ENGINEERING RESEARCH INSTITUTE
UNIVERSITY OF MICHIGAN
ANN ARBOR

PROGRESS REPORT

SIMPLE WAVES IN THE STEADY, SUPERSONIC, PLANE, ROTATIONAL FLOW OF A COMPRESSIBLE POLYTROPIC GAS

N. COBURN

Project 2201

ORDNANCE CORPS, U. S. ARMY, DETROIT ORDNANCE DISTRICT
CONTRACT NO. DA-20-018-ORD-13282
DA PROJECT NO. 599-01-004, ORD PROJECT NO. TB2-001(892)

September, 1954

TABLE OF CONTENTS

SIMPLE WAVES IN THE STEADY, SUPERSONIC, PLANE, ROTATIONAL FLOW OF A COMPRESSIBLE POLYTROPIC GAS

1. Introduction

The purpose of this paper is to express the characteristic relations for the steady, three-dimensional, supersonic motion of a polytropic gas in intrinsic form and to apply these relations to the study of simple waves. This means that the characteristic relations shall be written in such a form that they express relations between curvatures associated with the characteristic manifolds, the rate of change of the magnitude of the velocity, q, and the sound speed, c, with respect to displacements along the normal to the characteristic manifolds and along two independent directions in these manifolds. Then, by specifying the curvatures or the rate of change of q and c, degenerate characteristic manifolds such as generalized simple waves may be studied.

First, intrinsic forms of the characteristic relations for general three-dimensional nonisentropic rotational flows are obtained and canonical forms of these relations are determined. Then, application is made to the case of plane isentropic rotational flows. It is shown that for the limiting case when the Mach number of the flow is one, the bicharacteristics are always a single family of radial straight lines. For this type of rotational flow, the following conditions are satisfied: (1) the stream lines are orthogonal to the straight line bicharacteristics; and (2) the magnitude of the velocity and the sound speed are constant along a given stream line. Further, the sound speed and the magnitude of the velocity vary (for $\gamma = 5/3$) as the one-third power of the distance from the intersection of the radial lines. The flow is a vortex flow.

This leads to a study of simple waves of type I. These are defined by the condition that the bicharacteristics form a family of straight lines. It is shown that at least one such family exists for all γ, except $\gamma = 1$. The flows are vortex flows with the following properties: (1) the bicharacteristics are tangent lines to a circle, (2) this circle is the limiting line of the flow, (3) the sound speed varies (for $\gamma = 5/3$) as the one-third power of the distance measured along a bicharacteristic from the limiting line. Finally, the necessary and sufficient condition for the existence of simple waves of of type I is found. Present computations indicate that the above class of vortex flows are the only two-dimensional rotational flows with simple waves of type I.

Simple waves of type II are those for which the Mach number is constant along a bicharacteristic. It is shown that the above vortex flows, with $q = c$, are the only flows with simple waves of type II and straight line bicharacteristics. A detailed analysis of simple waves of type II is made. As a result, it is shown that a four-parameter family of simple waves of type II exists. The explicit determination of these simple waves depends on the solutions of four ordinary differential equations. One special case is studied where the bicharacteristics are spirals.

2. The Basic Relation in Terms of Characteristic Variables

Let x^j, $j = 1,2,3$, denote a Cartesian orthogonal coordinate system in Euclidean three-space and let us denote derivatives by the symbolism

$$\partial_j \equiv \frac{\partial}{\partial x^j} \quad . \tag{2.1}$$

In a Cartesian orthogonal coordinate system, covariant and contravariant quantities are equivalent. However, in order to use the Einstein summation convention of summing on repeated lower and upper indices, these two types of equivalent quantities shall be used.

Now, the equations of continuity, motion, and energy will be considered. If ρ denotes the specific density and v^j is the velocity vector, then the continuity relation is

$$v^j \partial_j \rho + \rho\, \partial_j v^j = 0 \, . \tag{2.2}$$

For nonisentropic flows of polytropic gases, the equations of motion may be expressed in the form [1]

$$\gamma \rho v^j \partial_j v_k + \partial_k(\rho c^2) = 0, \tag{2.3}$$

where c is the local sound speed which may be defined in terms of the pressure, p, and the specific entropy S by

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_S ,$$

and γ is the constant of the gas law

$$p = \rho^\gamma f(S) \quad .$$

In addition, an energy relation in the form of the Bernoulli equation is assumed

$$v^j \partial_j \left(\frac{c^2}{\gamma - 1} + \frac{q^2}{2} \right) = 0, \qquad (2.4)$$

where $q^2 = v^j v_j$ is the magnitude squared of the velocity vector.

By definition, the characteristic manifolds of the system (2.2), (2.3), and (2.4) are those surfaces along which discontinuities in the derivatives, $\partial_j \rho$, $\partial_j c$, $\partial_j v_k$, can occur. Using the methods of a previous paper[2], these surfaces can be easily determined. However, these methods are not pertinent to the present problem; hence, the procedure which follows will be by analogy. By eliminating ρ between (2.2) and (2.3) and then by eliminating $\partial_j c^2$ in the resulting equation through use of (2.4), the following relation is obtained

$$(v^j v^k - c^2 g^{jk}) \, \partial_j v_k = 0, \qquad (2.5)$$

where g^{jk} represents the metric tensor. Thus, the same basic equation (2.5) is valid in the nonisentropic as well as in the isentropic case[3].

If the symmetric tensor a^{jk} is defined by

$$a^{jk} = v^j v^k - c^2 g^{jk}, \qquad (2.6)$$

then for isentropic flows the characteristic manifolds of Equation (2.5) are determined by

$$a^{jk} n_j n_k = 0, \qquad (2.7)$$

where n_j is a unit vector orthogonal to one family of characteristic surfaces. From the known theory of characteristic manifolds[4], it follows that

$$v_j = c n_j + \sqrt{q^2 - c^2} \, t_j, \qquad (2.8)$$

where t_j is the unit vector tangent to the bicharacteristics. By use of (2.6) and (2.8), it is found that

$$a^{jk} n_k = c \sqrt{q^2 - c^2} \, t^j, \qquad (2.9)$$

$$a^{jk} t_k = c \sqrt{q^2 - c^2} \, n^j + (q^2 - 2c^2) \, t^j. \qquad (2.10)$$

It can be shown that (2.7) through (2.10) are valid in the nonisentropic case[2].

Now, a few basic relations from differential geometry are needed[5]. Since n_j is a unit vector orthogonal to ∞^1 surfaces, the following may be written;

$$\partial_j n_k = s_{jk} + n_j u_k , \quad s_{jk} n^k = u_k n^k = 0 , \tag{2.11}$$

where s_{jk} is the symmetric second fundamental tensor of the ∞^1 surfaces orthogonal to n_j and u_k is the curvature vector of the n_j congruence of curves. Further, let l_j denote a unit vector field which is orthogonal to both t_j and n_j so that l_j, n_j, and t_j form an orthogonal triple at each point.

In future work, the term, $g^{jk}\partial_j t_k$, will have to be evaluated in terms of the congruences, n_j and l_j and their curvature vectors, u_j and $\bar{u}_j$, respectively, where

$$l^j \partial_j l_k = \bar{u}_k . \tag{2.12}$$

From the basic decomposition of the metric tensor

$$g^{jk} = t^j t^k + n^j n^k + l^j l^k , \tag{2.13}$$

it can be found that

$$g^{jk}\partial_j t_k = t^j t^k \partial_j t_k + n^j n^k \partial_j t_k + l^j l^k \partial_j t_k . \tag{2.14}$$

In view of the relations

$$t^k t_k = 1, \quad l^k t_k = 0, \quad n_k t^k = 0, \tag{2.15}$$

it follows by differentiation that

$$t^k \partial_j t_k = 0, \quad l^k \partial_j t_k = - t_k \partial_j l^k , \quad n^k \partial_j t_k = -t_k \partial_j n^k . \tag{2.16}$$

The first relation of (2.16) shows that

$$t^j t^k \partial_j t_k = 0 .$$

Further, from (2.11), (2.12), and (2.16), it is seen that

$$n^j n^k \partial_j t_k = -t_k (n^j \partial_j n^k) = -t_k u^k ,$$

and

$$l^j l^k \partial_j t_k = -t_k (l^j \partial_j l^k) = -t_k \bar{u}^k .$$

Thus, Equation (2.14) may be expressed in the form

$$g^{jk} \partial_j t_k = -t_k (u^k + \bar{u}^k) . \tag{2.17}$$

In this paragraph the previously derived relations will be used to express the basic relation (2.5) in terms of rates of change of q and c with respect to displacements along t_j and n_j, respectively, and in terms of the curvatures s_{jk}, u_j, and $\bar{u}_j$. The following notation for directional derivatives will be used.

$$\frac{\partial}{\partial t} \equiv t^j\partial_j \qquad\qquad \frac{\partial}{\partial n} \equiv n^j\partial_j \qquad (2.18)$$

Thus, $\partial/\partial t$ represents rate of change with respect to displacement along t_j and $\partial/\partial n$ represents rate of change with respect to displacement along n_j. By differentiation of (2.8), it is found that

$$\partial_j v_k = c\,\partial_j n_k + \sqrt{q^2-c^2}\,\partial_j t_k + n_k\partial_j c + t_k\partial_j\sqrt{q^2-c^2}\,. \qquad (2.19)$$

Multiplying (2.19) by a^{jk} and using the relations (2.9) and (2.10), it can be shown that

$$a^{jk}\partial_j v_k = c\,a^{jk}\partial_j n_k + \sqrt{q^2-c^2}\,a^{jk}\partial_j t_k + c\sqrt{q^2-c^2}\left(\frac{\partial c}{\partial t} + \frac{\partial\sqrt{q^2-c^2}}{\partial n}\right) + (q^2-2c^2)\,\frac{\partial\sqrt{q^2-c^2}}{\partial t}\,. \qquad (2.20)$$

Next, the curvature terms, $a^{jk}\partial_j n_k$, of the above equation must be evaluated. From (2.6) and (2.8) the following result is obtained

$$a^{jk} = c^2(n^jn^k - g^{jk} - t^jt^k) + q^2t^jt^k + c\sqrt{q^2-c^2}(n^jt^k + n^jt^k)\,. \qquad (2.21)$$

Forming the scalar product of (2.21) with (2.11) it is seen that

$$a^{jk}\partial_j n_k = c\sqrt{q^2-c^2}\,t^ku_k + (q^2-c^2)\,s_{jk}t^jt^k - c^2M\,, \qquad (2.22)$$

where M is the mean curvature of the characteristic surfaces;

$$M = g^{jk}s_{jk}\,.$$

By use of the relations (2.11) and (2.6), it can be shown that

$$n^jn^k\partial_j t_k = -t^kn^j\partial_j n_k = t^ku_k\,,$$

$$n^jt^k\partial_j t_k = -t^jt^k\partial_j n_k = 0\,,$$

$$t^jn^k\partial_j t_k = -t^jt^k\partial_j n_k = -s_{jk}\,t^jt^k\,.$$

These last relations and (2.22) show that

$$a^{jk}\partial_j t_k = -c^2 t^k u_k - c\sqrt{q^2-c^2}\, s_{jk}t^j t^k - c^2 g^{jk}\partial_j t_k \quad . \qquad (2.23)$$

By use of (2.17), the above equation reduces to

$$a^{jk}\partial_j t_k = -c\sqrt{q^2-c^2}\, s_{jk}t^j t^k + c^2 t_k \bar{u}^k \quad . \qquad (2.24)$$

If (2.22) and (2.24) are substituted into (2.20) and the left hand side of the latter equation is equated to zero, an intrinsic form of the basic relation (2.5) is obtained in terms of characteristic variables

$$0 = c\sqrt{q^2-c^2}\left(\frac{\partial c}{\partial t} + \frac{\partial\sqrt{q^2-c^2}}{\partial n}\right) + (q^2-2c^2)\,\frac{\partial\sqrt{q^2-c^2}}{\partial t} + c^2\sqrt{q^2-c^2}\; t(u^k+\bar{u}^k) - c^3 M. \qquad (2.25)$$

3. The Intrinsic Conditions for Rotational (or Irrotational) Motion in Terms of Characteristic Variables

If e^{ijk} denotes the permutation tensor, then the vorticity vector, ω^j, is defined by

$$\omega^j = e^{jlk}\partial_l v_k \quad . \qquad (3.1)$$

Assuming that the ordered triad, t_j, n_j, and l_j, forms a right hand system, then the following cross-product relations are valid

$$l_j = e_{jpk}t^p n^k \ , \quad t_j = e_{jpk}n^p l^k \ , \quad n_j = e_{jpk}l^p t^k \ .$$

If each of the above equations is multiplied by e^{jmq} and the fact used that

$$e^{jmq}e_{jpk} = \delta^m_p\,\delta^q_k - \delta^m_k\,\delta^q_p \ ,$$

where δ^m_p is the Kronecker delta tensor, it can be shown that

$$\begin{aligned} e^{jmq}l_j &= t^m n^q - t^q n^m \ , \\ e^{jmq}t_j &= n^m l^q - n^q l^m \ , \qquad (3.2) \\ e^{jmq}n_j &= l^m t^q - l^q t^m \ . \end{aligned}$$

Forming the scalar product of (3.1) with the vectors t_j, n_j, and l_j, through use of (3.2) the following formulae are obtained

$$\omega^j t_j = (n^p l^k - n^k l^p)\, \partial_p v_k$$

$$\omega^j n_j = (l^p t^k - l^k t^p)\, \partial_p v_k \qquad (3.3)$$

$$\omega^j l_j = (t^p n^k - t^k n^p)\, \partial_p v_k \quad .$$

A lengthy but direct computation using (2.11), (2.16), (2.19) shows that (3.3) may be written as

$$\omega^j t_j = c\, l^k u_k - \frac{\partial c}{\partial l} + \sqrt{q^2-c^2}\,(n^p l^k - n^k l^p)\, \partial_p t_k \,, \qquad (3.4)$$

$$\omega^j n_j = \frac{\partial \sqrt{q^2-c^2}}{\partial l} + \sqrt{q^2-c^2}\, t^j t^k \partial_j l_k \,, \qquad (3.5)$$

$$\omega^j l_j = \frac{\partial c}{\partial t} - \frac{\partial \sqrt{q^2-c^2}}{\partial n} - \sqrt{q^2-c^2}\, g_{jk} t^j t^k - c\, t^k u_k \quad . \qquad (3.6)$$

In order to obtain an intrinsic formulation in terms of a characteristic variable of the left hand sides of (3.4) through (3.6), it should be noted that[6]

$$\partial_j h_o - T\partial_j S = e_{jlk} v^l \omega^k \, . \qquad (3.7)$$

Here, T is the absolute temperature, S is the specific entropy, and h_o is the stagnation enthalpy

$$h_o = \frac{c^2}{\gamma - 1} + \frac{q^2}{2} \,, \qquad (3.8)$$

where γ is the ratio of the specific heats of the polytropic gas. Since the stagnation enthalpy is constant along a stream line (see 2.4), with the aid of (2.8) it is found that

$$c\, \frac{\partial h_o}{\partial n} + \sqrt{q^2-c^2}\, \frac{\partial h_o}{\partial t} = 0 \qquad (3.9)$$

and S satisfies a similar relation.

To express (3.7) in intrinsic form, the scalar product of this relation with the vectors t_j, n_j, and l_j is found. Using (2.8), (3.2), and (3.4) through (3.6), the following formulae are obtained

$$\frac{\partial h_o}{\partial t} - T\frac{\partial S}{\partial t} = c\left(\frac{\partial c}{\partial t} - \frac{\partial\sqrt{q^2-c^2}}{\partial n} - \sqrt{q^2-c^2}\, s_{jk}\, t^j t^k - c\, t^k u_k\right), \quad (3.10)$$

$$\frac{\partial h_o}{\partial n} - T\frac{\partial S}{\partial n} = -\sqrt{q^2-c^2}\left(\frac{\partial c}{\partial t} - \frac{\partial\sqrt{q^2-c^2}}{\partial n} - \sqrt{q^2-c^2}\, s_{jk}\, t^j t^k - c\, t^k u_k\right) \quad (3.11)$$

$$\frac{\partial h_o}{\partial l} - T\frac{\partial S}{\partial l} = q\frac{\partial q}{\partial l} + (q^2-c^2)\, t^j t^k \partial_j l_k - c^2 l^k u_k - c\sqrt{q^2-c^2}\, n^j l^k\, (\partial_j t_k - \partial_k t_j). \quad (3.12)$$

Evidently, (3.11) is a consequence of (3.9) and (3.10).

The above equations can be described by saying that they form a system consisting of three equations (3.10), (3.12), and (2.25) in the unknowns

$$\frac{\partial q}{\partial t}, \quad \frac{\partial q}{\partial n}, \quad \frac{\partial q}{\partial l},$$

where h_o and S are prescribed functions which are constant along a stream line. The sound speed, c, is determined as a function of the magnitude of the velocity, q, by the Bernoulli relation (3.8). Thus, the roles of c and q may be interchanged. In the applications, the stream lines are unknown. Hence, the problem is to determine the functions h_o, S, and q so that two relations of the type (3.9) are satisfied (one equation in the derivatives of h_o and the other in the derivatives of S) and also equations (3.10), (3.12) and (2.25) are valid.

The curvature term,

$$K = n^p l^k(\partial_p t_k - \partial_k t_p), \quad (3.13)$$

in the right hand side of (3.12) will be briefly considered. Through use of (2.11) and (2.16), this term may be written as

$$K = -(n^p t^k \partial_p l_k - t^p l^k \partial_k n_p) = -n^p t^k \partial_p l_k + s_{kp} t^p l^k. \quad (3.14)$$

If the unit vector field, l_j, is orthogonal to ∞^1 surfaces (as is the case in plane and axial-symmetric flows) then

$$\partial_j l_k = r_{jk} + l_j \bar{u}_k, \quad l^k r_{jk} = l^k \bar{u}_k = 0,$$

where r_{jk} is the symmetric second fundamental tensor of these surfaces and $\bar{u}_k$ is the curvature vector of l_k congruence. In this case, (3.14) reduces to

$$K = -r_{jk}n^j t^k + s_{jk}t^j l^k . \tag{3.15}$$

However, if the vector field of the bicharacteristics, t_j, is orthogonal to ∞^1 surfaces (as in the case of plane flows) then

$$K = 0 . \tag{3.16}$$

4. Canonical Form of the System (2.25), (3.10), and (3.12)

In this section the equations (2.25), (3.10), and (3.12) shall be written so that one equation contains only the directional derivative, $\partial/\partial t$, a second equation contains only the directional derivative, $\partial/\partial n$, and the final equation contains only the directional derivative, $\partial/\partial l$.

If (3.10) is multiplied by $\sqrt{q^2-c^2}$ and the resulting equation is added to (2.25), the following relation is obtained

$$2c\sqrt{q^2-c^2}\,\frac{\partial c}{\partial t} + (q^2-2c^2)\,\frac{\partial\sqrt{q^2-c^2}}{\partial t} + c^2\sqrt{q^2-c^2}\;t_k\bar{u}^k - c(q^2-c^2)\;s_{jk}t^j t^k - c^3 M =$$

$$= \sqrt{q^2-c^2}\left(\frac{\partial h_o}{\partial t} - T\,\frac{\partial S}{\partial t}\right) . \tag{4.1}$$

The first two terms on the left hand side of (4.1) may be replaced by

$$2c\sqrt{q^2-c^2}\,\frac{\partial c}{\partial t} + (q^2-2c^2)\,\frac{\partial\sqrt{q^2-c^2}}{\partial t} \equiv (q^2-c^2)\,\frac{\partial}{\partial t}\left[\frac{q^2}{\sqrt{q^2-c^2}}\right] \tag{4.2}$$

Hence, (4.1) may be written as

$$\frac{\partial}{\partial t}\,\frac{q^2}{\sqrt{q^2-c^2}} = \frac{-c^2}{\sqrt{q^2-c^2}}\,t_k\bar{u} + c_{jk}t^j t^k + \frac{c^3}{q^2-c^2}\,M + \frac{1}{\sqrt{q^2-c^2}}\left(\frac{\partial h_o}{\partial t} - T\,\frac{\partial S}{\partial t}\right), \tag{4.3}$$

this is the first equation of the desired canonical system for $q \neq c$.

Now, the second equation of the desired system will be determined. First, (3.10) will be multiplied by $\sqrt{q^2-c^2}$ and the resulting equation will be subtracted from (2.25). It is found that

$$2c\sqrt{q^2-c^2}\frac{\partial\sqrt{q^2-c^2}}{\partial n} + (q^2-2c^2)\frac{\partial\sqrt{q^2-c^2}}{\partial t} + c^2\sqrt{q^2-c^2}\, t_k(2u^k+\bar{u}^k)$$

$$-c^3 M + c(q^2-c^2)\, s_{jk}\, t^j t^k = -\sqrt{q^2-c^2}\left(\frac{\partial h_o}{\partial t} - T\frac{\partial S}{\partial t}\right) \qquad (4.4)$$

Both S and h_o are constant along stream lines, and hence, by (3.9) the right hand side of (4.4) may be written as

$$-\sqrt{q^2-c^2}\left(\frac{\partial h_o}{\partial t} - T\frac{\partial S}{\partial t}\right) = c\left(\frac{\partial h_o}{\partial n} - T\frac{\partial S}{\partial n}\right). \qquad (4.5)$$

To evaluate the second terms on the left hand side of (4.4), note by use of (3.8) that

$$\frac{\partial\sqrt{q^2-c^2}}{\partial t} = \frac{1}{\sqrt{q^2-c^2}}\frac{\partial}{\partial t}\left[h_o - \frac{\gamma+1}{2(\gamma-1)}c^2\right]. \qquad (4.6)$$

Through use of (3.10), (4.6) reduces to equation

$$\frac{\partial\sqrt{q^2-c^2}}{\partial t} = \frac{1}{\sqrt{q^2-c^2}}\left\{\frac{2}{\gamma-1}\frac{\partial h_o}{\partial t} + \frac{\gamma+1}{\gamma-1}\left[T\frac{\partial S}{\partial t} - \frac{\partial\sqrt{q^2-c^2}}{\partial n} - c\sqrt{q^2-c^2}\, s_{jk}t^j t^k - c^2 t_k u^k\right]\right\}. \qquad (4.7)$$

Converting $\partial h_o/\partial t$, $\partial S/\partial t$ into $\partial h_o/\partial n$, $\partial S/\partial n$ through use of (3.8), it is found that (4.7) reduces to

$$\frac{\partial\sqrt{q^2-c^2}}{\partial t} = \frac{1}{\sqrt{q^2-c^2}}\left[\frac{2}{(\gamma-1)}\frac{c}{\sqrt{q^2-c^2}}\frac{\partial h_o}{\partial n} - \frac{\gamma+1}{\gamma-1}\left\{\frac{c}{\sqrt{q^2-c^2}}\frac{\partial S}{\partial n} + c\frac{\partial\sqrt{q^2-c^2}}{\partial n} + c\sqrt{q^2-c^2}\, s_{jk}t^j t^k + c^2 t_k u^k\right\}\right]. \qquad (4.8)$$

Substituting (4.5) and (4.8) into (4.4), the latter equation becomes

$$\frac{(\gamma-3)q^2+4c^2}{(\gamma-1)\sqrt{q^2-c^2}}\frac{\partial\sqrt{q^2-c^2}}{\partial n} = \frac{2q^2-(\gamma+3)c^2}{\gamma-1}s_{jk}t^j t^k + c^2 M + \frac{c}{(\gamma-1)\sqrt{q^2-c^2}}\left\{[(3-\gamma)q^2-4c^2]\,t^k u_k\right\} - c\sqrt{q^2-c^2}\,t^k\bar{u}_k + \frac{(\gamma-3)q^2-(\gamma-5)c^2}{(\gamma-1)(q^2-c^2)}\frac{\partial h_o}{\partial n} + \frac{2q^2-(\gamma+3)c^2}{(\gamma-1)(q^2-c^2)}T\frac{\partial S}{\partial n}. \qquad (4.9)$$

The equation (4.9) is the second of the desired canonical relations for $q \neq c$.

Evidently, Equation (3.12) is the third equation of the desired canonical system[7]. This equation can be written in the form

$$q \frac{\partial q}{\partial l} = \frac{\partial h_0}{\partial l} - T \frac{\partial S}{\partial l} + c^2 l^k u_k - (q^2-c^2)\, t^j t^k \partial_j l_k + c\sqrt{q^2-c^2}\, K, \tag{4.10}$$

where K is the curvature defined in (3.13).

5. The Case of Plane Flows

The congruence determined by the unit vector, l_j, consists of parallel straight lines perpendicular to the plane of the flow, and hence, $\bar{u}^h = 0$. These lines lie in one principal direction of the cylindrical characteristic surfaces. The principal normals of the bicharacteristics coincide with n_j, the unit normal to the characteristic surfaces, and hence the bicharacteristics are the curves in the second principal direction. Thus,

$$M = s_{jk} t^j t^k = -\kappa , \tag{5.1}$$

where κ is the curvature of the bicharacteristic curves. Further, if $\bar{\kappa}$ denotes the curvature of the normal congruence, n_k, then

$$\bar{u}_k = \bar{\kappa} t_k . \tag{5.2}$$

Finally, in the present case, the congruence determined by t_k is orthogonal to ∞^1 surfaces and thus the curvature K of (3.13) must vanish.

Now, the canonical equations (4.3) and (4.9) may be expressed in terms of an orthogonal Cartesian coordinate system, x, y, in the plane. It should be noted that the third canonical equation (4.10) is identically satisfied for plane flows. Consider two families of parameterized curves, α = constant denoting the family of bicharacteristic curves and β = constant denoting the orthogonal trajectories of these curves (and hence with tangent vector, n_j). With respect to these curves, the arc length element becomes

$$(ds)^2 = (A d\alpha)^2 + (B d\beta)^2 , \tag{5.3}$$

where A and B are metric coefficients. In addition, let $\Theta(\alpha,\beta)$ denote the angle between the ox-axis and the tangents to the bicharacteristic curves, α = constant. Then[8],

$$\frac{\partial x}{\partial \beta} = B \cos \Theta , \qquad \frac{\partial y}{\partial \beta} = B \sin \Theta , \tag{5.4}$$

$$\frac{\partial x}{\partial \alpha} = A \sin \Theta , \qquad \frac{\partial y}{\partial \alpha} = -A \cos \Theta , \tag{5.5}$$

$$\kappa = -\frac{1}{B}\frac{\partial \Theta}{\partial \beta} = -\frac{1}{AB}\frac{\partial B}{\partial \alpha} , \tag{5.6}$$

$$\bar{\kappa} = \frac{1}{A}\frac{\partial \Theta}{\partial \alpha} = -\frac{1}{AB}\frac{\partial A}{\partial \beta} , \tag{5.7}$$

$$\frac{\partial}{\partial \alpha}\left(\frac{1}{A}\frac{\partial B}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(\frac{1}{B}\frac{\partial A}{\partial \beta}\right) = 0 . \tag{5.8}$$

If (5.1) and (5.6) are substituted into (4.3), the following relation is obtained

$$\frac{q^2-c^2}{cq^2}\frac{\partial}{\partial \beta}\left(\frac{q^2}{\sqrt{q^2-c^2}}\right) = \frac{\partial \Theta}{\partial \beta} + \frac{\sqrt{q^2-c^2}}{cq^2}\left(\frac{\partial h_o}{\partial \beta} - \frac{\partial S}{\partial \beta}\right) . \tag{5.9}$$

This last relation may be written in the more symmetric form

$$\frac{\sqrt{q^2-c^2}}{c}\left[\frac{\partial}{\partial \beta} \ln \frac{q^2}{\sqrt{q^2-c^2}} - \frac{1}{q^2}\left(\frac{\partial h_o}{\partial \beta} - T\frac{\partial S}{\partial \beta}\right)\right] = \frac{\partial \Theta}{\partial \beta} . \tag{5.10}$$

This is the first canonical relation expressed in terms of net when $q \neq c$, $c \neq 0$. If the appropriate relations of the set (5.1) through (5.7) are substituted into (4.9), the latter equation becomes

$$\frac{(\gamma-3)q^2 + 4c^2}{\sqrt{q^2-c^2}}\frac{\partial \sqrt{q^2-c^2}}{\partial \alpha} = -2(q^2-2c^2)\frac{\partial}{\partial \alpha} \ln B - \frac{c[(\gamma-3)q^2+4c^2]}{\sqrt{q^2-c^2}}\frac{\partial \Theta}{\partial \alpha}$$

$$+ \frac{(\gamma-3)q^2-(\gamma-5)c^2}{\sqrt{q^2-c^2}}\frac{\partial h_o}{\partial \alpha} + \frac{2q^2-(\gamma+3)c^2}{q^2-c^2} T \frac{\partial S}{\partial \alpha}$$

(5.11)

The relation (5.11) expresses the second canonical equation when $q \neq c$ in terms of quantities associated with the α, β net. In addition, the conditions that h_o and S are constant along a stream line (see 3.9) lead to the equations

$$\frac{c}{A}\frac{\partial h_o}{\partial \alpha} + \frac{\sqrt{q^2-c^2}}{B}\frac{\partial h_o}{\partial \beta} = 0 \ , \qquad (5.12)$$

and

$$\frac{c}{A}\frac{\partial S}{\partial \alpha} + \frac{\sqrt{q^2-c^2}}{B}\frac{\partial S}{\partial \beta} = 0 \ . \qquad (5.13)$$

Since (5.12) and (5.13) each possess one independent integral, it follows that

$$S = S(h_o) \ . \qquad (5.14)$$

Finally, the equation (5.8) which expresses the vanishing of the Riemann tensor of the plane is valid. Thus, the basic system consists of Equations (5.8), (5.10), (5.11), (5.12), and (5.14).

Note that the previous equations can be used to determine a quasi-characteristic system for steady, nonisentropic, rotational, plane, supersonic flow of a polytropic gas ($T = c^2/\gamma R$). In this system of seven equations for the seven dependent variables, q, c, h_o, S, θ, A, B, and two independent variables α and β (the characteristic variable), two of the equations are

$$S = S(h_o)$$

$$h_o = \frac{q^2}{2} + \frac{c^2}{\gamma-1} \ ,$$

and of the remaining five first order partial differential equations, two equations are such that each equation contains derivatives with respect to only one variable (see(5.10), (5.11)), and three equations

$$\frac{c}{A}\frac{\partial h_o}{\partial \alpha} + \frac{\sqrt{q^2-c^2}}{B}\frac{\partial h_o}{\partial \beta} = 0 \ ,$$

$$\frac{\partial \theta}{\partial \beta} = \frac{1}{A}\frac{\partial B}{\partial \alpha} \ ,$$

$$\frac{\partial \theta}{\partial \alpha} = -\frac{1}{B}\frac{\partial A}{\partial \beta} \ ,$$

contain derivatives with respect to both α and β. By use of similar methods, two characteristic variables can be introduced and a full characteristic system obtained.[9]

6. Simple Waves of Type I in Plane Isentropic Rotational Flows and the Metric Coefficients

Simple waves of type I are defined by requiring that the family of bicharacteristics are straight lines or

$$\Theta = \Theta(\alpha) \quad . \tag{6.1}$$

Substituting (6.1) into (5.6) it is seen that

$$B = B(\beta) \quad . \tag{6.2}$$

On integrating (5.7), it is found that

$$A = -\Theta' (\alpha) \, \overline{B} \, (\beta) + g(\alpha) \quad , \tag{6.3}$$

where $\Theta'(\alpha)$ denotes the derivative of $\Theta(\alpha)$ with respect to α, $g(\alpha)$ is an arbitrary function of α, and

$$\overline{B}(\beta) = \int^{\beta} B(\beta) \, d\beta \quad . \tag{6.4}$$

By proper choice of scale factor along the bicharacteristics, α = constant,

$$B(\beta) = 1 \quad , \tag{6.5}$$

may be chosen. Then β is the distance along these straight lines and the metric coefficient, A, of (6.4) becomes

$$A = -\beta \, \Theta'(\alpha) + g(\alpha) \quad . \tag{6.6}$$

In section 8, the solution of the basic equations (5.10) through (5.12) is studied for the case when

$$B = 1 \; , \qquad A = a\alpha - \beta \; , \tag{6.7}$$

where a is a constant. These relations imply that the function, $g(\alpha)$, of (6.3) is linear in α and that

$$\Theta = \alpha \quad . \tag{6.8}$$

The net of α, β curves corresponding to these metric coefficients may be determined by integrating (5.4), (5.5). The result is

$$x = (\beta - a\alpha) \cos \alpha + a \sin \alpha + x_o \quad , \tag{6.9}$$

$$y = (\beta - a\alpha) \sin \alpha - a \cos \alpha + y_o \quad , \tag{6.10}$$

where the constants of integration have been denoted by x_o and y_o. The geometric meaning of the mapping relations (6.9) and (6.10) is as follows. Consider a circle of radius a with center at the origin ($x_o = y_o = 0$).

The tangent lines to this circle are the bicharacteristics, α = constant; the circle is the limiting line of the flow. This can be seen from the following figure.

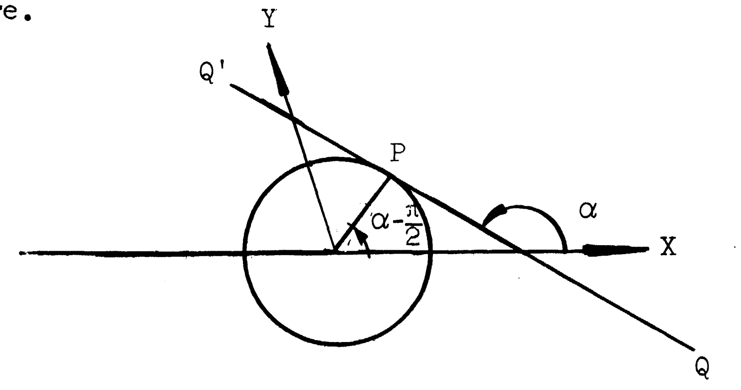

Fig. 1

The coordinates of the point P are

$$x_p = a \sin \alpha, \qquad y_p = a \cos \alpha.$$

If the distance PQ is denoted by r, where

$$r = a\alpha - \beta, \tag{6.11}$$

then the coordinates of the point Q are given by (6.9) and (6.10). The point Q' can be obtained by letting PQ' = r', where

$$r' = \beta - a\alpha . \tag{6.12}$$

However, only one of these two permissible mappings may be used in any given study of a flow. In particular, if a = 0, then the bicharacteristics are radial lines (a fan).

7. The Case of Rotational Plane Isentropic Flows at Mach Number One

Since the canonical relations (5.11) and (5.12) are valid only if the Mach number is greater than one ($q > c$), the basic relations (2.25), (3.11), and (5.12) must be used. The basic relation (2.25) in the case, $q = c$, leads to the condition

$$M = -\kappa = 0 . \tag{7.1}$$

Thus, the bicharacteristics must be straight lines or simple waves of type I. From (3.8), it can be seen that both c and q are constant along a stream line. The formula (2.8) shows that the stream lines are orthogonal to the bicharacteristics. Further, from (5.12) it follows that h_o is a function of

β only. The basic relation (3.11) is an identity and the relation (3.10) furnishes the equation (by use of (5.2) and (5.7))

$$\frac{dh_o}{d\beta} = c\,\frac{dc}{d\beta} - c^2\,\frac{B}{A}\,\frac{d\Theta}{d\alpha} \qquad (7.2)$$

By choosing a scale factor so that B = 1, and using the relation (6.3) for A, it is found that the second term of the right hand side of (7.2) becomes

$$c^2\,\frac{B}{A}\,\frac{d\Theta}{d\alpha} \equiv \frac{c^2\Theta'}{-\beta\Theta' + g(\alpha)} \quad .$$

Since c and h_o are functions of β, the relation (7.2) implies that

$$g(\alpha) = k\Theta' , \qquad (7.3)$$

where k is some constant. Thus, (7.2) becomes

$$\frac{dh_o}{d\beta} = \frac{dc}{d\beta} - \frac{c^2}{(k-\beta)} \qquad (7.4)$$

and the metric coeffcients are

$$B = 1, \qquad A = \Theta' (k-\beta) \; . \qquad (7.5)$$

By integration of (5.4) and (5.5) for the case of the metric coefficients (7.5), the following is obtained

$$x = r \cos\Theta , \qquad y = r \sin\Theta , \qquad (7.6)$$

where

$$r = \beta - k \; . \qquad (7.7)$$

For the present case of Mach number one, the relation (3.8) shows that

$$c^2 = q^2 = \frac{2(\gamma-1)}{(\gamma+1)}\,h_o \; . \qquad (7.8)$$

With the aid of (7.7) and (7.8), the relations (7.4) may be integrated and furnishes the equations

$$h_o = \bar{h}_o\, r^{(\gamma-1)} , \qquad c^2 = \frac{2(\gamma-1)}{(\gamma+1)}\,\bar{h}_o\, r^{(\gamma-1)} , \qquad (7.9)$$

where $\bar{h}_o$ is a constant. Thus, it is seen that for the case of Mach number one, the bicharacteristics form a family of radial straight lines along which h_o and c^2 vary according to the $(\gamma-1)$ power of the distance from the origin. The flow is a vortex flow in which the stream lines are the family of circles orthogonal to the radial lines[10]. It should be noted that for polytropic gases with $\gamma = 5/3$, c varies as the one-third power of the distance from the origin.

8. Properties of Plane Rotational Isentropic Flows with Simple Waves of Type I ($q > c$ and $\gamma \neq 1$): A Class of Simple Waves

Now some general properties of the solutions of Equations (5.10) through (5.12) will be considered before studying a particular solution of these equations for the cases when the metric coefficients are (6.5) and (6.7). Substituting (6.1) and (3.8) into (5.10), the following relation is obtained

$$(1-\gamma)\, c^2 \frac{\partial q^2}{\partial \beta} + [(\gamma-3)\, q^2 + 2c^2] \frac{\partial c^2}{\partial \beta} = 0 . \qquad (8.11)$$

Since (8.1) is homogeneous in c^2, q^2, a simple computation shows that

$$q^2 = f^2(\alpha)\, c^{2(\gamma-1)/(\gamma-3)} + c^2 , \quad \gamma \neq 1 , \qquad (8.2)$$

where $f(\alpha)$ is an unknown function of α. If (8.2) is substituted into (3.8) a relation between c and h_o is found, namely,

$$2(\gamma-1)\, h_o = (\gamma+1)\, c^2 + (\gamma-1)[f(\alpha)\, c^{(\gamma-3)/(\gamma-1)}]^2 , \quad \gamma \neq 1. \quad (8.3)$$

It should be noted that (8.2) and (8.3) are valid for <u>all</u> isentropic rotational flows with simple waves of type I. Also, in the remainder of the work, it will be assumed that

$$f(\alpha) \neq 0 .$$

This is due to the fact that when, $f(\alpha) = 0$, Equation (8.2) reduces to, $q = c$; the case discussed in Section 7.

It should be noted that for the irrotational isentropic case

$$h_o = \text{constant}, \qquad S = \text{constant}.$$

By use of (6.2) the second canonical relation (5.11) reduces to

$$\frac{\partial \sqrt{q^2-c^2}}{\partial \alpha} = -c \frac{\partial \theta}{\partial \alpha} \qquad (8.4)$$

The first canonical relation (5.10) and (3.8) lead to the well-known condition that c and q are constant along a given simple wave. Thus, the relation (8.4) may be written in the well-known form[11]

$$\frac{d\sqrt{q^2-c^2}}{d\theta} = -c\,. \tag{8.5}$$

The conditions (3.8) and (8.5) completely specify the family of simple waves.

It has been noted in the irrotational isentropic case that q and c are constant along a given bicharacteristic. This means that both of these quantities are functions of α only. For rotational isentropic flows, neither of these quantities, c and q, can be constant along a given bicharacteristic. This can be shown by noting that if one of the quantities q or c is a function of α, then use of (8.2) shows that the other of these quantities (and also h_o and by (8.3)) is a function of α. From (5.12), it follows that in this case, c vanishes and cavitation occurs. Thus, no continuous rotational flows consisting of simple waves can exist adjacent to a flow region in which the velocity and density are constant,[12] or adjacent to a simple wave region in the irrotational flow.

Another interesting question is, "What is the relation of simple waves as defined by (6.1) to degenerate mappings[13] of the hodograph plane?" In the irrotational isentropic plane case, the hodograph plane is determined by the two components of the velocity vector, v^j, $j = 1, 2$. For a degenerate mapping, each component of v^j is a function of only one variable, say μ. Since h_o of (3.8) is a constant, c and q are functions of μ. Thus, the angle between the velocity vector and the x-axis, and the Mach angle depend only on μ. So that angle θ (of Section 5) is a function only of μ. In short, the parameters α and μ can be identified. For the case of the rotational isentropic plane flow of a gas, the flow must be mapped in a hodograph space determined by v^j, $j = 1,2$ and c, the sound speed. In this case, the theory of degenerate mappings of the hodograph space into a curve is similar to that outlined above. Thus, in the preceding paragraph, it was shown that no such mapping with straight line bicharacteristics is possible.

Now, it shall be shown that a family of simple waves of type I exists for any isentropic flow of a polytropic gas, when $\gamma \neq 1$. This class of flows is characterized by the following: (1) magnitude of the velocity q, is constant along a stream line; hence, (2) the stream lines consist of concentric circles[10] (with center at the origin, see Fig. 1); and (3) the circle of Fig. 1 is a limiting line and the flow is a vortex flow.

To verify these results, a class of simple waves of type I is considered for a particular nonisentropic flow. It is assumed that the metric coefficients are given by (6.7). Further solutions of Equations (5.10) through (5.13) are sought such that c, q, and hence h_o, and also S are functions of the one variable

$$r = A = a\alpha - \beta . \qquad (8.6)$$

Equations (5.12) and (5.13) lead to the relation

$$q^2 = \left(1 + \frac{a^2}{r^2}\right) c^2 . \qquad (8.7)$$

By use of the Bernoulli relation (3.8), Equation (8.7), and the polytropic gas relation

$$T = c^2/\gamma R ,$$

the Equations (5.10) and (5.11) reduce to two ordinary first order differential equations in the dependent variables, c^2, S. In general, such a system possesses a unique solution (to within constants of integration) for $c^2(r)$ and $S(r)$.

By use of the substitution principle due to M. Munk and R. C. Prim[14], an isentropic flow with the same stream lines and Mach number as the above nonisentropic flow can be found. Hence, the Mach angle remains unaltered and the bicharacteristics are still straight lines (simple waves of type I). Further, the relation (8.7) is still valid. From this relation and (8.2), with $f(\alpha)$ replaced by a constant (say f), q and c can be determined as functions of the one variable, r. Then, the Bernoulli relation (3.8) determines h_o as a function of r. Further, a direct but lengthy computation (see Section 9) shows that for arbitrary values of the constants, a, f, Equations (8.2), (8.7), and (3.8) imply that

$$c^2 = c_o^2 \left(\frac{r}{r_o}\right)^{(\gamma-1)} , \qquad (8.8)$$

$$q^2 = c_o^2 \left[\left(\frac{r}{r_o}\right)^{(\gamma-1)} + \left(\frac{r}{r_o}\right)^{(\gamma-3)}\right] , \qquad (8.9)$$

$$2(\gamma+1)\, h_o = c_o^2 \left[(\gamma+1) \left(\frac{r}{r_o}\right)^{(\gamma-1)} + (\gamma-1) \left(\frac{r}{r_o}\right)^{(\gamma-3)}\right], \qquad (8.10)$$

where $$c_o^2 = f^{(\gamma-1)} , \qquad r_o = a . \qquad (8.11)$$

The bicharacteristics, α = constant, are determined by (6.9) and (6.10) (see Fig. 1).

For the important case, $\gamma = 5/3$, the formulas (8.8) through (8.10) reduce to

$$c^2 = c^2 (r/r_o)^{2/3} , \qquad (8.12)$$

$$q^2 = c_0^2 \left[(r/r_0)^{2/3} + (r/r_0)^{-4/3} \right] . \qquad (8.13)$$

$$2h_0 = c_0^2 \left[4(r/r_0)^{2/3} + (r/r_0)^{-4/3} \right] . \qquad (8.14)$$

Thus, on the limiting line, q and h_0 become infinite.

From the formulas (8.8) through (8.10), it is seen that q is constant when h_0 is constant. Thus, q is constant along the stream lines, r = constant. These curves are circles (see Fig. 1); the flow is a vortex flow.

9. The General Theory of Simple Waves of Type I in Plane, Rotational, Isentropic Flow of a Polytropic Gas, $\gamma = 5/3$

In this section, it shall be proved that the vortex flows of Section 8 are the only plane, rotational, supersonic flows with simple waves of type I.

To verify this result, consider (8.2), the integral of (5.10), (5.11) and (5.12). For $\gamma = 5/3$, (8.2) becomes

$$q^2 = \frac{f^2}{c^4} + c^2 . \qquad (9.1)$$

If a function $p(\alpha)$ and functions $\bar{c}(\alpha,\beta)$, $\bar{q}(\alpha,\beta)$, $\bar{h}_0(\alpha,\beta)$ are defined by

$$p^3(\alpha) = f(\alpha) , \qquad (9.2)$$

$$c = p\,\bar{c} , \quad q = p\,\bar{q} , \quad h_0 = p\,\bar{h}_0 , \qquad (9.3)$$

then, (9.1) leads to the relation

$$\bar{q}^2 = \bar{c}^{-4} + \bar{c}^2 . \qquad (9.4)$$

Similarily, (8.3) leads to

$$2\,\bar{h}_0 = \bar{c}^{-4} + 4\bar{c}^2 .$$

If the new dependent variables are introduced,

$$y = \bar{c}^3 , \qquad s = p'/p$$

and the metric coefficients (6.5) and (6.6) associated with simple waves of type I are used

$$B = 1, \qquad A = -\beta\theta' + g ,$$

then (5.11) and (5.12) reduce, respectively, to

$$\frac{\partial y}{\partial \alpha} = \theta' + \frac{12sy^3 - 9sy}{2(1-2y^2)} , \qquad (9.6)$$

$$\frac{\partial y}{\partial \beta} = -\frac{\theta' y}{A} + \frac{6sy^2}{A(1-2y^2)} . \qquad (9.7)$$

The theory for solving the overdetermined system, (9.6) and (9.7), is well known. The integrability condition,

$$\frac{\partial^2 y}{\partial \alpha \partial \beta} = \frac{\partial^2 y}{\partial \beta \partial \alpha} ,$$

is formed. This leads to a functional relation of the type

$$f\left(y, s, \theta, A, s', \theta'', \frac{\partial A}{\partial \alpha}\right) = 0 .$$

When this last relation is differentiated and compared with (9.6) and (9.7), then proper choice of s, θ', A, must lead to identities. This is due to the result shown in Section 8 that no relation of the type

$$k(y, s, \theta', s', \theta'') = 0$$

can exist for simple waves (that is, c cannot be a function of α only).

The details of this procedure are carried out for the case when

$$s = 0 , \; p(\alpha) = \text{constant} .$$

Then (9.6) and (9.7) reduce to

$$\frac{\partial y}{\partial \alpha} = \theta' , \qquad \frac{\partial y}{\partial \beta} = -\frac{\theta' y}{A} . \qquad (9.8)$$

The integrability condition of (9.8) leads to

$$\frac{\theta'}{A} \frac{\partial y}{\partial \alpha} + y \frac{\partial}{\partial \alpha}\left(\frac{\theta'}{A}\right) = 0 .$$

Substituting from (9.8) into this last equation, the following result is obtained

$$y = A\Big/\frac{d}{d\alpha}\left(\frac{g}{\theta'}\right) . \qquad (9.9)$$

This relation determines y (or $\bar{c}$) as a function of α, β. It must now be determined whether θ, g can be chosen so that (9.8) is satisfied. From (9.9) it follows that the second relation of (9.8) is identically satisfied. The first relation is satisfied if and only if

$$g = a\,\theta\,\theta' , \qquad (9.10)$$

where a is a constant. From (9.10) it follows that (9.9) can be written as

$$y = \bar{c}^3 = \frac{1}{a}\,(-\beta + a\,\theta) . \qquad (9.11)$$

From (9.3) and the fact that $p(\alpha)$ is constant, it is seen that

$$c^3 = \frac{p}{a}\,r ,$$

where $r = -\beta + a\,\theta$. Further, from (5.4) and (5.5), it follows that the metric coefficients

$$B = 1 , \quad A = -\beta\,\theta' + a\,\theta\,\theta'$$

determine the mapping discussed in Section 6 (see (6.9), (6.10)) with θ replacing the variable α. The flows are the vortex flows of Section 8.

The case when $p(\alpha)$ is not constant can be treated in the same manner. But in this general case where $s \neq 0$ (see (9.6), (9.7)), the computations are rather difficult. Present calculations indicated that no simple waves of type I exist in this case.

10. Simple Waves of Type II

This class of simple waves will be defined by the condition that the Mach number is constant along a bicharacteristic or

$$M = q/c = M(\alpha) . \qquad (10.1)$$

First, note that for irrotational flows of a polytropic gas, M is always constant along a simple wave. Again, for rotational flows of polytropic gases ($\gamma = 5/3$), it follows from (3.8) that

$$2h_0 = (M^2 + 3)\,c^2 . \qquad (10.2)$$

If simple waves are defined by the condition that the bicharacteristics are straight lines (see (6.1)), then (5.10) shows that when (10.1) is valid and $q > c$,

$$c = c(\alpha) .$$

Since h_0 is a function of only α, by (5.12) the flows are cavitation flows ($c = 0$). Thus, the vortex flows of Section 7 for $q = c$ are the only flows in which both the bicharacteristics are straight lines and the Mach number is constant along a bicharacteristic.

Consider the properties of the class of a simple wave of type II defined by (10.1). Expressing (5.10) in terms of the Mach number, M, and using the defining relation (10.1), the relation

$$3 \ln c + \lambda M^2 \ln \lambda + \theta + f(\alpha) = 0 \qquad (10.3)$$

is obtained, where $f(\alpha)$ is an arbitrary function of α and

$$\lambda = \sqrt{M^2-1}/M^2 . \qquad (10.4)$$

Now, (5.12) may be written as

$$B \frac{\partial}{\partial\alpha} [(M^2 + 3) c^2] + A\sqrt{M^2-1} \quad (M^2 + 3) \frac{\partial c^2}{\partial \beta} = 0 . \qquad (10.5)$$

Simplifying this last relation, the following equation is obtained

$$B \frac{\partial}{\partial\alpha} \ln [(M^2 + 3) c^2] + 2A\sqrt{M^2-1} \frac{\partial \ln c}{\partial\beta} = 0 . \qquad (10.6)$$

By use of (5.6) and (10.6), it is seen that

$$\frac{\partial}{\partial\alpha} \ln B = \frac{A}{B} \frac{\partial\theta}{\partial\beta} = - \frac{\frac{\partial}{\partial\alpha} [\ln (M^2 + 3)c^2]}{2\sqrt{M^2-1} \frac{\partial \ln c}{\partial\beta}} \frac{\partial\theta}{\partial\beta} . \qquad (10.7)$$

Use of (5.10) or (10.3) leads to the result

$$\frac{\partial}{\partial\alpha} \ln B = \frac{3}{2M^2} \frac{\partial}{\partial\alpha} [\ln (M^2 + 3) c^2] . \qquad (10.8)$$

The equation (5.11) can now be replaced by (see (10.8) or (10.3))

$$\frac{\partial}{\partial\alpha} (c \lambda M^2) = - \frac{9(M^2-2)}{4(M^2-3)} \lambda c \frac{\partial}{\partial\alpha} [\ln(M^2 + 3) c^2] + c \frac{\partial}{\partial\alpha} [3\lambda \ln c + \lambda M^2 \ln \lambda] +$$

$$+ c \frac{df}{d\alpha} + \frac{2M^2-5}{4\lambda M^2(M^2-3)} c \frac{\partial}{\partial\alpha} [(M^2+3) c^2] . \qquad (10.9)$$

The last relation may be written as

$$U \frac{\partial \ln c}{\partial\alpha} - 3 \frac{d\lambda}{d\alpha} (\ln c) + W = 0 , \qquad (10.10)$$

where

$$U = \lambda M^2 + \frac{9(M^2-2)\lambda}{2(M^2-3)} - 3\lambda - \frac{(2M^2-5)(M^2+3)}{2\lambda M^2(M^2-3)} \qquad (10.11)$$

$$W = \frac{d}{d\alpha}(\lambda M^2) + \frac{9(M^2-2)\lambda M}{2(M^2-3)(M^2+3)} - \frac{df}{d\alpha} - \frac{d}{d\alpha}(\lambda M^2 \ln\lambda) \frac{(2M^2-5)}{2\lambda M(M^2-3)} .$$

Since (10.10) is linear in (ln c), the solution of this equation is

$$\ln c = -e^{3\bar{\lambda}} \int \frac{W}{U} e^{-3\bar{\lambda}} d\alpha + e^{3\bar{\lambda}} g(\beta) , \qquad (10.12)$$

where $g(\beta)$ is an arbitrary function of β and

$$\bar{\lambda} = \int \frac{d\lambda}{U} .$$

The relation (10.8) can now be integrated to find the metric coefficient, B. It is found that

$$\ln B = \int \frac{3}{M(M^2+3)} dM + 3 \int \frac{1}{M^2} \left.\frac{\partial \ln c}{\partial M}\right)_{\beta} dM + h(\beta) ,$$

where $h(\beta)$ is an arbitrary function of β. By use of (10.12), this last formula becomes

$$\ln B = \int \frac{3}{M(M^2+3)} dM + 3 \int \frac{1}{M^2} \frac{d}{d\alpha} \left\{ e^{-3\bar{\lambda}} \int \frac{W}{U} e^{3\bar{\lambda}} d\alpha \right\} dM$$

$$+ 3g(\beta) \int \frac{1}{M^2} \frac{d}{d\alpha} (e^{3\bar{\lambda}}) d\alpha + h(\beta) . \qquad (10.13)$$

By use of (10.6) and (10.12), the metric coefficient, A, may be determined. Since the expression for this coefficient is complicated and will not be used in the remainder of the paper, it will not be discussed any further.

There remains one further problem: to determine the conditions on the functions $M(\alpha)$, $f(\alpha)$, $g(\beta)$, and $h(\beta)$ in order that the Riemann equation (5.8) may be satisfied. In view of the fact that $\theta(\alpha,\beta)$ is determined by (10.4) in terms of $M(\alpha)$, $f(\alpha)$, and $c(\alpha,\beta)$, it is necessary only to show that (5.6) and (5.7) are satisfied. Now (10.7) implies that (5.6) is satisfied. Hence, only (5.7) remains to be satisfied; this relation determines the conditions on the functions $M(\alpha)$, $f(\alpha)$, $g(\beta)$, and $h(\beta)$. By use of (10.3) and (10.12), (5.7) leads to

$$-\frac{1}{B}\frac{\partial A}{\partial \beta} = H(\alpha) + K(\alpha)\, g(\beta) \,, \tag{10.14}$$

where

$$H(\alpha) = -\frac{df}{d\alpha} - \frac{d}{d\alpha}(\lambda\, M^2 \ln \lambda) - 3\lambda\, e^{3\bar{\lambda}} \int \frac{W}{U} e^{-3\bar{\lambda}}\, d\alpha \,, \qquad K(\alpha) = -3\lambda\, e^{3\bar{\lambda}} \,. \tag{10.15}$$

Through use of (10.6), it is found that

$$-\frac{1}{B}\frac{\partial A}{\partial \beta} = \frac{1}{B}\frac{\partial}{\partial \beta}\left[\frac{B \frac{\partial}{\partial \alpha} \ln(M^2+3)\; c^2}{2\,\lambda\, M^2 \frac{\partial}{\partial \beta} \ln c}\right] . \tag{10.16}$$

Expanding the right-hand side of (10.16), through the use of (10.13) and (10.12) the following relation is obtained

$$-\frac{1}{B}\frac{\partial A}{\partial \beta} = Q\left[\frac{h'}{g'} + \frac{d}{d\beta}\left(\frac{1}{g'}\right)\right] + R\left[\frac{h'g}{g'} + \frac{d}{d\beta}\left(\frac{g}{g'}\right)\right] + 3Q\left[\int \frac{1}{M^2}\left(\frac{d}{d\alpha} e^{3\bar{\lambda}}\right) d\alpha\right]$$

$$+\; 3R\left[\int \frac{1}{M^2}\left(\frac{d}{d\alpha} e^{3\bar{\lambda}}\right) d\alpha\right] g \tag{10.17}$$

where

$$h' = \frac{dh}{d\beta} \,, \qquad g' = \frac{dg}{d\beta} \,,$$

$$Q(\alpha) = -\frac{1}{2\lambda\, e^{3\bar{\lambda}} M^2}\frac{d}{d\alpha}\left[2e^{3\bar{\lambda}} \int \frac{W}{U} e^{-3\bar{\lambda}}\, d\alpha + \ln(M^2+3)\right] , \tag{10.18}$$

$$R(\alpha) = \frac{1}{\lambda\, e^{3\bar{\lambda}} M^2}\frac{d}{d\alpha} e^{3\bar{\lambda}} \,.$$

From (10.14) and (10.17), find that $g(\beta)$, $h(\beta)$, $M(\alpha)$, and $f(\alpha)$ must be such that

$$\overline{H} + \overline{K}g = Q\left[\frac{h'}{g'} + \frac{d}{d\beta}\left(\frac{1}{g'}\right)\right] + R\left[\frac{h'g}{g'} + \frac{d}{d\beta}\left(\frac{g}{g'}\right)\right]. \qquad (10.19)$$

This is the basic condition on $M(\alpha)$, $f(\alpha)$, $g(\beta)$, and $h(\beta)$, where

$$\overline{H} = H - 3Q\int\frac{1}{M^2}\left(\frac{d}{d\alpha}\; e^{3\lambda}\right)d\alpha\ ,$$

$$\overline{K} = K - 3R\int\frac{1}{M^2}\left(\frac{d}{d\alpha}\; e^{3\lambda}\right)d\alpha\ . \qquad (10.20)$$

It shall be shown that a necessary and sufficient condition that (10.19) possess solutions for $M(\alpha)$, $f(\alpha)$, $g(\beta)$, $h(\beta)$ is that the following system of differential equations may be satisfied

$$Q = c_1 R + c_3 \overline{K}\ , \qquad \overline{H} = c_2 R + c_4 \overline{K}\ ,$$

$$g = c_3 l - c_4\ , \qquad p = -c_1 l + c_2\ , \qquad (10.21)$$

where c_1, c_2, c_3, and c_4 are arbitrary constants and (for $g' \neq 0$),

$$l = \frac{h'}{g'} + \frac{d}{d\beta}\left(\frac{1}{g'}\right),$$

$$p = \frac{h'g}{g'} + \frac{d}{d\beta}\left(\frac{g}{g'}\right). \qquad (10.22)$$

First, it will be shown that (10.21) is necessary. In terms of $l(\beta)$, $p(\beta)$, (10.19) may be written

$$\overline{H} + \overline{K}g = Ql + Rp. \qquad (10.23)$$

By differentiation of (10.23) with respect to β, the following equation is obtained,

$$0 = -\overline{K}g' + Ql' + Rp'\ , \qquad (10.24)$$

where primes denote differentiation with respect to β. Solving (10.24) for g' and differentiating the resulting equation with respect to α, it follows that

$$\left[\frac{d}{d\alpha}\left(\frac{Q}{K}\right)\right]\frac{dl}{d\beta} = \left[\frac{d}{d\alpha}\left(\frac{R}{K}\right)\right]\frac{dp}{d\beta}\ , \quad K \neq 0\ . \qquad (10.25)$$

Separating variables in (10.25), the equations

$$\frac{d}{d\alpha}\frac{Q}{\bar{K}} = c_1 \frac{d}{d\alpha}\frac{R}{\bar{K}} , \qquad \frac{dp}{d\beta} = - c_1 \frac{dl}{d\beta} , \qquad (10.26)$$

are obtained, where c_1 is an arbitrary constant. Integrating (10.26), it is found that

$$Q = c_1 R + c_3 \bar{K} , \qquad p = - c_1 l + c_2 , \qquad (10.27)$$

where c_2 and c_3 are arbitrary constants. Thus, two relations of the set (10.21) have been found. Substituting (10.27) into (10.23) results in

$$\frac{\bar{H}}{\bar{K}} - c_2 \frac{R}{\bar{K}} = - g + c_3 l . \qquad (10.28)$$

But (10.28) implies that

$$\bar{H} = c_2 R + c_4 \bar{K} , \qquad g = c_3 l - c_4 , \qquad (10.29)$$

where c_4 is an arbitrary constant. Relations (10.29) are the last two relations of the set (10.21).

By substituting (10.21) into (10.23), it is seen that the latter are satisfied. Thus, the conditions (10.21) are sufficient.

In addition to (10.21), other special solutions of (10.19) may be found. For instance, such a solution is

$$h = a\beta , \qquad g = \beta ,$$
$$\bar{K} = aR , \qquad \bar{H} = aQ + R , \qquad (10.30)$$

where a is a constant. In particular, if $a = 0$,

$$h = 0 , \qquad g = \beta ,$$
$$\bar{K} = 0 , \qquad \bar{H} = R , \qquad (10.31)$$

is a special solution. Thus, (10.13) leads to

$$\ln B = m(\alpha) + \beta\, n(\alpha) \qquad (10.32)$$

and (10.3) and (10.12), with $g(\beta) = \beta$, leads to

$$\theta = r(\alpha) + \beta\, s(\alpha) , \qquad (10.33)$$

where the particular functions m, n, r, and s are of no significance in the future work. In fact, from (10.32), (10.33), and (5.4), the geometric structure of the bicharacteristic, α = constant can immediately be determined. Integrating (5.4), it is found that

$$x - x_o = \bar{B} \cos(\theta - \psi) , \quad y - y_o = \bar{B} \sin(\theta - \psi), \tag{10.34}$$

where

$$\tan \psi = s/\sqrt{s^2 + n^2}$$

$$\bar{B} = B/\sqrt{s^2 + n^2}$$

and x_o, and y are functions of α. From (10.34), it follows that the bicharacteristics, α = constant, are spirals.

REFERENCES

1. The relation (2.3) is easily derived. See "Intrinsic Relations Satisfied by the Velocity and Vorticity Vectors", N. Coburn, Michigan Math J, I, No. 2 (1952).

2. "Discontinuities in Compressible Fluid Flows", N. Coburn, Mathematics Magazine (May-June, 1954). The theory of discontinuity manifolds for nonisentropic flows is similar to that of isentropic flows and was worked out by Mr. J. McCully (not published yet).

3. "Supersonic Flow and Shock Waves", R. Courant and K. O. Friedrichs, Interscience Publishers, see formula (102.06) p. 247 for the isentropic two-dimensional case.

4. "The Method of Characteristics in the Three-Dimensional Stationary Supersonic Flow of a Compressible Gas", N. Coburn and C. L. Dolph; Proc. of the First Symposium of Applied Math., 1947, Am. Math. Soc., 1947.

5. "Einführung in die Neuren Methoden der Differentialgeometrie", J. A. Schouten and D. J. Striuk, P. Noordhoff, Groningen, Batavia, Vol. II, 1938, p. 38.

6. See Reference 3, p 22, formula (14.03). The relations (3.7) are equivalent to the equations of motion. If h_o is eliminated from these equations through the use of (3.8), then the resulting equations, and (2.5), and the energy relation (2.4) form a system of five equations in the five unknowns, c, S, v^j.

7. "Supersonic Flow and Shock Waves", R. Courant and K. O. Friedrichs, Interscience Press, N. Y., (1948), p 40; "Characteristic Directions in Three-Dimensional Supersonic Flow", N. Coburn, Proc. Am. Math., 1, No. 2, 241-245 (April, 1950). It should be noted that the canonical system has one very important property; each equation possesses directional derivatives in only one direction. This is a highly desired property. Actually this property can also be extended to characteristic systems by replacing (3.11) by the equation corresponding to (3.10) for the second characteristic direction. But a price must be paid for this advantage; the right-hand sides of (4.3), (4.9), and (4.10) contain curvatures of the various manifolds. Finally, it should be noted that when the unit vectors t^j, n^j, and l^j do not form a triply orthogonal coordinate system, then it may be advantageous to use a full characteristic system, since the unit vectors t^j, n^j, and l^j (where l^j is a unit vector along the intersection of the characteristic manifolds) always determine such a coordinate system.

REFERENCES (cont.)

8. Relations (5.4) through (5.8) are well known in differnetial geometry. Equation (5.8) implies that the Riemann tensor of the plane vanishes; this equation is the integrability condition for Equations (5.6) and (5.7).

9. For another formulation see, "Characteristic Conditions for Three-Dimensional Flows with Vorticity", R. J. Clippinger and J. H. Giese, Ballistic Research Labs, No. 615.

10. "Steady Rotational Flow of an Ideal Gas", R. C. Prim, J. Rational Mech. Anal. I., No. 3, 425-497 (1952).

11. See Reference 3, p.265, formula (108.08).

12. In the irrotational case, discussed in Reference 3, p.60, the diametrically opposite theorem is shown to be valid.

13. See Reference 3, p 59-62. Also see "Compressible Flow with Degenerate Hodograph", J H Giese, Quart. Appl. Math., 9, 237-246, (1951).

14. See Reference 10, p 428. Also see "Vorticity and the Thermodynamic State in a Gas Flow", C. A. Truesdell, Memorial des Sci. Math., No. 119, 36-39, Gauthier-Villars, Paris (1952).

15. The Absolute Differential Calculus, T. Levi-Civita, Blackie and Son, London, 1929, p. 15.

DISTRIBUTION LIST

No. of Copies	Agency
10	Office of Ordnance Research Box CM, Duke Station Durham, North Carolina
1	Office, Chief of Ordnance Washington 25, D. C. Attn: ORDTB-PS
2	Commanding General Aberdeen Proving Grounds, Maryland Attn: BRL
1	Commanding Officer Redstone Arsenal Huntsville, Alabama
1	Commanding General Research and Engineering Command Army Chemical Center, Maryland
1	Chief, Ordnance Development Division National Bureau of Standards Washington 25, D.C.
1	Commanding Officer Watertown Arsenal Watertown 72, Massachusetts
1	Commanding Officer Engineer Research and Development Lab. Fort Belvoir, Virginia
1	Commander U. S. Naval Proving Ground Dahlgren, Virginia
1	Chief, Bureau of Ordnance (AD3) Department of the Navy Washington 25, D.C.
1	U. S. Naval Ordnance Laboratory White Oak, Silver Spring 19 Maryland Attn: Library Division

No. of Copies	Agency
1	Director National Bureau of Standards Washington 25, D.C.
1	Carona Laboratories National Bureau of Standards Corona, California
1	Commanding Officer Frankford Arsenal Bridesburg Station Philadelphia 37, Pennsylvania
1	Technical Information Service P.O. Box 62 Oak Ridge, Tennessee Attn: Reference Branch
1	The Director Naval Research Laboratory Washington 25, D.C. Attn: Code 2021
1	Jet Propulsion Laboratory California Institute of Technology 4800 Oak Grove Drive Pasadena 3, California
1	Director, Applied Physics Laboratory Johns Hopkins University 8621 Georgia Avenue Silver Spring 19, Maryland
2	Chief, Detroit Ordnance District 574 East Woodbridge Detroit 31, Michigan Attn: ORDEF-IM
1	Commanding General Air University Maxwell Air Force Base, Alabama Attn: Air University Library

No. of Copies	Agency
5	Document Service Center U. B. Building Dayton 2, Ohio Attn: DSC-SD
1	Commanding General Air Research and Development Command P. O. Box 1395 Baltimore 3, Maryland Attn: RDD
1	Commanding General Air Research and Development Command P. O. Box 1395 Baltimore 3, Maryland Attn: RDR
1	Commander U.S. Naval Ord. Test Station, Inyokern China Lake, California Attn: Technical Library
1	U. S. Atomic Energy Commission Document Library 19th and Constitution Avenue Washington 25, D. C.
1	Commanding General Air Material Command Wright-Patterson Air Force Base Dayton 2, Ohio Attn: F. N. Bubb, Chief Scientist Flight Research Laboratory
1	NAC for Aeronautics 1724 F Street, N.W. Washington 25, D.C. Attn: Mr. E. B. Jackson, Chief Office of Aeronautical Intel.
1	Scientific Information Section Research Branch Research and Development Division Office, Assistant Chief of Staff, G-4 Department of the Army Washington 25, D.C.

No. of Copies	Agency
1	Office of Naval Research Washington 25, D.C.
1	Canadian Joint Staff 1700 Massachusetts Ave., N.W. Washington 6, D.C. Attn: ORDGU-SE
1	Commanding General White Sands Proving Grounds Las Cruces, New Mexico