

Bayesian Estimation of Isotopic Age Differences¹

Rane L. Curl²

Isotopic dating is subject to uncertainties arising from counting statistics and experimental errors. These uncertainties are additive when an isotopic age difference is calculated. If large, they can lead to no significant age difference by "classical" statistics. In many cases, relative ages are known because of stratigraphic order or other clues. Such information can be used to establish a Bayes estimate of age difference which will include prior knowledge of age order. Age measurement errors are assumed to be log-normal and a noninformative but constrained bivariate prior for two true ages in known order is adopted. True-age ratio is distributed as a truncated log-normal variate. Its expected value gives an age-ratio estimate, and its variance provides credible intervals. Bayesian estimates of ages are different and in correct order even if measured ages are identical or reversed in order. For example, age measurements on two samples might both yield 100 ka with coefficients of variation of 0.2. Bayesian estimates are 22.7 ka for age difference with a 75% credible interval of [4.4, 43.7] ka.

KEY WORDS: Bayes estimate, dating, credible interval, stalagmite, log-normal.

INTRODUCTION

Many methods exist for dating rocks and sediments, among which those using radioactive isotopes are prominent. All methods are subject to errors arising from procedural and measurement imprecisions. Classic statistical methods for estimating age differences and their confidence intervals suffice when errors are small and age differences are large. However, if errors are large and ages are similar, classic methods lead to accepting the null hypothesis of zero age difference.

In some cases, age order is known. An important example, which was a primary motivation for this work, is dating of stalagmites by ²³⁰Th-²³⁴U disequilibrium measurements (e.g., Latham et al., 1986). Stratigraphically, lower samples are known a priori to be older than upper, regardless of date order found from isotope measurements. Bayesian statistical methods permit impos-

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²Department of Chemical Engineering, University of Michigan, Dow Building, Ann Arbor, Michigan, 48109-2136.

ing such information and estimating properly ordered age differences from even reversed-order data.

Bayesian methods and interpretations used here were adopted from Guttman et al. (1982).

DISTRIBUTION OF DATING ERRORS

A measured isotopic age x_i is assumed to be a sample from a log-normal distribution, with error log-variance σ_i^2 . Thus

$$\ln x_i \sim N(\ln a_i, \sigma_i^2) \quad (1)$$

where a_i is true age. Standard deviation σ_i may be estimated from dating statistics as the coefficient of variation of measured age.

Assuming log-normality assures that measured ages, and confidence limits, are non-negative. This usually will be true when variance arises primarily from one major error source such as counting statistics.

Two measured ages x_1 and x_2 are assumed independent (dating procedures have no covariance) with a joint probability density function (p.d.f.)

$$p(x_1, x_2 | a_1, a_2) = p(x_1 | a_1) p(x_2 | a_2) \quad (2)$$

where true ages are defined $a_2 > a_1 > 0$. Bayes' theorem gives a posterior p.d.f. for (a_1, a_2) as

$$p(a_1, a_2 | x_1, x_2) = \frac{p(x_1, x_2 | a_1, a_2) p(a_1, a_2)}{p(x_1, x_2)} \quad (3)$$

From the p.d.f. for a log-normal distribution (Eq. 1)

$$p(x_1, x_2 | a_1, a_2) \propto \frac{1}{x_1 x_2} \exp \left\{ -\frac{1}{2} \left[\left(\frac{1}{\sigma_1} \ln \frac{x_1}{a_1} \right)^2 + \left(\frac{1}{\sigma_2} \ln \frac{x_2}{a_2} \right)^2 \right] \right\} \quad (4)$$

BAYESIAN PRIOR

A noninformative but constrained prior p.d.f. is chosen to be

$$p(\ln a_1, \ln a_2) = k \quad \text{or} \quad (5)$$

$$p(a_1, a_2) = \frac{k}{a_1 a_2} \quad (6)$$

for $a_2 > a_1 > 0$, and zero otherwise. This forces non-negativity and the known age order but little more (it has infinite variance). Also, x_1 and x_2 , and hence

$p(x_1, x_2)$, are constants for a given pair of age measurements. From Eqs. 3, 4, and 6

$$p(a_1, a_2 | x_1, x_2) \propto \frac{1}{a_1 a_2} \exp \left\{ -\frac{1}{2} \left[\left(\frac{1}{\sigma_1} \ln \frac{x_1}{a_1} \right)^2 + \left(\frac{1}{\sigma_2} \ln \frac{x_2}{a_2} \right)^2 \right] \right\} \quad (7)$$

CONSTRAINED BAYES ESTIMATES

Conveniently, a_1 and a_2 may be estimated by means of transformed variables

$$u = \frac{1}{2^{1/2}} \ln \left(\frac{a_2}{a_1} \right) \quad (8a)$$

$$v = \frac{1}{2^{1/2}} \ln (a_1 a_2) \quad (8b)$$

whence Eq. 7 becomes

$$p(u, v | x_1, x_2) \propto \exp \left\{ -\frac{1}{2} \left[\left(\frac{v - u - 2^{1/2} \ln x_1}{2^{1/2} \sigma_1} \right)^2 + \left(\frac{v + u - 2^{1/2} \ln x_2}{2^{1/2} \sigma_2} \right)^2 \right] \right\} \quad (9)$$

from which (by integration and normalization)

$$p(u) = \exp \left[-\frac{\left[u - \frac{1}{2^{1/2}} \ln \left(\frac{x_2}{x_1} \right) \right]^2}{(\sigma_1^2 + \sigma_2^2)} \right] / \left[\pi(\sigma_1^2 + \sigma_2^2) \right]^{1/2} \Phi(\alpha) \quad u > 0 \quad (10)$$

a truncated normal distribution, where

$$\alpha = \frac{\ln (x_2/x_1)}{(\sigma_1^2 + \sigma_2^2)^{1/2}} \quad (11)$$

and $\Phi(\alpha)$ is the cumulative unit normal distribution function. Expected values for u and v , $\langle u \rangle$ and $\langle v \rangle$, from Eq. 10, are

$$\langle u \rangle = \frac{1}{2^{1/2}} \ln \left(\frac{x_2}{x_1} \right) + \frac{(\sigma_1^2 + \sigma_2^2)^{1/2} \exp(-\frac{1}{2} \alpha^2)}{2\pi^{1/2} \Phi(\alpha)} \quad (12a)$$

$$\langle v \rangle = \frac{1}{2^{1/2}} \ln(x_1 x_2) + \frac{(\sigma_2^2 - \sigma_1^2) \exp(-\frac{1}{2} \alpha^2)}{2[\pi(\sigma_1^2 + \sigma_2^2)]^{1/2} \Phi(\alpha)} \quad (12b)$$

Bayes age estimates \hat{a}_1 and \hat{a}_2 are obtained from Eq. 8 as

$$\hat{a}_1 = \exp\left(\frac{\langle v \rangle - \langle u \rangle}{2^{1/2}}\right) \quad (13a)$$

$$\hat{a}_2 = \exp\left(\frac{\langle v \rangle + \langle u \rangle}{2^{1/2}}\right) \quad (13b)$$

Credible intervals for age ratios are estimated from

$$\Pr(\underline{u} < \hat{u} < \bar{u}) = 1 - \beta \quad (14)$$

where \underline{u} and \bar{u} , lower $(1 - \beta/2)$ and upper $(\beta/2)$ percentage points for Eq. 10, are given by

$$\Phi(z) = 1 - (1 - \gamma) \Phi(\alpha) \quad \text{where} \quad (15)$$

$$z = \frac{\langle u \rangle - (1/2^{1/2}) \ln(x_2/x_1)}{[\frac{1}{2}(\sigma_1^2 + \sigma_2^2)]^{1/2}} \quad (16)$$

and $\gamma = \beta/2$ and $1 - \beta/2$ for \underline{u} and \bar{u} , respectively. Credible limits for $\hat{a}_2 - \hat{a}_1$ are calculated from the limits (\underline{u}, \bar{u}) with $\langle v \rangle$ and Eq. 13. Variance in $\hat{a}_2 - \hat{a}_1$ comes primarily from variance in u , and much less by variance in v , and therefore a joint credible region is unnecessary.

APPLICATIONS

Equations 11, 12, and 13 use four parameters $(x_1, x_2, \sigma_1, \sigma_2)$ determining two estimates (\hat{a}_1, \hat{a}_2) , obscuring the dependence of the a_i , and their difference, on the data. The general behavior is shown (Fig. 1) for $\sigma_1 = \sigma_2 = \sigma$.

Quantity u , and hence \hat{a}_2/\hat{a}_1 , is always greater than 1.0, even if $x_2/x_1 < 1.0$, agreeing with known age order. As expected, \hat{a}_2/\hat{a}_1 tends to x_2/x_1 as x_2/x_1 increases or σ decreases. Age estimates also yield $\hat{a}_1 = x_1$ if $\sigma_1 = 0$, and $\hat{a}_2 = x_2$ if $\sigma_2 = 0$.

Example 1 (Table 1) shows data for a stalagmite dated by Harmon and Curl (1978a,b). Two-sigma standard errors about measured ages overlapped, reducing the significance of classic age difference estimates [as pointed out by Gascoyne, 1978]. A Bayesian estimate for age difference is positive with positive credible limits.

Example 2 is from Latham et al. (1986) who regressed point age estimates

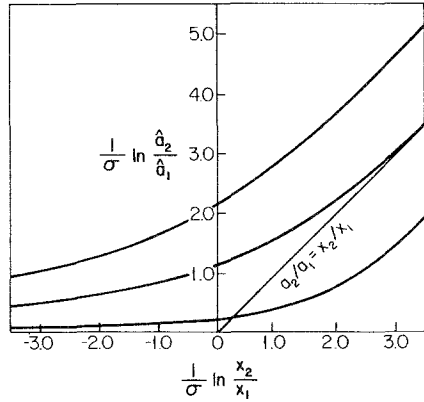


Fig. 1. Dependence of Bayes point and 75% credible-limit age-ratio estimates on measured age ratios for equal coefficients of variation.

versus position in a stalagmite, implicitly accepting age-order despite overlap of all adjacent (and many more distant) standard-error intervals. Despite large uncertainty in x_1 , $\hat{a}_2 - \hat{a}_1$ is nearly “classical.” Credible interval limits are positive.

Examples 3 to 7 are hypothetical cases to illustrate effects of relative x_1 and x_2 , and σ for $\sigma_1 = \sigma_2$. Examples 3 and 4 both have $x_2/x_1 = 2$, for small or large σ . Estimated age differences are classical. Examples 5 and 6 have x_2

Table 1. Examples of Bayesian Estimates of Age Differences

No.	<i>i</i>	X_i	σ_i	\hat{a}_i , ka	$\hat{a}_2 - \hat{a}_1$, ka	75% C.I., ka
1	1	126.0	0.209	124.0		
	2	207.0	0.241	212.0	88.0	[33.0, 147.]
2	1	0.045	2.30	0.024		
	2	0.525	0.64	0.550	0.526	[0.11, 1.75]
3	1	100.0	0.02	100.0		
	2	200.0	0.02	200.0	100.0	[95.2, 105.]
4	1	100.0	0.20	99.7		
	2	200.0	0.20	200.6	100.0	[53.5, 151.]
5	1	100.0	0.02	101.1		
	2	100.0	0.02	98.9	2.2	[0.40, 4.40]
6	1	100.0	0.20	112.0		
	2	100.0	0.20	89.3	22.6	[4.40, 43.7]
7	1	100.0	0.20	82.5		
	2	80.0	0.20	96.9	14.4	[2.50, 29.0]

$= x_1$, and example 7 has reversed age measurements ($x_2 < x_1$). Estimates \hat{a}_1 and \hat{a}_2 are ordered, $\hat{a}_2 - \hat{a}_1$ are positive, and 75% credible intervals are large.

DISCUSSION

At first, estimating ordered dates from reversed measurements appears startling. However, Bayesian estimates have only reasonable and desirable properties. These include:

1. $\hat{a}_2 > \hat{a}_1$: $\hat{a}_1 = \hat{a}_2$ cannot be accepted.
2. Classic estimates result for small σ_i or large x_2/x_1 .
3. Credible interval estimates have positive limits, but reflect increased uncertainty when $x_2 < x_1$.

Of course, no guard exists against reversed age measurements resulting from systematic errors in dating. To be properly interpreted, Bayesian estimates must be based on data subject only to (approximately log-normal) imprecision. However, systematic errors do not lead to contradictory ($\hat{a}_2 < \hat{a}_1$) results, but only to large credible intervals.

In view of these properties of Bayesian estimates of age difference, no reason exists for accepting a null hypothesis that $a_1 = a_2$ when it is known, a priori, to be false.

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