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Higher Eta-Invariants

JOHN LOTT*
Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, U.S.A.,
e-mail: lott@math.lsa.umich.edu

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Abstract. We define the higher eta-invariant of a Dirac-type operator on a nonsimply-connected closed manifold. We discuss its variational properties and how it would fit into a higher index theorem for compact manifolds with boundary. We give applications to questions of positive scalar curvature for manifolds with boundary, and to a Novikov conjecture for manifolds with boundary.

Key words. Eta-invariants, Dirac-type operators, higher index theorem, compact manifolds.

1. Introduction

The eta-invariant is a spectral invariant of Dirac-type operators on closed manifolds. It was introduced by Atiyah, Patodi and Singer [2] in order to prove an index theorem for elliptic operators on manifolds with boundary. Let W be an even-dimensional compact smooth spin manifold with boundary M. Give W a Riemannian metric which is a product near M. Let V be a Hermitian vector bundle with connection on W, also a product near the boundary. Denote the Dirac-type operator on W, acting on spinors which satisfy the APS boundary conditions, by Q_W, and the Dirac-type operator on M by Q_M. Suppose, for simplicity, that Q_M is invertible. Then the index theorem states

$$\mathrm{Index}(Q_W) = \int_W \hat{A}(W) \wedge \mathrm{Ch}(V) - \frac{1}{2}\eta(Q_M). \tag{1}$$

Note that while the left-hand side of (1) is a deformation-invariant, being the index of a Fredholm operator, neither term of the right-hand side of (1) is topological in nature. The integrand in (1) is a specific differential form on W. It is only the combination of the two terms on the right-hand side of (1) which has topological meaning.

By considering eta-invariants of Dirac-type operators coupled to flat vector bundles on M, one can also form the rho-invariant, an analytic expression with topological meaning [3]. We review some of this theory in Section 2.

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The index theorem (1) is a 'lower' index theorem, in that it does not involve the fundamental group of W. A 'higher' index theorem for closed manifolds is due to the work of Mischenko, Kasparov, Connes and Moscovici, and others. To state it, suppose X is a closed spin Riemannian manifold with fundamental group Γ. Let $\nu: X \to B\Gamma$ be the classifying map for the universal cover of X, defined up to homotopy. If one takes the fundamental group into account, one can refine the index of the Dirac-type operator to become a higher index living in the K-theory of the reduced group C^*-algebra $C_r^*(\Gamma)$. Under favorable conditions on Γ, such as Γ being hyperbolic [18], one can pair the higher index with the group cohomology of Γ, and the higher index theorem states

$$\langle \mathrm{Index}(Q_X), \tau \rangle = (\hat{A}(X) \cup \mathrm{Ch}(V) \cup \nu^*(\tau))[X], \tag{2}$$

for all $\tau \in H^*(B\Gamma; \mathbf{C})$.

In this paper we consider the 'higher' version of (1). That is, we want an index theorem for manifolds with boundary which involves the cohomology of the fundamental group of W.

Due to the nontopological nature of the integral in (1), it is clear that one first needs a way of proving (2) which gives the right-hand side as the integral of an explicit local expression over X. Using Quillen's theory of superconnections [36], we gave such a local expression in [31].

The next problem is to define a higher eta-invariant, an object which pairs with group cohomology. Our main interest is in the possible geometric and topological applications. There are some hints as to the right approach to the higher eta. First, there is an L^2-eta-invariant [15], which should be the pairing of the higher eta-invariant with $H^0(B\Gamma; \mathbf{C}) \cong \mathbf{C}$. The analog of (1) has been proven in this case [37]. Second, a higher rho-invariant has been defined for the signature operator by purely topological means [43].

An early approach to the index theorem of (2) when Γ is free Abelian, due to Lusztig, was to apply the families index theorem to a certain fibration which is canonically associated to X [34]. In the first half of this paper, we use this method to define the higher eta-invariant in some cases in which Γ is virtually Abelian i.e. has an Abelian subgroup of finite index. We have two reasons for using this approach. First, it involves 'commutative' analysis which may be more familiar to readers, thereby giving some justification for the noncommutative approach of the second half. Second, one obtains stronger results this way than for more general Γ. We initially consider the case when $\Gamma = \mathrm{F} \times \mathbf{Z}^k$, with F finite. The base of the above fibration is then $\hat{\mathrm{F}} \times \mathbf{T}^k$. An eta-form $\tilde{\eta}$, a differential form on the base of a fibration, was defined by Bismut and Cheeger [8]. In Sections 3.1.1–3.1.6, we analyze in detail this eta-form in the case of Lusztig's fibration. We look at how $\tilde{\eta}$ changes under conformal variations of the metric, and under arbitrary variations of the input data. In Section 3.1.7, we state a higher index theorem for manifolds with boundary, based on the results of [9], and give an application to the question of whether a closed positive-scalar-curvature (p.s.c) manifold can be the boundary of a p.s.c. manifold with a product metric near the boundary.

In Section 3.2, we consider the case when Γ is the semidirect product of $\mathbf{Z}^k$ and a finite group F. The space on which the eta-form lives turns out to be an orbifold of the type used in [5] in order to define 'delocalized' equivariant cohomology. In particular, the higher rho-invariant is a delocalized element of equivariant cohomology.

The second half of the paper is concerned with more general Γ. The idea is to work with a fibration as above, except that now the base is a noncommutative space B whose algebra of 'continuous functions' is $C_r^*(\Gamma)$. If $\mathscr{B}$ is a subalgebra of $C_r^*(\Gamma)$ consisting of 'smooth' functions, the 'homology' of B is taken to be the periodic cyclic cohomology of $\mathscr{B}$. The vector space of 'differential forms' on B is taken to be the universal graded differential algebra of $\mathscr{B}$, modulo its commutator. We start by reviewing some results on the cyclic cohomology of the group algebra $\mathbf{C}\Gamma$ in Section 4.1. We relate the results on semidirect product groups to cyclic cohomology in Section 4.2.

The main idea of this paper, along with [31], is to use superconnections in the context of noncommutative geometry. The paper [31] was concerned with expressing the Chern character of the higher index as an explicit closed differential form on B. In Section 4.3, we review some of the needed results of [31]. The higher eta-invariant $\tilde{\eta}$ is defined as a differential form on B in Section 4.4. To show that the formal expression for $\tilde{\eta}$ actually makes sense, we assume that the Dirac-type operator on the Γ-cover M' of M is invertible and that Γ is virtually nilpotent, i.e. of polynomial growth [21]. These technical conditions arise because unlike the Chern character, the higher eta-invariant involves heat kernels at arbitrarily large time, and unlike the L^2-eta-invariant, it involves heat kernels between arbitrarily distant points on M'. We use finite-propagation-speed methods to control these problems. The algebra $\mathscr{B}$ is taken to be the natural 'smooth' subalgebra of $C_r^*(\Gamma)$. In Section 4.5, we look at how $\tilde{\eta}$ changes as one varies the input data. As with the lower eta-invariant, we find that the variation is given by the integral of a local expression.

We define the higher rho-invariant to be the part of $\tilde{\eta}$ corresponding to nontrivial conjugacy classes in Γ. It is a closed differential form on B. In Section 4.6, we discuss the properties of the Chern character and the higher rho-invariant with respect to the periodicity operator in cyclic homology.

In Section 4.7, we consider the case of signature operators, and show how the higher eta- and rho-invariants can give a wider range of definition by making the signature operator on M' effectively invertible. Modulo technical conditions on Γ, our analytic higher rho-invariant is defined under the same circumstances as the topological higher rho-invariant of [43] and takes value in the same group. We propose, but do not prove, a higher index theorem for manifolds with boundary in Section 4.8. In Section 4.9 we use the higher eta-invariant to formulate a Novikov conjecture for manifolds with boundary. In Section 4.10 we look at the pairing of $\tilde{\eta}$

with 0-cocycles and 1-cocycles, where the formulas can be made more explicit. Finally, we conclude with some remarks.

2. The Lower Eta-Invariant

Let M^n be a connected closed smooth manifold. For purposes of exposition, suppose that the fundamental group Γ of M is finite. Then given a representation $\rho: \Gamma \to \mathrm{U}(N)$, there is an associated flat Hermitian $\mathbf{C}^N$-bundle $E_\rho = \widetilde{M} \otimes_\rho \mathbf{C}^N$ on M.

The input information needed to define the eta-invariant consists of

(1) A Riemannian metric on M.
(2) A Clifford module over M. For simplicity, we will assume that M is spin, n is odd and that the Clifford module is of the form $S \otimes V$, where S is the spinor bundle over M and V is a Hermitian vector bundle with connection.

There is a self-adjoint densely-defined Dirac-type operator Q_ρ acting on L^2-sections of $S \otimes V \otimes E_\rho$, with discrete spectrum.

DEFINITION 1 [2]. The eta-invariant is

$$\eta_\rho = \frac{2}{N\sqrt{\pi}} \int_0^\infty \mathrm{TR}(Q_\rho \, \mathrm{e}^{-s^2 Q_\rho^2}) \, \mathrm{d}s \in \mathbf{R}. \tag{3}$$

The integral in (3) is absolutely convergent [11]. Formally,

$$\eta_\rho = \frac{1}{N} \mathrm{TR}\left(\frac{Q}{|Q|}\right).$$

An important point about η_ρ is that if Q_ρ is invertible then as one varies the input information, the variation of η_ρ is given by the integral of a local expression on M [3]. (More generally, it is enough to assume that $\dim(\mathrm{Ker}(Q_\rho))$ is constant during the variation.)

A special case of geometric interest is when V is a vector bundle associated to the principal $\mathrm{Spin}(n)$-bundle of M by some representation σ of $\mathrm{Spin}(n)$. Then the Chern character $\mathrm{Ch}(V)$ is a polynomial in the Pontryagin classes and the Euler class of M, which can be computed from σ. Suppose that $\mathrm{Ch}(V)$ is a polynomial in the Pontryagin classes of M, i.e. does not involve the Euler class. Then the same is true for the index density $\hat{A}(M) \wedge \mathrm{Ch}(V)$, and it turns out that the local expression for the variation of η_ρ vanishes for conformal deformations of the Riemannian metric [3].

For general V, the locality of the expression for the variation of η_ρ implies that the variation is independent of ρ. Thus if ρ_1 and ρ_2 are two representations of Γ such that Q_{ρ_1} and Q_{ρ_2} are invertible, then the rho-invariant $\eta_{\rho_1} - \eta_{\rho_2}$ is a deformation-invariant.

Sometimes it is more convenient to look at the reduced eta-invariant

$$\eta'_\rho = \frac{\eta_\rho + \frac{1}{N}\dim(\mathrm{Ker}(Q_\rho))}{2}\left(\mathrm{mod}\,\frac{1}{N}\mathbf{Z}\right).$$

Then η'_ρ has a local expression for its variation, without qualifications, and so $\eta'_{\rho_1} - \eta'_{\rho_2}$ is a smooth invariant of the pair (M, V). It follows from the index theorem of [2] that if V is associated to the principal Spin(n)-bundle of M then $\eta'_{\rho_1} - \eta'_{\rho_2}$ gives a map from the bordism group $\Omega_n^{\mathrm{spin}}(B\Gamma)$ to $\mathbf{R}/(1/N_1N_2)\mathbf{Z}$. (As $\Omega_n^{\mathrm{spin}}(B\Gamma)$ is torsion, this map actually takes values in $\mathbf{Q}/(1/N_1N_2)\mathbf{Z}$.) However, in this paper we will always take rho-invariants to be real-valued.

Instead of considering representations of Γ, it will turn out to be useful to think of the eta-invariant as something computed on the universal cover $\tilde{M}$ of M. Let $\gamma \in \Gamma$ act on $\tilde{M}$ on the right by a diffeomorphism $R_\gamma \in \mathrm{Diff}(\tilde{M})$. Let $\tilde{V}$ be the pullback of V to $\tilde{M}$. Let $\tilde{Q}$ be the Dirac-type operator on L^2-sections of $\tilde{S} \otimes \tilde{V}$. Suppose that $\tilde{Q}$ is invertible. Then we can define an equivariant eta-invariant on $\tilde{M}$, a function from Γ to $\mathbf{C}$, by

$$\eta(\gamma) = \frac{2}{\sqrt{\pi}}\int_0^\infty \mathrm{TR}(R_\gamma^* \tilde{Q}\, \mathrm{e}^{-s^2\tilde{Q}^2})\,\mathrm{d}s \in \mathbf{C}. \tag{4}$$

The relationship between η_ρ and the η-function of (4) is simply that if χ_ρ is the character of the representation ρ then

$$\eta_\rho = \sum_{\gamma\in\Gamma} \frac{1}{N}\chi_\rho(\gamma)\eta(\gamma).$$

The evaluation $\eta(e)$ of η at the trivial element e has a local variation, and the variation of $\eta(\gamma)$ vanishes for $\gamma \neq e$.

Alternatively, we can define an element of the group algebra by

$$\eta = \sum_{\gamma\in\Gamma} \eta(\gamma)\gamma \in \mathbf{C}\Gamma. \tag{5}$$

The cyclic cohomology group $HC^0(\mathbf{C}\Gamma)$ is simply the vector space of traces on $\mathbf{C}\Gamma$, and decomposes according to the conjugacy classes of Γ:

$$HC^0(\mathbf{C}\Gamma) = \bigoplus_{\langle x\rangle\in\langle\Gamma\rangle} \mathbf{C}\mathscr{T}_{\langle x\rangle}, \tag{6}$$

where for a conjugacy class $\langle x\rangle \in \langle\Gamma\rangle$, the trace $\mathscr{T}_{\langle x\rangle}$ is given by

$$\mathscr{T}_{\langle x\rangle}\left(\sum_{\gamma\in\Gamma} c_\gamma\gamma\right) = \sum_{\gamma\in\langle x\rangle} c_\gamma.$$

We can think of the cohomology group $H^0(\Gamma; \mathbf{C}) = \mathbf{C}$ as being the summand $\mathbf{C}\mathscr{T}_{\langle e\rangle}$ in (6); although this identification may seem artificial at the moment, it is the zero-dimensional case of a general statement about the cyclic cohomology of group algebras, as will be discussed in Section 4.1. Then we can summarize the variational properties of the eta-invariant by saying that the pairings of the η of (5) with the elements of $H^0(\Gamma; \mathbf{C})$ have a local variation, while the pairings of η with the remaining elements of $HC^0(\mathbf{C}\Gamma)$ (i.e. those corresponding to a nontrivial conjugacy class) have vanishing variation. These statements are what we will generalize from the zero-dimensional case to higher dimensions in the second half of this paper.

3. Virtually Abelian Fundamental Groups

3.1. PRODUCT GROUPS

3.1.1. *The Basic Setup*

Let M^n be a connected closed smooth manifold with first Betti number k. The Albanese variety A of M is the k-torus $H_1(M; \mathbf{R})/H_1(M; \mathbf{Z})_{\mathrm{modTor}}$ [34]. Given a basis $\{e_i\}_{i=1}^k$ of $H_1(M; \mathbf{Z}^k)_{\mathrm{modTor}} \subset H_1(M; \mathbf{R}^k)$, let $\{v^i\}_{i=1}^k$ be an integral dual basis to $\{e_i\}_{i=1}^k$ in $(H_1(M; \mathbf{R}))^*$. The 1-forms $\{\mathrm{d}v^i\}_{i=1}^k$ in $\wedge^1(H_1(M; \mathbf{R}))$ descend to forms on A. Fix basepoints $m_0 \in M$ and $a_0 \in A$. There is a canonical homotopy class of basepoint-preserving maps from M to A constructed as follows: If $\{\omega^i\}_{i=0}^k$ are closed 1-forms on M which represent $\{v^i\}_{i=1}^k$ in $H^1(M; \mathbf{R}) \cong (H_1(M; \mathbf{R}))^*$, there is a map v_ω from M to A given by

$$v_\omega(m) = a_0 + \sum_i \left(\int_{m_0}^{m} \omega^i \right) e_i.$$

The desired canonical homotopy class is that of v_ω. Given a basepoint-preserving map $f: M \to A$ in this homotopy class, if we choose $\omega^i = f^* \mathrm{d}v^i$ then we recover f as $f = v_\omega$.

We will denote the dual torus to A by P, for Picard variety.

Note. The k-tori A and P will play very different roles in what follows. One should think of A as the classifying space $B\mathbf{Z}^k$, whereas P should be thought of as the dual group $\widehat{\mathbf{Z}^k}$.

There is a double fibration

$$M \xleftarrow{\pi_1} M \times P \xrightarrow{\pi_2} P$$

and a canonical line bundle E_0 on $M \times P$ given as follows: Let H be the Hermitian line bundle over $A \times P$ which is the quotient of $\mathbf{R}^k \times (\mathbf{R}^k)^* \times \mathbf{C}$ by the action of $\mathbf{Z}^k \times (\mathbf{Z}^k)^*$, where $(\gamma, \gamma^*) \in \mathbf{Z}^k \times (\mathbf{Z}^k)^*$ acts by

$$(v, v^*, z) \to (v + \gamma, v^* + \gamma^*, \mathrm{e}^{2\pi i v^*(\gamma)} z).$$

There is a canonical Hermitian connection on H given by the 1-form $-2\pi i \vec{v} \cdot \mathrm{d}\vec{v}^*$ on $\mathbf{R}^k \times (\mathbf{R}^k)^*$. Let $E_0 = (f \times \mathrm{Id})^* H$ be the pulled-back line bundle over $M \times P$, with the pulled-back connection.

Let F be a finite group. Suppose that the fundamental group of M is $\Gamma = \mathrm{F} \times \mathbf{Z}^k$. Let $\rho\colon \mathrm{F} \to \mathrm{U}(N)$ be a unitary representation of F. Let E_ρ be the flat Hermitian $\mathbf{C}^N$-bundle over M specified by ρ. Then we put L_ρ to be $\pi_1^* E_\rho \otimes E_0$, a $\mathbf{C}^N$-vector bundle over $M \times P$.

The input information needed to define the eta-form consists of

(1) A Hermitian connection on E_0, specified by the map $f\colon M \to A$.
(2) A Riemannian metric on M.
(3) A Clifford module over M. For simplicity, we will assume that M is spin, and that the Clifford module is of the form $S \otimes V$, where S is the spinor bundle over M and V is a Hermitian vector bundle with connection. (The analogous results when M is not spin will be straightforward.) If n is even then the Clifford module is $\mathbf{Z}_2$-graded by the grading on S, while if n is odd then the Clifford module is ungraded.

For each $p \in P$, the restriction of L_ρ to $\pi_2^{-1}(p)$ is a flat Hermitian bundle W_p over M, with twisting specified by ρ and p. Thus, we have a family of flat bundles over M parametrized by P.

Let $\mathscr{E}_\rho$ be the infinite-dimensional vector bundle on P such that $C^\infty(\mathscr{E}_\rho) = C^\infty(\pi_1^* S \otimes \pi_1^* V \otimes L_\rho)$. That is, the fiber of $\mathscr{E}_\rho$ over $p \in P$ is $C^\infty(S \otimes V \otimes W_p)$. The Hermitian connection on $\pi_1^* S \otimes \pi_1^* V \otimes L_\rho$ gives a Hermitian connection on $\mathscr{E}_\rho$, by horizontal differentiation, which we will denote by ∇. For each $p \in P$, there is a vertical Dirac-type operator Q_p acting on $C^\infty(S \otimes V \otimes W_p)$, with discrete real spectrum. These vertical operators fit together to give an operator Q acting on $C^\infty(\mathscr{E}_\rho)$. Fix a constant $\beta > 0$. We will abbreviate $\beta^{1/2} d$ by $\hat{d}$. Suppose that U_ρ is an open subset of P such that $\mathrm{Ker}(Q_p)$ forms a vector bundle over U_ρ as p varies in U_ρ.

3.1.2. *The Higher Eta-Invariant*

In what follows we use the superconnection formalism of Quillen [36], along with its extension to the odd-dimensional case [36, §5]. For the relevant notions, see [6, 7, 8, 36]. As for notation, an infinite-dimensional (super)trace will be written as (S)TR, while a finite-dimensional (super)trace will be written as $\mathrm{tr}_{(s)}$. We will write the Chern character of a (super)-vector bundle V as $\mathrm{Ch}_\beta(V) = \mathrm{tr}_{(s)}(\mathrm{e}^{-\beta F_V})$, where F_V is the curvature of a connection on V, and put $\mathrm{Ch}(V) = \mathrm{Ch}_1(V)$.

DEFINITION 2. The superconnection $D_s\colon C^\infty(\mathscr{E}_\rho) \to C^\infty(\mathscr{E}_\rho \otimes \wedge^*(P))$ is given by

$$D_s = \begin{cases} sQ + \nabla, & \text{if } n \text{ is even,} \\ s\sigma Q + \nabla, & \text{if } n \text{ is odd.} \end{cases} \tag{7}$$

DEFINITION 3 [7]. For $s > 0$, the Chern character $\mathrm{ch}_\rho(s) \in \Lambda^*(P)$ *of* $\mathscr{E}_\rho$, a closed form, is given by

$$\mathrm{ch}_\rho(s) = \begin{cases} \dfrac{1}{N}\mathrm{STR}(\mathrm{e}^{-\beta D_s^2}), & \text{if } n \text{ is even,} \\ \dfrac{1}{N}\mathrm{TR}_\sigma(\mathrm{e}^{-\beta D_s^2}), & \text{if } n \text{ is odd.} \end{cases} \tag{8}$$

DEFINITION 4 [8, 19]. The eta-form $\tilde{\eta}_\rho \in \Lambda^*(U_\rho)$ is given by

$$\tilde{\eta}_\rho = \begin{cases} \dfrac{\beta^{1/2}}{N} \displaystyle\int_0^\infty \mathrm{STR}(Q\,\mathrm{e}^{-\beta D_s^2})\,\mathrm{d}s, & \text{if } n \text{ is even,} \\ \dfrac{\beta^{1/2}}{N} \displaystyle\int_0^\infty \mathrm{TR}_\sigma(\sigma Q\,\mathrm{e}^{-\beta D_s^2})\,\mathrm{d}s, & \text{if } n \text{ is odd.} \end{cases} \tag{9}$$

Note. The integral in (9) is well-defined, as is shown in [6]. The reason for dividing by N in the definitions will become clear.

Let $\mathscr{R}_\beta$ be the rescaling operator on $\Lambda^*(P)$ which is multiplication by $\beta^{j/2}$ on $\Lambda^j(P)$. We will let $\vec{v}$ and $\vec{v}^*$ be the local coordinates on A and P, respectively, from Section 3.1.1.

PROPOSITION 1. *The differential forms* $\mathrm{ch}_\rho(s)$ *have a limit as* $s \to 0$, *given by*

$$\lim_{s\to 0} \mathrm{ch}_\rho(s) = \mathscr{R}_\beta\left(\int_M \hat{A}(M) \wedge \mathrm{Ch}(V) \wedge \mathrm{e}^{2\pi i\vec{\omega}\wedge \mathrm{d}\vec{v}^*}\right). \tag{10}$$

Proof. From [7], we have that

$$\lim_{s\to 0} \mathrm{ch}_\rho(s) = \mathscr{R}_\beta\left(\int_M \hat{A}(T^{\mathrm{vert}}(M \times P)) \wedge \mathrm{Ch}(\pi_1^* V \otimes L_\rho)\right).$$

As $T^{\mathrm{vert}}(M \times P) = \pi_1^* TM$,

$$\hat{A}(T^{\mathrm{vert}}(M \times P)) = \pi_1^* \hat{A}(M).$$

Now

$$\mathrm{Ch}(\pi_1^* V \otimes L_\rho) = \pi_1^* \mathrm{Ch}(V) \wedge \pi_1^* \mathrm{Ch}(E_\rho) \wedge \mathrm{Ch}(E_0).$$

As E_ρ is flat, $\mathrm{Ch}(E_\rho) = N$. It remains to compute $\mathrm{Ch}(E_0)$. As in [34], the curvature of H is

$$\mathrm{d}(-2\pi i\vec{v}\cdot \mathrm{d}\vec{v}^*) = -2\pi i\,\mathrm{d}\vec{v} \wedge \mathrm{d}\vec{v}^*.$$

Then the curvature of E_0 is

$$-2\pi i f^*\,\mathrm{d}\vec{v} \wedge \mathrm{d}\vec{v}^* = -2\pi i\vec{\omega} \wedge \mathrm{d}\vec{v}^*,$$

from which the proposition follows. □

Note. The right-hand side of (10) is a polynomial in the forms $\mathrm{d}\bar{v}^*$. The coefficients are higher indices. The index theorem for families [1, 6, 7] says that for all $s > 0$, $\mathrm{ch}_\rho(s)$ represents the Chern character $1/N\ \mathrm{Ch}_\beta(\mathrm{Index}(Q))$ of the index bundle $\mathrm{Index}(Q)$, and *a fortiori* so does the right-hand side of (10).

Let $\nabla_{\mathrm{Ker}(Q)}$ denote the Hermitian connection on $\mathrm{Ker}(Q)$ induced from its embedding in the Hilbert space $L^2(\mathscr{E}_\rho) = L^2(\pi_1^* S \otimes \pi_1^* V \otimes L_\rho)$.

PROPOSITION 2 [8, 19]. *The differential of $\tilde{\eta}_\rho$ on U_ρ is given by*

$$\hat{\mathrm{d}}\tilde{\eta}_\rho = \begin{cases} \mathscr{R}_\beta\left(\displaystyle\int_M \hat{A}(M) \wedge \mathrm{Ch}(V) \wedge \mathrm{e}^{2\pi i\bar{\omega}\wedge \mathrm{d}\bar{v}^*}\right) - \dfrac{1}{N}\mathrm{tr}_s(\mathrm{e}^{-\beta\nabla^2_{\mathrm{Ker}(Q)}}), & \text{if } n \text{ is even,} \\ \mathscr{R}_\beta\left(\displaystyle\int_M \hat{A}(M) \wedge \mathrm{Ch}(V) \wedge \mathrm{e}^{2\pi i\bar{\omega}\wedge \mathrm{d}\bar{v}^*}\right), & \text{if } n \text{ is odd.} \end{cases} \tag{11}$$

Proof. If n is even, then

$$\frac{\mathrm{d}(\mathrm{ch}_\rho(s))}{\mathrm{d}s} = -\frac{\beta}{N}\mathrm{d}\,\mathrm{STR}(Q\,\mathrm{e}^{-\beta D_s^2}).$$

Integrating with respect to s, we obtain

$$\hat{\mathrm{d}}\tilde{\eta}_\rho = \lim_{s\to 0} \mathrm{ch}_\rho(s) - \lim_{s\to\infty} \mathrm{ch}_\rho(s). \tag{12}$$

In the $s \to \infty$ limit, only the kernel of Q contributes to the supertrace in $\mathrm{ch}_\rho(s)$, and one has

$$\lim_{s\to\infty} \mathrm{ch}_\rho(s) = \frac{1}{N}\mathrm{STR}(\mathrm{e}^{-\beta\nabla^2_{\mathrm{Ker}(Q)}}).$$

Along with Proposition 1, this proves the even case. If n is odd, Equation (12) still holds, but

$$\lim_{s\to\infty} \mathrm{ch}_\rho(s) = \frac{1}{N}\mathrm{TR}_\sigma(\mathrm{e}^{-\beta\nabla^2_{\mathrm{Ker}(Q)}}) = 0. \qquad \square$$

We now look at what conclusions can be drawn about the eta-forms without having detailed information about the vector bundle $\mathrm{Ker}(Q)$. We will make successively weaker hypotheses, and will naturally get successively weaker conclusions.

We will want to see how $\tilde{\eta}_\rho$ changes as we vary the input data. The method to compute this is to consider the product bundle $\mathbf{R} \times M \times P \to \mathbf{R} \times P$. The $\mathbf{R}$ factor represents the parameter ε which controls the variation. Let us denote the corresponding eta-form on $\mathbf{R} \times P$ by $\tilde{\sigma}_\rho$. Then $\tilde{\sigma}_\rho \in \Lambda^*(\mathbf{R} \times P)$ can be written as

$$\tilde{\sigma}_\rho = \tilde{\eta}_\rho(\varepsilon) + \beta^{1/2}\,\mathrm{d}\varepsilon \wedge \tilde{\tilde{\eta}}_\rho(\varepsilon),$$

where $\tilde{\eta}_\rho$ and $\tilde{\tilde{\eta}}_\rho$ are forms on P. The differential of $\tilde{\sigma}_\rho$ on $\mathbf{R} \times P$ is given by

$$\hat{\mathrm{d}}\tilde{\sigma}_\rho = \hat{\mathrm{d}}\tilde{\eta}_\rho + \beta^{1/2}\,\mathrm{d}\varepsilon \wedge (\partial_\varepsilon \tilde{\eta}_\rho - \hat{\mathrm{d}}\tilde{\tilde{\eta}}_\rho). \tag{13}$$

Thus, the formulas for the differentials of eta-forms, applied to $\tilde{\sigma}_\rho$, will allow us to compute $\partial_\varepsilon \tilde{\eta}_\rho$ up to an exact form on P. If a quantity is independent of ε, we will say that it is a deformation invariant. (The reason that we do not say simply that it is an invariant is that there may be some restrictions on the operators parametrized by ε, such as invertibility.)

3.1.3. Q_p *Invertible for all* $p \in P$

In this section, we assume that the operators Q_p are invertible for all $p \in P$. We take $U_\rho = P$.

PROPOSITION 3. *The eta-form* $\tilde{\eta}_\rho$ *is closed.*

Proof. In this case Index(Q) is trivial, and so the higher indices of (10) vanish. The result then follows from Proposition 2. □

Thus $\tilde{\eta}_\rho$ represents a cohomology class $[\tilde{\eta}_\rho]$ in $H^*(P; \mathbf{C})$. A priori, this class depends on all of the choices made, namely

(1) The map $f\colon M \to A$.
(2) The Riemannian metric on M.
(3) The Hermitian connection on V.

PROPOSITION 4. *Suppose that the vector bundle V is associated to the principal* Spin(n)*-bundle on M. Suppose that the index density $\hat{A}(M) \wedge \mathrm{Ch}(V)$ is a polynomial in the Pontryagin classes of M, i.e. does not involve the Euler density. Then for fixed $f\colon M \to A$, $[\tilde{\eta}_\rho]$ is a conformal-deformation invariant.*

Proof. Let $g(\varepsilon)$ be a 1-parameter family of conformally equivalent metrics. Let $\tilde{V}$ be the corresponding vector bundle on $\mathbf{R} \times M$. Let $\tilde{\sigma}_\rho$ be the eta-form on $\mathbf{R} \times P$. By Proposition 2,

$$\hat{\mathrm{d}}\tilde{\sigma}_\rho = \mathscr{R}_\beta\left(\int_M \hat{A}(\mathbf{R} \times M) \wedge \mathrm{Ch}(\tilde{V}) \wedge \mathrm{e}^{2\pi i \bar{\omega} \wedge \mathrm{d}\bar{v}^*}\right).$$

Thus

$$\partial_\varepsilon \tilde{\eta}_\rho - \hat{\mathrm{d}}\tilde{\tilde{\eta}}_\rho = \beta^{-1/2} i(\partial_\varepsilon)\,\hat{\mathrm{d}}\tilde{\sigma}_\rho = \mathscr{R}_\beta\left(\int_M i(\partial_\varepsilon)(\hat{A}(\mathbf{R} \times M) \wedge \mathrm{Ch}(\tilde{V})) \wedge \mathrm{e}^{2\pi i \bar{\omega} \wedge \mathrm{d}\bar{v}^*}\right).$$

By hypothesis, $\hat{A}(\mathbf{R} \times M) \wedge \mathrm{Ch}(\tilde{V})$ is a polynomial in the Pontryagin classes $p_k \in \Lambda^{4k}(\mathbf{R} \times M)$, and it is known that this implies that $i(\partial_\varepsilon)(\hat{A}(\mathbf{R} \times M) \wedge \mathrm{Ch}(\tilde{V}))$ vanishes identically [16].

(To see this last point, it is enough to consider the Pontryagin forms $\mathrm{tr}\,\Omega^{2k}$ on $\mathbf{R} \times M$. If $\omega(\varepsilon)$ denotes the Riemannian connection 1-form, its curvature on $\mathbf{R} \times M$ is

$$\Omega = R(\varepsilon) + \mathrm{d}\varepsilon \wedge \partial_\varepsilon \omega,$$

where $R(\varepsilon)$ is the Riemannian curvature 2-form of M. Thus, $i(\partial_\varepsilon)\,\mathrm{tr}\,\Omega^{2k}$ is proportionate to $\mathrm{tr}(\partial_\varepsilon \omega \wedge R^{2k-1})$. In terms of a local orthonormal basis $\{\tau_i\}$, the change in ω under a conformal change of metric is of the form $\partial_\varepsilon \omega_{ij} = h_{,i}\tau_j - h_{,j}\tau_i$, for some function h on M [23]. Then $\mathrm{tr}(\partial_\varepsilon \omega \wedge R^{2k-1})$ is proportionate to

$$\sum_{ij} (h_{,i}\tau_j - h_{,j}\tau_i) \wedge (R^{2k-1})_{ij} = 2\sum_{ij} h_{,i}\tau_j \wedge (R^{2k-1})_{ij}.$$

However, $\Sigma_j \tau_j \wedge (R^{2k-1})_{ij}$ vanishes by the Bianchi identity.)

Therefore, $\partial_\varepsilon \tilde{\eta}_\rho$ is exact on P. □

PROPOSITION 5. *Suppose that ρ_1 and ρ_2 are two representations of* F *such that the corresponding families of Dirac-type operators are invertible on all of P. Then $[\tilde{\eta}_{\rho_1}] - [\tilde{\eta}_{\rho_2}]$ is independent of the choice of f and is a deformation invariant with respect to the Riemannian metric on M and the Hermitian connection on V.*

Proof. Consider a 1-parameter family of choices. From the corresponding families of Dirac-type operators, parametrized by $\mathbf{R} \times P$, and let $\tilde{\sigma}_{\rho_1}$ and $\tilde{\sigma}_{\rho_2}$ be their eta-forms. By Proposition 2, $\hat{\mathrm{d}}\tilde{\sigma}_{\rho_1}$ and $\hat{\mathrm{d}}\tilde{\sigma}_{\rho_2}$ are the same local expression on $\mathbf{R} \times P$, and so $\hat{\mathrm{d}}(\tilde{\sigma}_{\rho_1} - \tilde{\sigma}_{\rho_2}) = 0$. Then by (13), $\partial_\varepsilon(\tilde{\eta}_{\rho_1} - \tilde{\eta}_{\rho_2})$ is exact on P. Thus $[\tilde{\eta}_{\rho_1}] - [\tilde{\eta}_{\rho_2}]$ is a deformation-invariant with respect to the choices made. As any two choices of f can be joined by a path, and the invertibility of the operators Q_p is independent of the choice of f, the independence with respect to f follows. □

Note. In Proposition 5 we are interested only in the difference between $[\tilde{\eta}_{\rho_1}]$ and $[\tilde{\eta}_{\rho_2}]$. It is not really necessary to assume that both ρ_1 and ρ_2 are such that the corresponding families of Dirac-type operators are individually invertible on all of P. To be more general, suppose that $\{\rho_j\}$ are the irreducible representations of F and $\{c_j\}$ is a set of complex numbers such that $\Sigma_j c_j = 0$ and for all $p \in P$, $\Sigma_j (c_j/N_j)$ $\mathrm{TR}(\mathrm{e}^{-s^2 Q_p^2(\rho_j)})$ decreases exponentially as $s^2 \to \infty$. Then the same argument as in the proof of Proposition 5 gives that $\Sigma_j c_j [\tilde{\eta}_{\rho_j}]$ is a deformation invariant.

An important class of examples for which this more general invertibility sometimes holds is given by signature operators. The paper [43, §1] considers simple manifolds, meaning that if M' is the finite F-cover of M, the group F acts trivially on the twisted cohomology groups $H^*(M'; \mathbf{Q}\pi_1(M'))$. The analogous condition in our case would be that $\Sigma_j (c_j/N_j)\ \mathrm{TR}(\mathrm{e}^{-s^2 Q_p^2(\rho_j)})$ decreases exponentially as $s^2 \to \infty$ provided that the coefficient of the trivial representation vanishes. This condition is independent of the Riemannian metric on M. If in addition $\Sigma_j c_j = 0$ then $\Sigma_j c_j [\tilde{\eta}_{\rho_j}]$ is a deformation invariant. As any two Riemannian metrics can be joined by a path of Riemannian metrics, the deformation invariance implies complete invariance of $\Sigma_j c_j [\tilde{\eta}_{\rho_j}]$. That is, we have defined a smooth topological invariant of M. To put it another way, we have defined a higher rho-invariant which lies in $K^{\mathrm{F}}(pt.)/\{\text{trivial and regular representations}\} \otimes H^*(P; \mathbf{C})$. Presumably this coincides with the higher rho-invariant defined in [43, §1]. To show this, one would have to prove a families index theorem for fibrations whose fibers are singular spaces of the type used in [43].

EXAMPLE. A class of operators that fulfill the hypotheses of this section is given by Dirac operators on manifolds of positive scalar curvature. To see that $\tilde{\eta}_\rho$ can be nontrivial, let L be a spin spherical space form with fundamental group. F. Take M to be $L \times \mathbf{T}^l$, with the product metric and a spin structure induced from the given spin structure on L and any spin structure on $\mathbf{T}^l$. Take the vector bundle V to be trivial, so that one is considering the Dirac operator acting on spinors on M. The metric on M has positive scalar curvature, and so the Lichnerowicz formula implies that Q_p is invertible for all $p \in P$ [29]. By separation of variables, it is easy to see that

$$[\tilde{\eta}_\rho(M)] = \frac{\sqrt{\pi}}{2} \eta_\rho(L) \cdot \mathrm{Ch}_\beta(\mathbf{T}^l) \in H^*(P; \mathbf{C}).$$

Here $\eta_\rho(L) \in \mathbf{C}$ is the usual twisted eta-invariant of L and $\mathrm{Ch}_\beta(\mathbf{T}^l) \in H^*(P; \mathbf{C})$ is the Chern character of the index bundle for the family of twisted Dirac operators on $\mathbf{T}^l$. In particular, L and ρ can be chosen so that $\eta_\rho(L)$ is nonzero [20], and it follows from (10) that $\mathrm{Ch}_\beta(\mathbf{T}^l)$ is a nonzero element of $H^l(P; \mathbf{C})$.

3.1.4. $\mathrm{Ker}(Q_p)$ *Forms a Vector Bundle on P*

In this section we assume that the kernels of the operators Q_p form a vector bundle on P as p varies in P. If n is even then we cannot say anything without detailed information about the vector bundle $\mathrm{Ker}(Q)$. For example, one sees from Proposition 2 that there is no reason that $\tilde{\eta}_\rho$ should be closed. However, if n is odd then all of the results of the previous section go through.

PROPOSITION 6. *If n is odd, the eta-form $\tilde{\eta}_\rho$ is closed.*

Proof. In this case the right hand side of (11) is a polynomial in the variables $d\vec{v}^*$. However, the existence of $\tilde{\eta}_\rho$ means that this polynomial is an exact form on P. Thus, its coefficients must vanish. □

The proofs of the following propositions are virtually the same as in Section 3.1.3.

PROPOSITION 7. *Suppose that the vector bundle V is associated to the principal* $\mathrm{Spin}(n)$*-bundle on M. Suppose that the index density $\hat{A}(M) \wedge \mathrm{Ch}(V)$ is a polynomial in the Pontryagin classes of M. If n is odd then for fixed $f: M \to A$, $[\tilde{\eta}_\rho]$ is a conformal-deformation invariant.*

PROPOSITION 8. *Suppose that ρ_1 and ρ_2 are two representations of F such that the kernels of the corresponding families of Dirac-type operators form vector bundles on P. If n is odd then $[\tilde{\eta}_{\rho_1}] - [\tilde{\eta}_{\rho_2}]$ is independent of the choice of f and is a deformation invariant with respect to the Riemannian metric on M and the Hermitian connection on V.*

3.1.5. Q_p *is Invertible for* $p \in U_\rho$

In this section we assume that the operators Q_p are invertible when p lies in an open subset U_ρ of P. We can no longer conclude that $\tilde{\eta}_\rho$ is closed on U_ρ.

Let $i: U_\rho \to P$ be the embedding of U_ρ in P. The relative de Rham cohomology $H^*(P, U_\rho; \mathbf{C})$ is isomorphic to the homology of the complex

$$\cdots \xrightarrow{\hat{d}} \Lambda^{k-1}(P, U_\rho) \xrightarrow{\hat{d}} \Lambda^k(P, U_\rho) \xrightarrow{\hat{d}} \Lambda^{k+1}(P, U_\rho) \xrightarrow{\hat{d}} \cdots,$$

where

$$\Lambda^k(P, U_\rho) = \Lambda^k(P) \oplus \Lambda^{k-1}(U_\rho) \quad \text{and} \quad \hat{d}(\alpha, \alpha') = (\hat{d}\alpha, i^*(\alpha) - \hat{d}\alpha')\ [12].$$

Let $\tilde{C}$ denote

$$\mathscr{R}_\beta\left(\int_M \hat{A}(M) \wedge \mathrm{Ch}(V) \wedge \mathrm{e}^{2\pi i \tilde{\omega} \wedge \mathrm{d}\bar{v}^*}\right) \in \Lambda^*(P).$$

PROPOSITION 9. *The pair* $(\tilde{C}, \tilde{\eta}_\rho)$ *is a closed element of* $\Lambda^*(P, U_\rho)$. *Its class in* $H^*(P, U_\rho; \mathbf{C})$ *is independent of the choice of* f *and is a deformation invariant with respect to the Riemannian metric on* M *and the Hermitian connection on* V.

Proof. It is always true that $\hat{d}\tilde{C} = 0$, and it follows from Proposition 2 that $(\tilde{C}, \tilde{\eta}_\rho)$ is closed. Let ε parametrize a 1-parameter family of choices, and consider the forms

$$\mathscr{C} = \mathscr{R}_\beta\left(\int_M \hat{A}(\mathbf{R} \times M) \wedge \mathrm{Ch}(\tilde{V}) \wedge \mathrm{e}^{2\pi i \mathrm{d} f^*_{\sharp}(\bar{v} \cdot \mathrm{d}\bar{v}^*)}\right) \in \Lambda^*(\mathbf{R} \times P)$$

and $\tilde{\sigma}_\rho \in \Lambda^*(\mathbf{R} \times U_\rho)$. Decompose $\mathscr{C}$ as

$$\mathscr{C} = \tilde{C}(\varepsilon) + \beta^{1/2}\, \mathrm{d}\varepsilon \wedge \tilde{\tilde{C}}(\varepsilon).$$

Then the equations $\hat{d}\mathscr{C} = 0$ and $\hat{d}\tilde{\sigma}_\rho = i^*\mathscr{C}$ give

$$\partial_\varepsilon \tilde{C} - \hat{d}\tilde{\tilde{C}} = 0 \quad \text{and} \quad \partial_\varepsilon \tilde{\eta}_\rho - \hat{d}\tilde{\tilde{\eta}}_\rho = i^*\tilde{\tilde{C}}.$$

Thus $\partial_\varepsilon(\tilde{C}, \tilde{\eta}_\rho) = \hat{d}(\tilde{\tilde{C}}, -\tilde{\tilde{\eta}}_\rho)$. □

EXAMPLE. Take $M = S^1$. We will identify M with its Albanese variety A. Take Q to be the (tangential) signature operator: $i\partial_v$, acting on $\Lambda^0(M)$. Let us use the local coordinate $v^* \in [0, 1)$ on P, with $v^* = 0$ being the untwisted situation. Then Q_p is invertible for $v^* \in (0, 1)$. So $\tilde{\eta}$ is a 0-form on $(0, 1)$, which to $v^* \in (0, 1)$ assigns the corresponding twisted eta-invariant. A computation gives that

$$\tilde{\eta}(v^*) = \frac{\sqrt{\pi}}{2}(2v^* - 1).$$

Also $\tilde{C} = \sqrt{\pi\beta}\, \mathrm{d}v^*$, a 1-form defined on all of P. It is easy to check that $(\tilde{C}, \tilde{\eta})$ represents a generator for $H^1(P, (0, 1); \mathbf{C}) \cong \mathbf{C}$.

PROPOSITION 10. *Suppose that the vector bundle V is associated to the principal* Spin(n)*-bundle on M. Suppose that the index density $\hat{A}(M) \wedge \mathrm{Ch}(V)$ is a polynomial in the Pontryagin classes of M. Then for fixed $f: M \to A$, the class of $\tilde{\eta}_\rho$ in $\wedge^*(U_\rho)/\mathrm{Im}(\hat{\mathrm{d}})$ is a conformal-deformation invariant.*

Proof. The proof is the same as that of Proposition 7. □

PROPOSITION 11. *Suppose that ρ_1 and ρ_2 are two representations of* F *such that the corresponding families of Dirac-type operators are invertible on U_ρ. Then $\tilde{\eta}_{\rho_1} - \tilde{\eta}_{\rho_2}$ is a closed form on U_ρ. Its class in $H^*(U_\rho; \mathbf{C})$ is independent of the choice of f and is a deformation invariant with respect to the Riemannian metric on M and the Hermitian connection on V.*

Proof. By Proposition 2, $\tilde{\eta}_{\rho_1}$ and $\tilde{\eta}_{\rho_2}$ have the same differential on U_ρ. The proof of the deformation invariance of $[\tilde{\eta}_{\rho_1} - \tilde{\eta}_{\rho_2}]$ is the same as in Proposition 5. □

3.1.6. Ker(Q_p) *Forms a Vector Bundle on* U_ρ

In this section we assume that the kernels of the operators Q_p form a vector bundle on U_ρ as p varies in U_ρ. If n is even then we cannot say anything without detailed information about the vector bundle Ker(Q), but if n is odd then all of the results of the previous section go through.

PROPOSITION 12. *If n is odd then the pair $(\tilde{C}, \tilde{\eta}_\rho)$ is a closed element of $\wedge^*(P, U_\rho)$. Its class in $H^*(P, U_\rho; \mathbf{C})$ is independent of the choice of f and is a deformation invariant with respect to the Riemannian metric on M and the Hermitian connection on V.*

PROPOSITION 13. *Suppose that the vector bundle V is associated to the principal* Spin(n)*-bundle on M. Suppose that the index density $\hat{A}(M) \wedge \mathrm{Ch}(V)$ is a polynomial in the Pontryagin classes of M. If n is odd then for fixed $f: M \to A$, the class of $\tilde{\eta}_\rho$ in $\wedge^*(U_\rho)/\mathrm{Im}(\hat{\mathrm{d}})$ is a conformal-deformation invariant.*

PROPOSITION 14. *Suppose that ρ_1 and ρ_2 are two representations of* F *such that the kernels of the corresponding families of Dirac-type operators form vector bundles on U_ρ. If n is odd then $\tilde{\eta}_{\rho_1} - \tilde{\eta}_{\rho_2}$ is a closed form on U_ρ. Its class in $H^*(U_\rho; \mathbf{C})$ is independent of the choice of f and is a deformation invariant with respect to the Riemannian metric on M and the Hermitian connection on V.*

3.1.7. *Higher Index Theorem for Manifolds with Boundary*

Note that to define $\tilde{\eta}$, the group Γ does not really have to equal $\pi_1(M)$. It is enough just to have a homomorphism from $\pi_1(M)$ to Γ and a map f from M to the corresponding torus in the canonical homotopy class, and all of the previous steps go through.

We now suppose that M is the boundary of a compact spin manifold W, with a product metric near the boundary. (We no longer assume that M is connected.) Let

V be a Hermitian vector bundle with connection on W which is a product near the boundary. Take the map $f: W \to A$ from W to its Albanese variety to be constant in the normal direction near the boundary of W.

We will denote the fundamental group of W by Γ, and assume that it is the product of a finite group F and a free Abelian group. Then the inclusion of M into W gives a homomorphism from $\pi_1(M)$ to Γ, and f restricts to a map $f_M: M \to A$. Let $\rho: \mathrm{F} \to U(N)$ be a representation of F. Suppose that the twisted Dirac-type operators on M are all invertible. Then we can define the eta-form $\tilde{\eta}_\rho(M) \in \Lambda^*(P)$.

Let $Q_{W,\rho}$ denote the family of twisted Dirac-type operators on W, parametrized by P, with Atiyah–Patodi–Singer (APS) [2] boundary conditions. Then the index bundle $\mathrm{Index}(Q_{W,\rho})$ lies in $K^*(P)$ and from [9], its Chern character is given by

$$\frac{1}{N}\mathrm{Ch}_\beta(\mathrm{Index}(Q_{W,\rho}))$$

$$= \mathscr{R}_\beta\left(\int_W \hat{A}(W) \wedge \mathrm{Ch}(V) \wedge e^{2\pi i \bar{\omega}_W \wedge d\bar{v}\, \bar{t}_W}\right) - \tilde{\eta}_\rho(M) \in H^*(P; \mathbf{C}). \tag{14}$$

(This only seems to be proven when $\dim(W)$ is even; for remarks on the odd case, see [10].)

In particular, suppose that W has positive scalar curvature and that V is trivial, so that we are looking at the pure Dirac operator. Then M also has positive scalar curvature. The Bochner argument [29], applied to the manifold W with boundary, gives that $\mathrm{Index}(Q_{W,\rho}) = 0$. Thus if $\{\rho_j\}$ are the irreducible representations of F and $\{c_j\}$ is a sequence of numbers such that $\Sigma_j c_j = 0$ then $\Sigma_j c_j \tilde{\eta}_{\rho_j}(M)$, which is a sort of higher rho-invariant, vanishes in $H^*(P; \mathbf{C})$. So $[\Sigma_j c_j \tilde{\eta}_{\rho_j}(M)]$ is an obstruction to realizing M as the boundary of a manifold W with a positive-scalar-curvature metric which is a product near the boundary.

To see that this is a nonvacuous statement, let L be as in the Example of Section 3.1.3. Then L represents a torsion element in the bordism group $\Omega_*^{\mathrm{spin}}(B\mathrm{F})$, and so there is a positive integer c such cL is the boundary of a spin manifold W with fundamental group F; take any such W. Take M to be isometrically $cL \times \mathbf{T}^l$. Then M bounds $W \times \mathbf{T}^l$ and

$$\tilde{\eta}_\rho(M) = c\frac{\sqrt{\pi}}{2}\eta_\rho(L)\cdot \mathrm{Ch}_\beta(\mathbf{T}^l).$$

If L has a nontrivial rho-invariant in the ordinary sense [20] then we conclude that $W \times \mathbf{T}^l$ cannot have a positive-scalar-curvature metric which is a product near the boundary, with the boundary metric being the given one on M. (This example shows why we do not consider reduced rho-invariants. As cL is a boundary, its reduced rho-invariant vanishes, and we would not detect any obstruction to positive-scalar-curvature this way.)

3.2. SEMIDIRECT PRODUCT GROUPS

In this section we extend the results of the previous section on product groups to the case of a semidirect product of $\mathbf{Z}^k$ and a finite group F. That is, we assume that the fundamental group Γ of M fits into a split exact sequence

$$1 \to \mathbf{Z}^k \to \Gamma \to \mathrm{F} \to 1.$$

Let M' be the F-fold normal covering of M. We will let $\phi \in \mathrm{F}$ act on the right on M', by $R_\phi \in \mathrm{Diff}(M')$. Let A and P be the Albanese and Picard varieties of M'. The action of F on M' induces an action on P. We will denote the subset of P which is fixed by $\phi \in \mathrm{F}$ by P^ϕ.

It is known that the irreducible representations of Γ all arise as follows. Think of P as the dual group to $\mathbf{Z}^k$. Given $p \in P$, let r_p be the corresponding representation of $\mathbf{Z}^k$. Let F_p be the subgroup of F which fixes p. Let ρ_p be an irreducible representation of F_p. Then one forms the representation of Γ induced from the representation $r_p \cdot \rho_p$ of $\mathbf{Z}^k \cdot \mathrm{F}_p$ [28].

This motivates looking at the following space. (Unlike the preceding sections, we no longer look at representations of F.)

DEFINITION 5 [5]. $\hat{P} \subset P \times \mathrm{F}$ is given by

$$\hat{P} = \{(p, \phi) \in P \times \mathrm{F} : p\phi = p\} = \coprod_{\phi} (P^\phi, \phi).$$

F acts on $\hat{P}$ by $(p, \phi) \cdot \phi' = (p\phi', \phi'^{-1}\phi\phi')$. We will denote the space of F-invariant differential forms on $\hat{P}$ by $\wedge^*(\hat{P}/\mathrm{F})$, and the associated cohomology theory, the 'delocalized equivariant cohomology' [5] by

$$H^*(\hat{P}/\mathrm{F}; \mathbf{C}) = \left[\bigoplus_{\phi} H^*(P^\phi; \mathbf{C}) \right]^{\mathrm{F}} = [H^*(P; \mathbf{C})]^{\mathrm{F}} \oplus \left[\bigoplus_{\phi \neq e} H^*(P^\phi; \mathbf{C}) \right]^{\mathrm{F}}.$$

Fix $f: M' \to A$ in the canonical homotopy class. Let V be a Hermitian vector bundle with connection on M, and let V' be its pullback to M'.

Let W_p be the flat Hermitian line bundle on M' whose twisting is specified by $p \in P$. Let Q'_p be the Dirac-type operator acting on L^2-sections of $S' \otimes V' \otimes W_p$. Suppose that U is an open F-invariant subset of P such that $\mathrm{Ker}(Q'_p)$ forms an F-vector-bundle on U as p varies in U. Define $\hat{U}$ and U^ϕ as for P. If n is even, the Chern–Weil construction goes through to give a closed form $\mathrm{tr}_s(e^{-\beta\nabla^2_{\mathrm{Ker}(Q')}}) \in \wedge^*(\hat{U}/\mathrm{F})$.

Let $\mathscr{E}_\phi$ be the infinite-dimensional vector bundle on $(P^\phi, \phi) \subset \hat{P}$ whose fiber over (p, ϕ) is $C^\infty(S' \otimes V' \otimes W_p)$. The Dirac-type operators $\{Q'_p\}_{p \in P^\phi}$ fit together to give an operator $Q'_\phi: C^\infty(\mathscr{E}_\phi) \to C^\infty(\mathscr{E}_\phi)$. Let

$$\nabla'_\phi: C^\infty(\mathscr{E}_\phi) \to C^\infty(\mathscr{E}_\phi \otimes \wedge^1(P^\phi))$$

be the natural Hermitian connection. Using ∇'_ϕ and Q'_ϕ, form the superconnection $D'_{\phi,s}$ as in (7).

DEFINITION 6. For $s > 0$, the Chern character $\mathrm{ch}(s) \in \Lambda^*(\hat{P}/\mathrm{F})$ of $\mathscr{E}$, a closed form, is given on (P^ϕ, ϕ) by

$$\mathrm{ch}(s) = \begin{cases} \mathrm{STR}(R_\phi^* \mathrm{e}^{-\beta D'^2_{\phi,s}}), & \text{if } n \text{ is even,} \\ \mathrm{TR}_\sigma(R_\phi^* \mathrm{e}^{-\beta D'^2_{\phi,s}}), & \text{if } n \text{ is odd.} \end{cases}$$

DEFINITION 7. The eta-form $\tilde{\eta} \in \Lambda^*(\hat{U}/\mathrm{F})$ is given on (U^ϕ, ϕ) by

$$\tilde{\eta} = \begin{cases} \beta^{1/2} \displaystyle\int_0^\infty \mathrm{STR}(R_\phi^* Q' \mathrm{e}^{-\beta D'^2_{\phi,s}})\,\mathrm{d}s, & \text{if } n \text{ is even,} \\ \beta^{1/2} \displaystyle\int_0^\infty \mathrm{TR}_\sigma(R_\phi^* \sigma Q' \mathrm{e}^{-\beta D'^2_{\phi,s}})\,\mathrm{d}s, & \text{if } n \text{ is odd.} \end{cases}$$

The same arguments as in Section 3.1.2 give

PROPOSITION 15. *The differential forms* $\mathrm{ch}(s)$ *have a limit as* $s \to 0$, *given on* (P^ϕ, ϕ) *by*

$$\lim_{s \to 0} \mathrm{ch}(s) = \delta_{\phi,e} \mathscr{R}_\beta \left(\int_{M'} \hat{A}(M') \wedge \mathrm{Ch}(V') \wedge \mathrm{e}^{2\pi i \bar{\omega} \wedge \mathrm{d}\bar{v}^*} \right). \tag{15}$$

PROPOSITION 16. *The differential of* $\tilde{\eta}$ *is given on* (U^ϕ, ϕ) *by*

$$\hat{\mathrm{d}}\tilde{\eta} = \begin{cases} \delta_{\phi,e} \mathscr{R}_\beta \left(\displaystyle\int_{M'} \hat{A}(M') \wedge \mathrm{Ch}(V') \wedge \mathrm{e}^{2\pi i \bar{\omega} \wedge \mathrm{d}\bar{v}^*} \right) - \mathrm{tr}_s(\mathrm{e}^{-\beta \nabla^2_{\mathrm{Ker}(Q')}}), & \textit{if } n \textit{ is even,} \\ \delta_{\phi,e} \mathscr{R}_\beta \left(\displaystyle\int_{M'} \hat{A}(M') \wedge \mathrm{Ch}(V') \wedge \mathrm{e}^{2\pi i \bar{\omega} \wedge \mathrm{d}\bar{v}^*} \right), & \textit{if } n \textit{ is odd.} \end{cases} \tag{16}$$

It is now straightforward to extend the results of Section 3.1 to the case of semidirect products. For example, we given the extensions of Propositions 3–5. Assume that Q'_p is invertible for all $p \in P$.

PROPOSITION 17. *The eta-form* $\tilde{\eta}$ *is closed.*

Thus $\tilde{\eta}$ represents a class $[\tilde{\eta}]$ in $H^*(\hat{P}/\mathrm{F}; \mathbf{C})$.

PROPOSITION 18. *Suppose that the vector bundle* V *is associated to the principal* Spin(n)*-bundle on* M. *Suppose that the index density* $\hat{A}(M') \wedge \mathrm{Ch}(V')$ *is a polynomial in the Pontryagin classes of* M'. *Then for fixed* $f: M' \to A$, $[\tilde{\eta}]$ *is a conformal-deformation invariant.*

PROPOSITION 19. *The 'delocalized' part of* $[\tilde{\eta}]$, *that is, the part in* $[\oplus_{\phi \neq e} H^*(P^\phi; \mathbf{C})]^{\mathrm{F}}$, *is independent of the choice of* f *and is a deformation invariant with respect to the Riemannian metric on* M *and the Hermitian connection on* V.

Note. Suppose that M is a Riemannian spin manifold with positive scalar curvature. The same argument as in Section 3.1.7 gives that the delocalized part of $[\tilde{\eta}]$ is an obstruction to realizing M as the boundary of a spin manifold of positive scalar curvature with the same fundamental group, which is a product near the boundary.

Note. The $\tilde{\eta}_\rho$ of Section 3.1.2 is related to the $\tilde{\eta}$ of the present section by a Fourier transform on the group F. More precisely, suppose that $\Gamma = \mathrm{F} \times \mathbf{Z}^k$. Let $\langle \mathrm{F} \rangle$ denote the conjugacy classes of F. Then F acts trivially on P and

$$\hat{P} = P \times \mathrm{F}, \qquad \Lambda^*(\hat{P}/\mathrm{F}) \cong \bigoplus_{\langle \mathrm{F} \rangle} \Lambda^*(P), \qquad H^*(\hat{P}/\mathrm{F}; \mathbf{C}) \cong \bigoplus_{\langle \mathrm{F} \rangle} H^*(P; \mathbf{C}).$$

Let us write $\tilde{\eta} \in \Lambda^*(\hat{P}/\mathrm{F})$ as $\tilde{\eta} = \oplus_{\langle \phi \rangle \in \langle \mathrm{F} \rangle} \tilde{\eta}(\langle \phi \rangle)$, with each $\tilde{\eta}(\langle \phi \rangle)$ in $\Lambda^*(P)$. Given a representation $\rho\colon \mathrm{F} \to U(N)$, let χ_ρ denote its character. Then $\tilde{\eta}_\rho$ is given by

$$\tilde{\eta}_\rho = \frac{1}{N} \sum_{\phi \in \mathrm{F}} \chi_\rho(\phi) \tilde{\eta}(\langle \phi \rangle).$$

4. Noncommutative Eta-Invariants

4.1. CYCLIC COHOMOLOGY OF GROUP ALGEBRAS

Let $\mathscr{B}$ be an algebra over $\mathbf{C}$ with unit 1. As a vector space, the universal graded differential algebra of $\mathscr{B}$ is $\Omega_*(\mathscr{B}) = \oplus_{k=0}^{\infty} \Omega_k(\mathscr{B})$, with $\Omega_k(\mathscr{B}) = \mathscr{B} \otimes (\otimes^k(\mathscr{B}/\mathbf{C}))$. As a graded differential algebra, $\Omega_*(\mathscr{B})$ is generated by $\mathscr{B}$ and $\mathrm{d}\mathscr{B}$ with the relations

$$\mathrm{d}1 = 0, \qquad \mathrm{d}^2 = 0,$$
$$\mathrm{d}(\omega_k \omega_l) = (\mathrm{d}\omega_k)\omega_l + (-1)^k \omega_k (\mathrm{d}\omega_l) \quad \text{for } \omega_k \in \Omega_k(\mathscr{B}),\ \omega_l \in \Omega_l(\mathscr{B}).$$

It will be convenient to write an element ω_k of $\Omega_k(\mathscr{B})$ as a finite sum $\Sigma b_0\, \mathrm{d}b_1 \ldots \mathrm{d}b_k$.

The reduced cyclic homology $\overline{HC}_*(\mathscr{B})$ is the homology of the complex

$$\to \cdots \bar{C}^\lambda_{*+1}(\mathscr{B}) \xrightarrow{b} \bar{C}^\lambda_*(\mathscr{B}) \xrightarrow{b} \bar{C}^\lambda_{*-1}(\mathscr{B}) \to \cdots,$$

where $\bar{C}^\lambda_*(\mathscr{B})$ is the quotient of the space of cyclic chains $C^\lambda_*(\mathscr{B})$ by the subspace $\mathrm{span}\{b_0 \otimes \cdots \otimes b_* : b_i = 1 \text{ for some } i\}$. One has [26]

$$\overline{HC}_*(\mathscr{B}) \cong \mathrm{Cok}(HC_*(\mathbf{C}) \to HC_*(\mathscr{B})). \tag{17}$$

The homology $\bar{H}_*(\mathscr{B})$ of the differential complex $\bar{\Omega}_*(\mathscr{B}) = \Omega_*(\mathscr{B})/[\Omega_*(\mathscr{B}), \Omega_*(\mathscr{B})]$ is isomorphic to the subspace Ker(B) of $\overline{HC}_*(\mathscr{B})$ for $* > 0$ [17, 26]. (In the case $* = 0$, $\bar{H}_0(\mathscr{B}) \cong \mathrm{Ker}(\mathrm{B}\colon HC_0(\mathscr{B})\ (= \mathscr{B}/[\mathscr{B}, \mathscr{B}]) \to H_1(\mathscr{B}, \mathscr{B}))$.) Thus there is a pairing between the reduced cyclic cohomology $\overline{HC}^*(\mathscr{B})$ and $\bar{H}_*(\mathscr{B})$ for $* > 0$. This pairing comes from a pairing between $\overline{ZC}^*(\mathscr{B})$ and $\Omega_*(\mathscr{B})$; if $\mathscr{T} \in \overline{ZC}^k(\mathscr{B})$ is a reduced cyclic cocycle and $\Sigma b_0\, \mathrm{d}b_1 \ldots \mathrm{d}b_k \in \Omega_k(\mathscr{B})$ is a k-form then their pairing is $\Sigma \mathscr{T}(b_0, b_1, \ldots, b_k)$ [17]. (For $* = 0$, there is a pairing between $HC^0(\mathscr{B})$, the space of traces on $\mathscr{B}$, and $\Omega_0(\mathscr{B}) = \mathscr{B}$.)

Now let Γ be a discrete group. Let $\mathbf{C}\Gamma$ be the group algebra of Γ. Let $\langle\Gamma\rangle$ denote the conjugacy classes of Γ, and $\langle\Gamma\rangle'$ ($\langle\Gamma\rangle''$) those represented by elements of finite (infinite) order. For $x \in \Gamma$, let Z_x denote its centralizer in Γ and put $N_x = \{x\}\backslash Z_x$, the quotient of Z_x by the cyclic group generated by x. If x and x' are conjugate then N_x and $N_{x'}$ are isomorphic groups, and we will write $N_{\langle x\rangle}$ for their isomorphism class. Let $\mathbf{C}[z]$ be a polynomial ring in a variable z of degree 2. Then the cyclic cohomology of $\mathbf{C}\Gamma$ is given by [13]

$$HC^*(\mathbf{C}\Gamma) = \left(\bigoplus_{\langle x\rangle\in\langle\Gamma\rangle'} H^*(N_{\langle x\rangle};\mathbf{C})\otimes\mathbf{C}[z]\right)\oplus\bigoplus_{\langle x\rangle\in\langle\Gamma\rangle''} H^*(N_{\langle x\rangle};\mathbf{C}). \tag{18}$$

Let $S\colon H^*(N_{\langle x\rangle};\mathbf{C})\to H^{*+2}(N_{\langle x\rangle};\mathbf{C})$ be the Gysin homomorphism of the fibration $S^1\to BZ_x\to BN_x$. We will abbreviate even, odd by e, o. Put

$$T^{\mathrm{e,o}}(\langle x\rangle) = \lim(\cdots\to H^{*-2}(N_{\langle x\rangle};\mathbf{C})\xrightarrow{S} H^*(N_{\langle x\rangle};\mathbf{C})\xrightarrow{S} H^{*+2}(N_{\langle x\rangle};\mathbf{C})\to\cdots),$$

the inductive limit. Then the periodic cyclic cohomology of $\mathbf{C}\Gamma$ is given by [13]

$$PHC^{\mathrm{e,o}}(\mathbf{C}\Gamma) = \left(\bigoplus_{\langle x\rangle\in\langle\Gamma\rangle'} H^{\mathrm{e,o}}(N_{\langle x\rangle};\mathbf{C})\right)\oplus\bigoplus_{\langle x\rangle\in\langle\Gamma\rangle''} T^{\mathrm{e,o}}(\langle x\rangle). \tag{19}$$

In particular, $H^{\mathrm{e,o}}(\Gamma;\mathbf{C})$ is a direct summand of $PHC^{\mathrm{e,o}}(\mathbf{C}\Gamma)$, corresponding to $\langle x\rangle = \langle e\rangle$. Similar results hold for cyclic homology.

Note. $T^*(\langle x\rangle)$ often vanishes, for example if $N_{\langle x\rangle}$ has finite virtual cohomological dimension.

We will need explicit cocycles for $HC^*(\mathbf{C}\Gamma)$. Fix a representative $x\in\langle x\rangle$. Put

$$C_x^k = \{\tau\colon\Gamma^{k+1}\to\mathbf{C}\colon \tau \text{ is skew and for all } (\gamma_0,\gamma_1,\ldots,\gamma_k)\in\Gamma^{k+1} \text{ and } z\in Z_x,$$
$$\tau(z\gamma_0,z\gamma_1,\ldots,z\gamma_k) = \tau(\gamma_0,\gamma_1,\ldots,\gamma_k) \text{ and } \tau(x\gamma_0,\gamma_1,\ldots,\gamma_k) = \tau(\gamma_0,\gamma_1,\ldots,\gamma_k)\}.$$

Let δ be the usual coboundary operator:

$$\delta\tau(\gamma_0,\gamma_1,\ldots,\gamma_{k+1}) = \sum_{j=0}^{k+1}(-1)^j\tau(\gamma_0,\gamma_1,\ldots,\hat{\gamma}_j,\ldots,\gamma_{k+1}).$$

Denote the resulting cohomology groups by H_x^k. Then H_x^k is isomorphic to $H^k(N_{\langle x\rangle};\mathbf{C})$ and for each cocycle $\tau\in Z_x^k$, there is a cyclic cocycle $\mathscr{T}_\tau\in ZC^k(\mathbf{C}\Gamma)$ given by

$$\mathscr{T}_\tau(\gamma_0,\gamma_1,\ldots,\gamma_k) = \begin{cases} 0, & \text{if } \gamma_0\cdots\gamma_k\notin\langle x\rangle,\\ \tau(g,g\gamma_0,\ldots,g\gamma_0\cdots\gamma_{k-1}), & \text{if } \gamma_0\cdots\gamma_k = g^{-1}xg.\end{cases} \tag{20}$$

For $k>0$, these are in fact reduced cyclic cocycles.

4.2. PAIRING OF $[\tilde{\eta}]$ WITH CYCLIC COHOMOLOGY

We relate the results of Section 3 to the cyclic cohomology of $\mathbf{C}\Gamma$. Suppose that Γ is a semidirect product as in Section 3.2. If the operators Q_p are all invertible, we defined $[\tilde{\eta}] \in H^*(\hat{P}/\mathrm{F}; \mathbf{C})$. Thus to obtain numbers, we should pair $[\tilde{\eta}]$ with $H_*(\hat{P}/\mathrm{F}; \mathbf{C})$.

If F happens to be trivial then we are talking about $H_*(P; \mathbf{C})$. In this case, the relationship between P and the group Γ is given by Fourier transform. Namely, $\mathbf{C}\Gamma$ corresponds to certain analytic functions on P and the reduced group C^*-algebra $C_r^*(\Gamma)$ is isomorphic to $C(P)$. The algebra $C^\infty(P)$, which can be considered as a subalgebra of $C_r^*(\Gamma)$, has the same periodic cyclic cohomology as its subalgebra $\mathbf{C}\Gamma$, namely $PHC^{\mathrm{e,o}}(C^\infty(P)) = H_{\mathrm{e,o}}(P; \mathbf{C})$.

For general F, $\mathbf{C}\Gamma$ is the cross-product $\mathbf{C}\mathbf{Z}^k * \mathrm{F}$. The periodic cyclic cohomology of $\mathbf{C}\Gamma$ will be the same as that of the cross-product algebra $C^\infty(P) * \mathrm{F}$, and one has that $PHC^{\mathrm{e,o}}(C^\infty(P) * \mathrm{F}) = H_{\mathrm{e,o}}(\hat{P}/\mathrm{F}; \mathbf{C})$ [41]. As seen in (19), $PHC^{\mathrm{e,o}}(\mathbf{C}\Gamma)$ breaks up according to the conjugacy classes of Γ.

Thus in this case we obtain numbers by pairing $[\tilde{\eta}]$ with $PHC^{\mathrm{e,o}}(\mathbf{C}\Gamma)$. The 'delocalized' part of $[\tilde{\eta}]$ pairs with the part of $PHC^{\mathrm{e,o}}(\mathbf{C}\Gamma)$ coming from nontrivial conjugacy classes.

4.3. NONCOMMUTATIVE SUPERCONNECTIONS

The formal expressions for the higher Chern character and higher eta-invariant are the essentially the same as those of Section 3. However, the meanings of the symbols are somewhat different. We first review and extend some of the results of [31].

Let M^n be a connected closed oriented Riemannian manifold and let Γ be a finitely presented group. Let M' be a normal Γ-cover of M, with $\gamma \in \Gamma$ acting on the right by $R_\gamma \in \mathrm{Diff}(M')$. Let $\nu: M \to B\Gamma$ be the classifying map (defined up to homotopy) for the fibration $\Gamma \to M' \xrightarrow{\pi} M$. Let E be a Clifford module over M with Hermitian connection. For simplicity, we will assume that M is spin and that $E = S \otimes V$, where S is the spinor bundle of M and V is a Hermitian vector bundle with connection. If n is even then the Clifford module is $\mathbf{Z}_2$-graded by the grading on S, while if n is odd then the Clifford module is ungraded. Let E' be the pullback of E to M', with the pulled-back connection. Let Q' be the Dirac-type operator acting on L^2-sections of E', a densely-defined self-adjoint operator.

The results of [31] are valid for any finitely presented group Γ. However, in this paper we will assume hereafter that Γ has a finitely-presented nilpotent subgroup of finite index. Let $\|\cdot\|$ be a right-invariant word-length metric on Γ. The assumption on Γ is equivalent to saying that Γ is of polynomial growth with respect to $\|\cdot\|$ [21]. We will need this assumption in order to show that the formal expression for the higher eta-invariant is well-defined. The results from [31] which are given here are slightly modified in order to take this assumption on Γ into account.

Let $\mathscr{B}$ be the subalgebra of $C_r^*(\Gamma)$ consisting of elements whose entries die faster than any power in $\|\cdot\|$. That is,

$$\mathscr{B} = \{f: \Gamma \to \mathbf{C}: \forall q \in \mathbf{Z}, \sup_\gamma ((1 + \|\gamma\|)^q |f(\gamma)|) < \infty\}$$

It is a Fréchet locally m-convex algebra with unit, in the sense of [35]. One can define a completion $\hat{\Omega}_*(\mathscr{B})$ of $\Omega_*(\mathscr{B})$ which is a Fréchet graded differential algebra. The homology $\bar{H}_*(\mathscr{B})$ of the differential complex $\bar{\hat{\Omega}}_*(\mathscr{B}) = \hat{\Omega}_*(\mathscr{B})/\overline{[\hat{\Omega}_*(\mathscr{B}), \hat{\Omega}_*(\mathscr{B})]}$ pairs with the (topological) cyclic cohomology $HC^*(\mathscr{B})$ of $\mathscr{B}$, and in fact with the reduced cyclic cohomology for $* > 0$. It is shown in [24] that the periodic cyclic cohomology $PHC^{e,o}(\mathscr{B})$ is isomorphic to $PHC^{e,o}(\mathbf{C}\Gamma)$.

DEFINITION 8. $\mathscr{E} = (M' \times_\Gamma \mathscr{B}) \otimes E$, a vector bundle over M.

The fibers of $\mathscr{E}$ are right $\mathscr{B}$-modules, and there is a right $\mathscr{B}$-action on the space $C^\infty(\mathscr{E})$ of smooth sections of $\mathscr{E}$. If $\mathscr{F}$ is a Fréchet algebra containing $\mathscr{B}$, one can form the $\mathscr{B}$-vector bundle $\mathscr{E} \hat{\otimes}_\mathscr{B} \mathscr{F}$. We define $\mathrm{Hom}^\infty_\mathscr{B}(\mathscr{E}, \mathscr{E} \hat{\otimes}_\mathscr{B} \mathscr{F})$ to be the algebra of integral operators $T\colon C^\infty(\mathscr{E}) \to C^\infty(\mathscr{E} \hat{\otimes}_\mathscr{B} \mathscr{F})$ with smooth kernels $T(m_1, m_2) \in \mathrm{Hom}_\mathscr{B}(\mathscr{E}_{m_2}, \mathscr{E}_{m_1} \hat{\otimes}_\mathscr{B} \mathscr{F})$. That is, for $s \in C^\infty(\mathscr{E})$,

$$(Ts)(m_1) = \int_M T(m_1, m_2) s(m_2)\,\mathrm{dvol}(m_2) \in \mathscr{E}_{m_1} \hat{\otimes}_\mathscr{B} \mathscr{F}.$$

We denote $\mathrm{Hom}^\infty_\mathscr{B}(\mathscr{E}, \mathscr{E})$ by $\mathrm{End}^\infty_\mathscr{B}(\mathscr{E})$.

A $\mathscr{B}$-connection on $\mathscr{E}$ is a map $\nabla\colon C^\infty(\mathscr{E}) \to C^\infty(\mathscr{E} \hat{\otimes}_\mathscr{B} \hat{\Omega}_1(\mathscr{B}))$ with smooth integral kernel such that $\nabla(sb) = \nabla(s)b + s \hat{\otimes}_\mathscr{B} db$ for all $s \in C^\infty(\mathscr{E})$ and $b \in \mathscr{B}$.

One can define a Dirac-type operator acting on $C^\infty(\mathscr{E})$, which we will denote by Q. Then for all $T > 0$, there is a heat kernel $e^{-TQ^2} \in \mathrm{End}^\infty_\mathscr{B}(\mathscr{E})$.

DEFINITION 9. The superconnection D_s *is given by*

$$D_s = \begin{cases} sQ + \nabla, & \text{if } n \text{ is even,} \\ s\sigma Q + \nabla, & \text{if } n \text{ is odd.} \end{cases} \tag{21}$$

For $\beta > 0$, we define $e^{-\beta D_s^2} \in \mathrm{Hom}^\infty_\mathscr{B}(\mathscr{E}, \mathscr{E} \hat{\otimes}_\mathscr{B} \hat{\Omega}_*(\mathscr{B}))$ by a Duhamel expansion in ∇.

If n is (even) odd, one can define a (super)trace (S)TR: $\mathrm{Hom}^\infty_\mathscr{B}(\mathscr{E}, \mathscr{E} \hat{\otimes}_\mathscr{B} \hat{\Omega}_*(\mathscr{B})) \to \bar{\hat{\Omega}}_*(\mathscr{B})$.

DEFINITION 10. For $s > 0$, the Chern character $\mathrm{ch}(s) \in \bar{\hat{\Omega}}_*(\mathscr{B})$ of $\mathscr{E}$, a closed form, is given by

$$\mathrm{ch}(s) = \begin{cases} \mathrm{STR}(e^{-\beta D_s^2}), & \text{if } n \text{ is even,} \\ \mathrm{TR}_\sigma(e^{-\beta D_s^2}), & \text{if } n \text{ is odd.} \end{cases} \tag{22}$$

To make things more explicit, it will be convenient to work on M'. We now give the covering-space versions of the preceding definitions, which are adaptions of the results in [31]. Fix a basepoint $x_0 \in M'$ in each connected component of M'. For a multi-index α, let ∇^α denote repeated covariant differentiation on E'.

PROPOSITION 20 [31]. *There is an isomorphism between* $C^\infty(\mathscr{E})$ *and*

$$\{f \in C^\infty(M', E') \colon \forall q \in \mathbf{Z} \text{ and all multi-indices } \alpha,$$

$$\sup_x ((1 + d(x_0, x))^q |\nabla^\alpha f(x)|) < \infty\}.$$

PROPOSITION 21 [31]. *There is an isomorphism between the algebra* $\mathrm{End}^{\infty}_{\mathscr{B}}(\mathscr{E})$ *and the algebra of* Γ*-invariant integral operators* T *on* $L^2(M', E')$ *with smooth kernels* $T(x, y) \in \mathrm{Hom}(E'_y, E'_x)$ *such that for all* $q \in \mathbf{Z}$ *and all multi-indices* α *and* β,

$$\sup_{x,y} ((1 + \mathrm{d}(x, y))^q |\nabla^{\alpha}_x \nabla^{\beta}_y T(x, y)|) < \infty.$$

It follows from finite propagation speed estimates (see Equation (30)) that $\mathrm{e}^{-TQ'^2}$ defines an element of $\mathrm{End}^{\infty}_{\mathscr{B}}(\mathscr{E})$.

Let $\mathrm{tr}_{(s)}$ denote the local (super)trace on $\mathrm{End}(E'_x)$. Fix a function $\phi \in C^{\infty}_0(M')$ with the property that $\Sigma_{\gamma\in\Gamma} R^*_{\gamma}\phi = 1$.

PROPOSITION 22 [31]. *The* (*super*)*trace of an element* T *of* $\mathrm{End}^{\infty}_{\mathscr{B}}(\mathscr{E})$, *represented as in Proposition* 21, *is given by*

$$\mathrm{(S)TR}(T) = \sum_{\gamma\in\Gamma} \left[\int_{M'} \phi(x)\, \mathrm{tr}_{(s)}((R^*_{\gamma}T)(x, x))\, \mathrm{dvol}(x) \right] \gamma \ (\mathrm{mod}\, \overline{[\mathscr{B}, \mathscr{B}]}).$$

Similarly, an element f of $C^{\infty}(\mathscr{E} \hat{\otimes}_{\mathscr{B}} \hat{\Omega}_k(\mathscr{B}))$ can be written as $\Sigma f_{\gamma_1\ldots\gamma_k} \mathrm{d}\gamma_1 \ldots \mathrm{d}\gamma_k$, with each $f_{\gamma_1\ldots\gamma_k} \in C^{\infty}(M', E')$ a smooth rapidly decreasing section of E'. An element K of $\mathrm{Hom}^{\infty}_{\mathscr{B}}(\mathscr{E}, \mathscr{E} \hat{\otimes}_{\mathscr{B}} \hat{\Omega}_k(\mathscr{B}))$ can be represented by smooth rapidly decreasing kernels $K_{\gamma_1\ldots\gamma_k}(x, y) \in \mathrm{Hom}(E'_y, E'_x)$ such that $K = \Sigma K_{\gamma_1\ldots\gamma_k} \mathrm{d}\gamma_1 \ldots \mathrm{d}\gamma_k$ is Γ-invariant. Then

$$\mathrm{(S)TR}(K) = \sum_{\gamma_0,\ldots,\gamma_k\in\Gamma} \left[\int_{M'} \phi(x)\, \mathrm{tr}_{(s)}((R^*_{\gamma_0} K_{\gamma_1\ldots\gamma_k})(x, x))\, \mathrm{dvol}(x) \right] \gamma_0 \mathrm{d}\gamma_1 \ldots \mathrm{d}\gamma_k$$
$$(\mathrm{mod}\, \overline{[\hat{\Omega}_*(\mathscr{B}), \hat{\Omega}*(\mathscr{B})]}).$$

PROPOSITION 23 [31]. *For each function* $h \in C^{\infty}_0(M')$ *such that*

$$\sum_{\gamma\in\Gamma} R^*_{\gamma} h = 1, \tag{23}$$

there is a connection

$$\nabla: C^{\infty}(\mathscr{E}) \to C^{\infty}(\mathscr{E} \hat{\otimes}_{\mathscr{B}} \hat{\Omega}_1(\mathscr{B}))$$

given by

$$\nabla f = \sum_{\gamma\in\Gamma} h R^*_{\gamma} f \hat{\otimes}_{\mathscr{B}} \mathrm{d}\gamma$$

for all $f \in C^{\infty}(\mathscr{E})$.

PROPOSITION 24 [31]. *Define* $\mathrm{ch}(s)$ *using the connection of Proposition* 23. *Then* $\mathrm{ch}(s)$ *has a limit as* $s \to 0$ *given by the integral of a local expression on* M. *That is, there is a biform* $\omega \in \Lambda^*(M) \otimes \bar{\hat{\Omega}}_*(\mathscr{B})$, *closed in both factors, constructed from* h *such that*

$$\lim_{s\to 0} \mathrm{ch}(s) = \int_M \hat{A}(M) \wedge \mathrm{Ch}(V) \wedge \omega \in \bar{\hat{\Omega}}_*(\mathscr{B}). \tag{24}$$

We refer to [31] for the exact expression for ω. The important term in ω of degree k (with respect to $\mathscr{B}$) is a closed form on M with values in $\bar{\bar{\Omega}}_k(\mathscr{B})$, whose pullback to M' is given by

$$(-1)^k \frac{\beta^{k/2}}{k!} \sum_{\gamma_0 \dots \gamma_k = e} R^*_{\gamma_0} \mathrm{d}h \wedge \dots \wedge R^*_{\gamma_0 \dots \gamma_{k-1}} \mathrm{d}h \, \gamma_0 \mathrm{d}\gamma_1 \dots \mathrm{d}\gamma_k \in \Lambda^k(M') \otimes \bar{\bar{\Omega}}_k(\mathscr{B}).$$

There are other terms in degree k which are lower order forms on M, and arise because of the S operation in cyclic homology, as will become clear in Section 4.6. An important point is that the right-hand side of (24) has support on the forms spanned by $\{\gamma_0 \mathrm{d}\gamma_1 \dots \mathrm{d}\gamma_k : \gamma_0 \dots \gamma_k = e\}$.

COROLLARY 1 [31]. *Let $\mathscr{T}_\tau$ be a cyclic k-cocycle of* $\mathbf{C}\Gamma$ *constructed as in* (20). *Suppose that $\mathscr{T}_\tau$ extends to a cyclic cocycle of $\mathscr{B}$. Then for all $s > 0$, the pairing $\langle \mathrm{ch}(s), \mathscr{T}_\tau \rangle$ is well defined and independent of s. If $x \neq e$, then $\langle \mathrm{ch}(s), \mathscr{T}_\tau \rangle = 0$. If $x = e$, let $[\tau] \in H^k(B\Gamma; \mathbf{C})$ denote the cohomology class represented by τ. Then*

$$\langle \mathrm{ch}(s), \mathscr{T}_\tau \rangle = (-1)^k \beta^{k/2} \int_M \hat{A}(M) \wedge \mathrm{Ch}(V) \wedge \nu^*[\tau]. \tag{25}$$

Note. The right-hand side of (25) is a higher-index. The factor $(-1)^k$ arises because we are using a slightly different definition of $\mathscr{T}_\tau$ than [31].

4.4. THE HIGHER ETA-INVARIANT

We now wish to define the eta-form as a differential form on the noncommutative base space. In line with Section 4.2, the goal is to come up with a form which can be paired with the cyclic cohomology of $\mathbf{C}\Gamma$. In order to understand what are reasonable hypotheses under which to define $\tilde{\eta}$, it is worth reconsidering the discussion of Section 4.2. Suppose for simplicity that F is trivial. We have seen that the periodic cyclic cohomology of $\mathbf{C}\Gamma$ corresponds to the homology of P. Now a reasonable condition to define the pairing of a form with the homology of P is that the form should be defined on all of P. For the eta-form, this means that we need for $\mathrm{Ker}(Q_p)$ to form a vector bundle on P. This is equivalent to saying that $\dim(\mathrm{Ker}(Q_p))$ is constant in p. In other words, we rule out the possibility that an eigenvalue of Q_p goes from a nonzero value to zero, as p varies.

The way to generalize this condition to arbitrary fundamental groups can be seen by performing a Fourier transform over P. Namely, an element of the space $C^\infty(\mathscr{E})$ of Section 3.1.1 corresponds under Fourier transform to a section of the vector bundle E' of Section 4.3. One finds that the above condition on $\dim(\mathrm{Ker}(Q_p))$ is equivalent to the condition that Q'^2 has a Green's operator, i.e. that there is a gap between 0 and the nonzero L^2-spectrum of Q'^2. (The proof of this statement is similar to the arguments in [32, Section VI].) This last condition makes sense for arbitrary fundamental group.

Thus a reasonable requirement to define $\tilde{\eta}$ is for Q'^2 to have a Green's operator. In this case, there are general reasons to believe that (26) should make sense [6, Section 9.1]. However, in this paper we will look at the simpler situation in which Q'^2 actually has a bounded L^2-inverse i.e. that the infimum of the L^2-spectrum is strictly positive.

DEFINITION 11. Suppose that Q'^2 has a bounded L^2-inverse. The higher eta-invariant $\tilde{\eta} \in \tilde{\bar{\Omega}}_*(\mathscr{B})$ is

$$\tilde{\eta} = \begin{cases} \beta^{1/2} \displaystyle\int_0^\infty \mathrm{STR}(Q' \,\mathrm{e}^{-\beta D_s^2})\,\mathrm{d}s, & \text{if } n \text{ is even,} \\ \beta^{1/2} \displaystyle\int_0^\infty \mathrm{TR}_\sigma(\sigma Q' \,\mathrm{e}^{-\beta D_s^2})\,\mathrm{d}s, & \text{if } n \text{ is odd.} \end{cases} \tag{26}$$

It will easily follow from the proof of Proposition 25 that the integrand $\tilde{\eta}(s)$ of (26) is integrable on any compact interval of $(0, \infty)$. The problem is to show that it is integrable both near 0 and near ∞. The proof of the next proposition is slightly technical, and the reader may wish to omit it at first reading.

PROPOSITION 25. *$\tilde{\eta}(s)$ is absolutely integrable for large s.*

Proof. Let $\mu > 0$ be such that $[-\mu, \mu] \subset \mathbf{R}\backslash\mathrm{Spectrum}(Q')$. Let Θ be a smooth even function on $\mathbf{R}$ such that Θ is 0 on $[-\frac{1}{2}, \frac{1}{2}]$ and 1 on $\mathbf{R}\backslash(-1, 1)$. The idea of the proof is that for any function $g, g(Q') = g(Q')\Theta(Q'/\mu)$. This observation, along with finite propagation speed estimates, will allow us to prove the proposition.

For the purposes of the proof, we can assume that M' is connected. Let us recall the finite propagation speed estimate of [14]. Put $N = [n/4] + 1$. Let ε be a fixed sufficiently small positive number. If x and y are two points in M', put $R(x, y) = \min(0, \mathrm{d}(x, y) - \varepsilon)$. Let $f(r)$ be a Schwartz function on $\mathbf{R}$, with Fourier transform $\tilde{f}(p)$. Then Theorem 1.4 of [14] says

$$|f(Q')(x, y)| \leqslant \mathrm{const.} \sum_{j=0}^{2N} \int_{|p| \geqslant R(x,y)} |\tilde{f}^{(2j)}(p)|\,\mathrm{d}p. \tag{27}$$

Now for any integer $L \geqslant 0$,

$$\begin{aligned} |\tilde{f}^{(2j)}(p)| &= \mathrm{const.}|(\widetilde{r^{2j}f})(p)| \\ &\leqslant \mathrm{const.}(1 + p^2)^{-L} \left| \int_{-\infty}^{\infty} \left(1 - \frac{\mathrm{d}^2}{\mathrm{d}r^2}\right)^L (r^{2j} f(r))\,\mathrm{d}r \right| \\ &\leqslant \mathrm{const.}(1 + p^2)^{-L} \int_{-\infty}^{\infty} \left| \left(1 - \frac{\mathrm{d}^2}{\mathrm{d}r^2}\right)^L (r^{2j} f(r)) \right| \mathrm{d}r. \end{aligned}$$

Thus

$$|f(Q')(x, y)|$$

$$\leqslant \text{const.} \sum_{j=0}^{2N} \int_{R(x,y)}^{\infty} (1 + p^2)^{-L} \, \mathrm{d}p \int_{-\infty}^{\infty} \left| \left(1 - \frac{\mathrm{d}^2}{\mathrm{d}r^2}\right)^L (r^{2j} f(r)) \right| \mathrm{d}r. \tag{28}$$

In particular, suppose that $f(r) = r^a \Theta(r/\mu) \, \mathrm{e}^{-Tr^2}$ for some integer $a \geqslant 0$. Put

$$F_L(R) = \int_R^{\infty} (1 + p^2)^{-L} \, \mathrm{d}p.$$

(Note that $F_L(R)$ is $\mathrm{O}(R^{-2L+1})$ as $R \to \infty$.) Then we obtain

$$|(Q'^a \, \mathrm{e}^{-TQ'^2})(x, y)|$$

$$\leqslant \text{const.} \, F_L(R(x, y)) \sum_{A,B,C} \int_0^{\infty} r^A |\Theta^{(B)}(r/\mu)| T^C \, \mathrm{e}^{-Tr^2} \, \mathrm{d}r, \tag{29}$$

with A, B, and C being nonnegative integers and the sum over A, B and C being finite.

If instead we apply (27) with $f(r) = r^a \, \mathrm{e}^{-Tr^2}$ then we obtain [32, eqn. (9)]

$$|(Q'^a \, \mathrm{e}^{-TQ'^2})(x, y)|$$

$$\leqslant \text{const.} \left(\frac{R(x, y)^2}{T}\right)^{-(1/2)} [R(x, y)^{-a} + R(x, y)^{-a-4N} + R(x, y)^a T^{-a} +$$

$$+ R(x, y)^{a+4N} T^{-a-4N}] \exp\left[-\frac{R(x, y)^2}{4T}\right]. \tag{30}$$

As the integration in the Duhamel expansion involves all time, we will also need small-time bounds for the heat kernel. It follows from standard methods [40] that there is a $T_0 > 0$ such that for $0 < T \leqslant T_0$ and $\mathrm{d}(x, y) \leqslant 2\varepsilon$,

$$|(Q'^a \, \mathrm{e}^{-TQ'^2})(x, y)| \leqslant \text{const.} \, T^{-(n+a/2)} \exp\left[-\frac{\mathrm{d}^2(x, y)}{4.01T}\right]. \tag{31}$$

The strategy will be to use the estimate (29) when $T > T_0$, the estimate (30) when $T \leqslant T_0$ and $\mathrm{d}(x, y) > 2\varepsilon$, and the estimate (31) when $T \leqslant T_0$ and $\mathrm{d}(x, y) \leqslant 2\varepsilon$.

Consider the Duhamel expansion of $\tilde{\eta}(s)$. For simplicity, consider the case when n is even; the arguments are the same when n is odd. We have that

$$D_s^2 = s^2 Q'^2 + s(\nabla Q' + Q'\nabla) + \nabla^2,$$

where for $f \in C^{\infty}(M', E')$,

(1) $(\nabla Q' + Q'\nabla)(f) = \sum_{\gamma \in \Gamma} (\partial h) \, R_\gamma^* f \, \hat{\otimes}_{\mathscr{B}} \, \mathrm{d}\gamma$, with $\partial h = [Q', h]$ and

(2) $\nabla^2(f) = \sum_{\gamma, \gamma' \in \Gamma} h(R_\gamma^* h) \, R_{\gamma\gamma'}^* f \, \hat{\otimes}_{\mathscr{B}} \, \mathrm{d}\gamma \, \mathrm{d}\gamma'$.

To show that $\tilde{\eta}(s)$ is integrable, it is enough to only consider the component of a fixed degree, say k. Only a finite number of terms of the Duhamel expansion will contribute to this degree. Consider a typical term, such as

$$\beta^{1/2}\mathrm{STR}\Big((-1)^k \int_0^\infty \dots \int_0^\infty \delta\Big(\beta - \sum_{j=0}^k u_j\Big) Q' \,\mathrm{e}^{-u_0 s^2 Q'^2} s(\nabla Q' + Q'\nabla) \times \\ \mathrm{e}^{-u_1 s^2 Q'^2} s(\nabla Q' + Q'\nabla) \dots s(\nabla Q' + Q'\nabla)\,\mathrm{e}^{-u_k s^2 Q'^2}\Big)\,\mathrm{d}u_k \dots \mathrm{d}u_0. \tag{32}$$

Written out explicitly, this will be

$$\beta^{1/2} \sum_{\gamma_0,\dots,\gamma_k} (-1)^k \int_0^\infty \dots \int_0^\infty \delta\Big(\beta - \sum_{j=0}^k u_j\Big) \int_{M'} \phi(x_0) \times \\ \mathrm{tr}_s[R^*_{\gamma_0} Q' \,\mathrm{e}^{-u_0 s^2 Q'^2} s(\partial h)\, R^*_{\gamma_1}\,\mathrm{e}^{-u_1 s^2 Q'^2} s(\partial h)\, R^*_{\gamma_2} \dots s(\partial h)\, R^*_{\gamma_k}\,\mathrm{e}^{-u_k s^2 Q'^2}] \\ (x_0, x_0)\,\mathrm{dvol}(x_0)\,\mathrm{d}u_k \dots \mathrm{d}u_0 \gamma_0\, \mathrm{d}\gamma_1 \dots \mathrm{d}\gamma_k \tag{33}$$

$$= \beta^{1/2} \sum_{\gamma_0,\dots,\gamma_k} (-1)^k \int_0^\infty \dots \int_0^\infty \delta\Big(\beta - \sum_{j=0}^k u_j\Big) \int_{M'} \phi(x_0) \times \\ \mathrm{tr}_s[Q' \,\mathrm{e}^{-u_0 s^2 Q'^2} \{sR^*_{\gamma_0}(\partial h)\}\,\mathrm{e}^{-u_1 s^2 Q'^2} \{sR^*_{\gamma_0\gamma_1}(\partial h)\} \dots s\{R^*_{\gamma_0\dots\gamma_{k-1}}(\partial h)\} \times \\ \mathrm{e}^{-u_k s^2 Q'^2} R^*_{\gamma_0\dots\gamma_k}]\,(x_0, x_0)\,\mathrm{dvol}(x_0)\,\mathrm{d}u_k \dots \mathrm{d}u_0\, \gamma_0\, \mathrm{d}\gamma_1 \dots \mathrm{d}\gamma_k \tag{34}$$

$$= \beta^{1/2} \sum_{\gamma_0,\dots,\gamma_k} (-1)^k \int_0^\infty \dots \int_0^\infty \delta\Big(\beta - \sum_{j=0}^k u_j\Big) \int_{M'} \dots \int_{M'} \phi(x_0)\,\mathrm{tr}_s \\ [Q' \,\mathrm{e}^{-u_0 s^2 Q'^2}(x_0, x_1)\{s(\partial h)(x_1\gamma_0)\}\,\mathrm{e}^{-u_1 s^2 Q'^2}(x_1, x_2)\{s(\partial h)(x_2\gamma_0\gamma_1)\} \dots \\ \{s(\partial h)(x_k\gamma_0 \dots \gamma_{k-1})\}\,\mathrm{e}^{-u_k s^2 Q'^2}(x_k\gamma_0 \dots \gamma_k, x_0)] \\ \mathrm{dvol}(x_k) \dots \mathrm{dvol}(x_0)\,\mathrm{d}u_k \dots \mathrm{d}u_0\, \gamma_0\, \mathrm{d}\gamma_1 \dots \mathrm{d}\gamma_k. \tag{35}$$

Let us change variable to $v_j = u_j s^2$, to obtain

$$\beta^{1/2} s^{-k-2} \sum_{\gamma_0,\dots,\gamma_k} (-1)^k \int_0^\infty \dots \int_0^\infty \delta\Big(\beta - s^{-2} \sum_{j=0}^k v_j\Big) \int_{M'} \dots \int_{M'} \phi(x_0)\,\mathrm{tr}_s \\ [Q' \,\mathrm{e}^{-v_0 Q'^2}(x_0, x_1)\,\{(\partial h)(x_1\gamma_0)\}\,\mathrm{e}^{-v_1 Q'^2}(x_1, x_2)\,\{(\partial h)(x_2\gamma_0\gamma_1)\} \dots \\ \{(\partial h)(x_k\gamma_0 \dots \gamma_{k-1})\}\,\mathrm{e}^{-v_k Q'^2}(x_k\gamma_0 \dots \gamma_k, x_0)] \\ \mathrm{dvol}(x_k) \dots \mathrm{dvol}(x_0)\,\mathrm{d}v_k \dots \mathrm{d}v_0\, \gamma_0\, \mathrm{d}\gamma_1 \dots \mathrm{d}\gamma_k. \tag{36}$$

It is enough to show that the coefficient of $\gamma_0\, \mathrm{d}\gamma_1 \dots \mathrm{d}\gamma_k$ in (36) decays faster than any power of $1 + \Sigma_{j=0}^k \|\gamma_j\|$.

We will divide the integration domain of the $\{v_j\}_{j=0}^k$ into 2^{j+1} pieces according as to whether each v_j is less than or equal to, or greater than T_0. First, consider the contribution to the coefficient of $\gamma_0 \mathrm{d}\gamma_1 \dots \mathrm{d}\gamma_k$ from the piece having all of the $\{v_j\}_{j=0}^k$

greater than T_0. Using (29), its norm will be bounded above by

$$\text{const.}\left\{\sum_{\gamma_0,\ldots,\gamma_k}\int_{M'}\cdots\int_{M'}|\phi(x_0)|F_L(R(x_0,x_1))|\partial h(x_1\gamma_0)|F_L(R(x_1,x_2))\right.$$
$$|\partial h(x_2\gamma_0\gamma_1)|\ldots|\partial h(x_k\gamma_0\ldots\gamma_{k-1})|F_L(R(x_k\gamma_0\ldots\gamma_k,x_0))\,\mathrm{dvol}(x_k)\ldots$$
$$\left.\mathrm{dvol}(x_0)\right\}\times$$
$$\left\{s^{-k-2}\int_{T_0}^{\infty}\cdots\int_{T_0}^{\infty}\int_0^{\infty}\cdots\int_0^{\infty}\delta\left(\beta-s^{-2}\sum_{j=0}^{k}v_j\right)\sum_{\vec{A},\vec{B},\vec{C},}\prod_{j=0}^{k}r_j^{A_j}\left|\Theta^{(B_j)}\left(\frac{r_j}{\mu}\right)\right|\right.$$
$$\left.v_j^{C_j}\,\mathrm{e}^{-v_jr_j^2}\,\mathrm{d}r_k\ldots\mathrm{d}r_o\mathrm{d}v_k\ldots\mathrm{d}v_0\right\},\tag{37}$$

where the sum over $\vec{A},\vec{B},\vec{C}$ is a sum over a finite set.

Let S be a compact set which contains $\mathrm{supp}(\phi)$ and $\mathrm{supp}(h)$. Then the first factor in (37) is

$$\sum_{\gamma_0,\ldots,\gamma_k}\int_{M'}\cdots\int_{M'}|\phi(y_0)|F_L(R(y_0\gamma_0,y_1))|\partial h(y_1)|F_L(R(y_1\gamma_1,y_2))\times$$
$$|\partial h(y_2)|\ldots|\partial h(y_k)|F_L(R(y_k\gamma_k,y_0))\,\mathrm{dvol}(y_k)\ldots\mathrm{dvol}(y_0)$$
$$=\sum_{\gamma_0,\ldots,\gamma_k}\int_S\cdots\int_S|\phi(y_0)|F_L(R(y_0\gamma_0,y_1))|\partial h(y_1)|F_L(R(y_1\gamma_1,y_2))|\partial h(y_2)|\ldots$$
$$|\partial h(y_k)|F_L(R(y_k\gamma_k,y_0))\,\mathrm{dvol}(y_k)\ldots\mathrm{dvol}(y_0).\tag{38}$$

The fact that Γ is of polynomial growth implies that for large enough L, (38) is finite.

As Θ and its derivatives vanish on $[-\frac{1}{2},\frac{1}{2}]$, the second factor in (37) is bounded above by

$$\text{const.}\,s^{-k-2}\int_{T_0}^{\infty}\cdots\int_{T_0}^{\infty}\int_{\mu/2}^{\infty}\cdots\int_{\mu/2}^{\infty}\delta\left(\beta-s^{-2}\sum_{j=0}^{k}v_j\right)\sum_{\vec{A},\vec{C},}\prod_{j=0}^{k}r_j^{A_j}v_j^{C_j}\,\mathrm{e}^{-v_jr_j^2}$$
$$\mathrm{d}r_k\ldots\mathrm{d}r_o\mathrm{d}v_k\ldots\mathrm{d}v_0$$
$$\leqslant\text{const.}\,s^{-k-2}\int_{T_0}^{\infty}\cdots\int_{T_0}^{\infty}\int_0^{\infty}\cdots\int_0^{\infty}\delta\left(\beta-s^{-2}\sum_{j=0}^{k}v_j\right)\sum_{\vec{A},\vec{C},}\prod_{j=0}^{k}$$
$$\left(x_j+\frac{\mu}{2}\right)^{A_j}v_j^{C_j}\exp\left[-v_j\left(\frac{\mu^2}{4}+\mu x_j\right)\right]\mathrm{d}x_k\ldots\mathrm{d}x_o\mathrm{d}v_k\ldots\mathrm{d}v_0$$
$$\leqslant\text{const.}\,s^{-k-2}\int_{T_0}^{\infty}\cdots\int_{T_0}^{\infty}\delta\left(\beta-s^{-2}\sum_{j=0}^{k}v_j\right)\sum_{\vec{A},\vec{C},}\prod_{j=0}^{k}v_j^{C_j-1}\left(\frac{1}{v_j}+\frac{\mu}{2}\right)^{A_j}$$
$$\exp\left(-\frac{v_j\mu^2}{4}\right)\mathrm{d}v_k\ldots\mathrm{d}v_0$$

$$= \text{const.}\, s^{-k-2} \exp\left(-\frac{\beta s^2 \mu^2}{4}\right) \int_{T_0}^{\infty} \cdots \int_{T_0}^{\infty} \delta\left(\beta - s^{-2} \sum_{j=0}^{k} v_j\right) \sum_{\vec{A},\vec{C},} \prod_{j=0}^{k} v_j^{C_j - 1} \left(\frac{1}{v_j} + \frac{\mu}{2}\right)^{A_j} \mathrm{d}v_k \ldots \mathrm{d}v_0. \tag{39}$$

For large s, the exponential term in (39) will dominate the rest, which will grow at most polynomially in s. Thus the contribution of the piece with all v_j's greater than T_0 will decay rapidly in s.

Now let us look at the contributions from the pieces with $v_j \leqslant T_0$ for some v_j. For simplicity, let us consider the piece with $v_0 \leqslant T_0$ and $v_j > T_0$ for $j > 0$; the estimates of the other pieces will be similar. Its contribution to (36) is

$$\begin{aligned}&\beta^{1/2} s^{-k-2} \sum_{\gamma_0, \ldots, \gamma_k} (-1)^k \int_0^{T_0} \int_{T_0}^{\infty} \cdots \int_{T_0}^{\infty} \delta\left(\beta - s^{-2} \sum_{j=0}^{k} v_j\right) \int_{M'} \cdots \int_{M'} \phi(x_0)\, \mathrm{tr}_s \\ &[Q' \mathrm{e}^{-v_0 Q'^2}(x_0, x_1) \{(\partial h)(x_1 \gamma_0)\} \mathrm{e}^{-v_1 Q'^2}(x_1, x_2) \{(\partial h)(x_2 \gamma_0 \gamma_1)\} \ldots \\ &\{(\partial h)(x_k \gamma_0 \ldots \gamma_{k-1})\} \mathrm{e}^{-v_k Q'^2}(x_k \gamma_0 \ldots \gamma_k, x_0)]\, \mathrm{dvol}(x_k) \ldots \\ &\mathrm{dvol}(x_0)\, \mathrm{d}v_k \ldots \mathrm{d}v_0 \gamma_0 \mathrm{d}\gamma_1 \ldots \mathrm{d}\gamma_k.\end{aligned} \tag{40}$$

Using (29), the norm of the coefficient of $\gamma_0\, \mathrm{d}\gamma_1 \ldots \mathrm{d}\gamma_k$ in (40) will be bounded above by

$$\begin{aligned}&\text{const.} \left\{ \sum_{\gamma_0, \ldots, \gamma_k} \int_{M'} \cdots \int_{M'} \left(\int_0^{T_0} |Q' \mathrm{e}^{-v_0 Q'^2}(x_0, x_1)|\, \mathrm{d}v_0 \right) |\phi(x_0)|\, |\partial h(x_1 \gamma_0)| \right. \\ &F_L(R(x_1, x_2))\, |\partial h(x_2 \gamma_0 \gamma_1)| \ldots |\partial h(x_k \gamma_0 \ldots \gamma_{k-1})|\, F_L(R(x_k \gamma_0 \ldots \gamma_k, x_0)) \\ &\left. \mathrm{dvol}(x_k) \ldots \mathrm{dvol}(x_0) \right\} \\ &\left\{ s^{-k-2} \sup_{v_0 \in [0, T_0]} \int_{T_0}^{\infty} \cdots \int_{T_0}^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} \delta\left(\beta - s^{-2} \sum_{j=0}^{k} v_j\right) \sum_{\vec{A},\vec{B},\vec{C},} \prod_{j=1}^{k} r_j^{A_j} \left| \Theta^{(B_j)}\left(\frac{r_j}{\mu}\right) \right| \right. \\ &\left. v_j^{C_j} \mathrm{e}^{-v_j r_j^2}\, \mathrm{d}r_k \ldots \mathrm{d}r_1\, \mathrm{d}v_k \ldots \mathrm{d}v_1 \right\}.\end{aligned} \tag{41}$$

Integrating (31) gives that for $\mathrm{d}(x_0, x_1) \leqslant 2\varepsilon$,

$$\begin{aligned}&\int_0^{T_0} |Q' \mathrm{e}^{-v_0 Q'^2}(x_0, x_1)|\, \mathrm{d}v_0 \\ &\quad \leqslant \text{const.} \int_0^{T_0} T^{-((n+1)/2)} \mathrm{e}^{-(d^2/4.01 T)}\, \mathrm{d}T = \text{const.} \int_{T_0^{-1}}^{\infty} r^{(n-3)/2} \mathrm{e}^{-r d^2/4.01}\, \mathrm{d}r \\ &\quad = \text{const.} \int_{T_0^{-1}}^{\infty} (T_0^{-1} + x)^{(n-3)/2} \exp\left[-\frac{(T_0^{-1} + x)\mathrm{d}^2}{4.01} \right] \mathrm{d}x\end{aligned}$$

$$\leqslant \text{const.}\, \frac{1}{\mathrm{d}^2}\left(\frac{1}{T_0} + \frac{4.01}{\mathrm{d}^2}\right)^{(n-3)/2} \mathrm{e}^{-(d^2/4.01 T_0)}, \tag{42}$$

where $\mathrm{d} = \mathrm{d}(x_0, x_1)$. Similarly, integrating (30) gives that for $\mathrm{d}(x_0, x_1) \geqslant 2\varepsilon$,

$$\int_0^{T_0} |Q' \mathrm{e}^{-v_0 Q'^2}(x_0, x_1)| \, \mathrm{d}v_0$$

$$\leqslant \text{const.}\, \frac{1}{R^3}\left(\frac{1}{T_0} + \frac{4}{R^2}\right)^{-5/2}\left[R^{-1} + R^{-1-4N} + R\left(\frac{1}{T_0} + \frac{4}{R^2}\right) + R^{1+4N} \times \left(\frac{1}{T_0} + \frac{4}{R^2}\right)^{1+4N}\right] \mathrm{e}^{-(R^2/4T_0)}, \tag{43}$$

where $R = \mathrm{d}(x_0, x_1) - \varepsilon$. Equations (42) and (43) show that $\int_0^{T_0} |Q' \mathrm{e}^{-v_0 Q'^2}(x_0, x_1)| \, \mathrm{d}v_0$ is locally integrable in x_0, and decays faster than any power in $\mathrm{d}(x_0, x_1)$ as $\mathrm{d}(x_0, x_1) \to \infty$. Then for large enough L, the first term in (41) is finite. The second term in (41) can be bounded as in Equation (39), and so we obtain that the contribution of the piece with $v_0 \leqslant T_0$ and $v_j > T_0$ for $j > 0$ is integrable for large s. It should be clear that the same arguments will apply to rest of the 2^{j+1} pieces of the integration domain. Also, one can check that the same arguments apply to the other terms in the Duhamel expansion. □

PROPOSITION 26. *$\tilde{\eta}(s)$ is absolutely integrable for small s.*

Proof. The method of proof will be as in [6, Section 10.5]. (Our labels s and t are the opposite of [6].) We will cross the noncommutative base space with $\mathbf{R}$. That is, we consider the algebra $\tilde{\mathscr{B}} = C^\infty(\mathbf{R}) \otimes \mathscr{B}$ and the graded differential algebra $\Lambda^*(\mathbf{R}) \otimes \hat{\Omega}_*(\mathscr{B})$. Let $\mathscr{M}$ be $C^\infty(\mathbf{R}) \otimes C^\infty(\mathscr{E})$, a $\tilde{\mathscr{B}}$-module. Let t be a coordinate for $\mathbf{R}$ and consider the superconnection $\tilde{D}_s$, acting on $\mathscr{M}$, given by

$$\tilde{D}_s = D_{ts} + \mathrm{d}t \wedge \partial_t.$$

Define $\widetilde{\mathrm{ch}}(s) \in \Lambda^*(\mathbf{R}) \otimes \bar{\hat{\Omega}}_*(\mathscr{B})$ as in Definition 10. Then it follows as in [6, Section 10.5] that

$$\widetilde{\mathrm{ch}}(s) = \mathrm{ch}(ts) - \beta^{1/2} s \, \mathrm{d}t \wedge \tilde{\eta}(ts). \tag{44}$$

As in (24), one can compute the asymptotics of the left-hand side of (44) as $s \to 0$. One finds that there is a Taylor's series expansion with s^0-term given by the right-hand side of (24). In particular, the $\mathrm{d}t$-term of (44) starts at order s^1, and so $\tilde{\eta}(ts)$ has a finite limit as $s \to 0$. Taking $t = 1$, the proposition follows. □

4.5. VARIATIONAL PROPERTIES OF $\tilde{\eta}$

A priori, $\tilde{\eta}$ depends on the choices made in its definition, namely

(1) the function h,

(2) the Riemannian metric on M, and
(3) the hermitian connection on V.

To understand this dependence, first let us do some formal calculations. For simplicity, suppose that n is even. Consider a smooth 1-parameter family of input information, parametrized by a real number ε. As elements of $\bar{\bar{\Omega}}_*(\mathscr{B})$, we have the equalities

$$\frac{\mathrm{d}\,\mathrm{ch}(s)}{\mathrm{d}s} = -\beta^{1/2}\,\mathrm{d}\tilde{\eta}(s) \tag{45}$$

and

$$\frac{\mathrm{d}\,\mathrm{ch}(s)}{\mathrm{d}\varepsilon} = -\beta\,\mathrm{d}\,\mathrm{STR}\left(\frac{\mathrm{d}D_s}{\mathrm{d}\varepsilon}\,\mathrm{e}^{-\beta D_s^2}\right). \tag{46}$$

Then

$$\mathrm{d}\frac{\mathrm{d}\tilde{\eta}(s)}{\mathrm{d}\varepsilon} = \mathrm{d}\,\beta^{1/2}\frac{\mathrm{d}}{\mathrm{d}s}\left(\mathrm{STR}\left(\frac{\mathrm{d}D_s}{\mathrm{d}\varepsilon}\,\mathrm{e}^{-\beta D_s^2}\right)\right). \tag{47}$$

This makes it plausible that

$$\frac{\mathrm{d}\tilde{\eta}(s)}{\mathrm{d}\varepsilon} = \beta^{1/2}\frac{\mathrm{d}}{\mathrm{d}s}\left(\mathrm{STR}\left(\frac{\mathrm{d}D_s}{\mathrm{d}\varepsilon}\,\mathrm{e}^{-\beta D_s^2}\right)\right)\ (\mathrm{mod}\ \mathrm{Im}(d)), \tag{48}$$

which is in fact true, as one can check that

$$\begin{aligned}&\frac{\mathrm{d}\tilde{\eta}(s)}{\mathrm{d}\varepsilon} - \beta^{1/2}\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{STR}\left(\frac{\mathrm{d}D_s}{\mathrm{d}\varepsilon}\,\mathrm{e}^{-\beta D_s^2}\right)\\ &\quad = \mathrm{d}\left(\int_0^\beta \mathrm{STR}\left(\mathrm{e}^{-uD_s^2}Q\,\mathrm{e}^{-(\beta-u)D_s^2}\frac{\mathrm{d}D_s}{\mathrm{d}\varepsilon}\right)\mathrm{d}u\right).\end{aligned} \tag{49}$$

(Recall that in defining $\bar{\bar{\Omega}}_*(\mathscr{B})$ we quotient out by the commutator.)

From (49), we obtain that

$$\frac{\mathrm{d}\tilde{\eta}}{\mathrm{d}\varepsilon} = \beta^{1/2}\left(\lim_{s\to\infty} - \lim_{s\to 0}\right)\mathrm{STR}\left(\frac{\mathrm{d}D_s}{\mathrm{d}\varepsilon}\,\mathrm{e}^{-\beta D_s^2}\right)\ (\mathrm{mod}\ \mathrm{Im}(d)). \tag{50}$$

One can justify the formal manipulations in Equations (45)–(50) by the estimates used in the proof of Proposition 25. With our assumption that the operators $Q'(\varepsilon)$ are all invertible, we have

$$\lim_{s\to\infty}\mathrm{STR}\left(\frac{\mathrm{d}D_s}{\mathrm{d}\varepsilon}\,\mathrm{e}^{-\beta D_s^2}\right) = 0.$$

Thus

$$\frac{\mathrm{d}\tilde{\eta}}{\mathrm{d}\varepsilon} = -\beta^{1/2}\lim_{s\to 0}\mathrm{STR}\left(\frac{\mathrm{d}D_s}{\mathrm{d}\varepsilon}\,\mathrm{e}^{-\beta D_s^2}\right)\ (\mathrm{mod}\ \mathrm{Im}(d)),$$

which, being a small-time limit, is given by the integral of a local expression on M. Note that this is essentially the same argument as was used at the end of Section 3.1.2. The small-time limit can be calculated as in [31], and we will simply state the result.

PROPOSITION 27. *Consider the product bundle* $\mathbf{R} \times M$, *with vertical metrics given by* $g(\varepsilon)$, *and pulled-back vector bundle* $\tilde{V}$. *Let* $\tilde{h}$ *be the function on* $\mathbf{R} \times M'$ *corresponding to* $h(\varepsilon)$. *There is a biform* $\omega' \in \Lambda^*(\mathbf{R} \times M) \otimes \bar{\hat{\Omega}}_*(\mathscr{B})$, *closed in both factors, constructed from* $\tilde{h}$ *such that*

$$\frac{\mathrm{d}\tilde{\eta}}{\mathrm{d}\varepsilon} = \int_M i(\partial_\varepsilon)(\hat{A}(\mathbf{R} \times M) \wedge \mathrm{Ch}(\tilde{V}) \wedge \omega') \ (\mathrm{mod}\ \mathrm{Im}(d)). \tag{51}$$

The important term in ω' of degree k (with respect to $\mathscr{B}$) is a closed form on $\mathbf{R} \times M$ whose pullback to $\mathbf{R} \times M'$ is given by

$$\beta^{k/2} \sum_{\gamma_0 \cdots \gamma_k = e} R^*_{\gamma_0} \mathrm{d}\tilde{h} \wedge \ldots \wedge R^*_{\gamma_0 \cdots \gamma_{k-1}} \mathrm{d}\tilde{h}\, \gamma_0 \mathrm{d}\gamma_1 \ldots, \mathrm{d}\gamma_k \in \Lambda^k(\mathbf{R} \times M') \otimes \bar{\hat{\Omega}}_k(\mathscr{B}).$$

The right-hand side of (51) has support on the forms spanned by

$$\{\gamma_0 \mathrm{d}\gamma_1 \ldots \mathrm{d}\gamma_k \ (\mathrm{mod}\ \mathrm{Im}(d)) \colon \gamma_0 \ldots \gamma_k = e\}.$$

COROLLARY 2. *Let* $\mathscr{T}_\tau$ *be a cyclic cocycle of* $\mathbf{C}\Gamma$ *constructed as in* (20). *Suppose that* $\mathscr{T}_\tau$ *extends to a cyclic cocycle of* $\mathscr{B}$. *Suppose that the vector bundle* V *is associated to the principal Spin(n)-bundle on M. Suppose that the index density* $\hat{A}(M) \wedge \mathrm{Ch}(V)$ *is a polynomial in the Pontryagin classes of M. Then for fixed h,* $\langle \tilde{\eta}, \mathscr{T}_\tau \rangle$ *is a conformal-deformation invariant.*

Proof. As in the proof of Proposition 4, $i(\partial_\varepsilon)(\hat{A}(\mathbf{R} \times M) \wedge \mathrm{Ch}(\tilde{V}))$ vanishes identically. □

COROLLARY 3. *Let* $\mathscr{T}_\tau$ *be a cyclic cocycle of* $\mathbf{C}\Gamma$ *constructed as in* (20). *Suppose that* $\mathscr{T}_\tau$ *extends to a cyclic cocycle of* $\mathscr{B}$. *If* $x \neq e$ *then* $\langle \tilde{\eta}, \mathscr{T}_\tau \rangle$ *is independent of h and is a deformation invariant with respect to the Riemannian metric on M and the Hermitian connection on V.*

Proof. This follows directly from Proposition 27. □

EXAMPLE. Take M as in the Example of Section 3.1.3, with the Dirac operator. Then the cyclic cohomology of $\mathbf{C}\Gamma$ is given by

$$HC^*(\mathbf{C}\Gamma) = \bigoplus_{\langle F \rangle} HC^*(\mathbf{C}\mathbf{Z}^l).$$

Under Fourier transform, an element $\mathscr{T}_\sigma$ of $HC^k(\mathbf{C}\mathbf{Z}^l)$ becomes a sum of a closed k-current on $\mathbf{T}^l$ and lower-dimensional homology classes on $\mathbf{T}^l$ [17]. Let Φ denote the corresponding total class in $H_*(\mathbf{T}^l)$. Let $\langle f \rangle$ be a conjugacy class in F. Let $\mathscr{T}_\tau$ be the cyclic cocycle on $\mathbf{C}\Gamma$ formed from $\langle f \rangle$ and $\mathscr{T}_\sigma$. It follows from separation of variables that

$$\langle \tilde{\eta}, \mathcal{T}_\tau \rangle = \beta^{k/2} \frac{\sqrt{\pi}}{2} \eta_L(f) \langle \Phi, [\mathbf{T}^l] \rangle.$$

Here $\eta_L(f)$ is the eta-invariant of (4) and $[\mathbf{T}^l]$ is the fundamental class of $\mathbf{T}^l$ in cohomology.

4.6. PAIRING WITH PERIODIC CYCLIC COHOMOLOGY

We saw in Corollary 3 that we obtain deformation invariants of (M, V) by pairing $\tilde{\eta}$ with certain cyclic cocycles of $\mathscr{B}$. This gives a generalization of the rho-invariant of [3], which corresponds to the special case of pairing with 0-cocycles. More precisely, $\bar{\hat{\Omega}}_*(\mathscr{B})$ breaks up into a sum of subcomplexes labeled by the conjugacy classes of Γ, and we can write

$$\tilde{\eta} = \bigoplus_{\langle x \rangle \in \langle \Gamma \rangle} \tilde{\eta}(\langle x \rangle). \tag{52}$$

We define the higher rho-invariant by

$$\tilde{\rho} = \bigoplus_{\langle x \rangle \neq \langle e \rangle} \tilde{\eta}(\langle x \rangle). \tag{53}$$

Integrating (45) with respect to s, we have

$$\beta^{1/2}\, \mathrm{d}\tilde{\eta} = -\left(\lim_{s \to \infty} - \lim_{s \to 0}\right) \mathrm{ch}(s).$$

As Q' is invertible, $\lim_{s \to \infty} \mathrm{ch}(s) = 0$. From Proposition 24, $\lim_{s \to 0} \mathrm{ch}(s)$ has support on the subcomplex corresponding to the trivial conjugacy class. Thus $\tilde{\rho}$ is closed, and so represents an element of $\bar{\bar{H}}_*(\mathscr{B})$. By Proposition 27, the class of $\tilde{\rho}$ in $\bar{\bar{H}}_*(\mathscr{B})$ is a deformation invariant of (M, V). The pairing of $\tilde{\rho}$ with reduced cyclic cocycles of $\mathbf{C}\Gamma$ was described in Corollary 3. It does not immediately pass to a pairing with the *periodic* cyclic cohomology, mostly because of a problem with numerical factors.

First, let us discuss periodicity in reduced cyclic cohomology. From the dual equation to (17),

$$\overline{HC}^*(\mathscr{B}) \cong \mathrm{Ker}(HC^*(\mathscr{B}) \to HC^*(\mathbf{C})), \tag{54}$$

the S-operator on cyclic cohomology passes to an operator on reduced cyclic cohomology. However, it does not generally have a simple expression as an operator on reduced cochains. Of course, if $\mathscr{B}$ is an augmented algebra then there is a simple expression. More generally, suppose that $\mathscr{B}$ is an algebra with a trace Tr. Given $\phi \in C^k_\lambda(\mathscr{B})$, define $\mathrm{Tr} \cdot \phi \in C^{k+1}(\mathscr{B}, \mathscr{B}^*)$ by

$$(\mathrm{Tr} \cdot \phi)(b_0, \ldots, b_{k+1}) = \mathrm{Tr}(b_0)\phi(b_1, \ldots, b_{k+1}).$$

Using the notation of [17], define $S\phi \in C_\lambda^{k+2}(\mathscr{B})$ by

$$S\phi = A\left[\frac{1}{k+3}(\sigma\sharp\phi) + b'(\mathrm{Tr}\cdot\phi)\right] \tag{55}$$

and define $\tilde{S}\phi \in C_\lambda^{k+2}(\mathscr{B})$ by

$$\tilde{S}\phi = -\frac{1}{(k+1)(k+2)}S\phi. \tag{56}$$

Note that because $Ab' = bA$, the term that we have added to the usual expression for the S-operator is a cyclic coboundary. Then one can check that S and $\tilde{S}$ extend to operators on reduced cyclic cohomology. Similarly, there are operators S and $\tilde{S}$ in reduced cyclic homology. Periodicity in reduced cyclic homology will refer to invariance with respect to the $\tilde{S}$-operator. In particular, for the various group algebras which we consider, there is a trace Tr given by evaluation at the identity element.

We now consider the relationship between the Chern character and periodicity. (We will loosely speak of the Chern character of a module as an element of $\overline{HC}_*(\mathscr{B})$, although this is not strictly true for the term of degree zero.) In general, the Chern character is not $\tilde{S}$-invariant. For example, in the case of a finite right projective $\mathbf{Z}_2$-graded module $\mathscr{E}$, putting $Q = 0$, we have

$$\mathrm{Ch}_\beta(\mathscr{E}) = \sum_{j=0}^{\infty}(-1)^j\frac{\beta^j}{j!}\mathrm{tr}_s((\nabla^2)^k), \tag{57}$$

which as an element of $\overline{HC}_*(\mathscr{B})$ is not $\tilde{S}$-invariant. However, one can easily modify this expression by defining $\mathrm{Ch}^{\mathrm{per}}(\mathscr{E})$ to be

$$\mathrm{Ch}^{\mathrm{per}}(\mathscr{E}) = \int_0^\infty \mathrm{e}^{-\beta}\mathrm{Ch}_{\beta^2}\,\mathrm{d}\beta.$$

Then

$$\mathrm{Ch}^{\mathrm{per}}(\mathscr{E}) = \sum_{j=0}^{\infty}(-1)^j\frac{(2j)!}{j!}\mathrm{tr}_s((\nabla^2)^k), \tag{58}$$

which is $\tilde{S}$-invariant [26].

Similarly, in the case of an ungraded finite right-projective module, assume that the self-adjoint operator Q is invertible. Give the module a $\mathbf{Z}_2$-grading by the positive and negative spectral subspaces of Q. Then $\tilde{\eta}_\beta(\mathscr{E})$ is closed, and its class in $\overline{HC}_*(\mathscr{B})$ is $(\sqrt{\pi}/2)\mathrm{Ch}_\beta(\mathscr{E})$ [8, 30]. Thus

$$\tilde{\eta}^{\mathrm{per}}(\mathscr{E}) = \int_0^\infty \mathrm{e}^{-\beta}\tilde{\eta}_{\beta^2}\,\mathrm{d}\beta$$

is $\tilde{S}$-invariant.

This motivates the following definitions:

DEFINITION 12.

$$\mathrm{ch}^{\mathrm{per}}(s) = \int_0^\infty \mathrm{e}^{-\beta} \mathrm{ch}_{\beta^2}(s)\, \mathrm{d}\beta \in \bar{\bar{H}}_*(\mathscr{B})$$

and

$$\tilde{\rho}^{\mathrm{per}} = \int_0^\infty \mathrm{e}^{-\beta} \tilde{\rho}_{\beta^2}\, \mathrm{d}\beta \in \bar{\bar{H}}_*(\mathscr{B}).$$

As the dependence of $\mathrm{ch}_{\beta^2}(s)$ and $\tilde{\rho}_{\beta^2}$, as elements of $\bar{\bar{H}}_*(\mathscr{B})$, on β is simply given by a nonnegative power of β in each degree, it is clear that the β-integral makes sense.

PROPOSITION 28. *As an element of* $\overline{HC}_*(\mathscr{B})$, $\mathrm{ch}^{\mathrm{per}}(s)$ *is* $\tilde{S}$*-invariant.*

Proof. First suppose that n is even. With our assumption on Γ, the index theorem of [31] applies, and so $\mathrm{ch}_\beta(s) = \mathrm{Ch}_\beta(\mathrm{Index}(Q))$. The $\tilde{S}$-invariance of $\mathrm{ch}^{\mathrm{per}}(s)$ then follows from the above discussion of the finite-dimensional case i.e. equation (58). In fact, the Proposition is true for all finitely-presented Γ, regardless of the existence of an index theorem, and so we now give an alternative proof. The class of $\mathrm{ch}_{\beta^2}(s)$ in $\overline{HC}_*(\mathscr{B})$ equal to the $s \to 0$ limit, which was given in Proposition 24. Let us write $\mathrm{ch}_{\beta^2}(0)$ as

$$\mathrm{ch}_{\beta^2}(0) = \sum_{j=0}^{\infty} (-1)^j \beta^{2j} \mathrm{ch}^{[2j]}(0),$$

with $\mathrm{ch}^{[2j]}(0) \in \overline{HC}_{2j}(\mathscr{B})$. Using the expression for ω derived in [31], one can check that

$$S\, \mathrm{ch}^{[2j]}(0) = \mathrm{ch}^{[2j-2]}(0). \tag{59}$$

We will not give the (uninteresting) computation here. It follows that

$$\tilde{S}\, \mathrm{ch}^{[2j]}(0) = -\frac{1}{(2j-1)(2j)} \mathrm{ch}^{[2j-2]}(0), \tag{60}$$

and so

$$\mathrm{ch}^{\mathrm{per}}(0) = \sum_{j=0}^{\infty} (-1)^j (2j)!\, \mathrm{ch}^{[2j]}(0),$$

and, hence, also $\mathrm{ch}^{\mathrm{per}}(s)$, are $\tilde{S}$-invariant.

If n is odd, consider $S^1 \times M$. Now $\pi_1(S^1 \times M) = \mathbf{Z} \times \pi_1(M)$, and the algebra of rapidly decaying elements of $C_r^*(\mathbf{Z} \times \pi_1(M))$ is isomorphic to $C^\infty(S^1) \otimes \mathscr{B}$. A separation of variables argument shows that the image of $\mathrm{ch}_{\beta^2}^{S^1 \times M}(s)$ under the natural map $t: \overline{HC}_*(C^\infty(S^1) \otimes \mathscr{B}) \to \overline{HC}_*(C^\infty(S^1)) \otimes \overline{HC}_*(\mathscr{B})$, is given by

$$t(\mathrm{ch}_{\beta^2}^{S^1 \times M}(s)) = \beta\eta \otimes \mathrm{ch}_{\beta^2}^M(s) = \sum_{j=0}^{\infty} \beta\eta \otimes (-1)^{j+1}\beta^{2j+1}\mathrm{ch}^{[2j+1]}(s), \tag{61}$$

where η is a generator of $\overline{HC}_1(C^\infty(S^1))$ and we put

$$\mathrm{ch}_{\beta^2}^M(s) = \sum_{j=0}^{\infty} (-1)^{j+1}\beta^{2j+1}\mathrm{ch}^{[2j+1]}(s). \tag{62}$$

As the S operator is simply obtained by taking tensor products with the cyclic homology of $\mathbf{C}$, it commutes with t, and

$$S(\eta \otimes \mathrm{ch}^{[2j+1]}(s)) = \eta \otimes S(\mathrm{ch}^{[2j+1]}(s)).$$

Applying (59) to $S^1 \times M$ then gives

$$S(\mathrm{ch}^{[2j+1]}(s)) = \mathrm{ch}^{[2j-1]}(s), \tag{63}$$

and so

$$\tilde{S}(\mathrm{ch}^{[2j+1]}(s)) = -\frac{1}{(2j)(2j+1)}\mathrm{ch}^{[2j-1]}(s). \tag{64}$$

It follows that the periodic Chern character of M,

$$\mathrm{ch}^{\mathrm{per}}(s) = \sum_{j=0}^{\infty} (-1)^{j+1}(2j+1)!\,\mathrm{ch}^{[2j+1]}(s)$$

is $\tilde{S}$-invariant. □

We expect that as an element of $\overline{HC}_*(\mathscr{B})$, $\tilde{\rho}^{\mathrm{per}}$ is also $\tilde{S}$-invariant. In view of the truth of this statement in the finite-dimensional case, this seems likely to be true, but we do not see how to prove it.

4.7. HIGHER ETA-INVARIANTS FOR THE SIGNATURE OPERATOR

We now consider the case when Q' is the signature operator on M' if n is even, or the tangential signature operator if n is odd. We have only defined the higher eta-invariant for invertible operators in this paper, and so as it stands, the higher eta-invariant for a signature operator would probably never be defined. However, there are various ways to make the obstructions to invertibility cancel, in order to obtain an effectively invertible operator. This is somewhat similar to how the Ray–Singer torsion becomes a topological invariant for a pair of homotopy-equivalent manifolds (the Whitehead torsion) or for a flat acyclic bundle (the Reidemeister torsion).

To be more precise, suppose that M_1 and M_2 are closed smooth oriented Riemannian manifolds with a smooth orientation-preserving homotopy equivalence $\alpha: M_2 \to M_1$. Let Γ be a finitely presented virtually nilpotent group and let

$\alpha': M'_2 \to M'_1$ be a lift of α to normal Γ-covers. First, suppose that α is a submersion. Consider the complex

$$\cdots \xrightarrow{\mathrm{d}} \Lambda^{k-1}(M'_1, M'_2) \xrightarrow{\mathrm{d}} \Lambda^{k}(M'_1, M'_2) \xrightarrow{\mathrm{d}} \Lambda^{k+1}(M'_1, M'_2) \xrightarrow{\mathrm{d}} \cdots,$$

where

$$\Lambda^k(M'_1, M'_2) = \Lambda^k(M'_1) \oplus \Lambda^{k-1}(M'_2) \quad \text{and} \quad \mathrm{d}(\omega_1, \omega_2) = (\mathrm{d}\omega_1, (\alpha')^*\omega_1 - \mathrm{d}\omega_2).$$

Then the homotopy-equivalence of M_1 and M_2 implies that the relative (tangential) signature operator is L^2-invertible on $\Lambda^*(M'_1, M'_2)$ [33]. If $h \in C_0^\infty(M'_1)$ satisfies (23) then we can form a superconnection from the pair $(h, (\alpha')^*h)$ as in Section 4.3, and define a relative higher eta-invariant $\tilde{\eta}(M_1, M_2)$. As the invertibility of the relative signature operator is independent of the Riemannian metrics, it follows that the relative higher rho-invariant $\tilde{\rho}(M_1, M_2)$ is independent of all choices made.

If α is not a submersion then there are technical problems with the above definition, as the operator $(\alpha')^*$, defined on smooth forms, may not be L^2-bounded, or even closable. However, as in [22], one can put $N_2 = M_2 \times B^N$, where B^N is a ball of large dimension, and find a submersion $\sigma: N_2 \to M_1$ such that $\sigma(m_2, 0) = \alpha(m_2)$. Then $(\sigma')^*$ is a bounded operator from $\Lambda^*(M'_1)$ to $\Lambda^*(N'_2)$. However, we will want to consider forms on N'_2 which satisfy absolute boundary conditions, and the forms in the image of $(\sigma')^*$ may not satisfy these. But there is a bounded Γ-homotopy T from the smooth forms on N'_2 to those which satisfy absolute boundary conditions. So we consider the complex

$$\cdots \xrightarrow{\mathrm{d}} \Lambda^{k-1}(M'_1, N'_2) \xrightarrow{\mathrm{d}} \Lambda^{k}(M'_1, N'_2) \xrightarrow{\mathrm{d}} \Lambda^{k+1}(M'_1, N'_2) \xrightarrow{\mathrm{d}} \cdots,$$

where

$$\Lambda^k(M'_1, N'_2) = \Lambda^k(M'_1) \oplus \Lambda^{k-1}_{\mathrm{abs}}(N'_2) \quad \text{and} \quad \mathrm{d}(\omega_1, \omega_2) = (\mathrm{d}\omega_1, T(\sigma')^*\omega_1 - \mathrm{d}\omega_2).$$

We then define the relative higher rho-invariant as before. In any case, we get a smooth topological invariant of the pair of homotopic manifolds. This can be compared with the higher rho-invariant defined for a homotopy equivalence in [43, §2] by means of an analysis of the surgery exact sequence.

Another possible cancellation mechanism can be seen from the fact that the lower signature of an even-dimensional manifold can be computed from $\Lambda^{n/2}(M)$, and the lower eta-invariant of an odd-dimensional manifold can be computed from $\mathrm{Im}(d^*) \subset \Lambda^{(n-1)/2}(M)$. That is, there is a cancellation outside of a certain subspace of $\Lambda^*(M)$.

To extend this cancellation mechanism, suppose first that M is a smooth closed oriented Riemannian manifold of even dimension n. The integrand $\tilde{\eta}(s)$ of (26) is always integrable near $s = 0$, and the question is the large-s integrability. Suppose that the Laplacian has a bounded L^2-inverse on $\Lambda^{n/2}(M')$. (This condition is a homotopy invariant of M [22], and as Γ satisfies the Strong Novikov Conjecture [27], it implies that the higher signatures of M vanish.) Then there are no integrability problems in defining $\tilde{\eta}$. To see this, let P be the projection operator onto $\overline{\Lambda^{n/2}(M') \oplus \mathrm{d}\,\Lambda^{n/2}(M')} \oplus \overline{\mathrm{d}^*\,\Lambda^{n/2}(M')}$. Then Q' commutes with P, but the connection ∇ will not commute with P, and so we cannot say that $\tilde{\eta}$ can be computed from $\mathrm{Im}(P)$. However, as the question of large-s integrability is independent of the choice of connection, we can homotop our superconnection from D_s to $D_s' = PD_sP + (1 - P)D_s(1 - P)$, without affecting the integrability question. Now $\tilde{\eta}$, defined using D_s', decomposes as $\tilde{\eta}_{\mathrm{Im}(P)} + \tilde{\eta}_{\mathrm{Ker}(P)}$. As Q' is invertible on $\mathrm{Im}(P)$, there is no problem with the large-s integrability of $\tilde{\eta}_{\mathrm{Im}(P)}$. However, $\tilde{\eta}_{\mathrm{Ker}(P)}$ vanishes for algebraic reasons. To see this, define the operator W on $\mathrm{Ker}(P)$ to be multiplication by $\mathrm{sign}(k - (n/2))$ on $\mathrm{Ker}(P) \cap \Lambda^k(M')$ [38]. Then W is an invertible odd operator which commutes with Q' and D_s'. Thus

$$\begin{aligned}
&\mathrm{STR}(Q'\,\mathrm{e}^{-\beta D_s^2}) \\
&\quad = \mathrm{STR}(W^{-1}WQ'\,\mathrm{e}^{-\beta D_s^2}) = -\mathrm{STR}(WQ'\,\mathrm{e}^{-\beta D_s^2}W^{-1}) \\
&\quad = -\mathrm{STR}(WW^{-1}Q'\,\mathrm{e}^{-\beta D_s^2}) = -\mathrm{STR}(Q'\,\mathrm{e}^{-\beta D_s^2}) = 0.
\end{aligned} \tag{65}$$

This implies that $\tilde{\eta}_{\mathrm{Ker}(P)}$ vanishes.

Again, the higher rho-invariant $\tilde{\rho}$ is independent of all choices made, and is a smooth topological invariant of M.

If n is odd, a similar argument shows that it is enough to assume that the Laplacian has a bounded L^2-inverse on $\mathrm{Im}(d^*) \subset \Lambda^{(n-1)/2}(M')$.

Finally, suppose that $\Gamma = \mathrm{F} \times \Gamma_0$, with F a finite group. In analogy to Section 3.1.3, suppose that the Laplacian has a bounded L^2-inverse on the orthogonal complement to the F-invariant forms in $\Lambda^{n/2}(M')$ or $\mathrm{Im}(d^*) \subset \Lambda^{(n-1)/2}(M')$. Then $\tilde{\eta}$ will be well-defined as long as we only look at it away from the trivial representation of F.

4.8. CONJECTURAL HIGHER INDEX THEOREM FOR MANIFOLDS WITH BOUNDARY

We now suppose that M is the boundary of a compact spin manifold W, with a product metric near the boundary. Let W' be a normal cover of W with virtually nilpotent covering group Γ. Let V be a Hermitian vector bundle with connection on W which is a product near the boundary. Let $h \in C_0^\infty(W')$ be a function which is constant in the normal direction near the boundary, such that $\Sigma_{\gamma\in\Gamma} R_\gamma^* h = 1$. Let $M' = \partial W'$ be the Γ-cover of M. Suppose that the Dirac-type operator on M' is invertible. Using the restriction of h to M', we can define $\tilde{\eta}_M \in \bar{\Omega}_*(\mathscr{B})$.

Let Q_W' be the Dirac-type operator acting on a $C_r^*(\Gamma)$-Hilbert module of spinors on W', with APS boundary conditions. The analysis of [37] shows that Q_W' gives an unbounded $KK(\mathbf{C}, C_r^*(\Gamma))$-cycle in the sense of [4]. Thus $\mathrm{Index}(Q_W')$ is well-defined in $K_*(C_r^*(\Gamma))$. As $K_*(C_r^*(\Gamma)) \cong K_*(\mathscr{B})$ [25], there is a Chern character $\mathrm{Ch}_{\mathscr{B}}(\mathrm{Index}(Q_W')) \in \bar{H}_*(\mathscr{B})$.

CONJECTURE 1.

$$\mathrm{Ch}_\beta(\mathrm{Index}(Q'_W)) = \int_W \hat{A}(W) \wedge \mathrm{Ch}(V) \wedge \omega - \tilde{\eta}_M \in \bar{\bar{H}}_*(\mathscr{B}) \tag{66}$$

As evidence for this conjecture, we note that it follows from Proposition 27 that the right-hand side of (66) is deformation-invariant. Furthermore, (66) has been proven when paired with the trivial 0-cocycle [37] for general finitely-presented Γ. It should be possible to prove the conjecture by combining the methods of [31] and [37].

As an application, consider the case when V is trivial, so that one has the pure Dirac operator. As in Section 3.1.7, a consequence of the conjecture would be that the higher rho-invariant gives an obstruction to extending a positive scalar curvature metric from the boundary of a compact spin manifold to the entire manifold, so as to have a product metric near the boundary.

4.9. HIGHER SIGNATURES FOR MANIFOLDS WITH BOUNDARY

We refer to [44] for a survey of the Novikov conjecture. Let us just recall the statement. For simplicity, we will work with smooth oriented manifolds, and all homotopy equivalences will be assumed to be smooth and orientation-preserving. Let W be a closed manifold and let $\nu: W \to B\Gamma$ be a continuous map into the classifying space of a finitely presented group Γ. The L-class of W can be taken to lie in $H^*(W; \mathbf{C})$ and its Poincaré dual $*L$ then lies in $H_*(W; \mathbf{C})$. One version of the Novikov conjecture is that $\nu_*(*L) \in H_*(B\Gamma; \mathbf{C})$ is a homotopy invariant of W. (Instead of considering all such Γ, one can equally well just take Γ to be $\pi_1(W)$, which is a more standard form of the conjecture.)

If W is now a manifold with boundary M, there are various possible Novikov conjectures. For the simplest one, let Γ' and Γ be finitely presented groups with a homomorphism from Γ' to Γ such that one has a commutative diagram of continuous maps:

$$\begin{array}{ccc} M & \longrightarrow & W \\ \downarrow & & \downarrow \\ B\Gamma' & \longrightarrow & B\Gamma \end{array}.$$

Let $\nu: (W, M) \to (B\Gamma, B\Gamma')$ be the corresponding map of pairs. The L-class still defines an element of $H^*(W; \mathbf{C})$, and its Poincaré dual $*L$ now lies in $H_*(W, M; \mathbf{C})$. Then one can conjecture that $\nu_*(*L) \in H_*(B\Gamma, B\Gamma'; \mathbf{C})$ is a homotopy invariant of the pair (W, M) [44]. This can be considered to be a relative Novikov conjecture, in that it involves two groups. As pointed out in [44], the relative Novikov conjecture would follow if one knew the truth of the Novikov conjecture for Γ and the Borel conjecture for Γ'.

That the relative Novikov conjecture is not completely satisfactory can be seen by considering the case when W and M have the same fundamental group $\Gamma = \Gamma'$. Then $H_*(B\Gamma, B\Gamma'; \mathbf{C})$ is the 0-vector space, and so $\nu_*(*L)$ vanishes. However, the ordinary signature is a nontrivial homotopy invariant of the pair (W, M). Thus there are more homotopy invariants than those detected by the statement of the relative Novikov conjecture.

We wish to propose an absolute Novikov conjecture for manifolds with boundary, in that it only involves one group Γ. For the same technical reasons as before, we will assume that Γ is virtually nilpotent. So let $\nu: W \to B\Gamma$ be a continuous map. There is an induced map $\nu_M: M \to B\Gamma$ and corresponding normal covering M'. Assume that M is such that the Laplacian, acting on middle-dimensional (or middle two-dimensional) forms on M' is invertible, as discussed in Section 4.7. (One could also consider the case when $\Gamma = \mathrm{F} \times \Gamma_0$, as discussed there.) Then $\tilde{\eta}_M \in \bar{\bar{\Omega}}_*(\mathscr{B})$ is well-defined. Let ω be the biform of Proposition 24. Now

$$\int_W L(W) \wedge \omega - \tilde{\eta}_M \tag{67}$$

represents a class in $\bar{\bar{H}}_*(\mathscr{B})$ which is a smooth topological invariant of the pair (W, M). Let $\tau \in Z^*(\Gamma; \mathbf{C})$ be a group cocycle and form the corresponding cyclic cocycle $\mathscr{T}_\tau$ as in (20), with $x = e$. If $\mathscr{T}_\tau$ extends to a cyclic cocycle of the algebra $\mathscr{B}$ then we obtain a higher signature $\sigma(W, M, [\tau]) \in \mathbf{C}$ by pairing the form (67) with $\mathscr{T}_\tau$ via the pairing of Section 4.1.

CONJECTURE 2. *$\sigma(W, M, [\tau])$ is a homotopy-invariant of the pair (W, M).*

Notes

(1) Upon integrating (67) over β as in Definition 12, one presumably obtains an element of $PHC_{\mathrm{e,o}}(\mathscr{B})$, say Σ. (One way to prove this would be to show that (67) is the Chern character of an index, as in the proof of Proposition 28.) As $PHC_{\mathrm{e,o}}(\mathscr{B})$ is isomorphic to $PHC_{\mathrm{e,o}}(\mathbf{C}\Gamma)$ [24], the description of Section 4.1 shows that Σ breaks up according to conjugacy classes of Γ into a part in $H_*(\Gamma; \mathbf{C})$ and a part outside of $H_*(\Gamma; \mathbf{C})$. As ω is concentrated on the trivial conjugacy class, the part of Σ outside of $H_*(\Gamma; \mathbf{C})$ is simply the negative of the higher rho-invariant of M. By the higher signature $\sigma(W, M, \Gamma)$, we will mean the part of Σ in $H_*(\Gamma; \mathbf{C})$. Then one can rephrase Conjecture 2 as saying that $\sigma(W, M, \Gamma)$ is a homotopy-invariant of the pair (W, M).

(2) As a consequence of Conjecture 2, we would get a Novikov additivity for the higher signature of a closed manifold which is split along a codimension-1 submanifold satisfying the conditions on M.

(3) When Γ is trivial and $\tau \in Z^0(\Gamma; \mathbf{C})$ is given by $\tau(e) = 1$ then it follows from [2] that $\sigma(W, M, [\tau])$ is the ordinary signature of the pair (W, M), which does satisfy the conjecture.

(4) The relative Novikov invariant is the image of $\sigma(W, M, \Gamma)$ under the map $H_*(B\Gamma; \mathbf{C}) \to H_*(B\Gamma; B\Gamma'; \mathbf{C})$.

4.10. PAIRINGS OF $\tilde{\eta}$ WITH 0-COCYCLES AND 1-COCYCLES

4.10.1. 0-*Cocycles*

Let n be odd and let $\mathscr{T}$ be a 0-cocycle on $\mathscr{B}$, that is, a trace on $\mathscr{B}$. Then

$$\langle \tilde{\eta}, \mathscr{T} \rangle = \left\langle \beta^{1/2} \int_0^\infty \mathrm{TR}_\sigma(\sigma Q \, \mathrm{e}^{-\beta s^2 Q^2}) \, \mathrm{d}s, \mathscr{T} \right\rangle = \frac{\sqrt{\pi}}{2} \left\langle \mathrm{TR}\left(\frac{Q}{|Q|} \right), \mathscr{T} \right\rangle$$

$$= \frac{\sqrt{\pi}}{2} \sum_{\gamma \in \Gamma} \left\langle \int_{M'} \phi(m) \, \mathrm{tr}\left(\left(R_\gamma^* \frac{Q'}{|Q'|} \right)(m, m) \right) \mathrm{dvol}(m) \gamma, \mathscr{T} \right\rangle$$

$$= \frac{\sqrt{\pi}}{2} \sum_{\gamma \in \Gamma} \int_{M'} \phi(m) \, \mathrm{tr}\left(\frac{Q'}{|Q'|}(m\gamma, m) \right) \mathrm{dvol}(m) \mathscr{T}(\gamma).$$

We can relax the smoothness condition on ϕ, and take ϕ to be the characteristic function of a fundamental domain $\mathscr{F}$ in M', to obtain

$$\langle \tilde{\eta}, \mathscr{T} \rangle = \frac{\sqrt{\pi}}{2} \sum_{\gamma \in \Gamma} \int_{\mathscr{F}} \mathrm{tr}\left(\frac{Q'}{|Q'|}(m\gamma, m) \right) \mathrm{dvol}(m) \mathscr{T}(\gamma). \tag{68}$$

As a special case, if we take $\mathscr{T}$ to be obtained from the character of a finite-dimensional representation ρ of Γ, we get $\sqrt{\pi}/2$ times the lower eta-invariant for ρ. On the other hand, if we take $\mathscr{T}$ to be the standard trace obtained from evaluation at the identity element of Γ, we get $\sqrt{\pi}/2$ times the L^2-eta-invariant of [15].

More generally, following the discussion of Section 2, given an element x of Γ, let $\mathscr{T}_x$ be the 0-cocycle obtained by pairing with the characteristic function of $\langle x \rangle$ in Γ. Then

$$\langle \tilde{\eta}, \mathscr{T}_x \rangle = \frac{\sqrt{\pi}}{2} \sum_{\gamma \in \langle x \rangle} \int_{\mathscr{F}} \mathrm{tr}\left(\frac{Q'}{|Q'|}(m\gamma, m) \right) \mathrm{dvol}(m). \tag{69}$$

If $x \neq e$ then $\langle \tilde{\eta}, \mathscr{T}_x \rangle$ is deformation-invariant.

4.10.2. 1-*Cocycles*

Let n be even and let $\mathscr{T}$ be a 1-cocycle on $\mathscr{B}$. Then

$$\langle \tilde{\eta}, \mathscr{T} \rangle$$

$$= \left\langle \beta^{1/2} \int_0^\infty \int_0^\beta \mathrm{STR}(Q \, \mathrm{e}^{-us^2 Q^2}(-s[\nabla, Q]) \, \mathrm{e}^{-(\beta - u)s^2 Q^2}) \, \mathrm{d}u \, \mathrm{d}s, \mathscr{T} \right\rangle$$

$$= \left\langle \beta^{1/2} \int_0^\infty \int_0^\beta \mathrm{STR}((-s[\nabla, Q])Q\, \mathrm{e}^{-\beta s^2 Q^2})\, \mathrm{d}u\, \mathrm{d}s, \mathscr{T} \right\rangle$$

$$= \left\langle -\frac{\beta^{1/2}}{2} \int_0^\infty \mathrm{STR}([\nabla, Q]Q\, \mathrm{e}^{-\beta s^2 Q^2})\, \mathrm{d}(\beta s^2), \mathscr{T} \right\rangle$$

$$= \left\langle -\frac{\beta^{1/2}}{2} \mathrm{STR}([\nabla, Q]Q^{-1}), \mathscr{T} \right\rangle$$

$$= -\frac{\beta^{1/2}}{2} \sum_{\gamma_0, \gamma_1 \in \Gamma} \left\langle \int_{M'} \phi(m)\, \mathrm{tr}_s((R^*_{\gamma_0}(\partial h) R^*_{\gamma_1} Q'^{-1})(m, m))\, \mathrm{dvol}(m) \gamma_0\, \mathrm{d}\gamma_1, \mathscr{T} \right\rangle$$

$$= -\frac{\beta^{1/2}}{2} \sum_{\gamma_0, \gamma_1 \in \Gamma} \int_{M'} \phi(m)\, \mathrm{tr}_s((\partial h)(m\gamma_0) Q'^{-1}(m\gamma_0\gamma_1, m))\, \mathrm{dvol}(m) \mathscr{T}(\gamma_0, \gamma_1).$$

Again, we can take ϕ to be the characteristic function of a fundamental domain, to obtain

$$\langle \tilde{\eta}, \mathscr{T} \rangle = -\frac{\beta^{1/2}}{2} \sum_{\gamma_0, \gamma_1 \in \Gamma} \int_{\mathscr{F}} \mathrm{tr}_s((\partial h)(m\gamma_0) Q'^{-1}(m\gamma_0\gamma_1, m))\, \mathrm{dvol}(m) \mathscr{T}(\gamma_0, \gamma_1). \quad (70)$$

Given an element x of Γ, let τ be a cocycle constructed as in Section 4.1, such that $\mathscr{T}_\tau$ extends to a 1-cocycle of $\mathscr{B}$. Let $\{g_j\}$ be a sequence in Γ such that $\{g_j^{-1} x g_j\}$ parametrizes $\langle x \rangle$. Then

$$\langle \tilde{\eta}, \mathscr{T} \rangle = -\frac{\beta^{1/2}}{2} \sum_{g_j} \sum_{\gamma_0 \in \Gamma} \int_{\mathscr{F}} \mathrm{tr}_s((\partial h)(m\gamma_0) Q'^{-1}(m g_j^{-1} x g_j, m))\, \mathrm{dvol}(m) \tau(g_j, g_j\gamma_0)$$

$$= -\frac{\beta^{1/2}}{2} \sum_{g_j} \sum_{\gamma_0 \in \Gamma} \int_{\mathscr{F} g_j^{-1}} \mathrm{tr}_s((\partial h)(m g_j \gamma_0) Q'^{-1}(m x, m))\, \mathrm{dvol}(m) \tau(g_j, g_j\gamma_0)$$

$$= -\frac{\beta^{1/2}}{2} \sum_{g_j} \sum_{\gamma \in \Gamma} \int_{\mathscr{F} g_j^{-1}} \mathrm{tr}_s((\partial h)(m\gamma) Q'^{-1}(m x, m))\, \mathrm{dvol}(m) \tau(g_j, \gamma).$$

If $x \neq e$, this is deformation-invariant. If $x = e$, then $\tau(e, \gamma) = \mu(\gamma)$ for some group homomorphism $\mu \colon \Gamma \to (\mathbf{C}, +)$, and

$$\langle \tilde{\eta}, \mathscr{T} \rangle = -\frac{\beta^{1/2}}{2} \sum_{\gamma \in \Gamma} \int_{\mathscr{F}} \mathrm{tr}_s((\partial h)(m\gamma) Q'^{-1}(m, m))\, \mathrm{dvol}(m) \mu(\gamma). \quad (71)$$

If we put $A = \Sigma_{\gamma \in \Gamma} \mu(\gamma) R^*_\gamma(\mathrm{d}h) \in \Lambda^1(M')$ then

$$R^*_g A = \sum_{\gamma \in \Gamma} \mu(\gamma) R^*_{g\gamma}(\mathrm{d}h) = \sum_{\gamma \in \Gamma} \mu(g^{-1}\gamma) R^*_\gamma(\mathrm{d}h) = \sum_{\gamma \in \Gamma} (\mu(\gamma) - \mu(g)) R^*_\gamma(\mathrm{d}h) = A.$$

Thus the integrand of (71) is Γ-invariant, and so (71) can be written as the integral of a smooth quantity on M.

4.11. REMARKS

(1) It would be desirable to weaken the assumptions that Q' is invertible and that the group Γ is virtually nilpotent. This latter assumption is very strong, and we hope that it can be weakened to a statement that, roughly, one can prove the Strong Novikov Conjecture for Γ. This would be more consistent with the results of [43] for the signature operator.

(2) The higher eta-invariant described in this paper can be viewed as fitting into a $(\mathbf{C}, \mathscr{B})$-bivariant theory in the sense of [27]. One should be able to extend this to a $(C^\infty(M), \mathscr{B})$-bivariant eta-invariant using the equations of [30]. This would give a higher rho-invariant which pairs with both the cyclic cohomology of $\mathbf{C}\Gamma$ and the de Rham cohomology of M. The $(C^\infty(M), \mathbf{C})$-bivariant eta-invariant is considered in [42].

(3) As the higher rho-invariant of this paper lies in cyclic homology, it is natural to guess that it related to something which is defined in K-theory. Recall that the Chern character of the index of a Dirac-type operator on M also takes value in cyclic homology, but in the part corresponding to the trivial conjugacy class, as can be seen from (24). In contrast, the higher rho-invariant takes value in the complementary part, as seen in (53). Thus the higher rho-invariant gives complementary information to the higher index. This seems to be related to the fact that when a group Γ has torsion, the assembly map from $KO_*(B\Gamma)$ to $K_*(C_r^*(\Gamma))$ is generally neither injective nor surjective, even if Γ is finite [39]. In this latter case, the (reduced) lower rho-invariant detects $\mathbf{Q}/\mathbf{Z}$ factors in $KO_*(B\Gamma)$.

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