Blow-Up of Test Fields Near Cauchy Horizons

J. RAUCH[*]
Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, U.S.A.

and

A. D. RENDALL
Max-Planck-Institut für Astrophysik, Karl-Schwarzschild-Str. 1, 8046 Garching bei München, Germany

Abstract. The behaviour of test fields near a compact Cauchy horizon is investigated. It is shown that solutions of nonlinear wave equations on Taub spacetime with generic initial data cannot be continued smoothly to both extensions of the spacetime through the Cauchy horizon. This is proved using an energy method. Similar results are obtained for the spacetimes of Moncrief containing a compact Cauchy horizon and for more general matter models.

Mathematics Subject Classifications (1991). 35Q75, 35B60.

1. Introduction

The cosmic censorship hypothesis is one of the most important open problems in general relativity. Penrose [1] has presented a plausibility argument in favour of strong cosmic censorship and it is of interest to ask how this could be turned into a rigorous proof. The basic idea of the argument is that if a spacetime contains a Cauchy horizon, then a generic small perturbation of the initial data on a partial Cauchy surface should lead to an accumulation of energy or infinite blueshift and that, as a result, the Cauchy horizon would be replaced by a curvature singularity. There are various ways in which one could attempt to capture an effect of this kind mathematically. One is to attempt to show the blow-up of solutions of the Einstein equations along appropriate curves using some kind of geometrical optics approximation [2]. Another approach is to attempt to use energy inequalities. These are usually employed in the proofs of existence theorems to show that certain quantities remain bounded. However, such inequalities can also be used to prove blow-up in certain circumstances. In this Letter, a procedure of this kind will be used successfully, in a situation simpler than that of the above discussion.

In order to make progress, we make some simplifying assumptions. Instead of the Einstein equations themselves, we study test fields, i.e. solutions of various hyperbolic

equations on a *fixed spacetime background*. The goal is to prove that generic solutions of these equations blow up near a Cauchy horizon of the background spacetime. The relevance of such a result to the original problem is twofold. On the one hand, one would expect, from a physical point of view, that if a spacetime is such as to cause matter fields propagating through it to become singular, this would lead in a more realistic description, where the matter fields are coupled to the Einstein equations, to a spacetime singularity as well. On the other hand, the sudy of test fields may lead to the discovery of mathematical ideas which will also be important when it comes to solving the full problem. It follows from the latter point that theorems concerning test fields will be particularly interesting if they use techniques which have a good chance of being generalised. An advantage of the energy method is that it has a wide potential range of applicability. Its disadvantage is that the information obtained concerns global integral quantities; it is difficult to localise singularities which occur.

The basic idea of the energy method will now be recalled. Suppose that a spacetime has been chosen along with a kind of test field whose propagation on this spacetime is to be studied. Let $T^{\alpha\beta}$ denote the energy-momentum tensor and let S_t be a leaf of a foliation of part of spacetime by compact spacelike hypersurfaces. The leaves are indexed by real numbers t lying in some interval. The assumption of compactness is not essential but is convenient and also sufficient for most of the applications in this paper. Let G be a compact region bounded by the hypersurfaces S_t and $S_{t'}$. Denote the normal vector field to the foliation by n^{α}. Then Stokes' theorem and the divergence-free nature of the energy-momentum tensor imply the identity

$$\int_{S_{t'}} T_{\alpha\beta} n^{\alpha} p^{\beta} = \int_{S_t} T_{\alpha\beta} n^{\alpha} p^{\beta} + \int_G T^{\alpha\beta} p_{\alpha;\beta} \tag{1}$$

for any smooth vector field p^{α}. Of couse, this identity is not useful for the Einstein equations themselves. The gravitational field does not have an energy-momentum tensor. In that case, however, it is possible to envisage replacing $T^{\alpha\beta}$ by the Bel–Robinson tensor, a strategy which was used to good effect in [3].

In the following section, it will be shown how the above strategy can be used to obtain information about scalar wave equations on the Taub spacetime. Section 3 contains generalisations of this to a class of spacetimes with compact Cauchy horizons studied by Moncrief, to other kinds of matter models and to certain spacetimes with noncompact Cauchy horizons.

2. Scalar Wave Equations on the Taub Spacetime

This section is concerned with nonlinear wave equations of the form

$$\Box u = m^2 u + \lambda u^3, \tag{2}$$

where m and λ are non-negative real numbers. They will be studied on the spacetime with metric

$$\begin{aligned} ds^2 = {} & -U^{-1}\,dt^2 + (2l)^2 U(d\psi + \cos\theta\,d\phi)^2 + \\ & + (t^2 + l^2)(d\theta^2 + \sin^2\theta\,d\phi^2). \end{aligned} \tag{3}$$

The notation here coincides with that used by Hawking and Ellis [4], p. 170. The function U depends only on t, and l is a positive constant. Equation (3) defines a nondegenerate metric so long as U is positive. The Taub spacetime is given by the particular function $U(t) = (l^2 - 2mt - t^2)/(t + l^2)$ and is a solution of the vacuum Einstein equations. All that is required for our blowup theorem is that U is strictly positive on an interval (t_-, t_+) and is zero with nonvanishing derivatives at the endpoints. The Taub spacetime is defined on the manifold $M = S^3 \times (t_-, t_+)$. It is well known (see [4]) that $t = t_+$ is only a coordinate singularity and that the Taub spacetime can be extended through this hypersurface in two distinct ways. This gives rise to the Taub-NUT spacetime for which $t = t_+$ is a Cauchy horizon. Let M_1 denote the manifold where one extension is defined and M_2 the other. If

$$\psi' := \psi + (1/2l)\int U^{-1}(t)\,dt, \tag{4}$$

then (t, ψ', θ, ϕ) are regular coordinates on M_1 (apart from the trivial singularities due to the use of polar coordinates). Similarly $\psi'' = \psi - (1/2l)\int U^{-1}(t)\,dt$ defines together with t, θ and ϕ a regular coordinate system on M_2. The explicit form of the linear wave equation in Taub space is

$$-Uu_{tt} - U_t u_t + [(t^2 + l^2)^{-1}\cot^2\theta + (4l^2 U)^{-1}]u_{\psi\psi} + (t^2 + l^2)^{-1}u_{\theta\theta}$$
$$+ (t^2 + l^2)^{-1}\csc^2\theta u_{\phi\phi} - 2(t^2 + l^2)^{-1}\csc\theta\cot\theta u_{\theta\phi} - (t^2 + l^2)^{-1}u_\theta = 0. \tag{5}$$

Consider for a moment the possibility that u only depends on t. Then (5) reduces to

$$Uu_{tt} + U_t u_t = 0, \tag{6}$$

with the explicit solution

$$u = C_1 + C_2 \int U^{-1}(t)\,dt. \tag{7}$$

Hence there are special solutions of (5) which blow up near the Cauchy horizon, i.e. as $t \to t_+$. For the linear equation (5), this is enough to show that the generic solution blows up (since it is always possible to add ε times the data for this singular solution to any data whose corresponding solution does not blow up), but this no longer works for a nonlinear equation. It is necessary to use methods which are more flexible.

The energy identity (1) will now be applied to Equation (2) on the Taub spacetime. The energy-momentum tensor associated to the Equation (2) is given by

$$T^{\alpha\beta} = \nabla^\alpha u \nabla^\beta u - (1/2)g^{\alpha\beta}(g^{\gamma\delta}\nabla_\gamma u \nabla_\delta u + m^2 u^2 + (1/2)\lambda u^4). \tag{8}$$

That it is divergence-free follows directly from (2). The foliation chosen is that given by the hypersurfaces of constant t. Choose $\partial/\partial\psi$ as the vector field p^α. This extends smoothly to both M_1 and M_2, since it is equal to $\partial/\partial\psi'$ and $\partial/\partial\psi''$ in the regions where the relevant coordinate patches overlap. This is a Killing vector and so the volume contribution in (1) vanishes. It follows that $\int_{S_t} T_{\alpha\beta} n^\alpha p^\beta$ is independent of t. To see what this means, the integral is evaluated explicitly. The normal vector n^α to the foliation is given in the coordinates used in (3) by $U^{1/2}\partial/\partial t$. Hence,

$$T_{\alpha\beta} n^\alpha p^\beta = U^{1/2} u_t u_\psi. \tag{9}$$

Next, note that the expression $\sin\theta\, d\psi\, d\theta\, d\phi$ defines a smooth volume element on each leaf of the foliation which extends regularly to M_1 and M_2. Also, the volume element induced on S_t by the given metric is $2lU^{1/2}(t^2 + l^2)\sin\theta\, d\psi\, d\theta\, d\phi$. Thus, the integral can be written in the form

$$\int_{S_t} T_{\alpha\beta} n^\alpha p^\beta = l^2 U^2 \int \left(\frac{\partial u}{\partial t} + \frac{1}{2lU} \frac{\partial u}{\partial \psi} \right)^2 - \left(\frac{\partial u}{\partial t} - \frac{1}{2lU} \frac{\partial u}{\partial \psi} \right)^2 dV, \tag{10}$$

where $dV = (t^2 + l^2)\sin\theta\, d\psi\, d\theta\, d\phi$. Now in the overlap of the coordinate patches, the vector field $\partial/\partial t - (2lU)^{-1}\partial/\partial\psi$ on M corresponds to the regular vector field $\partial/\partial t$ on M_1 and similarly $\partial/\partial t + (2lU)^{-1}\partial/\partial\psi$ corresponds to $\partial/\partial t$ on M_2. It follows that if a solution u of (2) had C^1 extensions to both M_1 and M_2 the right-hand side of (10) would tend to zero as $t \to t_+$. But we have already seen that this expression is constant. Thus, this extendibility property can only hold for very special initial data, proving the following theorem.

THEOREM 1. *Let t_0 be a number in the interval (t_-, t_+) and suppose that the initial data (u, u_t) for Equation (2) are given on the spacelike hypersurface $t = t_0$ in Taub spacetime. Then if*

$$\int_{S_{t_0}} u_\psi u_t \sin\theta\, d\psi\, d\theta\, d\phi \neq 0, \tag{11}$$

the corresponding solution of (2) cannot extend in a C^1 manner to both M_1 and M_2.

Remarks 1. The condition (11) is generic in the sense that it is fulfilled for an open dense set of initial data in the uniform topology.

2. The particular form of the nonlinearity in (2) was chosen because it ensures that for given initial data, there exists a corresponding solution globally on M[5]. The theorem would still hold for a more general nonlinearity but in that case blow-up could in principle occur even before the Cauchy horizon.

3. Further Examples

The existence of a large class of solutions of the vacuum Einstein equations containing compact Cauchy surfaces has been shown by Moncrief [6]. These are

somewhat similar to the Taub-NUT spacetime but differ by the fact that they have partial Cauchy surfaces diffeomorphic to $K \times S^1$, where K is a compact surface, instead of S^3. The form of the metric is

$$\mathrm{d}s^2 = \mathrm{e}^{-2\phi} N^2 \, \mathrm{d}t^2 + g_{ab} \, \mathrm{d}x^a \, \mathrm{d}x^b] + t^2 \, \mathrm{e}^{2\phi} (\mathrm{d}x^3 + \beta_a \, \mathrm{d}x^a)^2. \tag{12}$$

Here Latin indices take the values 1 and 2 and x^3 is the coordinate on S^1. The metric coefficients do not depend on x^3. The function N is assumed to vanish nowhere. This form of the metric becomes singular at $t = 0$. However, as in the case of the Taub spacetime, it is possible to extend the metric through the hypersurface $t = 0$ in two different ways to manifolds M_1 and M_2 provided a certain additional condition is satisfied. This is seen by introducing new coordinates

$$x^{3'} = x^3 - \log t \quad \text{and} \quad x^{3''} = x^3 + \log t$$

respectively, and $t' = t^2$. The condition for the existence of the extensions is then that $(N^2 - \mathrm{e}^{4\phi})/t'$ is a smooth function of the new coordinates. The spacetimes constructed by Moncrief are all analytic. This condition plays no role in the present considerations, where finite differentiability of the metric (12) is sufficient. It will be shown that the above arguments concerning solutions of (2) on Taub space can easily be adapted to this case.

The equivalent of (6) in this case is $tu_{tt} + u_t = 0$ (which can be read off from the formulae in [5]) with solution $u = C_1 + C_2 \log t$. For the application of the energy identity choose the foliation $t = \text{const}$ and let the vector field p^α be given by $\partial/\partial x^3$. Since $\partial/\partial x^3$ is a Killing vector, there is no volume contribution in (1). The normal vector n^α to the foliation is $\mathrm{e}^\phi N^{-1} \partial/\partial t$. The volume element induced on S_t by the spacetime metric is

$$t \, \mathrm{e}^\phi [g_{11} g_{22} - g_{12}^2 + 2 g_{12} \beta_1 \beta_2 - g_{11} \beta_2^2 - g_{22} \beta_1^2]^{1/2} \, \mathrm{d}x^1 \, \mathrm{d}x^2 \, \mathrm{d}x^3. \tag{13}$$

Now the metric

$$\mathrm{e}^{-2\phi} [-N^2 \, \mathrm{d}t^2 + g_{ab} \, \mathrm{d}x^a \, \mathrm{d}x^b] + \mathrm{e}^{2\phi} (\mathrm{d}x^3 + \beta_a \, \mathrm{d}x^a)^2 \tag{14}$$

is regular even at $t = 0$ without doing a coordinate change. Hence, the expression obtained from (13) by omitting the factor t is a regular volume element, call it $\mathrm{d}V$. Hence,

$$\int_{S_t} T_{\alpha\beta} n^\alpha p^\beta$$

$$= \frac{1}{2} \int \mathrm{e}^{-\phi} N^{-1} t^4 \left[\left(\frac{1}{t} \frac{\partial u}{\partial t} + \frac{1}{t^2} \frac{\partial u}{\partial x^3} \right)^2 - \left(\frac{1}{t} \frac{\partial u}{\partial t} - \frac{1}{t^2} \frac{\partial u}{\partial x^3} \right)^2 \right] \mathrm{d}V$$

must tend to zero as $t \to 0$ if the solution u has a C^1 extension to both M_1 and M_2. On the other hand, the energy identity shows that this expression is independent of time. Thus, we get the following analogue of Theorem 1.

THEOREM 2. *Let t_0 be a positive real number and suppose that the initial data (u, u_t) are given for equation* (2) *on the spacelike hypersurface $t = t_0$ in the spacetime defined by the metric* (12) *on the manifold $K \times S^1 \times (0, \infty)$, where K is a compact surface. Suppose furthermore that the condition given above for the existence of extensions M_1 and M_2 of the spacetime through $t = 0$ is satisfied. Then if*

$$\int_{S_{t_0}} e^{-\phi} N^{-1} u_t u_{x^3} \, dV \neq 0 \tag{15}$$

the corresponding solution of (2) *cannot extend in a C^1 manner to both M_1 and M_2.*

The fact that in this theorem the Cauchy horizon occurs in the past of the initial hypersurface (if the direction of increasing t is chosen as the future direction) in contrast to the situation in Theorem 1, is merely a matter of notational convenience. In [5], Moncrief analyses solutions of Equation (2) on spacetimes of the form (2) with regard to their behaviour near the Cauchy horizon. His results do not seem to be directly comparable with those of this Letter.

Next, we consider the possibility of applying the energy method to other matter models. For simplicity only, the case of Taub spacetime will be written out but the arguments generalise to Moncrief's spacetimes. The assumption that a solution only depends on t has no natural analogue for geometric objects more complicated than scalars and so the cheap construction of solutions which blow up on the Cauchy horizon in the scalar case is not available. However, the energy method still works, as will now be demonstrated. Denote the vectors $\partial/\partial t + (2lU)^{-1} \partial/\partial \psi$ and $\partial/\partial t - (2lU)^{-1} \partial/\partial \psi$ by n_1^α and n_2^α, respectively. Then

$$n^\alpha = (1/2)(n_1^\alpha + n_2^\alpha). \tag{16}$$

Hence

$$2T_{\alpha\beta} n^\alpha p^\beta = T_{\alpha\beta} n_1^\alpha p^\beta + T_{\alpha\beta} n_2^\alpha p^\beta. \tag{17}$$

If the solution extends smoothly to M_1, then the first term on the right-hand side of (16) is bounded while a smooth extension to M_2 implies the boundedness of the second term. So the kind of argument used twice already implies that the following theorem holds.

THEOREM 3. *Let t_0 be a number in the interval (t_-, t_+) and suppose that initial data are given on the spacelike hypersurface $t = t_0$ in Taub spacetime for some matter field with energy-momentum tensor $T^{\alpha\beta}$. Then if*

$$\int_{S_{t_0}} T_{\alpha\beta} n^\alpha p^\beta \neq 0. \tag{18}$$

the solution of the matter field equation corresponding to the given initial data cannot extend to both M_1 and M_2 in a way which would imply the continuous extendibility of the energy-momentum tensor.

Remark. The extendibility property is stated here in a somewhat indirect way. For most matter models commonly used in general relativity, the energy-momentum tensor is expressible pointwise in terms of the matter variables and their first covariant derivatives. Thus, C^1 extendibility of the matter variables implies continuous extendibility of $T^{\alpha\beta}$.

The only thing which varies from one matter model to another in applying Theorem 3 is the explicit form of (18) in terms of the matter variables. For a Maxwell field $F^{\alpha\beta}$, for instance, it takes the form

$$\int_{S_{t_0}} (\csc^2\theta\, F_{03}F_{13} + F_{02}F_{12} - \csc\theta\cot\theta\, F_{01}F_{13}) \neq 0. \tag{19}$$

The condition for a Yang–Mills field is identical to this except that the expression on the left-hand side of (19) is then matrix-valued and the condition (18) is given by the vanishing of its trace.

The methods used above can also be used to obtain information about test fields on certain spacetimes containing noncompact Cauchy horizons. Consider spacetimes of the Moncrief form where the condition that the surface K be compact is dropped. Then spacetimes will be obtained which contain Cauchy surfaces diffeomorphic to $K \times S^1$. If compactly supported data for some test field are given on a hypersurface $t = $ const, then the support of the corresponding solution on the globally hyperbolic region (i.e. the part of the spacetime before the Cauchy horizon) will have compact closure in each of the extensions. Also the intersection of this support with the other hypersurfaces of constant t are compact and so the identity (1) holds. Here it has been assumed that the domain of dependence for the matter field equations is limited by the light cones. A sufficient condition for this to be true is that $T^{\alpha\beta}$ satisfies the dominant energy condition. (For a discussion of this point, see [4], p. 94.) Thus, the same arguments as before apply. To sum up, suppose that condition that K be compact is removed from the hypotheses made on the spacetime in Theorem 2 and a matter field is considered which satisfies the dominant energy condition (for example, the scalar field with energy-momentum tensor given by (8)). Let compactly supported initial data for this matter field be given on $t = t_0$ satisfying (18). Then the solution which evolves from these initial data cannot have extensions to both M_1 and M_2 which are regular enough to imply that the energy-momentum has continuous extensions in both cases. Unfortunately, there are naturally occurring Cauchy horizons (such as that of the Reissner–Nordström solution) which are not covered by this argument since their Cauchy horizons have a different topology and compactly supported initial data can be smeared over the whole horizon by the time evolution.

References

1. Penrose, R., Singularities and time-asymmetry, in S. Hawking and W. Israel (eds), *General Relativity: An Einstein Centenary Survey*, Cambridge University Press, Cambridge, 1979.

2. Joly, J. L., Metivier, G., and Rauch, J., Resonant one-dimensional geometric optics, *J. Funct. Anal.* **114**, 106–231 (1993);
   Joly, J. L., Metivier, G., and Rauch, J., Remarques sur l'optique géometrique nonlinéaire multidimensionelle, Séminaire Equations aux Dérivées Partielles, Ecole Polytechnique, 1990–1991, exposé no. 1;
   Schochet, S., Fast singular limits of hyperbolic equations, Preprint.
3. Christodoulou, D. and Klainerman, S., The global nonlinear stability of the Minkowski space, Preprint.
4. Hawking, S. W. and Ellis, G. F. R. *The Large-Scale Structure of Space-Time*, Cambridge University Press, Cambridge, 1973.
5. Moncrief, V., The asymptotic behaviour of nonlinear waves near a cosmological Cauchy horizon, *J. Math. Phys.* **30**, 1760–1768 (1989).
6. Moncrief, V., Neighbourhoods of Cauchy horizons in cosmological spacetimes with one Killing field, *Ann. Phys.* **141**, 83–103 (1982).