



## Ionization and Scattering for Short-Lived Potentials

*Dedicated to the memory of Moshé Flato*

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**Abstract.** We consider perturbations of a model quantum system consisting of a single bound state and continuum radiation modes. In many problems involving the interaction of matter and radiation, one is interested in the effect of time-dependent perturbations. A time-dependent perturbation will couple the bound and continuum modes causing ‘radiative transitions’. Using techniques of time-dependent resonance theory, developed in earlier work on resonances in linear and nonlinear Hamiltonian dispersive systems, we develop the scattering theory of short-lived ( $\mathcal{O}(t^{-1-\epsilon})$ ) spatially localized perturbations. For weak perturbations, we compute (to second order) the ionization probability, the probability of transition from the bound state to the continuum states. These results can also be interpreted as a calculation, in the paraxial approximation, of the energy loss resulting from wave propagation in a waveguide in the presence of a localized defect.

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### 1. Introduction

The interactions of matter with electromagnetic fields play a major role in many fields of physics, chemistry, and engineering [2]. The study of laser interactions with atoms and electron microscopes, as well as the sudden approximation in quantum mechanics, lead us to consider the time-dependent Schrödinger equation with a perturbing potential which is short lived, that is, localized in time. A typical model of this type is

$$i\partial_t\phi(t, \mathbf{x}) = (-\Delta_{\mathbf{x}} + V(\mathbf{x}))\phi(t, \mathbf{x}) + \beta(t, \mathbf{x})\phi(t, \mathbf{x}), \quad (1.1)$$

where  $V(\mathbf{x})$  is localized in  $\mathbf{x} \in \mathbb{R}^d$  and  $\beta$  is localized in  $\mathbf{x}$  and  $t \in \mathbb{R}$ .

Another important area of application is in the propagation of waves in optical waveguides (e.g. optical fibers) in the *paraxial approximation*. Let  $\mathbf{x}$  denote the coordinates which are transverse to the guiding structure and let  $z$  denote the coordinate along the direction of propagation in the waveguide. Then,

$\phi = \phi(z, \mathbf{x})$ , the slowly varying envelope of the electric field, solves an equation of the form (1.1) where  $z$  plays the role of the time variable. The potential  $V(\mathbf{x})$  models transverse variations in the refractive index and may have localized modes associated with it. The perturbation,  $\beta(z, \mathbf{x})$  models perturbations in the refractive index which vary along the waveguide. A ‘short-lived’ (in  $z$ ) potential corresponds to a local ‘defect’ or local perturbation in the refractive index [7].

We consider a general class of problems including (1.1) in which the unperturbed Hamiltonian,  $H_0$ , has one bound state  $\psi_0$ ,  $\|\psi_0\| = 1$ . The perturbation,  $\beta(t)$ , has the effect of coupling the bound state to continuum (dispersive) modes and leads to energy transfer. Our aim in this Letter is the analysis of the large time behavior of such systems where  $\beta(t)$  is symmetric *short lived* and localized; see (2.4). The quantities  $P(t) \equiv |\langle \phi(t), \psi_0 \rangle|^2$  and  $1 - P(t)$  can be interpreted, respectively, as the probability that the system is in the state  $\psi_0$  and the ionization probability. In the context of optical waveguides,  $P(t)$  is a measure of the amount of energy trapped by the defect and  $1 - P(t)$  the amount of energy radiated away.

Our results are stated precisely in Theorems 2.1 and 2.2. Theorem 2.1 states that any solution of the initial value problem converges as  $t \rightarrow \pm\infty$  to an asymptotic bound state  $e^{-i\lambda_0 t} A(\pm\infty)\psi_0$  plus a purely dispersive part. The perturbation is not assumed to be small in Theorem 2.1. Theorem 2.2 gives a more detailed description of the solution as  $t \rightarrow \infty$  and in particular gives, in the case of a small perturbation  $\beta(t)$ , a useful expression for the asymptotic bound state component. Suppose the perturbed dynamical system is initialized in the state  $\psi_0$ . Theorem 2.2 implies

$$P(\infty) = e^{-2\Gamma} (1 + \mathcal{O}(\|h_\beta\|_{L^1}^2)), \quad (1.2)$$

where  $\|h_\beta\|_{L^1}$  is an integrated measure of the strength of the perturbation; see (2.4). This gives an explicit and useful expression for the ionization probability for weak perturbations. Note that its validity does not rely on pointwise smallness of  $\beta(t)$ . The quantity  $\Gamma$  is given by the formula

$$\begin{aligned} \Gamma &= \|P_c \hat{\beta}(\lambda_0 - H_0)\psi_0\|^2 \\ &= \int \gamma(\mu) \, d\mu, \end{aligned} \quad (1.3)$$

where

$$\gamma(\mu) = |\mathcal{F}_{H_0}[\hat{\beta}(\mu)\psi_0](\lambda_0 - \mu)|^2, \quad (1.4)$$

where  $\mathcal{F}_{H_0}$  denotes the (generalized) Fourier transform with respect the continuous spectral part of the operator  $H_0$ ; see also (2.17). The integration in (1.3) is over the support of  $\hat{\beta}(\mu)$ . This formula is an analogue of the *Fermi golden rule*<sup>\*</sup> appropriate for perturbations which are localized in  $t$ .

A  $T$ -periodic in perturbation may be thought of as a sequence of  $t$ -localized defects. After encountering  $N$  defects we have  $P(NT) \sim \exp(-2N\Gamma_{\text{one defect}})$ . Thus

<sup>\*</sup>See [10, 11] and references cited therein.

roughly, neglecting dispersive component of the solution, we have exponential decay as  $|t|$  increases. The results of [11, 6] (see also the exactly solvable model considered in [3, 9]) show that this is valid for large but finite  $t$ . For example, in [11] it is proved for  $\beta(t, x) = \cos(\mu_0 t)\beta(x)$  that  $P(t) \sim \exp(-2\gamma(-\mu_0)|t|)$ , for  $|t| \sim \mathcal{O}(\beta^{-2})$ . For a large class of perturbations with discrete time-frequency content (periodic, almost periodic in  $t$ ), the very large time behavior ( $t \gg \mathcal{O}(\beta^{-2})$ ) is characterized by algebraic decay to zero with increasing  $t$  [11, 6].

The methods of this Letter are a natural extension of the techniques of time dependent resonance theory developed in [10–12] to the case of time-localized perturbations. These methods can be extended to the case where  $H_0$  has multiple bound states [6].

## 2. Preliminaries and Results

We assume that the system under consideration is given by the following Hamiltonian:

$$H(t) \equiv H_0 + \beta(t), \quad (2.1)$$

and satisfies the Schrödinger equation

$$i \frac{\partial \phi}{\partial t} = H(t)\phi. \quad (2.2)$$

We assume that  $H_0$  is a self-adjoint operator densely defined on a Hilbert space  $\mathcal{H}$  and having exactly one bound state, with eigenvalue  $\lambda_0$ , and normalized eigenfunction  $\psi_0$ :

$$H_0\psi_0 = \lambda_0\psi_0, \quad \|\psi_0\| = 1. \quad (2.3)$$

The projection  $P_c = I - \langle \cdot, \psi_0 \rangle \psi_0$  is onto the continuous part of  $H_0$ . We define ‘weights’  $w_+$  and  $w_-$  satisfying

- (i)  $w_+ \geq cI, \quad c > 0,$
- (ii)  $w_- \in \mathcal{L}(\mathcal{H}),$
- (iii)  $w_- w_+ P_c = P_c = P_c w_- w_+.$

We also assume that  $w_+\psi_0 \in \mathcal{H}$ . This is the setting used in [11].

**EXAMPLE 2.1.** The example to keep in mind is  $\mathcal{H} = L^2(\mathbb{R}^n)$ ,  $H_0 = -\Delta + V(x)$ , with  $V(x)$  sufficiently regular and decaying rapidly as  $|x| \rightarrow \infty$ . Typical weights chosen in this case are:  $w_{\pm} = \langle x \rangle^{\pm\sigma}$ , where  $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$  and  $\sigma$  is sufficiently large and positive. In this case, the estimates (2.6)–(2.7) hold; see, for example, [10].

$\beta(t)$  is assumed to be a symmetric and bounded operator, such that

$$h_\beta(t) \equiv \|w_+\beta(t)w_+\|_{B(\mathcal{H})} \in L^1(\mathbb{R}; dt). \tag{2.4}$$

For simplicity we'll assume that  $\beta(t)$  is supported for  $t > 0$  and extend it as an even operator-valued function on  $\mathbb{R}$ . In the above example, one often has in applications a perturbation of the form  $\beta(t, x) \equiv \sum_j \beta_j(t)V_j(x)$ .

*Remark and Warning:* Introduce the Fourier transform of the operator  $\hat{\beta}$  defined for any  $v \in \mathcal{H}$ :

$$\beta(t)v = \int_{\mathbb{R}} e^{i\mu t} \hat{\beta}(\mu)v \, d\mu, \quad \hat{\beta}(\mu)v = (2\pi)^{-1} \int_{\mathbb{R}} e^{-i\mu t} \beta(t)v \, dt, \tag{2.5}$$

Using the spectral theorem, one can then define  $\hat{\beta}(A)$ , where  $A$  is a self-adjoint operator; we shall use this below with  $A = \lambda_0 - H_0$ . In applications, as in the example above, one often has  $\beta(t) = \beta(t, x)$  and therefore  $\hat{\beta}(A) = \hat{\beta}(A, x)$  may be, for example, a pseudo-differential operator. In our general setting, we shall simply write  $\hat{\beta}(A)$  with the understanding that this is *not* simply a function of the operator  $A$ . Thus, in general,  $\hat{\beta}(H_0)\psi_0 \neq \hat{\beta}(\lambda_0)\psi_0$ .

We also assume that  $H_0$  satisfies the following:

*Local energy decay:* For some  $r > 1$ ,

$$\|w_- e^{-iH_0 t} P_c(H_0)f\|_2 \leq c\langle t \rangle^{-r} \|w_+ f\|_2. \tag{2.6}$$

Theorem 2.1 and most of Theorem 2.2 rely mainly on the estimate (2.6) alone. The assertion in Theorem 2.2 concerning the rate of approach of  $P(t)$  to its asymptotic value requires an additional technical assumption. In particular, we shall use the following estimate which holds in the situation where  $f$  may not lie in the domain of  $w_+$ :

*Weak local energy decay:*

$$\int_{\mathbb{R}} \|w_- e^{-iH_0 t} P_c(H_0)f\|_2^2 \, dt \leq C\|f\|_2^2. \tag{2.7}$$

*Remark 2.1.* The estimate (2.7) is applied with  $f \equiv \hat{\beta}(\lambda_0 - H_0)\psi_0$ . Consider the setting of Example 2.1,  $H_0 = -\Delta + V(x)$ . Let  $\sigma = 2 + \theta$ ,  $\theta > 0$ ,  $|V(x)| \leq C\langle x \rangle^{-\sigma}$  and assume that zero is neither an eigenvalue nor a resonance of  $H_0$ . Then, (2.7) is a consequence of results in [1].

*Remark 2.2.* Improvements in decay, which are of an intermediate character between (2.7) and (2.6) can be obtained from appropriate assumptions on commutators of weights with  $\hat{\beta}(\lambda_0 - H_0)$ . For Example 2.1 by assuming that the

commutators:

$$\left[ \langle x \rangle, \dots \left[ \langle x \rangle, \hat{\beta}(\lambda_0 - H_0) \right] \right], \quad (j \leq k \text{ times}) \tag{2.8}$$

are bounded operators on  $\mathcal{H}$ , we can obtain improved decay:

$$\| \langle x \rangle^{-\sigma} e^{-iH_0 t} P_c(H_0) f \|_2 \leq c \langle t \rangle^{-\min\{k, \frac{\sigma}{2}\}} \| \langle x \rangle^k f \|_2. \tag{2.9}$$

The above boundedness assumption on commutators amounts to spatial localization of  $\beta(t, x)$  and smoothness of  $\hat{\beta}(\mu, x)$  in  $\mu$ . These are ensured by the assumption:

$$\int_{\mathbb{R}} \langle t \rangle^k \| \langle x \rangle^\sigma \beta(t, \cdot) \|_{L^2} dt < \infty. \tag{2.10}$$

*Remark 2.3.* For the choice of  $r$  appearing in estimates (2.6), the integral

$$I_r[h_\beta](t) \equiv \int_0^t \langle t - s \rangle^{-r} h_\beta(s) ds, \tag{2.11}$$

arises. Since  $r > 1$ , by Young's inequality

$$\| I_r[h_\beta] \|_{L^1} \leq c \| h_\beta \|_{L^1} \text{ and, furthermore, } \| I_r[h_\beta] \|_{L^\infty} \leq \| h_\beta \|_{L^1}.$$

Under the above conditions it is known that  $\phi(t)$  exists for all times  $t$ , for any given  $\phi(x, 0) \in \mathcal{D}(H_0)$ , the domain of  $H_0$ .

Furthermore, since  $\beta$  is symmetric, we have the conservation of energy

$$\| \phi(t) \| = \| \phi(0) \|. \tag{2.12}$$

We will prove the following theorem:

**THEOREM 2.1.** *Consider the initial value problem for (2.2) with data  $\phi(0) \in D(\mathcal{H})$ . The large time behavior of  $\phi(t)$  is a linear combination of free waves and the bound state  $\psi_0$  (orthogonal to  $\phi(t)$  for all  $t$ ). In particular, there exists a complex number  $A(\infty)$  and a vector  $\phi_\pm \in \mathcal{H}$  such that*

$$\phi(t) - \left[ e^{-i\lambda_0 t} A(\infty) \psi_0 + e^{-iH_0 t} P_c(H_0) \phi_+ \right] \tag{2.13}$$

tends to zero in  $\mathcal{H}$  as  $t \rightarrow \pm\infty$ .

The following result gives a more detailed picture of the asymptotic behavior and provides a useful expression for the small  $\beta$  (in  $\| h_\beta \|_{L^1}$ ) behavior.

**THEOREM 2.2.** *For  $t \geq 0$ , and  $\|h_\beta\|_{L^1} \leq 1$  we have*

$$\begin{aligned} \phi(t) &= e^{-i(\lambda_0 t + \int_0^\infty \langle \psi_0, \beta(s) \psi_0 \rangle ds + \Lambda)} e^{-\Gamma} \times \\ &\quad \times \left( a_\infty + \mathcal{O}\left(\int_t^\infty h_\beta(s) ds\right) \right) \psi_0 + \\ &\quad + \phi_d(t), \end{aligned} \tag{2.14}$$

where

$$a_\infty = a(0) + \mathcal{O}(\|\phi_d(0)\| \|h_\beta\|_{L^1} + \|\phi(0)\| \|h_\beta\|_{L^1}^2) \tag{2.15}$$

and  $\phi_d(t)$  is dispersive and satisfies the decay estimate

$$\|w_- \phi_d(t)\| \leq c_1 \|w_+ \phi_d(0)\| \langle t \rangle^{-r} + c_2 \|\phi(0)\| I_r[h_\beta](t). \tag{2.16}$$

$\Gamma$  and  $\Lambda$  are given by

$$\begin{aligned} \Gamma &= \pi^2 \int_{\mathbb{R}} d\mu \langle P_c \hat{\beta}(\mu) \psi_0, \delta(H_0 - (\lambda_0 - \mu)) P_c \hat{\beta}(\mu) \psi_0 \rangle \\ &= \|P_c \hat{\beta}(\lambda_0 - H_0) \psi_0\|^2, \end{aligned} \tag{2.17}$$

$$\Lambda = \pi \text{ P.V. } \int_{\mathbb{R}} d\mu \langle P_c \hat{\beta}(\mu) \psi_0, (H_0 - (\lambda_0 - \mu))^{-1} P_c \hat{\beta}(\mu) \psi_0 \rangle. \tag{2.18}$$

Finally, for  $P(t)$  (see the Introduction) we have that there exists  $p_\infty \geq 0$  such that

$$P(t) = e^{-2\Gamma} p_\infty + \mathcal{O}\left(\int_t^\infty h_\beta(s) \alpha_\beta(s) ds\right), \tag{2.19}$$

where

$$p_\infty = 1 + \mathcal{O}(\|h_\beta\|_{L^1}^2) \text{ and } \alpha_\beta(t) \in L^2(\mathbb{R}; dt). \tag{2.20}$$

### 3. Decomposition and Proof of Theorem 2.1

Our first aim is to derive a system of equations, equivalent to (2.1), for the coupled evolution of the bound state and dispersive channels [10–12]. A natural basis to use is that of the unperturbed dynamics.

Let

$$\phi(t) \equiv a(t) \psi_0 + \phi_d(t), \quad \langle \psi_0, \phi_d(t) \rangle = 0 \tag{3.1}$$

with  $H_0 \psi_0 = \lambda_0 \psi_0$ .

Then, from (2.1) we get

$$i\partial_t a(t) \psi_0 + i\partial_t \phi_d = H_0 \phi_d + \lambda_0 a(t) \psi_0 + \beta a(t) \psi_0 + \beta \phi_d. \tag{3.2}$$

Taking the scalar product with  $\psi_0$ , we get

$$i\partial_t a(t) = \lambda_0 a(t) + \langle \psi_0, \beta \psi_0 \rangle a(t) + \langle \psi_0, \beta \phi_d \rangle. \quad (3.3)$$

Letting

$$a(t) \equiv e^{-i\lambda_0 t} A(t), \quad (3.4)$$

we have

$$i\partial_t A(t) = \langle \psi_0, \beta \psi_0 \rangle A(t) + e^{i\lambda_0 t} \langle \psi_0, \beta \phi_d \rangle \quad (3.5)$$

and, by applying  $P_c$  to (3.3), we get

$$i\partial_t \phi_d = H_0 \phi_d + P_c \beta \phi_d + e^{-i\lambda_0 t} A(t) P_c \beta \psi_0 \quad (3.6)$$

and  $P_c \phi_d = \phi_d$ .

Our aim is to prove that  $A(\infty) \equiv \lim_{t \rightarrow \infty} A(t)$  exists.

Solving Equation (3.6), we get

$$\begin{aligned} \phi_d(t) &= e^{-iH_0 t} \phi_d(0) - i \int_0^t e^{-iH_0(t-s)} P_c \beta(s) \phi_d(s) \, ds - \\ &\quad - i \int_0^t e^{-iH_0(t-s)} e^{-i\lambda_0 s} A(s) P_c \beta(s) \psi_0 \, ds \\ &\equiv \phi_0 + \phi_1 + \phi_2. \end{aligned} \quad (3.7)$$

We estimate  $\|w_- \phi_d(t)\|$  by considering each term individually. First,  $\phi_0(t) = e^{-iH_0 t} \phi_d(0)$  satisfies local decay since  $\phi_d(0) = P_c \phi_d(0) = P_c \phi_0$ . Thus,

$$\|w_- \phi_0(t)\| \leq c \|w_+ \phi_d(0)\| \langle t \rangle^{-r}. \quad (3.8)$$

Estimating  $\phi_1$ , we have

$$\begin{aligned} \|w_- \phi_1(t)\| &\leq \|w_- \int_0^t e^{-iH_0(t-s)} P_c \beta(s) \phi_d(s) \, ds\|_{L^2} \\ &\leq c \int_0^t \langle t-s \rangle^{-r} \|w_+ \beta(s) w_+ \cdot w_- \phi_d(s)\|_{B(\gamma t)} \, ds \\ &\leq c \|\phi_d(0)\| I_r[h_\beta](t). \end{aligned} \quad (3.9)$$

We have also used that  $\|\langle x \rangle^{-\sigma} \phi_d(s)\|_{L^2} \leq \|\phi_d(0)\|$  is bounded as a consequence of  $L^2$  conservation and orthogonality:

$$\|\phi(t)\|^2 = |A(t)|^2 + \|\phi_d(t)\|^2 = \|\phi(0)\|^2.$$

Finally, we estimate the local decay norm of  $\phi_2$  in a similar way:

$$\begin{aligned} \|w_- \phi_2(t)\| &\leq c \int_0^t \langle t-s \rangle^{-r} |A(s)| \|w_+ \beta(s) \psi_0\| \, ds \\ &\leq \left( c \int_0^t \langle t-s \rangle^{-r} h_\beta(s) \, ds \right) \times \left( \sup_{a < s \leq t} |A(s)| \right) \end{aligned} \tag{3.10}$$

$$\leq c \|\phi(0)\| \, I_r[h_\beta](t). \tag{3.11}$$

Therefore, we have local energy decay of  $\phi_d$ :

$$\|w_- \phi_d(t)\| \leq c_1 \|w_+ \phi_d(0)\| \langle t \rangle^{-r} + c_2 \|\phi(0)\| \, I_r[h_\beta](t). \tag{3.12}$$

To study the detailed asymptotic behavior of  $\phi_d(t)$ , we rewrite (3.7) as

$$\begin{aligned} \phi_d(t) &- e^{-iH_0 t} \phi_+ \\ &= e^{-iH_0 t} \left( i \int_t^\infty e^{iH_0 s} P_c [\beta(s) \phi_d(s) + e^{-i\lambda_0 s} A(s) \beta(s) \psi_0] \, ds \right), \end{aligned}$$

where

$$\begin{aligned} \phi_+ &\equiv \phi_d(0) - i \int_0^\infty e^{iH_0 s} P_c \beta(s) \phi_d(s) \, ds - \\ &- i \int_0^\infty e^{iH_0 s} e^{-i\lambda_0 s} A(s) P_c \beta(s) \psi_0 \, ds. \end{aligned} \tag{3.13}$$

It follows that

$$\|\phi_d(t) - e^{-iH_0 t} \phi_+\| = \mathcal{O} \left( \int_t^\infty h_\beta(s) \, ds \right). \tag{3.14}$$

If one additionally has asymptotic completeness for  $H_0$  [8], then there exists  $\bar{\phi}_+ \in \mathcal{H}$  such that

$$\|e^{-iH_0 t} \phi_+ - e^{i\Delta t} \bar{\phi}_+\| \rightarrow 0. \tag{3.15}$$

To see that  $A(\infty) \equiv \lim_{t \uparrow \infty} A(t)$  exists, we solve (3.5) for  $A(t)$  to get

$$\begin{aligned} A(t) &= e^{-i \int_0^t \langle \psi_0, \beta(s) \psi_0 \rangle \, ds} A(0) + \\ &+ e^{-i \int_0^t \langle \psi_0, \beta(s) \psi_0 \rangle \, ds} \left[ \int_0^t e^{i \int_0^s \langle \psi_0, \beta(u) \psi_0 \rangle \, du} J(s) \, ds \right], \end{aligned} \tag{3.16}$$

where  $J(s) = e^{i\lambda_0 s} \langle \psi_0, \beta(s) \phi_d(s) \rangle$ .

Since  $\langle \psi_0, \beta(s) \psi_0 \rangle$  is integrable in  $s$ , and  $\phi_d(s)$  is bounded (even decaying), the existence of the limit  $A(\infty)$  follows.



#### 4. Proof of Theorem 2 – Ionization Probability

Our aim now is to find  $A(\infty)$  to second order in  $\beta$ , for  $\beta$  small in the appropriate sense. However, the expressions we get can be used as the starting point to finding the ionization probability for the general case ( $\beta$  large).

From Equations (3.5) and (3.7) we have

$$\partial_t A(t) = -i\langle \psi_0, \beta(t)\psi_0 \rangle A(t) - i e^{i\lambda_0 t} \sum_{j=0}^2 \langle \psi_0, \beta(t)\phi_j(t) \rangle. \quad (4.1)$$

We now prove the following proposition:

**PROPOSITION 4.1.**

$$\partial_t A(t) = -g(t)A(t) + R(t), \quad (4.2)$$

where

$$\begin{aligned} g(t) = & i\langle \psi_0, \beta(t)\psi_0 \rangle + \pi \langle e^{-i(\lambda_0 - H_0)t} P_c \beta(t)\psi_0, P_c \hat{\beta}(\lambda_0 - H_0)\psi_0 \rangle - \\ & - i \text{PV} \int d\mu \langle e^{-i\mu t} \beta(t)\psi_0, (H_0 - (\lambda_0 - \mu))^{-1} P_c \hat{\beta}(\mu)\psi_0 \rangle \end{aligned} \quad (4.3)$$

and

$$\begin{aligned} R(t) = & -i e^{i\lambda_0 t} \sum_{j=0}^1 \langle \psi_0, \beta(t)\phi_j(t) \rangle - \\ & - i e^{i\lambda_0 t} A(0) \langle \beta(t)\psi_0, e^{-iH_0 t} \mathcal{T}(0)\psi_0 \rangle - \\ & - i e^{i\lambda_0 t} \left\langle \beta(t)\psi_0, e^{-iH_0 t} \int_0^t e^{i(H_0 - \lambda_0)s} \mathcal{T}(s) \dot{A}(s) ds \psi_0 \right\rangle. \end{aligned} \quad (4.4)$$

*Proof.* The proposition is proved by expanding the  $j = 2$  term in the sum in (4.1).

$$\begin{aligned} \langle \psi_0, \beta(t)\phi_2(t) \rangle &= \left\langle \psi_0, \beta(t) (-i) \int_0^t e^{-iH_0(t-s)} e^{-i\lambda_0 s} P_c \beta(s)\psi_0 ds \right\rangle \\ &= -i \left\langle \beta(t)\psi_0, e^{-iH_0 t} \int_0^t ds e^{iH_0 s} e^{-i\lambda_0 s} A(s) P_c \int d\mu e^{i\mu s} \hat{\beta}(\mu)\psi_0 \right\rangle \\ &= -i \left\langle \beta(t)\psi_0, e^{-iH_0 t} \int_0^t ds \int d\mu e^{i(H_0 - (\lambda_0 - \mu))s} A(s) P_c \hat{\beta}(\mu)\psi_0 \right\rangle. \end{aligned}$$

Now it could be that for some  $\mu$  in the support of  $\hat{\beta}$  that  $\lambda_0 - \mu$  is in the continuous spectrum of  $H_0$ . For such  $\mu$ , there is a resonance and we now compute its effect. First,

we introduce the operator

$$\mathcal{T}(t) = \lim_{\varepsilon \downarrow 0} \int e^{i\mu t} (H_0 - (\lambda_0 - \mu) - i\varepsilon)^{-1} P_c \hat{\beta}(\mu) d\mu \quad (4.5)$$

The resonance effects are calculated using integration by parts:

$$\begin{aligned} & -ie^{i\lambda_0 t} \langle \psi_0, \beta(t)\phi_2(t) \rangle \\ &= -ie^{i\lambda_0 t} \lim_{\varepsilon \downarrow 0} -i \left\langle \beta(t)\psi_0, e^{-iH_0 t} \int_0^t ds \int d\mu e^{i(H_0 - (\lambda_0 - \mu) - i\varepsilon)s} A(s) P_c \hat{\beta}(\mu) \psi_0 \right\rangle \\ &= ie^{i\lambda_0 t} \lim_{\varepsilon \downarrow 0} \left\langle \beta(t)\psi_0, e^{-iH_0 t} \int_0^t ds \int d\mu \frac{d}{ds} e^{i(H_0 - (\lambda_0 - \mu) - i\varepsilon)s} \times \right. \\ &\quad \left. \times (H_0 - (\lambda_0 - \mu) - i\varepsilon)^{-1} A(s) P_c \hat{\beta}(\mu) \psi_0 \right\rangle \\ &= ie^{i\lambda_0 t} \left\langle \beta(t)\psi_0, e^{-iH_0 t} \int d\mu e^{i(H_0 - (\lambda_0 - \mu))t} \times \right. \\ &\quad \left. \times (H_0 - (\lambda_0 - \mu) - i0)^{-1} A(t) P_c \hat{\beta}(\mu) \psi_0 \right\rangle \\ &\quad - ie^{i\lambda_0 t} \left\langle \beta(t)\psi_0, e^{-iH_0 t} A(0) \int d\mu (H_0 - (\lambda_0 - \mu) - i0)^{-1} P_c \hat{\beta}(\mu) \psi_0 \right\rangle \\ &\quad - ie^{i\lambda_0 t} \left\langle \beta(t)\psi_0, e^{-iH_0 t} \int d\mu \int_0^t ds e^{i(H_0 - (\lambda_0 - \mu))s} \times \right. \\ &\quad \left. \times (H_0 - (\lambda_0 - \mu) - i0)^{-1} \dot{A}(s) P_c \hat{\beta}(\mu) \psi_0 \right\rangle \\ &= i \langle \beta(t)\psi_0, \mathcal{T}(t)\psi_0 \rangle A(t) \\ &\quad - ie^{i\lambda_0 t} A(0) \langle \beta(t)\psi_0, e^{-iH_0 t} \mathcal{T}(0)\psi_0 \rangle \\ &\quad - ie^{i\lambda_0 t} \left\langle \beta(t)\psi_0, e^{-iH_0 t} \int_0^t e^{i(H_0 - \lambda_0)s} \mathcal{T}(s) \dot{A}(s) ds \psi_0 \right\rangle \\ &\equiv S_1 + S_2 + S_3. \end{aligned} \quad (4.6)$$

*Remark 4.1.* Operators of the type appearing in the integrand of  $\mathcal{T}(t)$  arise in [6, 10, 11], where the time-frequency content of the perturbation is discrete. In this Letter, singular local energy decay estimates were required. The main difference here is that the perturbation  $\beta$  contains a continuum of time frequencies and therefore it is possible that for some  $\mu$  in the support of  $\hat{\beta}$ , we have that  $-\lambda_0 + \mu$  is at a threshold energy of  $H_0$  (e.g. endpoint of the continuous spectrum). Typically, for such energies weaker local energy decay estimates hold and in the above cited works on quantum resonances, and Hamiltonian systems parametrically forced by  $t$ -periodic and almost periodic potentials, the discreteness of the frequency spectrum essentially required that resonances with the continuous spectrum be bounded away from thresholds. Here, however, the  $\mu$  integral has a smoothing effect and we require no such hypothesis; see Proposition 4.2 below.

The explicit evaluation of  $S_1$  and estimation of  $S_2 - S_3$  will proceed using the identity

$$(H_0 - (\lambda_0 - \mu) - i0)^{-1} = \text{PV}(H_0 - (\lambda_0 - \mu))^{-1} + i\pi\delta(H_0 - (\lambda_0 - \mu)). \quad (4.7)$$

and the the following proposition:

**PROPOSITION 4.2.** *Assume  $w_+\chi \in \mathcal{H}$ . Then,*

$$\begin{aligned} \mathcal{T}(t)\chi &= i\pi e^{i(\lambda_0 - H_0)t} P_c \hat{\beta}(\lambda_0 - H_0)\chi - \\ &\quad - i \int_{\mathbb{R}} e^{i(\lambda_0 - H_0)(t-\tau)} P_c \text{sgn}(t - \tau)\beta(\tau) \, d\tau \, \chi. \end{aligned} \quad (4.8)$$

*Proof.* By (4.7)

$$\begin{aligned} \mathcal{T}(t)\chi &= i\pi \int_{\mathbb{R}} e^{i\mu t} \delta(\mu - (\lambda_0 - H_0)) \hat{\beta}(\mu)\psi_0 \, d\mu + \\ &\quad + \text{PV} \int_{\mathbb{R}} e^{i\mu t} \hat{\beta}(\mu)(\mu - (\lambda_0 - H_0))^{-1} \, d\mu \\ &= i\pi e^{i\lambda_0 t} e^{-iH_0 t} \hat{\beta}(\lambda_0 - H_0)\psi_0 + \text{HT}[\hat{\beta}(\cdot + t)](\lambda_0 - H_0), \end{aligned}$$

where  $\text{HT}[f](x) \equiv \text{PV} \int f(y)/(x - y) \, dy$  is the Hilbert transform.

Evaluation of the latter term using the Plancherel identity and the fact that  $H[\hat{f}][\xi] = i \text{sgn}(\xi)$  yields (4.8).

We use properties of the Fourier transform, (2.5), and the identity (4.7).

We now apply this proposition to the evaluation and estimation of of the terms (4.6). Consider first  $S_1$ . We have

$$\begin{aligned} S_1 &= -\pi \langle e^{-i(\lambda_0 - H_0)t} P_c \beta(t)\psi_0, P_c \hat{\beta}(\lambda_0 - H_0)\psi_0 \rangle A(t) + \\ &\quad + \left\langle \beta(t)\psi_0, \text{PV} \int_{\mathbb{R}} d\mu e^{i\mu t} (H_0 - (\lambda_0 - \mu))^{-1} P_c \hat{\beta}(\mu)\psi_0 \right\rangle A(t). \end{aligned} \quad (4.9)$$

The two terms in  $S_1$  are incorporated in the definition of  $g(t)$ , displayed in (4.3), while the terms  $S_2$  and  $S_3$  are included in  $R(t)$ . This proves Proposition 4.1.

We next estimate  $R(t)$ . Before stating the estimate we employ Proposition 4.2 to obtain the following useful expressions for  $S_2$  and  $S_3$  which we then estimate. Beginning with  $S_2$ , we have

$$\begin{aligned} S_2 &= -ie^{i\lambda_0 t} A(0) \langle \beta(t)\psi_0, e^{-iH_0 t} \mathcal{T}(0)\psi_0 \rangle \\ &= \pi A(0) \langle \beta(t)\psi_0, e^{i(\lambda_0 - H_0)t} P_c \hat{\beta}(\lambda_0 - H_0)\psi_0 \rangle + \\ &\quad + A(0) \left\langle \beta(t)\psi_0, \int_{\mathbb{R}} e^{i(\lambda_0 - H_0)(t-\tau)} P_c \text{sgn}(\tau)\beta(\tau) \, d\tau \, \psi_0 \right\rangle. \end{aligned}$$

Therefore,

$$|S_2| \leq c|A(0)|h_\beta(t) \left( \|w_- e^{-iH_0 t} P_c \hat{\beta}(\lambda_0 - H_0) \psi_0\| + \int_{\mathbb{R}} \langle t - \tau \rangle^{-r} h_\beta(\tau) d\tau \right). \quad (4.10)$$

For  $S_3$  we have

$$\begin{aligned} S_3 &= \pi e^{i\lambda_0 t} (A(t) - A(0)) \langle \beta(t) \psi_0, e^{-iH_0 t} P_c \hat{\beta}(\lambda_0 - H_0) \psi_0 \rangle - \\ &\quad - e^{i\lambda_0 t} \left\langle \beta(t) \psi_0, e^{-iH_0 t} \int_0^t \dot{A}(s) ds \times \right. \\ &\quad \left. \times \int_{\mathbb{R}} e^{-i\lambda_0(s-\tau)} e^{iH_0(s-\tau-t)} P_c \operatorname{sgn}(s-\tau) \beta(\tau) d\tau ds \psi_0 \right\rangle. \end{aligned} \quad (4.11)$$

Estimation of  $S_3$  gives

$$\begin{aligned} |S_3| &\leq c \|\phi(0)\| h_\beta(t) \\ &\quad \times \left( \|w_- e^{-iH_0 t} P_c \hat{\beta}(\lambda_0 - H_0) \psi_0\| + h_\beta(t) \int_{\mathbb{R}} I_r[h_\beta](\omega) h_\beta(\omega - t) d\omega \right). \end{aligned} \quad (4.12)$$

By weak local energy decay (2.7)

$$\alpha_\beta(t) \equiv \|w_- e^{-iH_0 t} P_c \hat{\beta}(\lambda_0 - H_0) \psi_0\| \in L^2(\mathbb{R}; dt). \quad (4.13)$$

Note also that  $\alpha_\beta$  also satisfies the following elementary uniform bound:

$$\begin{aligned} \alpha_\beta(t) &\leq \|\hat{\beta}(\lambda_0 - H_0) \psi_0\| \\ &\leq \left\| \int e^{-i(\lambda_0 - H_0)t} \beta(t) \psi_0 dt \right\| \leq \int \|\beta(t) \psi_0\| dt \\ &\leq c \|h_\beta\|_{L^1}. \end{aligned}$$

These considerations imply the following proposition:

**PROPOSITION 4.3.**

$$\begin{aligned} |R(t)| &\leq C_1 \|\phi_d(0)\| \langle t \rangle^{-r} I_r[h_\beta](t) + C_2 \|\phi(0)\| h_\beta(t) I_r[h_\beta](t) + \\ &\quad + C_3 \|\phi(0)\| h_\beta(t) \times \\ &\quad \times \left( \min\{\alpha_\beta(t), \|h_\beta\|_{L^1}\} + \int_{\mathbb{R}} \langle t - \tau \rangle^{-r} h_\beta(\tau) d\tau + \right. \\ &\quad \left. + \int_{\mathbb{R}} I_r[h_\beta](\omega) h_\beta(\omega - t) d\omega \right). \end{aligned} \quad (4.14)$$

Using the above bound for  $R(t)$ , we now analyze the equation for  $A(t)$ . From (59) we have

$$A(t) = e^{-\int_0^t g(s) ds} \left( A(0) + \int_0^t e^{\int_0^\xi g(\tau) d\tau} R(\xi) d\xi \right), \quad (4.15)$$

where  $g(t)$  is given by (4.3). We now seek a simple relation between the initial bound state amplitude,  $A(0)$ , and the asymptotic bound state amplitude,  $A(\infty)$  by re-expressing  $A(t)$  in (4.15) as its asymptotic value plus a decaying perturbation.

$$A(t) = A(\infty) + \rho(t),$$

where

$$A(\infty) = e^{-\int_0^\infty g(s) ds} \left( A(0) + \int_0^\infty e^{\int_0^s g(s_1) ds_1} R(s) ds \right), \quad (4.16)$$

and  $\rho(t)$  decays as  $t \rightarrow \infty$  and is given explicitly by

$$\begin{aligned} \rho(t) = & e^{-\int_0^\infty g(s) ds} \left( e^{\int_t^\infty g(s) ds} - 1 \right) \left( A(0) + \int_0^\infty e^{\int_0^s g(s_1) ds_1} R(s) ds \right) - \\ & - e^{-\int_0^\infty g(s) ds} \int_t^\infty e^{\int_0^s g(s_1) ds_1} R(s) ds. \end{aligned}$$

From (4.16), we see that it is necessary to evaluate the integral of  $g$  to obtain  $A(\infty)$ , and the ionization probability,  $1 - P(\infty)$ . We do this using the expression for  $g$  in Proposition 4.1. The result is given in

**PROPOSITION 4.4.**

$$\int_0^\infty g(s) ds = \frac{i}{2} \int_{\mathbb{R}} dt \langle \psi_0, \beta(t) \psi_0 \rangle + \Gamma - i\Lambda, \quad (4.17)$$

where

$$\begin{aligned} \Gamma &= \pi^2 \int_{\mathbb{R}} d\mu \langle P_c \hat{\beta}(\mu) \psi_0, \delta(H_0 - (\lambda_0 - \mu)) P_c \hat{\beta}(\mu) \psi_0 \rangle \\ &= \pi^2 \| P_c \hat{\beta}(\lambda_0 - H_0) \psi_0 \|^2, \end{aligned} \quad (4.18)$$

$$\Lambda = \pi \text{PV} \int_{\mathbb{R}} d\mu \langle P_c \hat{\beta}(\mu) \psi_0, (H_0 - (\lambda_0 - \mu))^{-1} P_c \hat{\beta}(\mu) \psi_0 \rangle. \quad (4.19)$$

Therefore,

$$\begin{aligned} A(\infty) &= e^{-\Gamma} e^{-i(\frac{1}{2} \int_{\mathbb{R}} \langle \psi_0, \beta(s) \psi_0 \rangle ds + \Lambda)} \times \\ &\quad \times [A(0) + \mathcal{O}(\|\phi(0)\| \|h_\beta\|_{L^1}^2)], \\ |A(\infty)| &= e^{-\Gamma} [|A(0)| + \mathcal{O}(\|\phi(0)\| \|h_\beta\|_{L^1}^2)]. \end{aligned} \quad (4.20)$$

Finally, the assertions concerning the rate of convergence of  $\langle \phi(t), \psi_0 \rangle$  to an asymptotic bound state and of  $P(t)$  to its asymptotic value, are obtained by estimating the above expressions for  $A(t) - A(\infty)$  and of  $|A(t)| - |A(\infty)|$ . These estimates follow from (4.14) and an estimation of  $\exp(\int_t^\infty g(s) ds) - 1$  and  $\exp(\int_t^\infty \Re g(s) ds) - 1$  using local energy and weak local energy decay estimates (2.6), (2.7).

This concludes the proof of Theorem 2.2.

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### References

1. Ben-Artzi, M. and Klainerman, S.: Decay and regularity for the Schrödinger equation, *J. Anal. Math.* **58** (1992), 25–37.
2. Cohen-Tannoudji, C., Dupont-Roc, J. and Grynberg, G.: *Atom-Photon Interactions: Basic Processes and Applications*, Wiley, New York, 1992.
3. Costin, O., Lebowitz, J. L. and Rokhlenko, A.: Exact results for the ionization of a model atom, Rutgers University preprint, 1999.
4. Galindo, A. and Pascual, P.: *Quantum Mechanics II*, Springer-Verlag, New York, 1991.
5. Howland, J. S.: Stationary scattering theory for time dependent Hamiltonians, *Math. Ann.* **207** (1974), 315–335.
6. Kirr, E. and Weinstein, M. I.: Parametrically excited Hamiltonian partial differential equations, In preparation.
7. Marcuse, D.: *Theory of Dielectric Optical Waveguides*, *Quantum Electronics – Theory and Applications*, Academic Press, New York, 1974.
8. Reed, M. and Simon, B.: *Methods of Modern Mathematical Physics, Volume 3: Scattering Theory*, Academic Press, New York, 1978.
9. Rokhlenko, A. and Lebowitz, J. L.: Ionization of a model atom by perturbations of the potential, Rutgers University preprint, 1998.
10. Soffer, A. and Weinstein, M. I.: Time dependent resonance theory, *Geom. Funct. Anal.* **8** (1998), 1086–1128.
11. Soffer, A. and Weinstein, M. I.: Nonautonomous Hamiltonians, *J. Statist. Phys.* **93** (1998), 359–391.
12. Soffer, A. and Weinstein, M. I.: Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations, *Invent. Math.* **136** (1999), 9–74.