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MEAN VALUES OF MULTIPLICATIVE FUNCTIONS

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Dedicated to Professor András Sárközy on the occasion of his 60th birthday

Abstract

Let $f(n)$ be a totally multiplicative function such that $|f(n)| \le 1$ for all n, and let $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ be the associated Dirichlet series. A variant of Halász's method is developed, by means of which estimates for $\sum_{n=1}^{N} f(n)/n$ are obtained in terms of the size of $|F(s)|$ for s near 1 with $\Re s > 1$. The result obtained has a number of consequences, particularly concerning the zeros of the partial sum $U_N(s) = \sum_{n=1}^{N} n^{-s}$ of the series for the Riemann zeta function.

1. Introduction

Let $f(n)$ be a multiplicative function such that $|f(n)| \le 1$ for all n. Then the associated Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} f(n)n^{-s} \tag{1}$$

is absolutely convergent for $\sigma > 1$. (We write $s = \sigma + it$.) In 1968, Halász [1] showed that if for every $T > 0$, $F(s) = o\big(1/(\sigma-1)\big)$ as $\sigma \to 1^{+}$, uniformly for $|t| \le T$, then $S_0(x) = \sum_{n\le x} f(n) = o(x)$. One may note that Halász's theorem, together with the information that $\zeta(1+it) \ne 0$, yields the estimate $\sum_{n\le x} \mu(n) = o(x)$, which is equivalent to the Prime Number Theorem. Later, Halász [2] established a sharp quantitative form of his theorem. After further refinements of Montgomery [5] and Tenenbaum [8], this takes the following form.

THEOREM 1. *Suppose that $f(n)$ is a multiplicative function such that $|f(n)| \le 1$ for all n, and let $F(s)$ and $S_0(x)$ be defined as above. For $\alpha > 0$*

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put

$$M_0(\alpha) = \Big(\sum_{k=-\infty}^{\infty} \max_{\substack{|t-k|\le 1/2 \\ 1+\alpha\le\sigma\le 2}} \Big| \frac{F(\sigma+it)}{\sigma+it} \Big|^2 \Big)^{1/2}.$$

Then for $x \ge 3$,

$$S_0(x) \ll \frac{x}{\log x} \int_{1/\log x}^{1} M_0(\alpha)\alpha^{-1}\, d\alpha. \tag{2}$$

Since $|F(2)| \asymp 1$ it follows that $M_0(\alpha) \gg 1$ and hence in the most favorable circumstance Theorem 1 gives the estimate

$$S_0(x) \ll \frac{x \log\log x}{\log x}.$$

To see that this is sharp, take $f(n)$ to be the totally multiplicative function determined by the equations

$$f(p) = \begin{cases} e(\phi_p) & \text{when } \sqrt{x} < p \le x, \\ i & \text{otherwise,} \end{cases}$$

where the ϕ_p are at our disposal. Then by comparing $F(s)$ with $\exp(i\log\zeta(s))$ it follows that $|S_0(u)| \gg u/\log u$ when $2 \le u \le \sqrt{x}$, and that $M_0(\alpha) \ll 1$. Moreover,

$$S_0(x) = \sum_{\substack{n\le x \\ p|n \Rightarrow p\le x}} f(n) + \sum_{\sqrt{x}<p\le x} f(p) S_0(x/p),$$

so that by choosing the ϕ_p appropriately we have

$$|S_0(x)| = \Big| \sum_{\substack{n\le x \\ p|n \Rightarrow p\le x}} f(n) \Big| + \sum_{\sqrt{x}<p\le x} |S_0(x/p)| \gg \sum_{\sqrt{x}<p\le x} \frac{x}{p\log(2x/p)} \gg \frac{x\log\log x}{\log x}.$$

Thus in particular we see that the integral in (2) cannot be replaced by $M_0(1/\log x)$.

In this paper we consider similar estimates for the partial sum

$$S_1(x) = \sum_{n\le x} \frac{f(n)}{n}$$

in terms of the quantity

$$M_1(\alpha) = \Big(\sum_{k=-\infty}^{\infty} \max_{\substack{|t-k|\le 1/2 \\ 1+\alpha\le\sigma\le 2}} \Big| \frac{F(\sigma+it)}{\sigma-1+it} \Big|^2 \Big)^{1/2}$$

for $\alpha > 0$.

THEOREM 2. *Suppose that $x \ge 3$, that $f(n)$ is a totally multiplicative function such that $|f(n)| \le 1$ for all n, and that $S_1(x)$ and $M_1(\alpha)$ are defined as above. Then*

$$S_1(x) \ll \frac{1}{\log x} \int_{1/\log x}^{1} M_1(\alpha)\alpha^{-1}\, d\alpha. \tag{3}$$

In Theorem 1 the upper bound obtained is smaller than the trivial bound $S_0(x) \ll x$ by at best $(\log\log x)/\log x$, but for S_1 we are more successful. The trivial upper bound is $S_1(x) \ll \log x$, and in the most favorable circumstances we obtain an upper bound that is smaller than this by a factor $(\log x)^{-2}\log\log x$. Because (3) is comparatively farther from the trivial, its proof is more delicate. The hypothesis that f is totally multiplicative could be relaxed to requiring merely that f be multiplicative, but then the proof would become even more complicated. The restriction to totally multiplicative functions is not a hindrance below, since our intended applications pertain to totally multiplicative functions.

It is well-known that the hypothesis that $F(\sigma) = o\big(1/(\sigma-1)\big)$ as $\sigma \to 1^+$ does not imply that $S_0(x) = o(x)$, even when $f(n)$ is a totally multiplicative unimodular function. (For example, if $f(n) = n^i$ then $F(s) = \zeta(s-i)$, $|F(\sigma)|$ is uniformly bounded for $\sigma \ge 1$, but $S_0(x) \sim x^{1+i}/(1+i)$.) In contrast, by elementary reasoning we may estimate $M_1(\alpha)$ in terms of $|F(\sigma)|$, and hence Theorem 2 provides an estimate of S_1 in terms of $|F(\sigma)|$.

THEOREM 3. *Suppose that $x \ge 3$, and that $1 + \frac{1}{\log x} \le \sigma \le 2$. If $f(n)$ is a totally multiplicative function such that $|f(n)| \le 1$ for all n, then*

$$S_1(x) \ll |F(\sigma)|(\sigma-1)\big((\sigma-1)^{-4/\pi} + \log x\big). \tag{4}$$

It is instructive to compare this with the Hardy–Littlewood Tauberian theorem, which (in one form) asserts that if $f(n) \ll 1$ and $F(\sigma) = o(1/(\sigma-1))$ as $\sigma \to 1^+$, then $S_1(x) = o(\log x)$ as $x \to \infty$. The same conclusion is seen in (4), under more stringent hypotheses. The advantage of Theorem 3 is that it is quantitatively more precise. For example, a quantitative form of the Hardy–Littlewood Tauberian theorem (see Ingham [4]) asserts that if $f(n) \ll 1$ and $F(\sigma) \ll 1$ then $S_1(x) \ll (\log x)/\log\log x$. This is only slightly better than the trivial bound, but it is best possible (take $f(n) = \cos\big((\log\log n)^2\big)$). By comparison, in the more restricted situation of Theorem 3 we have the much better bound $S_1(x) \ll (\log x)^{1-\pi/4}$. Seen in this light, Theorems 1–3 are quantitative Tauberian theorems whose hypotheses are of an arithmetic nature.

By taking $\sigma = 1 + 1/\log x$ in Theorem 3, we see in particular that

$$S_1(x) \ll |F(1 + 1/\log x)|(\log x)^{4/\pi - 1}. \tag{5}$$

This estimate is sharp, as may be seen by letting f be the totally multiplicative function for which $f(p) = b(\frac{1}{2\pi}\log p)$ where $b(u)$ has period 1 and $b(u) = ie^{i\pi u}$ for $0 \le u \le 1$. In this case,

$$S_1(x) \sim c_1 x^i (\log x)^{2/\pi - 1}$$

as $x \to \infty$, and

$$F(\sigma) \sim c_2(\sigma - 1)^{2/\pi}$$

as $\sigma \to 1^+$. It may be further shown that

$$F(\sigma + i) \sim c_3(\sigma - 1)^{-2/\pi}$$

as $\sigma \to 1^+$, and that $M_1(\alpha) \asymp \alpha^{-2/\pi}$ in this situation. Here the c_j are non-zero complex constants.

The estimates (4) and (5) do not hold if f is merely assumed to be multiplicative instead of totally multiplicative. To see this, suppose that f is the multiplicative function defined by the relations

$$f(2^k) = -1,$$
$$f(p^k) = p^{ki} \qquad (p > 2)$$

for $k > 0$. Then $|F(\sigma)| \asymp (\sigma - 1)$, so the right hand side of (5) is $\asymp (\log x)^{4/\pi - 2} = o(1)$, while in actuality $S_1(x) \sim cx^i$ as $x \to \infty$, with $c \neq 0$.

Let

$$U_N(s) = \sum_{n=1}^{N} n^{-s}. \tag{6}$$

Turán [10] proved that $U_N(s) \neq 0$ in the half-plane $\sigma \geq 1 + 2(\log\log N)/\log N$, for all large N. By introducing the estimate of Theorem 3 into Turán's argument, we obtain the following stronger result.

THEOREM 4. *Suppose that $U_N(s)$ is given by (6). There is a constant N_0 such that if $N > N_0$, then $U_N(s) \neq 0$ whenever*

$$\sigma \geq 1 + \Big(\frac{4}{\pi} - 1\Big)\frac{\log\log N}{\log N}.$$

In the opposite direction, Montgomery [6] has shown that for each $c < 4/\pi - 1$ there is an $N_0(c)$ such that if $N > N_0(c)$ then $U_N(s)$ has zeros in the half-plane $\sigma > 1 + c(\log\log N)/\log N$.

As an application of Theorem 1, we consider the behaviour of

$$T(x, n) = \sum_{\substack{m|n \\ m \leq x}} \mu(m).$$

THEOREM 5. *In the above notation,*

$$T(x, n) \ll x(\log x)^{-1+1/\pi}$$

uniformly for $x \geq 2$, $n \geq 1$.

It is not hard to see that $\max_n |T(x, n)| = \Omega\big(x(\log x)^{-1+1/\pi}\big)$, but Hall and Tenenbaum [3] have shown more, namely that $\max_n |T(x, n)| \gg x(\log x)^{-1+1/\pi}$. Thus the upper bound above is sharp for all x.

2. Proof of Theorem 2

We first note that $M_1(\alpha) \gg 1$ uniformly for $0 < \alpha \le 1$, since

$$|F(2)| \ge \prod_p \Big(1 + \frac{1}{p^2}\Big)^{-1} > 0.$$

From this we see that we may assume that $x \ge x_0$, since the implicit constant may be adjusted to deal with the range $3 \le x \le x_0$. If we multiply both sides of (3) by $\log x$ then the right hand side is an increasing function of x. Also, $|S_1(x)|\log x$ is increasing in each interval $[n, n+1)$. Thus if the equation $|S(x)|\log x = V$ has a root then it has a least root. Hence it suffices to prove (3) when x is a member of the set

$$\mathfrak{S} = \{x \ge x_0 : x_0 \le y \le x \Rightarrow |S(y)|\log y < |S(x)|\log x\}.$$

Multiply both sides of the identity

$$\log x = \log n + \frac{(\log n)\log x/n}{\log x} + \frac{(\log x/n)^2}{\log x}$$

by $f(n)/n$ and sum over $n \le x$ to obtain the relation

$$\begin{aligned} S_1(x)\log x &= \sum_{n\le x} \frac{f(n)}{n}\log n + \frac{1}{\log x}\sum_{n\le x}\frac{f(n)}{n}(\log n)\log x/n \\ &\quad + \frac{1}{\log x}\sum_{n\le x}\frac{f(n)}{n}(\log x/n)^2 \\ &= T_1 + T_2 + T_3, \end{aligned} \tag{7}$$

say. (This is equivalent to integrating the inverse Mellin transform by parts *twice*.)

Our first step is to show that if $x \in \mathfrak{S}$ then

$$T_1(x) \ll \int_1^x |S(u)|\,\frac{du}{u} + |S(x)|\log\log x. \tag{8}$$

We write $\log n = \sum_{d|n}\Lambda(d)$, and invert the order of summation. Since f is totally multiplicative, we find that

$$T_1(x) = \sum_{d\le x}\frac{f(d)\Lambda(d)}{d}\sum_{m\le x/d}\frac{f(m)}{m}.$$

Since $|f(d)| \le 1$ for all d, it follows that

$$T_1(x) \ll \sum_{d\le x}\frac{\Lambda(d)}{d}|S(x/d)|. \tag{9}$$

We take $h = x/\log x$, and observe that if $x - h \le v \le x$, then trivially

$$\begin{aligned} T_1(x) - T_1(v) &= \sum_{v<n\le x}\frac{f(n)\log n}{n} \\ &\ll hx^{-1}\log x, \end{aligned}$$

so that

$$T_1(x) \ll hx^{-1}\log x + \frac{1}{h}\int_{x-h}^{x} |T_1(v)|\,dv.$$

By (9) this is

$$\ll 1 + \frac{1}{h}\int_{x-h}^{x} \sum_{d\le v} \frac{\Lambda(d)}{d}|S_1(v/d)|\,dv.$$

Since $S(u) = 1$ for $1 \le u < 2$, it follows that the sum over $x/2 < d \le x-h$ is $\gg 1$ and hence the second term above is $\gg 1$. Thus the above is

$$\ll \frac{1}{h}\sum_{d\le x}\frac{\Lambda(d)}{d}\int_{x-h}^{x} |S_1(v/d)|\,dv$$

$$\ll \frac{1}{h}\sum_{d\le x}\Lambda(d)\int_{\frac{x-h}{d}}^{\frac{x}{d}} |S_1(u)|\,du$$

(10) $$\ll \frac{1}{h}\int_{1}^{x} |S_1(u)|\Big(\sum_{\frac{x-h}{u}\le d\le \frac{x}{u}}\Lambda(d)\Big)\,du.$$

Write this integral as $\int_1^y + \int_y^x$ where $y = x/(\log x)^2$. Suppose first that $1 \le u \le y$. Then

$$\sum_{\frac{x-h}{u}<d\le\frac{x}{u}}\Lambda(d) \ll \sum_{k\le 2\log x}\frac{1}{k}\Big(\log\frac{x}{u}\Big)\Big(\pi\Big(\big(\frac{x}{u}\big)^{1/k}\Big) - \pi\Big(\big(\frac{x-h}{u}\big)^{1/k}\Big)\Big).$$

Here the last factor counts the number of primes in an interval $\mathfrak{I} = \mathfrak{I}(x,h,u,k)$. This interval is contained in an interval $\mathfrak{I}'$ of length $\ll h/(ku)$. By applying the Brun–Titchmarsh inequality to $\mathfrak{I}'$ we see that the number of primes in question is $\ll h/\big(ku\log(4h/(ku))\big)$. Hence the above is

$$\ll \sum_{k\le 2\log x}\frac{1}{k}\Big(\log\frac{x}{u}\Big)\frac{h}{ku\log(4h/(ku))} \ll h/u.$$

Hence

(11) $$\int_1^y \ll h\int_1^y |S_1(u)|\,\frac{du}{u}.$$

To treat the remaining range we appeal to our assumption that $x \in \mathfrak{S}$. Since

$\log u \asymp \log x$ in this range, it follows that $S_1(u) \ll |S_1(x)|$. On the other hand,

$$\int_y^x \sum_{\frac{x-h}{u}<d\le\frac{x}{u}} \Lambda(d)\,du \ll \sum_{d\le x/y} \Lambda(d) \int_{\frac{x-h}{d}}^{\frac{x}{d}} du \ll h \sum_{d\le(\log x)^2} \Lambda(d)/d \ll h\log\log x,$$

and hence

$$\int_y^x \ll h|S_1(x)|\log\log x.$$

On inserting this and (11) in (10), we obtain (8).

Next we show that

$$\int_e^x |S_1(u)|\log u \frac{du}{u} \ll M_1\Big(\frac{2}{\log x}\Big)\log x. \tag{12}$$

By the Cauchy–Schwarz inequality the integral here is

$$\le (\log x)^{1/2}\Big(\int_2^x |S_1(u)|^2(\log u)^2\frac{du}{u}\Big)^{1/2},$$

so it suffices to show that

$$\int_e^\infty |S_1(u)|^2(\log u)^2\frac{du}{u^{1+2\alpha}} \ll \alpha^{-1}M_1(\alpha)^2. \tag{13}$$

On writing

$$S_1(u)\log u = \sum_{n\le u}\frac{f(n)}{n}\log n + \sum_{n\le u}\frac{f(n)}{n}\log u/n,$$

we see that this integral is

$$\ll \int_1^\infty \Big|\sum_{n\le u}\frac{f(n)}{n}\log n\Big|^2\frac{du}{u^{1+2\alpha}} + \int_1^\infty \Big|\sum_{n\le u}\frac{f(n)}{n}\log u/n\Big|^2\frac{du}{u^{1+2\alpha}}.$$

By Plancherel's identity this is

$$\ll \int_{-\infty}^\infty \Big|\frac{F'(1+\alpha+it)}{\alpha+it}\Big|^2 dt + \int_{-\infty}^\infty \Big|\frac{F(1+\alpha+it)}{(\alpha+it)^2}\Big|^2 dt. \tag{14}$$

To treat the first of these integrals we break the range of integration into intervals of length 1 and write $F' = F\cdot F'/F$. Thus the integral is

$$\ll \sum_k \Big(\max_{|t-k|\le 1/2}\Big|\frac{F(1+\alpha+it)}{\alpha+it}\Big|^2\Big)\int_{k-1/2}^{k+1/2}\Big|\frac{F'}{F}(1+\alpha+it)\Big|^2 dt$$

$$\ll M_1(\alpha)^2 \sup_k \int\limits_{k-1/2}^{k+1/2} \Big|\frac{F'}{F}(1+\alpha+it)\Big|^2\,dt.$$

Thus it suffices to show that

$$\int\limits_{T-1/2}^{T+1/2} \Big|\frac{F'}{F}(1+\alpha+it)\Big|^2\,dt \ll \alpha^{-1}$$

uniformly for $0<\alpha\le 1$. To this end we recall that if $|a_n|\le b_n$ for all n then

$$\int\limits_{T-U}^{T+U} \Big|\sum_n a_n n^{-it}\Big|^2\,dt \le 3\int\limits_{-U}^{U} \Big|\sum_n b_n n^{-it}\Big|^2\,dt. \tag{15}$$

This is a refined form of an inequality used by Halász [1], [2]. For a simple proof see Montgomery [7, pp. 131–132]. Since

$$\frac{F'}{F}(s) = -\sum_{n=1}^{\infty} f(n)\Lambda(n)n^{-\sigma-it},$$

it follows by (15) that

$$\int\limits_{T-1/2}^{T+1/2} \Big|\frac{F'}{F}(1+\alpha+it)\Big|^2\,dt \le 3\int\limits_{-1/2}^{1/2} \Big|\frac{\zeta'}{\zeta}(1+\alpha+it)\Big|^2\,dt \ll \int\limits_{-1/2}^{1/2} |\alpha+it|^{-2}\,dt \ll \alpha^{-1}$$

and hence that

$$\int\limits_{-\infty}^{\infty} \Big|\frac{F'(1+\alpha+it)}{\alpha+it}\Big|^2\,dt \ll \alpha^{-1}M_1(\alpha)^2. \tag{16}$$

The second integral in (14) is

$$\ll \sum_k \Big(\max_{|t-k|\le 1/2}\Big|\frac{F(1+\alpha+it)}{\alpha+it}\Big|^2\Big)\int\limits_{k-1/2}^{k+1/2} |\alpha+it|^{-2}\,dt \ll \alpha^{-1}M_1(\alpha)^2. \tag{17}$$

On combining these estimates we obtain (13), and with it (12).

Let $J(x)$ denote the left hand side of (12). By integrating by parts we see that

$$\int\limits_{e}^{x} |S_1(u)|\,\frac{du}{u} \le \frac{J(x)}{\log x} + \int\limits_{e}^{x} \frac{J(u)}{u(\log u)^2}\,du.$$

By (12) this is

$$\ll M_1\Big(\frac{2}{\log x}\Big) + \int_{1/\log x}^{1} M_1(2\alpha)\alpha^{-1}\,d\alpha.$$

But $M_1(\alpha)$ is decreasing, so the above is

$$\ll \int_{1/\log x}^{1} M_1(\alpha)\alpha^{-1}\,d\alpha.$$

On combining this with (8), we find that

$$T_1(x) \ll \int_{1/\log x}^{1} M_1(\alpha)\alpha^{-1}\,d\alpha + |S_1(x)|\log\log x. \tag{18}$$

We now treat T_2, as defined in (7). Clearly

$$T_2\log x = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{F'(s+1)}{s^2}x^s\,ds$$

for any $\alpha > 0$. For $1/\log x \le \alpha \le 2/\log x$ this is

$$\ll \int_{-\infty}^{\infty} \frac{|F'(1+\alpha+it)|}{|\alpha+it|^2}\,dt.$$

By the Cauchy–Schwarz inequality this integral is

$$\ll \bigg(\alpha^{-1}\int_{-\infty}^{\infty} \Big|\frac{F'(1+\alpha+it)}{\alpha+it}\Big|^2\,dt\bigg)^{1/2}.$$

Thus by (16) we see that

$$T_2 \ll \frac{M_1(\alpha)}{\alpha\log x}$$

uniformly for $1/\log x \le \alpha \le 2/\log x$, and hence

$$T_2 \ll \int_{1/\log x}^{2/\log x} M_1(\alpha)\alpha^{-1}\,d\alpha. \tag{19}$$

We treat T_3 similarly. For $\alpha > 0$ we have

$$T_3\log x = \frac{1}{\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{F(s+1)}{s^3}x^s\,ds.$$

If $1/\log x \le \alpha \le 2/\log x$ then this is

$$\ll \int_{-\infty}^{\infty} \frac{|F(1+\alpha+it)|}{|\alpha+it|^3}\,dt.$$

By the Cauchy–Schwarz inequality this is

$$\ll \Big(\frac{1}{\alpha}\int_{-\infty}^{\infty} \Big|\frac{F(1+\alpha+it)}{(\alpha+it)^2}\Big|^2\,dt\Big)^{1/2}.$$

This integral is the second integral in (14), which is majorized in (17). Thus

$$T_3 \ll \frac{M_1(\alpha)}{\alpha \log x}$$

uniformly for $1/\log x \le \alpha \le 2/\log x$, and hence

$$T_3 \ll \int_{1/\log x}^{2/\log x} M_1(\alpha)\alpha^{-1}\,d\alpha.$$

On combining (18), (19), and the above in (7), we find that

$$S_1(x)\log x \ll \int_{1/\log x}^{1} M_1(\alpha)\alpha^{-1}\,d\alpha + |S_1(x)|\log\log x.$$

But $\log\log x = o(\log x)$, so the last term on the right is small compared with the left hand side for $x \ge x_0$. Thus we have (3), and the proof is complete.

3. Proof of Theorem 3

We first establish two lemmas.

LEMMA 1. *Suppose that $f(n)$ is a totally multiplicative function such that $|f(n)| \le 1$ for all n, and for $\sigma > 1$ let $F(s)$ be defined as in* (1). *If $1 < \sigma_1 \le \sigma_2 \le 2$ then*

$$\frac{\sigma_1 - 1}{\sigma_2 - 1} \ll \Big|\frac{F(\sigma_2)}{F(\sigma_1)}\Big| \ll \frac{\sigma_2 - 1}{\sigma_1 - 1}.$$

PROOF. The quotient in question is

$$\asymp \exp\Big(\Re \sum_p f(p)\big(p^{-\sigma_2} - p^{-\sigma_1}\big)\Big).$$

Since $|f(p)| \le 1$, this is

$$\le \exp\left(\sum_p p^{-\sigma_1} - p^{-\sigma_2}\right) \asymp \frac{\zeta(\sigma_1)}{\zeta(\sigma_2)} \asymp \frac{\sigma_2 - 1}{\sigma_1 - 1}.$$

The lower bound is proved similarly.

LEMMA 2. *Let $f(n)$ and $F(s)$ be as in the preceding lemma. If $1 < \sigma \le 2$ and $|t| \le 2$ then*

$$\frac{F(\sigma+it)}{F(\sigma)} \ll \Big(1 + \frac{|t|}{\sigma - 1}\Big)^{4/\pi}.$$

If $1 < \sigma \le 2$ and $|t| \ge 2$ then

$$\frac{F(\sigma+it)}{F(\sigma)} \ll \Big(\frac{\log|t|}{\sigma - 1}\Big)^{4/\pi}.$$

PROOF. We may suppose that $t > 0$. Since

$$\frac{F'}{F}(s) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)f(n)}{n^s} \ll -\frac{\zeta'}{\zeta}(\sigma) \ll \frac{1}{\sigma-1},$$

it follows that $|F(\sigma+it)| \asymp |F(\sigma)|$ when $0 \le t \le \sigma - 1$. As for $t \ge \sigma - 1$, we note that the quotient in question has modulus

$$\asymp \exp\left(\Re \sum_p \frac{f(p)}{p^\sigma}(p^{-it} - 1)\right)$$

$$\le \exp\left(2\sum_p \frac{1}{p^\sigma}|\sin(\frac{t}{2}\log p)|\right). \tag{20}$$

Suppose that $\sigma - 1 \le t \le 2$. Since $|\sin x| \le x$, the sum over $p \le e^{1/t}$ is

$$\ll t \sum_{p \le e^{1/t}} \frac{\log p}{p} \ll 1.$$

Since $|\sin x| \le 1$, the sum over $p \ge e^{1/(\sigma-1)}$ is

$$\ll \sum_{p > e^{1/(\sigma-1)}} p^{-\sigma} \ll 1.$$

The remaining sum is

$$\le \sum_{e^{1/t} < p \le e^{1/(\sigma-1)}} \frac{\left|\sin\left(\frac{t}{2}\log p\right)\right|}{p}. \tag{21}$$

Put

$$L(y) = \sum_{p \le y} \frac{1}{p}, \qquad I(y) = \int_0^y |\sin u|\, du.$$

Thus

$$L(y) = \log\log y + c + O((\log 2y)^{-2}), \qquad I(y) = \frac{2}{\pi}y + O(1).$$

It follows by partial summation that the sum (21) is

$$\frac{2}{\pi}\log\frac{t}{\sigma-1} + O(1)$$

when $\sigma - 1 \le t \le 2$. This gives the stated result in this case.

Now suppose that $t \ge 2$. We write

$$|\sin\pi\theta| = \sum_{k=-\infty}^{\infty} c_k e(k\theta) \tag{22}$$

where $c_k = 2\pi^{-1}(1-4k^2)^{-1}$. The quantity (20) is

$$\asymp \exp\left(2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^\sigma\log n}|\sin(\frac{t}{2}\log n)|\right) = \exp\left(2\sum_{k=-\infty}^{\infty}c_k\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\log n}n^{-\sigma+ikt}\right)$$
$$= \prod_{k=-\infty}^{\infty}|\zeta(\sigma-ikt)|^{2c_k}.$$

The term $k=0$ contributes an amount $\asymp(\sigma-1)^{-4/\pi}$. We let C be a constant such that

$$|\zeta(\sigma+it)| \ge \frac{1}{C\log t}$$

uniformly for $\sigma \ge 1$, $t \ge 2$. The existence of such a C is assured, for example, by (3.11.18) of Titchmarsh [9]. Since $c_k < 0$ when $k \ne 0$, the product above is

$$\ll (\sigma-1)^{-4/\pi}\prod_{k=1}^{\infty}(C\log\pi kt)^{-4c_k}.$$

Moreover $\log\pi kt \ll (\log\pi k)(\log t)$ and $\sum_{k=1}^{\infty}|c_k|\log\log k < \infty$, so the above is

$$\ll (\sigma-1)^{-4/\pi}(\log t)^{-4\sum_{k=1}^{\infty}c_k}.$$

By evaluating (22) at $\theta = 0$ we see that $\sum_{k=1}^{\infty}c_k = -1/\pi$. Thus the proof of Lemma 2 is complete.

We now use the lemmas to show that if $0<\alpha\le\beta\le 1$, $|t|\le 1/2$, then

$$\frac{F(1+\beta+it)}{\beta+it} \ll |F(\sigma)|\Big(\alpha^{-2}(\sigma-1)+\alpha^{-1}(\sigma-1)^{1-4/\pi}+\alpha^{1-4/\pi}(\sigma-1)^{-1}\Big). \tag{23}$$

We also show that if k is a non-zero integer, $0<\alpha\le\beta\le 1$, $|t-k|\le 1/2$, then

$$F(1+\beta+it) \ll |F(\sigma)|\big(\log 2|k|\big)^{4/\pi}\Big(\alpha^{-1}(\sigma-1)^{1-4/\pi}+\alpha^{1-4/\pi}(\sigma-1)^{-1}\Big). \tag{24}$$

From these estimates it follows immediately that

$$M_1(\alpha) \ll |F(\sigma)|\Big(\alpha^{-2}(\sigma-1)+\alpha^{-1}(\sigma-1)^{1-4/\pi}+\alpha^{1-4/\pi}(\sigma-1)^{-1}\Big),$$

and then Theorem 3 follows by applying Theorem 2.

We prove (23) first. Suppose that $\sigma - 1 \le \beta$. By Lemma 2 we see that

$$\frac{F(1+\beta+it)}{\beta+it} \ll |F(1+\beta)|\frac{\left(1+\frac{|t|}{\beta}\right)^{4/\pi}}{|\beta+it|} \asymp |F(1+\beta)|\beta^{-4/\pi}(\beta+|t|)^{4/\pi-1}$$
$$\ll |F(1+\beta)|\beta^{-4/\pi}$$

since $|t| \le 1/2$. As $\sigma - 1 \le \beta$, by Lemma 1 this is

$$\ll |F(\sigma)|\frac{\beta}{\sigma-1}\beta^{-4/\pi} \ll |F(\sigma)|\alpha^{1-4/\pi}(\sigma-1)^{-1}$$

since $\beta \ge \alpha$. This gives (23) in this case. Suppose alternatively that $\beta \le \sigma - 1$. Then by Lemma 1,

$$\frac{F(1+\beta+it)}{\beta+it} \ll \frac{|F(\sigma+it)|}{|\beta+it|}\cdot\frac{\sigma-1}{\beta} \ll \frac{|F(\sigma+it)|}{|\alpha+it|}\cdot\frac{\sigma-1}{\alpha}$$

since $\beta \ge \alpha$. By Lemma 2 this is

$$\ll |F(\sigma)|\Big(1+\frac{|t|}{\sigma-1}\Big)^{4/\pi}\frac{\sigma-1}{|\alpha+it|\alpha}.$$

If $|t| \le \beta - 1$ then the product of the last two factors is $\ll (\sigma-1)\alpha^{-2}$, while if $\sigma - 1 \le |t| \le 1/2$ then the product of the last two factors is $\ll (\sigma-1)^{1-4/\pi}\alpha^{-1}$. Thus we have (23) in all cases.

We now derive (24). If $\sigma - 1 \le \beta$, then by Lemma 2

$$F(1+\beta+it) \ll |F(1+\beta)|\Big(\frac{\log 2|k|}{\beta}\Big)^{4/\pi}.$$

By Lemma 1 this is

$$\ll |F(\sigma)|\cdot\frac{\beta}{\sigma-1}\Big(\frac{\log 2|k|}{\beta}\Big)^{4/\pi} \ll |F(\sigma)|(\log 2|k|)^{4/\pi}\alpha^{1-4/\pi}(\sigma-1)^{-1}$$

since $\beta \ge \alpha$. This gives (24) in this case. Alternatively, suppose that $\beta \le \sigma - 1$. Then by Lemma 1 we see that

$$F(1+\beta+it) \ll |F(\sigma+it)|\cdot\frac{\sigma-1}{\beta} \ll |F(\sigma+it)|\cdot\frac{\sigma-1}{\alpha}$$

since $\beta \ge \alpha$. By Lemma 2 this is

$$\ll |F(\sigma)|(\log 2|k|)^{4/\pi}\alpha^{-1}(\sigma-1)^{1-4/\pi}.$$

Thus we have (24) in all cases, and the proof is complete.

4. Proof of Theorem 4

We adopt the notation of Theorem 3. By integrating by parts we see that

$$\sum_{n>N} f(n)n^{-\sigma} = -S_1(N)N^{1-\sigma} + (\sigma-1)\int_N^\infty S_1(u)u^{-\sigma}\,du$$

for $\sigma > 1$. By (3) it follows that the above is

$$\ll |F(\sigma)|(\sigma-1)\Big((\sigma-1)^{-4/\pi}+\log N\Big)N^{1-\sigma}$$

when $1+\frac{1}{\log N}\le\sigma\le 2$. Since

$$U_N(s)=\zeta(s)-\sum_{n>N} n^{-s},$$

by taking $f(n)=n^{-it}$ we deduce that

$$U_N(s)=\zeta(s)\Big(1+O\big((\log\log N)^{1-4/\pi}\big)\Big)$$

uniformly for

$$\sigma\ge 1+\Big(\frac{4}{\pi}-1\Big)\frac{\log\log N}{\log N}.$$

Since $\zeta(s)\neq 0$ in this half-plane, it follows that $U_N(s)\neq 0$, and the proof is complete.

5. Proof of Theorem 5

We apply Theorem 1 with $f(m)=\mu(m)$ when $m|n$, $f(m)=0$ otherwise. Then

$$F(s)=\prod_{p|n}(1-p^{-s})$$

and we require an estimate for this that is uniform in n.

Lemma 3. *Suppose that* $1<\sigma\le 2$. *If* $|t|\le 2$ *then*

$$\prod_{p|n}\big(1-p^{-s}\big)\ll 1+\Big(\frac{|t|}{\sigma-1}\Big)^{1/\pi}. \tag{25}$$

If $|t|\ge 2$ *then*

$$\prod_{p|n}\big(1-p^{-s}\big)\ll(\sigma-1)^{-1/\pi}\log|t|. \tag{26}$$

Proof. Put $G(s)=\prod_{p|n}\big(1+p^{-s}\big)^{-1}$. Since $|F(s)|\asymp|G(s)|$ uniformly for $\sigma\ge 1$, it suffices to estimate $|G(s)|$. We may suppose that $t\ge 0$. Clearly $0<G(\sigma)\le 1$. Since

$$\frac{G'}{G}(s)=-\sum_{p|n}\frac{\log p}{p^s+1}\ll-\frac{\zeta'}{\zeta}(\sigma)\ll\frac{1}{\sigma-1},$$

it follows that $G(s) \ll 1$ if $0 \le t \le \sigma - 1$. Now suppose that $t \ge \sigma - 1$. We observe that

$$|G(s)| \asymp \exp\Big(-\sum_{p|n} p^{-\sigma}\cos(t\log p)\Big) \le \exp\Big(\sum_p p^{-\sigma} g(\frac{1}{2\pi} t\log p)\Big) \tag{27}$$

where $g(x) = -\min(0, \cos 2\pi x)$. Suppose that $\sigma - 1 \le t \le 2$, and put $X = \exp(1/t)$, $Y = \exp(1/(\sigma-1))$. We observe that $\sum_{p>Y} p^{-\sigma} \ll 1$, and that $\sum_{p\le Y}\big(p^{-1} - p^{-\sigma}\big) \ll 1$. Hence

$$G(s) \ll \exp\Big(\sum_{X<p\le Y} p^{-1} g(\frac{1}{2\pi} t\log p)\Big). \tag{28}$$

Put

$$L(y) = \sum_{p\le y}\frac{1}{p}, \qquad I(y) = \int_0^y g(u)\,du.$$

Thus

$$L(y) = \log\log y + c + O((\log 2y)^{-2}), \qquad I(y) = \frac{1}{\pi} y + O(1).$$

It follows by partial summation that the sum in (28) is

$$= \frac{1}{\pi}\log\frac{t}{\sigma - 1} + O(1)$$

when $\sigma - 1 \le t \le 2$. Thus we have (25).

Now suppose that $t \ge 2$. We write $g(x) = \sum_k \widehat{g}(k) e(kx)$, and note that $\widehat{g}(\pm 1) = -1/4$, $\widehat{g}(2k) = \pi^{-1}(-1)^{k+1}(4k^2-1)^{-1}$, and that $\widehat{g}(k) = 0$ otherwise. The expression (27) is

$$\asymp \exp\Big(\sum_{n=1}^\infty \frac{\Lambda(n)}{n^\sigma \log n} g(\frac{1}{2\pi} t\log n)\Big) = \exp\Big(\sum_{k=-\infty}^\infty \widehat{g}(k)\sum_{n=1}^\infty \frac{\Lambda(n)}{\log n} n^{-\sigma+ikt}\Big)$$

$$= \prod_{k=-\infty}^\infty |\zeta(\sigma - ikt)|^{\widehat{g}(k)}.$$

The term $k = 0$ contributes an amount $\asymp (\sigma - 1)^{-1/\pi}$. From (3.5.1) and (3.11.18) of Titchmarsh [9] we know that there is a constant C such that

$$\frac{1}{C\log t} \le |\zeta(\sigma + it)| \le C\log t$$

uniformly for $\sigma \ge 1$, $t \ge 2$. Hence the product above is

$$\ll (\sigma - 1)^{-1/\pi}(\log t)^A$$

where

$$A = \sum_{k\ne 0} |\widehat{g}(k)| = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^\infty \frac{1}{4k^2 - 1} = \frac{1}{2} + \frac{1}{\pi} < 1.$$

Thus we have (26), and the proof is complete.

By Lemma 3 we see that $M_0(\alpha) \ll \alpha^{-1/\pi}$ for $0 < \alpha \leq 1$. Thus Theorem 5 follows from Theorem 1.

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