]$ and on $(T - \delta, T + \delta)$. If $\{r_t\}$ is nonconstant on $(T - \delta, T]$, then $\{r_t\}$ is monotonic on $[\bar{t}, T + \delta)$ and we have completed the proof of claim (2). Otherwise, by Lemma 10.7, either $r_t = b$ for all $t \in [\bar{t}, T]$ or $P_t = P_T$ for all $t \in (T - \delta, T]$. If $r_t = b$ for all $t \in [\bar{t}, T]$, then $\{r_t\}$ is clearly monotonic on $[\bar{t}, T + \delta)$ and we are again done.

If $P_t = P_T$ for all $t \in (T - \delta, T]$, then $T \neq \bar{T}$ and b_T^+ and b_T^- must be disjoint. We may then choose neighborhoods N^+ and N^- of b_T^+ and b_T^-, such that no geodesic in N^+ intersects any geodesic in N^- and no geodesic in N^+ or N^- intersects P_0. We choose $\delta_1 > 0$ so that if $t_1, t_2 \in (\bar{T} - \delta_1, \bar{T}]$ then P_{t_1} and P_{t_2} are either equal or their intersection is in N^-. Also we choose $\delta_2 > 0$ so that if $t_1, t_2 \in [T, T + \delta_2)$, then P_{t_1} and P_{t_2} are either equal or their intersection is in N^+. Let $\delta_0 = \min(\delta_1, \delta_2, \delta)$.

We first show that b_T^- separates r_T from b_T^+. By the definition of $\bar{T}$, $r_t \neq r_T$ for any $t \in (\bar{T} - \delta_0, \bar{T})$. Thus, r_t separates b and r_T in P_0. So r_T is in the interior of H_t for any $t \in (\bar{T} - \delta_0, \bar{T})$. Since b_T^+ and b_T^- are bending lines they are on the same side of r_t^T in P_T. Thus b_T^+ is in the interior of H_t^C. Therefore r_t^T separates r_T and b_T^+ in P_T. Since r_t^T tends to b_T^- as $t \to \bar{T}^-$, b_T^- separates r_T and b_T^+.

If $r_{t_1} = r_T$ for some $t_1 \in (T, T + \delta_0)$, then, by the monotonicity of $\{r_t\}$ on $(T - \delta_0, T + \delta_0)$, $r_t = r_T$ for all $t \in [T, t_1)$ which would imply that r_t is monotonic on $[\bar{t}, t_1)$, which would contradict the maximality of T. Suppose that $t \in (T, T + \delta_0)$. Since b_T^- separates r_T and b_T^+ in P_T and r_t^T lies in N^+, b_T^- separates r_T and r_t^T in P_T. Thus, r_T is in the interior of H_t^C. If r_t separates b from r_T on P_0, then b is in the interior of H_t. This contradicts the fact that b is a bending line. Thus, for all $t \in (T, T + \delta_0)$, r_T separates b and r_t on P_0. Therefore, $\{r_t\}$ is monotonic on $(\bar{t}, T + \delta_0)$. This completes the proof of claim (2).

We now prove claim (1) by induction on the number of local roofs. If (P, Q) is a local roof over α then $i(\alpha, \beta_\Gamma)_P^Q \leqslant \theta < \pi$ by the definition of intersection number. Assume now that we have established claim (1) for any arc which is covered by $n - 1$ local roofs and that α is covered by n local roofs with the ith having boundary support planes $P_{t_{i-1}}$ and P_{t_i}, so that $P_{t_0} = P_0$ and $P_{t_n} = P_k$. Let $\theta_{i,j}$ be the exterior dihedral angle between P_{t_i} and P_{t_j} and let $r_{t_i} = P_0 \cap P_{t_i}$. It follows from the definition of the bending measure and our inductive assumption, that

$$i(\alpha, \beta_\Gamma)_P^Q \leqslant \theta_{0,n-1} + \theta_{n-1,n}.$$

If $P_{t_n} = P_{t_{n-1}}$, then $\theta_{n-1,n} = 0$ and so

$$i(\alpha, \beta_\Gamma)_P^Q = i(\alpha, \beta_\Gamma)_P^{P_{n-1}} \leqslant \theta_{0,n-1} = \theta$$

If $P_{t_n} \neq P_{t_{n-1}}$, then we consider the ridge lines $r_{t_{n-1}}$ and r_{t_n}. If $r_{t_{n-1}} = r_{t_n}$ then, as $P_{t_n} \neq P_{t_{n-1}}$, Lemma 10.7 implies that $r_t = b$ for all $t \in (\bar{t}, k]$. Thus, $\{P_t |\ t \in [0, k]\}$ sweeps out an arc in $\Sigma(b)$ with total angle θ and $i(\alpha, \beta_\Gamma)_P^Q = \theta$.

If $r_{t_{n-1}} \neq r_{t_n}$ then either $r_{t_{n-1}}$ separates b and r_{t_n} or $r_{t_{n-1}} = b$. If $r_{t_{n-1}}$ separates b and r_{t_n} then, by Lemma 10.6, the half-spaces H_0, $H_{t_{n-1}}$, and H_{t_n} satisfy Lemma 10.5, so $\theta_{0,n-1} + \theta_{n-1,n} \leqslant \theta_{0,n}$ and therefore $i(\alpha, \beta_\Gamma)_P^Q \leqslant \theta_{0,n} = \theta$.

Now consider the case with $r_{t_{n-1}} \neq r_{t_n}$ and $r_{t_{n-1}} = b$. Let $r = P_{t_{n-1}} \cap P_{t_n}$. To apply Lemma 10.5, we need to show that b is in the interior of $H_{t_n}^C$ and that r_{t_n} is in the interior of $H_{t_{n-1}}$. Since b is a bending line which does not meet H_{t_n}, b lies in the interior of $H_{t_n}^C$. Since $r_{t_{n-1}} \neq r_{t_n}$, Lemma 10.7 implies that r_{t_n} is not a bending line. Choose $t_a \in [t_{n-1}, t_n]$, such that r_{t_a} separates b and r_{t_n}. Thus, r_{t_n} is in the interior of H_{t_a} and for all $t \in [t_{n-1}, t_a]$, $r_t = P_t \cap P_0$ is between b and r_{t_a}, so $P_t \cap r_{t_n} = \emptyset$. Considering the half-spaces H_t for $t \in [t_{n-1}, t_a]$, we note that r_{t_n} is in the interior of H_{t_a} and $P_t \cap r_{t_n} = \emptyset$ for all $t \in [t_{n-1}, t_a]$. Therefore, by continuity, r_{t_n} is in the interior of H_t for all $t \in [t_{n-1}, t_a]$. In particular, r_{t_n} is in the interior of $H_{t_{n-1}}$. Applying Lemma 10.5 we have that $\theta_{0,n-1} + \theta_{n-1,n} \leqslant \theta_{0,n}$ and therefore, again, $i(\alpha, \beta_\Gamma)_P^Q \leqslant \theta_{0,n} = \theta$. □

References

1. Ahlfors, L. V.: Finitely generated Kleinian groups, *Amer. J. Math.* **86** (1964), 413–429.
2. Bonahon, F.: Bouts des variétés hyperboliques de dimension 3, *Ann. of Math.* **124** (1986), 71–158.
3. Bonahon, F.: The geometry of Teichmüller space via geodesic currents, *Invent. Math.* **92** (1988), 139–162.
4. Bridgeman, M.: Average bending of convex pleated planes in hyperbolic three-space, *Invent. Math.* **132** (1998), 381–391.
5. Buser, P.: *Geometry and Spectra of Compact Riemann Surfaces*, Birkhäuser, Basel, 1992.
6. Canary, R. D.: The conformal boundary and the boundary of the convex core, *Duke Math. J.* **106** (2001), 193–207.
7. Epstein, D. B. A. and Marden, A.: Convex hulls in hyperbolic space, a theorem of Sullivan, and measured pleated surfaces, In: *Analytical and Geometrical Aspects of Hyperbolic Space*, Cambridge Univ. Press, 1987, pp. 113–253.
8. Epstein, D. B. A., Marden, A. and Markovic, V.: Quasiconformal homeomorphisms and the convex hull boundary, Preprint.
9. Epstein, D. B. A., Marden, A. and Markovic, V.: Complex angle scaling, Preprint.
10. Epstein, D. B. A. and Markovic, V.: The logarithmic spiral: a counterexample to the $K = 2$ conjecture, Preprint.
11. McMullen, C. T.: Complex earthquakes and Teichmüller theory, *J. Amer. Math. Soc.* **11** (1998), 283–320.
12. Sigmund, K.: On dynamical systems with the specification property, *Trans. Amer. Math. Soc.* **190** (1974), 285–299.
13. Thurston, W. P.: The geometry and topology of 3-manifolds, Lecture Notes, Princeton University, 1979.
14. Thurston, W. P.: Minimal stretch maps between hyperbolic surfaces, Preprint, available at: http://front.math.ucdavis.edu/math.GT/9801058.