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Upper Bound on the Free Energy of the Spin 1/2 Heisenberg Ferromagnet*

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Abstract. The authors obtain an upper bound on the free energy of the spin 1/2 Heisenberg ferromagnet. The zero field bound is, at low temperature, similar to the formula given by the magnon approximation. That is, its functional dependence on temperature is the same but the constant is different.

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1. Introduction

Let $\mathbb{Z}^d$ be the integer lattice in d dimensions. For integer $L = 1, 2, \dots$, let Λ be the intersection of $\mathbb{Z}^d$ with a 'cube' of side L,

$$\Lambda = [0, L]^d \cap \mathbb{Z}^d. \tag{1.1}$$

We impose periodic boundary conditions on Λ by identifying endpoints. Then Λ becomes a d-dimensional torus with volume $|\Lambda| = L^d$. We shall be interested in the spin $\frac{1}{2}$ nearest neighbor Heisenberg ferromagnet on Λ.

To each $R \in \Lambda$, we attach a spin vector $\bar{S}(R) = (S_x(R), S_y(R), S_z(R))$ where the components of $\bar{S}(R)$ are given in terms of the Pauli spin matrices $\sigma_x, \sigma_y, \sigma_z$ by

$$S_x(R) = \tfrac{1}{2}\sigma_x, \qquad S_y(R) = \tfrac{1}{2}\sigma_y, \qquad S_z(R) = \tfrac{1}{2}\sigma_z. \tag{1.2}$$

It easily follows that

$$\bar{S}(R)^2 = S_x(R)^2 + S_y(R)^2 + S_z(R)^2 = \tfrac{3}{4}. \tag{1.3}$$

The spin $\frac{1}{2}$ Heisenberg Hamiltonian on Λ is then given by

$$\mathfrak{H}_\Lambda = \tfrac{1}{2} \sum_{R \in \Lambda, \delta} [(\bar{S}(R) - \bar{S}(R + \delta))^2 - 1], \tag{1.4}$$

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where δ ranges over all vectors in the lattice with length 1. The Hamiltonian (1.4) has been normalized to make the ground state zero. The number operator corresponding to $\mathfrak{H}_\Lambda$ is $\mathfrak{N}_\Lambda$, where

$$\mathfrak{N}_\Lambda = \sum_{R \in \Lambda} [S_z(R) + \tfrac{1}{2}]. \tag{1.5}$$

Thus $\mathfrak{N}_\Lambda$ counts the number of sites R with $S_z(R) = \frac{1}{2}$. In particular, the vacuum $\mathfrak{N}_\Lambda = 0$ consists of all sites R taking the value $S_z(R) = -\frac{1}{2}$. It is well known that $\mathfrak{N}_\Lambda$ commutes with $\mathfrak{H}_\Lambda$, whence $[\mathfrak{N}_\Lambda, \mathfrak{H}_\Lambda] = 0$.

The physical free energy per unit volume at inverse temperature $\beta > 0$ and field h, $-\infty < h < \infty$, is given by $-\beta^{-1} f_\Lambda$, where

$$f_\Lambda(\beta, h) = |\Lambda|^{-1} \log \operatorname{Tr} \exp[-\beta(\mathfrak{H}_\Lambda + h\mathfrak{N}_\Lambda)]. \tag{1.6}$$

For simplicity of notation, we shall refer from here on to f_Λ and its analogues as free energy. The thermodynamic limit of f_Λ,

$$f(\beta, h) = \lim_{|\Lambda| \to \infty} f_\Lambda(\beta, h) \tag{1.7}$$

is known to exist [9] and gives a continuous convex function of β and h. We shall be concerned with the zero field free energy $f(\beta, 0)$ at large β.

The standard method of understanding heuristically the low temperature Heisenberg model is the magnon approximation [2]. Let $H_{\Lambda,N}$ be the Hamiltonian $\mathfrak{H}_\Lambda$ restricted to the space $\mathfrak{N}_\Lambda = N$, $N = 0, 1, 2, \ldots$. Then in view of our normalization of $\mathfrak{H}_\Lambda$, $H_{\Lambda,0}$ is just the zero Hamiltonian acting on the complex numbers $\mathbb{C}$. Let $L^2(\Lambda)$ be the Hilbert space of functions ψ,

$$L^2(\Lambda) = \left\{ \psi : \Lambda \to \mathbb{C} \;\middle|\; \sum_{R \in \Lambda} |\psi(R)|^2 < \infty \right\}. \tag{1.8}$$

Then $H_{\Lambda,1}$ is the negative lattice Laplacian acting on $L^2(\Lambda)$, i.e. for $\psi \in L^2(\Lambda)$

$$H_{\Lambda,1}\psi(R) = 2\,\mathrm{d}\psi(R) - \sum_\delta \psi(R + \delta). \tag{1.9}$$

The magnon approximation consists then of assuming that the low temperature Heisenberg model at nonnegative field behaves like the free Bose gas generated by $H_{\Lambda,1}$. Since it is possible to calculate the free energy of a Bose gas precisely one expects from the magnon approximation that the Heisenberg free energy (1.7) satisfies

$$\lim_{\beta \to \infty} \beta^{d/2} f(\beta, 0) = C_d, \tag{1.10}$$

where

$$C_d = \frac{-1}{(2\pi)^d} \int_{\mathbb{R}^d} \log[1 - e^{-k^2}]\, dk. \tag{1.11}$$

We prove here a lower bound on the zero field free energy of the Heisenberg model which is of magnon form (1.10). Thus, we have

THEOREM 1.1. *Let c_d be defined by*

$$c_d = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{-k^2}\,\mathrm{d}k = (2\sqrt{\pi})^{-d}. \tag{1.12}$$

Then there is the inequality

$$\liminf_{\beta\to\infty} \beta^{d/2} f(\beta, 0) \geqslant \tfrac{1}{2} c_d. \tag{1.13}$$

In dimension $d = 3$, the constant $C_d = 0.0301$, which is to be compared to our bound $\frac{1}{2}c_d = 0.0112$.

To prove Theorem 1.1, we actually consider a model which lies between the Ising model and the Heisenberg. This is the reason why we get the constant c_d occurring in (1.13) instead of C_d as in (1.10). Indeed, for the intermediate model, the constant c_d would be the correct free gas lower bound (see the remark after Theorem 1.2). Unfortunately, in (1.13) we cannot even go all the way to c_d.

The configuration space for the Heisenberg N particle Hamiltonian $H_{\Lambda,N}$ is [3]

$$\Lambda^N_{1/2} = \{\mathbf{R} = (R_1, R_2, \ldots, R_N) \in \Lambda^N \mid R_i \neq R_j, 1 \leqslant i < j \leqslant N\}. \tag{1.14}$$

The boundary of $\Lambda^N_{1/2}$ is

$$\partial\Lambda^N_{1/2} = \Lambda^N \backslash \Lambda^N_{1/2}. \tag{1.15}$$

The Hamiltonian $H_{\Lambda,N}$ is then the lattice Neumann Laplacian on $\Lambda^N_{1/2}$, $N = 2, 3, \ldots$. Thus, $H_{\Lambda,N}$ is a self-adjoint operator on the Hilbert space $L^2(\Lambda^N_{1/2})$,

$$L^2(\Lambda^N_{1/2}) = \left\{\psi : \Lambda^N_{1/2} \to \mathbb{C} \;\middle|\; \sum_{\mathbf{R}\in\Lambda^N_{1/2}} |\psi(\mathbf{R})|^2 < \infty\right\}. \tag{1.16}$$

The quadratic form defining $H_{\Lambda,N}$ is given by

$$\begin{aligned}&\langle\psi, H_{\Lambda,N}\psi\rangle\\ &\quad= \sum_{i=1}^{N} \sum_{\mathbf{R}}^{(i)} |\psi(R_1, \ldots, R_{i-1}, R_i + \delta, R_{i+1}, \ldots, R_N) - \psi(R_1, \ldots, R_N)|^2,\end{aligned} \tag{1.17}$$

where the second $\Sigma^{(i)}$ in (1.17) is taken over all pairs

$$\mathbf{R} = (R_1, \ldots, R_N) \quad\text{and}\quad \mathbf{R}_i(\delta) = (R_1, \ldots, R_{i-1}, R_i + \delta, R_{i+1}, \ldots, R_N)$$

such that $\mathbf{R}, \mathbf{R}_i(\delta)$ are in $\Lambda^N_{1/2}$. The N particle Heisenberg Hamiltonian is then $H_{\Lambda,N}$ acting on *symmetric* functions $\psi(R_1, \ldots, R_N)$. Thus, the Heisenberg model corresponds to Bose–Einstein statistics [3]. Let us consider the system corresponding to Maxwell–Boltzmann statistics. Thus we consider $H_{\Lambda,N}$ acting on the full L^2 space

(1.16) and then the Maxwell–Boltzmann free energy is

$$F_\Lambda(\beta, h) = |\Lambda|^{-1} \log\left\{\sum_{N=0}^{\infty} \frac{e^{-\beta h N}}{N!} \operatorname{Tr} \exp(-\beta H_{\Lambda,N})\right\}. \tag{1.18}$$

Note that N is restricted to $N \leqslant |\Lambda|$ in the above sum.

This Maxwell–Boltzmann model lies between the Ising model and the Heisenberg in the sense that

$$\mathscr{F}_\Lambda(\beta, h) < F_\Lambda(\beta, h) \leqslant f_\Lambda(\beta, h), \tag{1.19}$$

where $\mathscr{F}_\Lambda(\beta, h)$ is the Ising model free energy

$$\mathscr{F}_\Lambda(\beta, h) = \sum_{\substack{\sigma(R) = \pm 1/2 \\ R \in \Lambda}} \exp\left\{-\beta\left[\sum_{R,\delta} (1/4 - \sigma(R)\sigma(R+\delta)) + h \sum_{R \in \Lambda} (\sigma(R) + 1/2)\right]\right\}. \tag{1.20}$$

The inequalities (1.19) are easily seen from the representation (given in [3]) of the partition functions in terms of random walk expansions. In fact, the first inequality was explicitly stated in [3].

One can prove the existence of the thermodynamic limit for the Maxwell–Boltzmann model just as for the Heisenberg model. Let $F(\beta, h)$ be the infinite volume free energy per unit volume. The finite volume magnetization $M_\Lambda(\beta, h)$ is

$$M_\Lambda(\beta, h) = 1 + \frac{2}{\beta} \partial F_\Lambda(\beta, h)/\partial h. \tag{1.21}$$

Since F_Λ has the symmetry

$$F_\Lambda(\beta, h) = F_\Lambda(\beta, -h) - \beta h, \tag{1.22}$$

it follows that $M_\Lambda(\beta, 0) = 0$. It is not difficult to show that the thermodynamic magnetization $M(\beta, h) = \lim_{\Lambda \to \infty} M_\Lambda(\beta, h)$ satisfied $M(\beta, 0) = 0$ for small β: We have the following estimates on thermodynamic free energy and magnetization for *large* β.

THEOREM 1.2. *With* c_d *as in* (1.12), *there are the inequalities*

$$\liminf_{\beta \to \infty} \beta^{d/2} F(\beta, 0) \geqslant \tfrac{1}{2} c_d, \tag{1.23}$$

$$\liminf_{\beta \to \infty} \beta^{d/2} |M(\beta, 0) - 1| \geqslant c_d. \tag{1.24}$$

Remark. Observe that the bounds in Theorem 1.2 are not free gas lower bounds. The free gas lower bounds from (1.23) would correspond to replacing $c_d/2$ by c_d on the right-hand side of the inequality. Inequality (1.24) is known to exist for the Heisenberg model [2] and is proved by using Mermin–Wagner-type arguments [4, 9].

Evidently, the inequality (1.23) implies Theorem 1.1. It is natural to ask if upper bounds corresponding to (1.23) and (1.24) exist. This is a much more difficult question than the one we have answered. The reason is that the Heisenberg process, viewed as a random walk [3], differs from the free system in that the walks have been slowed down. The estimates (1.23), (1.24) and the Mermin–Wagner argument [9] are rigorous versions of this intuition. To obtain upper bounds, one needs to estimate the degree of slowing down or subdiffusivity. Remarkably, an upper bound of the form (1.24) is known to exist for the Heisenberg antiferromagnet. This was proved in [4] using the method of Gaussian domination. The constant there is also off, by a factor of 3/2, coming from an overcounting of degrees of freedom.

We conjecture that upper bounds of the form (1.23), (1.24) are *not* correct in dimension $d = 1$ but are correct in dimension $d \geqslant 3$. For $d = 1, F_\Lambda = f_\Lambda$ and the model can be exactly solved [7, 10]. However, we can find no reference to estimating $f(\beta, 0)$ for large β. We expect that in $d = 1$, there is the inequality

$$\liminf_{\beta \to \infty} f(\beta, 0) > 0. \tag{1.25}$$

To explain why we make these conjectures let $G(\beta, \rho)$ be the thermodynamic free energy

$$G(\beta, \rho) = \lim_{\substack{\Lambda \to \infty \\ N/|\Lambda| = \rho}} |\Lambda|^{-1} \log\left\{\frac{1}{N!} \operatorname{Tr} \exp(-\beta H_{\Lambda, N})\right\}, \tag{1.26}$$

where $0 < \rho < 1$. Then

$$F(\beta, h) = \sup_{0 < \rho < 1} [-\beta h \rho + G(\beta, \rho)], \tag{1.27}$$

and $G(\beta, \rho)$ is a concave function of ρ satisfying the conditions

$$G(\beta, \rho) = G(\beta, 1 - \rho), \qquad G(\beta, 0) = 0. \tag{1.28}$$

It follows from (1.28) that $G(\beta, \rho)$ as a function of ρ increases as ρ goes from 0 to $\frac{1}{2}$ and then decreases. Let $\rho(\beta)$ be defined by

$$\rho(\beta) = \inf\{\rho > 0 : G(\beta, \rho) = \sup_{0 < \rho' < 1} G(\beta, \rho')\}. \tag{1.29}$$

Then $\rho(\beta)$ is related to the zero field magnetization $M(\beta, 0)$ by

$$M(\beta, 0) = 1 - 2\rho(\beta). \tag{1.30}$$

Now in $d = 1$ there is no phase transition [9], whence $M(\beta, 0) = 0$ and so $\rho(\beta) = \frac{1}{2}$. However, if $\rho = \frac{1}{2}$ then one expects that particles on average can diffuse only a finite distance, whence $G(\beta, \frac{1}{2}) = 0(1)$ as $\beta \to \infty$.

Our conjecture in $d \geqslant 3$ is based on the expectation that the Heisenberg ferromagnet has a phase transition for $d \geqslant 3$. There appears to be a relation between this problem and the problem of random walk in random environment [1, 5]. It has been recently shown in a remarkable paper [1] that random walk in a weak random environment is diffusive for $d \geqslant 3$. This suggests for our situation that if g is the quantity corresponding to (1.26) for the Heisenberg model, then

$$\lim_{\beta \to \infty} \beta^{d/2} g(\beta, \rho) = c < \infty \quad \text{for } 0 < \rho < \varepsilon, \tag{1.31}$$

in $d \geqslant 3$, where ε is a sufficiently small number. Now suppose that an upper bound of the form (1.23) does not hold in $d \geqslant 3$. Then, of course, it does not hold for the Heisenberg model. We conclude from (1.31) that in the Heisenberg model $\rho(\beta) \geqslant \varepsilon > 0$ for large β, whence the zero field magnetization does not converge to 1 as $\beta \to \infty$. Thus, if (1.23) does not hold in $d \geqslant 3$, there is either no phase transition in the Heisenberg ferromagnet or the nature of the transition is quite different from what has formally been assumed.

The anisotropic Heisenberg ferromagnet at low temperature is already well understood, and it has been shown in [6, 8] that a phase transition does exist in $d \geqslant 3$. The existence of a phase transition in the Heisenberg isotropic antiferromagnet has been given in [4].

5. Proof of Theorem 1.2

For $N \geqslant 1$, we consider the set

$$Q^N = \{\xi = (\mathbf{R}, R') \mid \mathbf{R} \in \Lambda_{1/2}^N, R' \in \Lambda\}, \tag{2.1}$$

and the Hilbert space

$$L^2(Q^N) = \left\{\psi : Q^N \to \mathbb{C} \;\middle|\; \sum_{\xi} |\psi(\xi)|^2 < \infty\right\}. \tag{2.2}$$

Evidently, $L^2(Q^N)$ is just a tensor product,

$$L^2(Q^N) = L^2(\Lambda_{1/2}^N) \otimes L^2(\Lambda). \tag{2.3}$$

Hence, the operator $H_{\Lambda,N} \otimes 1 + 1 \otimes H_{\Lambda,1}$ acts on it in a natural way.

There is also another operator which acts naturally on $L^2(Q^N)$. The set Q^N can be decomposed into $N+1$ disjoint subsets

$$Q^N = Q_0^N \cup \bigcup_{r=1}^{N} Q_r^N, \tag{2.4}$$

where

$$\begin{aligned} Q_0^N &= \{\xi = (\xi_1, \xi_2, \ldots, \xi_{N+1}) \mid \xi_i \neq \xi_j, 1 \leqslant i < j \leqslant N+1\}, \\ Q_r^N &= \{\xi = (\xi_1, \ldots, \xi_{N+1}) \mid \xi_i \neq \xi_j, 1 \leqslant i < j \leqslant N, \xi_r = \xi_{N+1}\}. \end{aligned} \tag{2.5}$$

It is clear that Q_0^N is identical to $\Lambda_{1/2}^{N+1}$ and that Q_r^N, $r = 1, \ldots, N$ are each isomorphic to $\Lambda_{1/2}^N$. The decomposition (2.4) leads to a decomposition of $L^2(Q^N)$,

$$L^2(Q^N) = \bigoplus_{r=0}^{N} L^2(Q_r^N). \tag{2.6}$$

Now $H_{\Lambda,N+1}$ acts naturally on $L^2(Q_0^N)$ and $H_{\Lambda,N}$ on each of $L^2(Q_r^N)$, $r = 1, \dots, N$. Let us denote $H_{\Lambda,N}$ acting on $L^2(Q_r^N)$ by $H_{\Lambda,N}(r)$, $r = 1, \dots, N$. Hence, the operator

$$H_{\Lambda,N+1} \oplus \sum_{r=1}^{N} H_{\Lambda,N}(r) \tag{2.7}$$

also acts on the space $L^2(Q^N)$.

LEMMA 2.1. *The following operator inequality holds on* $L^2(Q^N)$.

$$H_{\Lambda,N} \otimes 1 + 1 \otimes H_{\Lambda,1} \geqslant H_{\Lambda,N+1} \oplus \sum_{r=1}^{N} H_{\Lambda,N}(r). \tag{2.8}$$

Proof. We have for $\psi \in L^2(Q^N)$,

$$\begin{aligned} &\left\langle \psi, \left[H_{\Lambda,N+1} \oplus \sum_{r=1}^{N} H_{\Lambda,N}(r) \right] \psi \right\rangle \\ &\quad = \sum_{i=1}^{N+1} \sum_{\xi, \xi_i(\delta) \in Q_0^N} |\psi(\xi_i(\delta)) - \psi(\xi)|^2 + \\ &\quad\quad + \sum_{r=1}^{N} \sum_{i=1}^{N} \sum_{\xi, \xi_i(\delta) \in Q_r^N} |\psi(\xi_i(\delta)) - \psi(\xi_i)|^2, \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} &\xi_i(\delta) = (\xi_1, \dots, \xi_{i-1}, \xi_i + \delta, \xi_{i+1}, \dots, \xi_N, \xi_{N+1} + \delta) \quad \text{if } \xi \in Q_i^N, 1 \leqslant i \leqslant N, \\ &\xi_i(\delta) = (\xi_1, \dots, \xi_{i-1}, \xi_i + \delta, \xi_{i+1}, \dots, \xi_{N+1}) \\ &\quad\quad \text{if } i = N+1 \quad \text{or} \quad \xi \in Q^N \backslash Q_i^N, 1 \leqslant i \leqslant N. \end{aligned} \tag{2.10}$$

For $\xi = (\xi_1, \dots, \xi_{N+1}) \in Q^N$, let us write

$$\xi' = (\xi_1, \dots, \xi_N), \quad \xi = (\xi', \xi_{N+1}), \quad \xi' \in \Lambda_{1/2}^N \tag{2.11}$$

and define

$$\xi_i'(\delta) \quad \text{for } 1 \leqslant i \leqslant N$$

by

$$\xi_i'(\delta) = (\xi_1, \dots, \xi_{i-1}, \xi_i + \delta, \xi_{i+1}, \dots, \xi_N). \tag{2.12}$$

Let $\xi \in Q_i^N$, $1 \leqslant i \leqslant N$. From (2.10), we have the elementary inequality

$$\begin{aligned} &|\psi(\xi_i(\delta)) - \psi(\xi)|^2 \\ &\quad \leqslant |\psi(\xi_i'(\delta), \xi_{N+1} + \delta) - \psi(\xi', \xi_{N+1} + \delta)|^2 + \\ &\quad\quad + |\psi(\xi_i'(\delta), \xi_{N+1}) - \psi(\xi', \xi_{N+1})|^2 + \\ &\quad\quad + |\psi(\xi', \xi_{N+1} + \delta) - \psi(\xi', \xi_{N+1})|^2 + \\ &\quad\quad + |\psi(\xi_i'(\delta), \xi_{N+1} + \delta) - \psi(\xi_i'(\delta), \xi_{N+1})|^2. \end{aligned} \tag{2.13}$$

Substituting (2.13) into (2.9) we have that

$$\left\langle \psi, \left[H_{\Lambda,N+1} \oplus \sum_{r=1}^{N} H_{\Lambda,N}(r) \right] \psi \right\rangle$$
$$\leqslant \sum_{i=1}^{N} \sum_{\substack{\xi', \xi'_i(\delta) \in \Lambda_{1/2}^N \\ \xi_{N+1} \in \Lambda}} |\psi(\xi'_i(\delta), \xi_{N+1}) - \psi(\xi', \xi_{N+1})|^2 +$$
$$+ \sum_{\substack{\xi' \in \Lambda_{1/2}^N \\ \xi_{N+1} \in \Lambda}} |\psi(\xi', \xi_{N+1} + \delta) - \psi(\xi', \xi_{N+1})|^2$$
$$= \langle \psi, [H_{\Lambda,N} \otimes 1 + 1 \otimes H_{\Lambda,1}] \psi \rangle. \tag{2.14}$$

COROLLARY 2.1. *For $N \geqslant 1$ and $\beta > 0$, there is the trace inequality*

$$\operatorname{Tr} \exp[-\beta H_{\Lambda,N+1}]$$
$$\geqslant [\operatorname{Tr} \exp[-\beta H_{\Lambda,1}] - N] \operatorname{Tr} \exp[-\beta H_{\Lambda,N}]. \tag{2.15}$$

Proof. This follows immediately from (2.8).

We turn to the proof of Theorem 1.1. We have from (1.9) that

$$\lim_{\Lambda \to \infty} |\Lambda|^{-1} \operatorname{Tr} \exp[-\beta H_{\Lambda,1}]$$
$$= \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} e^{-\beta \varepsilon(k)} \, dk = \beta^{-d/2} \gamma_d(\beta), \tag{2.16}$$

where

$$\varepsilon(k) = 4 \sum_{i=1}^{d} \sin^2\left(\frac{k_i}{2}\right). \tag{2.17}$$

It follows that

$$\lim_{\beta \to \infty} \gamma_d(\beta) = (2\sqrt{\pi})^{-d}. \tag{2.18}$$

Hence, for large Λ and N satisfying

$$N/|\Lambda| \leqslant \gamma_d(\beta)/2\beta^{d/2}, \tag{2.19}$$

we have

$$\operatorname{Tr} \exp[-\beta H_{\Lambda,N}] \geqslant [\gamma_d(\beta)/2\beta^{d/2}]^N |\Lambda|^N. \tag{2.20}$$

For $N = 0, 1, 2, \ldots$ let $a_{N,\Lambda}$ be

$$a_{N,\Lambda} = \frac{1}{N!} \operatorname{Tr} \exp[-\beta H_{\Lambda,N}]. \tag{2.21}$$

Then, from (1.27), we have that

$$\begin{aligned} F(\beta, 0) &\geqslant \lim_{\substack{\Lambda \to \infty \\ N/|\Lambda| = \gamma_d(\beta)/2\beta^{d/2}}} |\Lambda|^{-1} \log a_{N,\Lambda} \\ &\geqslant \lim_{\substack{\Lambda \to \infty \\ N/|\Lambda| = \gamma_d(\beta)/2\beta^{d/2}}} |\Lambda|^{-1} \log\left\{\frac{1}{N!}[\gamma_d(\beta)/2\beta^{d/2}]^N|\Lambda|^N\right\} \\ &= \gamma_d(B)/2\beta^{d/2} \end{aligned} \tag{2.22}$$

by Stirling's formula. This proves (1.23).

To prove (1.24), we observe that $a_{N,\Lambda}$ is increasing as a function of N provided $N/|\Lambda| \leqslant \gamma_d(\beta)/2\beta^{d/2}$. It follows from this and (1.29) that

$$\rho(\beta) \geqslant \gamma_d(\beta)/2\beta^{d/2}. \tag{2.23}$$

The inequality (1.24) then follows from (1.30).

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