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Integral Equations and Operator Theory

Spectrum of the Kerzman–Stein Operator for Model Domains

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Abstract. For a domain $\Omega \subset \mathbb{C}$, the Kerzman-Stein operator is the skew-hermitian part of the Cauchy operator acting on $L^2(b\Omega)$, which is defined with respect to Euclidean measure. In this paper we compute the spectrum of the Kerzman-Stein operator for three domains whose boundaries consist of two circular arcs: a strip, a wedge, and an annulus. We also treat the case of a domain bounded by two logarithmic spirals.

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1. Introduction

For a smooth domain $\Omega \subset\subset \mathbb{C}^n$, Kerzman and Stein studied in [8] a certain compact operator A in relation to the Szegő projection. Let C be the Cauchy operator on Ω, defined for an integrable function f according to

$$Cf(z) = \frac{1}{2\pi i}\int_{b\Omega}\frac{f(w)\,dw}{w-z} \quad \text{for } z \in \Omega,$$

then using the nontangential limit of the integral for $Cf(z)$ when $z \in b\Omega$. For rather general domains (eg., piecewise-smooth boundary), C is a bounded projection from $L^2(b\Omega)$ onto the Hardy space $H^2(b\Omega)$. Moreover, if the boundary is smooth and has finite length, then the operator $A = C - C^*$ is compact, and the Szegő projection can be expressed as the composition of bounded operators $S = C(I+A)^{-1}$.[1]

A problem, suggested by Kerzman in [7], is to relate the spectrum of A to the geometry of $b\Omega$. It is known that A is identically zero precisely when the boundary

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[1]The Szegő projection is the *orthogonal* projection from L^2 to H^2, where H^2 is the space of square-integrable functions that extend holomorphically to Ω. A measures how close is C to being the Szegő projection.

has constant curvature. This happens since the Kerzman-Stein kernel, defined by

$$A(z,w) = \frac{1}{2\pi i}\left[\frac{T_w}{w-z} - \frac{\overline{T}_z}{\overline{w}-\overline{z}}\right] \text{ for } z,w \in b\Omega,$$

is zero precisely when there exists a circular arc that is tangent to the curve at both points. Here, T_w is the unit tangent vector at $w \in b\Omega$, so if the arclength measure is ds, then $dw = T_w\, ds_w$.

Here we extend the result and compute the spectrum of A for domains bounded by either two circular arcs, or by two loxodromes that meet only at their poles. Since $A(z,w)$ vanishes for z and w on a common circular arc, it follows that A acts diagonally for the circular arc domains—it maps a function supported on one circular arc to a function supported on the other. For the domains bounded by two loxodromes, A can be expressed using convolutions in the inversive arclength coordinates. If the domain is bounded by more than two arcs, then A is a sum of operators, and the interference prevents us from writing A in a concise fashion.

Since A is Mobius invariant we restrict our attention to the strip, wedge, annulus, and logarithmic sector. In [12], Singh treated the case of an annulus for a problem that is equivalent to ours. For completeness we include that example here as well. The equivalence between problems was outlined by Burbea in [2].

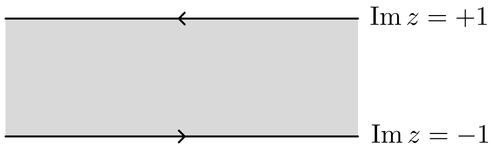
There is also closely related work by Feldman, Krupnik, and Spitkovsky in [4]. They computed the norm of the Cauchy singular operator for contours that include the case of finitely many parallel lines or concentric circles. In those situations, they obtain matrix representations that are essentially the same as ones given here for the strip and annulus. Moreover, there is a general relationship between the norm of a projection operator and the norm of its skew-hermitian part. This is described by Gerisch in [5], for instance.

We remark that there is another general symmetry in the kernel, namely, $A(z,w) = \overline{A(z,w)}\, T_w\, \overline{T_z}$. From this it follows that if f is an eigenfunction of A corresponding to $+i\lambda$, then $\overline{f}\,\overline{T}$ is an eigenfunction corresponding to $-i\lambda$. In general, the imaginary spectrum of A is symmetric with respect to the origin. A final symmetry occurs between a domain and its complement. The spectrum is the same for each, since reversing the orientation of the boundary corresponds to multiplying $A(z,w)$ by -1.

We hope that the examples given here will help initiate a study of the spectrum of A which is analagous to the theory for the Fredholm eigenvalues. See the work of Schiffer and others in [1, 10, 11, 13], for instance.

2. Strip

The boundary of the strip consists of the two lines $\operatorname{Im} z = \pm 1$, which can be parameterized by $s \in \mathbb{R} \to s \pm i$, respectively. Notice that if $z = s \pm i$, then $T_z = \mp 1$. We identify f and $g = Af \in L^2(b\Omega)$ with the vectors $\begin{bmatrix} f_+ \\ f_- \end{bmatrix}, \begin{bmatrix} g_+ \\ g_- \end{bmatrix} \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$, where, for instance, $f_\pm(s) = f(s \pm i)$ for $s \in \mathbb{R}$.



Since the kernel $A(z, w)$ vanishes if z and w belong to the same line, the value of $g_\pm(s)$ depends only on the values of $f_\mp(t)$ for $t \in \mathbb{R}$, respectively. We then compute,

$$g_\pm(s) = \frac{1}{2\pi i}\int_{\mathbb{R}}\left(\frac{\pm 1}{(t \mp i) - (s \pm i)} - \frac{\mp 1}{(t \pm i) - (s \mp i)}\right) f_\mp(t)\, dt$$
$$= \frac{\pm i}{\pi}\int_{\mathbb{R}}\left(\frac{s-t}{(s-t)^2+4}\right) f_\mp(t)\, dt.$$

So A involves convolution with the function $k(s) = (\pm i/\pi)\, s\,(s^2+4)^{-1}$. The transform of this function can be found using residues, but since the integral is standard, we refer to formula 3.723–3 from [6], page 418, and find

$$\frac{\pm i}{\pi}\int_{\mathbb{R}} \frac{s}{s^2+4}\, e^{-is\xi}\, ds = \frac{\pm 2}{\pi}\int_0^\infty \frac{s\cdot \sin(s\xi)}{s^2+4}\, ds = \pm \operatorname{sgn} \xi \cdot e^{-2|\xi|}.$$

Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ be the Fourier and inverse Fourier transforms, defined for $h \in L^2(\mathbb{R})$ by

$$\mathcal{F}\, h(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} h(s)\, e^{-is\xi}\, ds,$$

and

$$\mathcal{F}^{-1}h(s) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} h(\xi)\, e^{+is\xi}\, d\xi.$$

If $\phi \in L^2(\mathbb{R})$ is the real function $\phi(\xi) = \operatorname{sgn}\xi \cdot e^{-2|\xi|}$, then the Kerzman-Stein operator acts according to

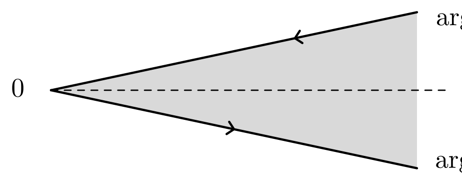
$$\begin{bmatrix}\mathcal{F}g_+\\ \mathcal{F}g_-\end{bmatrix} = \begin{bmatrix} 0 & +\phi \\ -\phi & 0\end{bmatrix}\cdot\begin{bmatrix}\mathcal{F}f_+\\ \mathcal{F}f_-\end{bmatrix} = \begin{bmatrix}+\phi\,\mathcal{F}f_-\\ -\phi\,\mathcal{F}f_+\end{bmatrix},$$

where $Ag = f$, and we are using the identifications of f, g with $f_\pm, g_\pm$ described above.

For a fixed $\xi \in \mathbb{R}$, the eigenvalues of the 2×2 matrix are $\pm i|\phi(\xi)|$, so the continuous spectrum of A is precisely the range of these two functions, that is, the interval along the imaginary axis $[-i, +i]$. The L^2-operator norm of A is precisely $\max_\xi |\phi(\xi)| = 1$, and since ϕ vanishes only at zero, the nullspace of A is trivial.

3. Wedge

The boundary of the wedge consists of the rays $\arg z = \pm\theta$ $(0 < \theta < \pi/2)$, which can be parameterized by $s \in \mathbb{R}^+ \to s\, e^{\pm i\theta}$, respectively. Notice that if $z = s\, e^{\pm i\theta}$, then $T_z = \mp e^{\pm i\theta}$. We identify f and $g = Af \in L^2(b\Omega)$ with the



vectors $\begin{bmatrix} f_+ \\ f_- \end{bmatrix}, \begin{bmatrix} g_+ \\ g_- \end{bmatrix} \in L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^+)$, where, for instance, $f_\pm(s) = f(s\,e^{\pm i\theta})$ for $s \in \mathbb{R}^+$.

Then, since the kernel $A(z,w)$ vanishes when z and w belong to the same line, the value of $g_\pm$ depends only on the values of $f_\mp(t)$ for $t \in \mathbb{R}^+$, respectively. We compute,

$$\begin{aligned} g_\pm(s) &= \frac{1}{2\pi i}\int_0^\infty \left(\frac{\pm e^{\mp i\theta}}{t\,e^{\mp i\theta} - s\,e^{\pm i\theta}} - \frac{\mp e^{\mp i\theta}}{t\,e^{\pm i\theta} - s\,e^{\mp i\theta}} \right) f_\mp(t)\,dt \\ &= \frac{\pm e^{\mp i\theta}\cos\theta}{\pi i}\int_0^\infty \frac{t-s}{s^2+t^2-2st\cos 2\theta}\, f_\mp(t)\,dt. \end{aligned}$$

Next, the change of coordinate $s = e^u$ induces an isometry $\Lambda : L^2(\mathbb{R}^+) \to L^2(\mathbb{R})$ given by $h \to h'$ where $h'(u) = h(e^u)\,e^{u/2}$. In terms of the new coordinate,

$$\begin{aligned} & g_\pm(e^u)e^{u/2} \\ &\quad = \frac{\pm e^{\mp i\theta}\cos\theta}{\pi i}\int_{-\infty}^{+\infty} \frac{e^v - e^u}{e^{2u}+e^{2v}-2\,e^{u+v}\cos 2\theta}\cdot e^{(u+v)/2}\cdot f_\mp(e^v)e^{v/2}\,dv, \end{aligned}$$

and the kernel is again a convolution kernel since

$$\frac{(e^v - e^u)\cdot e^{(u+v)/2}}{e^{2u}+e^{2v}-2\,e^{u+v}\cos 2\theta} = \frac{-\sinh[(u-v)/2]}{\cosh(u-v)-\cos 2\theta}.$$

In particular, A involves convolution with the function

$$k(u) = \frac{\pm e^{\mp i\theta}\cos\theta}{\pi i}\,\frac{-\sinh(u/2)}{\cosh u - \cos 2\theta},$$

whose transform can be found using residues, or by referring to formula 3.984–3 from [6], page 506. We find

$$\frac{\pm e^{\mp i\theta}\cos\theta}{\pi i}\int_{\mathbb{R}} \frac{-\sinh(u/2)}{\cosh u - \cos 2\theta}\,e^{-iu\xi}\,du = \frac{\pm e^{\mp i\theta}\sinh[\xi(\pi-2\theta)]}{\cosh(\xi\pi)} \quad \text{for } \xi \in \mathbb{R}.$$

So if $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier and inverse Fourier transforms, as in the previous section, and if $\phi \in L^2(\mathbb{R})$ is given by $\phi(\xi) = e^{-i\theta}\sinh[\xi(\pi-2\theta)]/\cosh(\xi\pi)$, then the Kerzman-Stein operator acts according to

$$\begin{bmatrix} \mathcal{F}\Lambda g_+ \\ \mathcal{F}\Lambda g_- \end{bmatrix} = \begin{bmatrix} 0 & +\phi \\ -\overline{\phi} & 0 \end{bmatrix}\cdot\begin{bmatrix} \mathcal{F}\Lambda f_+ \\ \mathcal{F}\Lambda f_- \end{bmatrix} = \begin{bmatrix} +\phi\,\mathcal{F}\Lambda f_- \\ -\overline{\phi}\,\mathcal{F}\Lambda f_+ \end{bmatrix},$$

where $Ag = f$, and we are using the identifications of f, g with $f_\pm, g_\pm$, and the isometry Λ, described above.

Again, for fixed $\xi \in \mathbb{R}$, the eigenvalues of the 2×2 matrix are $\pm i\,|\phi(\xi)|$, so the continuous spectrum of A is precisely the range of these two functions. In Figure 1 we show a graph of the real-valued function $(\xi, \theta) \to e^{i\theta}\phi(\xi)$. Since ϕ vanishes

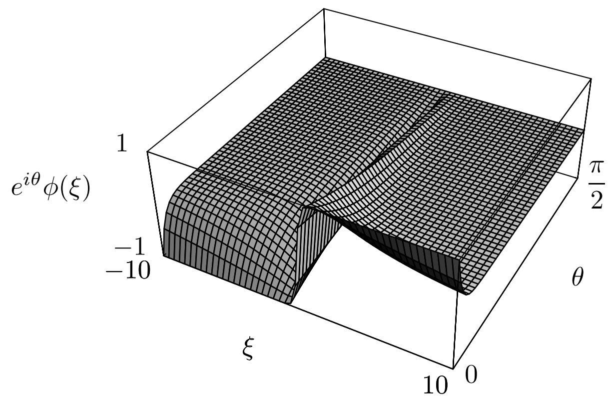


FIGURE 1. Graph of $(\xi, \theta) \to e^{i\theta}\phi(\xi)$ for the wedge, $0 < \theta < \pi/2$.

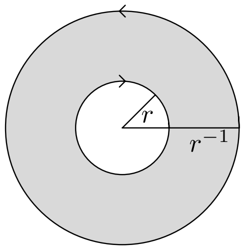
only at zero, it follows that the null-space of A is trivial for all θ. We also point out that the L^2 operator norm of A approaches 0 as $\theta \to \pi/2$, and it approaches 1 as $\theta \to 0$.

There is an interesting continuity property suggested by these limiting cases. As $\theta \to \pi/2$, the wedge tends toward a half-plane, and as mentioned above, the norm for the wedge tends to 0, which is precisely the norm for a half-plane. Likewise, as $\theta \to 0$, the wedge tends toward a strip, and as mentioned above, the norm for the wedge tends to 1, which is precisely the norm for a strip. The manner in which the wedge becomes a strip can be pictured as follows—for $\theta > 0$, let the vertex V_θ of the wedge lie on the negative real axis, so that the boundary of the wedge also passes through $\pm i$. Then as $\theta \to 0$, we have $V_\theta \to -\infty$, and the wedge tends toward a strip.

4. Annulus

The boundary of the annulus consists of the circles $|z| = r^{\pm 1}$ $(0 < r < 1)$, which are parameterized by $s \in [0, 2\pi] \to e^{is}r^{\pm 1}$, respectively. If $z = e^{is}r^{\pm 1}$, then $T_z = \mp i\, e^{is}$. We identify f and $g = Af \in L^2(b\Omega)$ with the vectors $\begin{bmatrix} f_+ \\ f_- \end{bmatrix}, \begin{bmatrix} g_+ \\ g_- \end{bmatrix} \in L^2([0, 2\pi]) \times L^2([0, 2\pi])$, where $f_\pm(s) = f(e^{is}r^{\pm 1})$ for $s \in [0, 2\pi]$.

The kernel $A(z, w)$ vanishes when z and w belong to the same circle, so the value of $g_\pm(s)$ depends only on the values of $f_\mp(t)$ for $t \in [0, 2\pi]$, respectively. We



compute,

$$g_\pm(s) \;=\; \frac{1}{2\pi i}\int_0^{2\pi}\left(\frac{\pm i\, e^{it}}{e^{it}r^{\mp 1}-e^{is}r^{\pm 1}} - \frac{\pm i\, e^{-is}}{e^{-it}r^{\mp 1}-e^{-is}r^{\pm 1}}\right) f_\mp(t)\,(r^{\mp 1}dt).$$

The kernel in this expression can be rewritten as

$$\frac{\pm r^{\mp 1}}{2\pi}\left[\frac{1}{r^{\mp 1}-e^{i(s-t)}r^{\pm 1}} + \frac{1}{r^{\pm 1}-e^{i(s-t)}r^{\mp 1}}\right] = \frac{\pm r^{\mp 1}}{2\pi}\sum_{j=0}^{\infty} r^{2j+1}\Big[e^{ij(s-t)} - e^{-i(j+1)(s-t)}\Big].$$

For $n \geq 0$, consider the function $f^{n,1} \in L^2(b\Omega)$ given by $f^{n,1}_\pm(s) = e^{\pm i\pi/4}\, r^{\mp 1/2}\, e^{ins}$. We find

$$g_\pm(s) \;=\; \frac{\pm r^{\mp 1}}{2\pi}\int_0^{2\pi}\sum_{j=0}^{\infty} r^{2j+1}\Big[e^{ij(s-t)} - e^{-i(j+1)(s-t)}\Big]\, e^{\mp i\pi/4} r^{\pm 1/2}\, e^{int}\, dt$$
$$= \pm r^{\mp 1}\, r^{2n+1}\, e^{\mp i\pi/4}\, r^{\pm 1/2}\, e^{ins} \;=\; -i\, r^{2n+1}\, f^{n,1}_\pm(s),$$

so $f^{n,1}$ is an eigenvector of A corresponding to the eigenvalue $-i\, r^{2n+1}$.

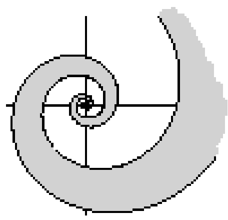
Having found one eigenfunction, the symmetry of the annulus provides another. In general, linear transformations $\mu : \Omega \to \Omega'$ induce isometries $L^2(b\Omega) \leftarrow L^2(b\Omega')$ according to $h(\mu(z))\sqrt{\mu'(z)} \leftarrow h = h(z)$, and these isometries commute with $A = A_{b\Omega}$ and $A_{b\Omega'}$. So for an annulus, the map $\mu(z) = 1/z$ determines an eigenfunction $f^{n,2}$ according to $f^{n,2}(z) = f^{n,1}(1/z)\sqrt{1/z^2}$. Then, using these eigenfunctions, we find two more using the involution described in the introduction. In particular, $f^{n,3} = \overline{f^{n,1}\,T}$ and $f^{n,4} = \overline{f^{n,2}\,T}$ are eigenfunctions corresponding to $+i\, r^{2n+1}$. For $n \geq 0$ these are all recorded in the following table.

Eigenvalue	Eigenfunction
$-i\, r^{2n+1}$	$f^{n,1}_\pm(s) \;=\; e^{\pm i\pi/4}\, r^{\mp 1/2}\, e^{ins}$
$-i\, r^{2n+1}$	$f^{n,2}_\pm(s) \;=\; e^{\mp i\pi/4}\, r^{\mp 1/2}\, e^{-i(n+1)s}$
$+i\, r^{2n+1}$	$f^{n,3}_\pm(s) \;=\; e^{\pm i\pi/4}\, r^{\mp 1/2}\, e^{-i(n+1)s}$
$+i\, r^{2n+1}$	$f^{n,4}_\pm(s) \;=\; e^{\mp i\pi/4}\, r^{\mp 1/2}\, e^{ins}$

Notice that as n ranges over the nonnegative integers and $k = 1, 2, 3, 4$, the $f^{n,k}$ form a complete basis for $L^2(b\Omega)$, so we've determined all the eigenvectors. As before, the null-space of A is trivial.

5. Logarithmic Sector

The boundary of the logarithmic sector consists of the logarithmic spirals that can be parameterized by $s \in \mathbb{R}^+ \to z(s) = s\, e^{ia \log s} e^{\pm i\theta}/(1 + ia)$, respectively, where $a > 0$ and $0 < \theta < \pi/2$ are fixed. Notice that if $z = z(s)$, then $T_z = \mp\, e^{ia \log s} e^{\pm i\theta}$.



Again, we identify f and $g = Af \in L^2(b\Omega)$ with $\begin{bmatrix} f_+ \\ f_- \end{bmatrix}, \begin{bmatrix} g_+ \\ g_- \end{bmatrix} \in L^2(\mathbb{R}^+) \times L^2(\mathbb{R}^+)$, where $f_\pm(s) = f(s\, e^{ia \log s} e^{\pm i\theta}/(1 + ia))$ for $s \in \mathbb{R}^+$.

In this case, $A(z, w)$ does not vanish for z and w on the same spiral, so when computing $g_\pm$ we consider contributions from both $f_\pm$ and $f_\mp$. Moreover, it will be simpler to work in the inversive arclength coordinate $u = \sqrt{a}\, \log s$, rather than the Euclidean arclength coordinate s.[2] So, as for the wedge, we use an isometry $\Lambda : L^2(\mathbb{R}^+) \to L^2(\mathbb{R})$ given by $h \to h'$ where $h'(u) = h(e^{u/\sqrt{a}})\, e^{u/(2\sqrt{a})}/\sqrt[4]{a}$. The effect of this coordinate change on the kernel is the multiplicative factor $e^{(u+v)/(2\sqrt{a})}/\sqrt{a}$.

After simplifying, we find

$$\begin{aligned} \Lambda g_\pm(u) = \\ &\pm \int_{-\infty}^{+\infty} \frac{e^{-i\sqrt{a}(u-v)/2}}{2\pi\sqrt{a}} \cdot \operatorname{Im}\left[\frac{1 + ia}{\sinh[(u - v)(1 + ia)/(2\sqrt{a})]}\right] \Lambda f_\pm(v)\, dv \\ &\mp \int_{-\infty}^{+\infty} \frac{e^{-i\sqrt{a}(u-v)/2 \mp i\theta}}{2\pi i\sqrt{a}} \cdot \operatorname{Re}\left[\frac{1 + ia}{\sinh[(u - v)(1 + ia)/(2\sqrt{a}) \pm i\theta]}\right] \Lambda f_\mp(v) dv. \end{aligned}$$

So A involves convolution with the functions

$$k_1(u) = \pm\frac{1}{2\pi} \frac{e^{-i\sqrt{a}u/2}}{\sqrt{a}} \operatorname{Im}\left[\frac{1 + ia}{\sinh[u(1 + ia)/(2\sqrt{a})]}\right],$$

[2]For a nice study of the inversive geometry of plane curves, see Patterson [9] or Cairns and Sharpe [3].

and

$$k_2(u) = \mp\frac{1}{2\pi i}\,\frac{e^{-i\sqrt{a}u/2\mp i\theta}}{\sqrt{a}}\,\mathrm{Re}\left[\frac{1+ia}{\sinh[u(1+ia)/(2\sqrt{a})\pm i\theta]}\right].$$

The transform of k_1 can be found using formula 4.111–1 from [6], page 511. We omit the lengthy computation, but, after simplifying, we find

$$\begin{aligned}\phi_1(\xi) &\stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{-i\sqrt{a}u/2}}{\sqrt{a}}\ \mathrm{Im}\left[\frac{1+ia}{\sinh[u(1+ia)/(2\sqrt{a})]}\right] e^{-iu\xi}\,du \\ &= \frac{i\,\sin(2\pi ad)}{\cosh(2\pi d)+\cos(2\pi ad)},\end{aligned}$$

where $d = \sqrt{a}(\xi+\sqrt{a}/2)/(1+a^2)$. The transform of k_2 can be found using formulas 3.983–5,6, page 506. After simplifying, we find that

$$\begin{aligned}\phi_2(\xi) &\stackrel{\text{def}}{=} \frac{-1}{2\pi i}\int_{-\infty}^{+\infty} \frac{e^{-i\sqrt{a}u/2-i\theta}}{\sqrt{a}}\mathrm{Re}\left[\frac{1+ia}{\sinh[u(1+ia)/(2\sqrt{a})+i\theta]}\right] e^{-iu\xi}\,du \\ &= \frac{e^{-i\theta}}{2}\left[\frac{e^{(\pi-2\theta)d(1-ia)}}{\cosh(\pi d(1-ia))} - \frac{e^{-(\pi-2\theta)d(1+ia)}}{\cosh(\pi d(1+ia))}\right] \\ &= e^{(2\theta-\pi)iad-i\theta}\cdot\frac{e^{\pi iad}\sinh 2(\pi-\theta)d - e^{-\pi iad}\sinh 2\theta d}{\cosh 2\pi d+\cos 2a\pi d},\end{aligned}$$

where $d = \sqrt{a}(\xi+\sqrt{a}/2)/(1+a^2)$.

We summarize what we have shown so far. Let ϕ_1 and $\phi_2 \in L^\infty(\mathbb{R})$ be defined as above, and let $\mathcal{F}$ and $\mathcal{F}^{-1}$ be the Fourier transforms, as in the previous sections. Then the Kerzman-Stein operator acts according to

$$\begin{bmatrix}\mathcal{F}\Lambda g_+\\ \mathcal{F}\Lambda g_-\end{bmatrix} = \begin{bmatrix}+\phi_1 & +\phi_2\\ -\overline{\phi_2} & -\phi_1\end{bmatrix}\cdot\begin{bmatrix}\mathcal{F}\Lambda f_+\\ \mathcal{F}\Lambda f_-\end{bmatrix},$$

where $Ag = f$, and we are using the identifications of f, g with $f_\pm, g_\pm$, and the isometry Λ, described above.

Notice that ϕ_1 is purely imaginary. So, for fixed $\xi \in \mathbb{R}$, the eigenvalues of the 2×2 matrix are $\pm i\sqrt{|\phi_1(\xi)|^2+|\phi_2(\xi)|^2}$, and the continuous spectrum of A is the range of these functions as ξ spans $\mathbb{R}$. Since ϕ_1 vanishes only at discrete points, we know that the null-space of A is trivial. In Figures 2 and 3 we illustrate the spectrum for certain values of a and θ. In Figure 2 we show a graph of $(\xi,\theta) \to \sqrt{|\phi_1(\xi')|^2+|\phi_2(\xi')|^2}$ for $a = 4$, using $\xi' = \xi - \sqrt{a}/2$, and in Figure 3 we show a graph of $(\xi, a) \to \sqrt{|\phi_1(\xi')|^2+|\phi_2(\xi')|^2}$ for $\theta = \pi/4$.

Observe that large values of a correspond to large operator norms for A. In particular, if ξ is chosen so that $\pi ad = \pi/2$, then $|\phi_1(\xi)| = 0$, and

$$|\phi_2(\xi)| = \frac{\sinh\frac{\pi-\theta}{a}+\sinh\frac{\theta}{a}}{\cosh\frac{\pi}{a}-1} = \frac{\cosh\frac{\pi-2\theta}{2a}}{\sinh\frac{\pi}{2a}}.$$

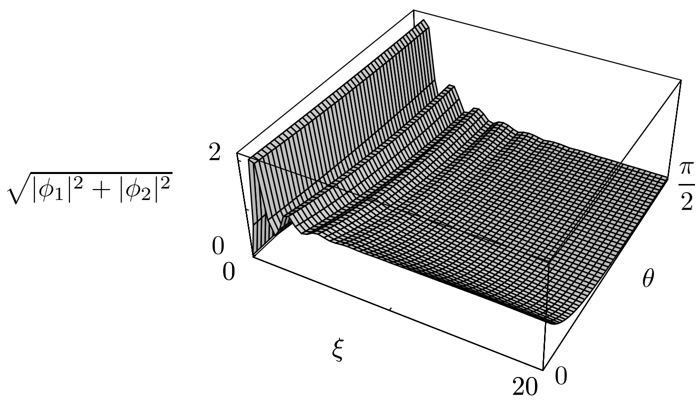


FIGURE 2. Graph of $(\xi,\theta) \to \sqrt{|\phi_1(\xi')|^2+|\phi_2(\xi')|^2}$ with $a = 4$.

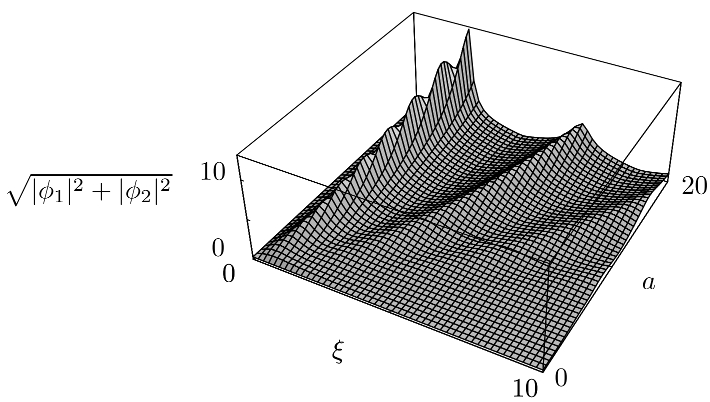


FIGURE 3. Graph of $(\xi, a) \to \sqrt{|\phi_1(\xi')|^2+|\phi_2(\xi')|^2}$ with $\theta = \pi/4$.

It follows that

$$\|A\| = \max\sqrt{|\phi_1|^2+|\phi_2|^2} \geq \frac{\cosh\frac{\pi-2\theta}{2a}}{\sinh\frac{\pi}{2a}} \geq \frac{1}{\sinh\frac{\pi}{2a}}.$$

This estimate is best for large values of a.

We also point out the continuity property of the spectrum at the endpoint $a = 0$. If $a = 0$, then in fact, the logarithmic sector is a wedge. Following a linear change of coordinate, $\xi \to d = \sqrt{a}(\xi + \sqrt{a}/2)/(1 + a^2)$, we find from the first expression for ϕ_2 that $\phi_2(d) \to \phi(d)$ as $a \to 0$, where ϕ was defined for the wedge in the previous section; we also find that $\phi_1(d) \to 0$. So the description given in this section passes continuously to the one given for the wedge, as $a \to 0$.

To finish, we show a graph of $(a, \theta) \to \|A\|$. This function was computed by numerically solving for $\max_\xi \sqrt{|\phi_1|^2 + |\phi_2|^2}$ for an array of pairs (a, θ).

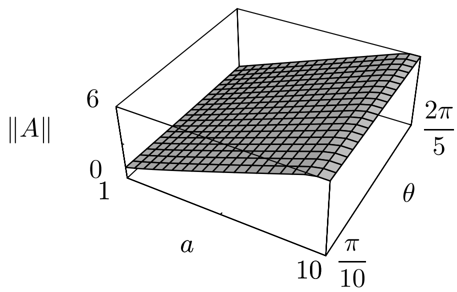


FIGURE 4. Graph of $(a, \theta) \to \|A\|$ for logarithmic sectors.

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