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e to the A, in a New Way, Some More to Say

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Abstract. Expressions are given for the exponential of a Hermitian matrix, A. Replacing A by iA these are explicit formulas for the Fourier transform of e^{iA}. They extend to any size A the previous results for the 2×2, 3×3, and 4×4 cases. The expressions are elegant and should prove useful.

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The support of the Fourier transform of e^{iA} was established by E. Nelson in [1]. (That is the Fourier transform of each entry of the matrix e^{iA} in terms of the entries of the Hermitian matrix A.) But I believe this result of E. Nelson is very little known in the mathematical community at large. In further work [2–4] the transform was exhibited in the 2×2 case, and presented in some unwieldy forms in higher dimensions. In a previous paper, [5], explicit formulas were obtained for $2 \times 2, 3 \times 3$ and 4×4 matrices. We here treat the general case.

Let A be an $r \times r$ Hermitian matrix. We write

$$\mathrm{Det}(1 - A) = \sum_{j=0}^{r} P_j(A), \tag{1}$$

where $P_j(A)$ is homogeneous of degree j in the entries of A. The formulas we obtain for the exponential of A are as follows:

$$(e^A)_{\alpha\beta} = \frac{1}{\Gamma(r)} \sum_{j=0}^{r} P_j(A) \frac{d^{r-j}}{ds^{r-j}} \left(\int d\Omega e^{s \mathrm{Tr}(AW)} W_{\alpha\beta} s^r \right) \Bigg|_{s=1} \tag{2}$$

or

$$(e^A)_{\alpha\beta} = \frac{1}{\Gamma(r)} \sum_{j=0}^{r} P_j(A) \frac{d^{r-j}}{ds^{r-j}} \left(\int d\Omega e^{s\langle A\mathbf{n}, \bar{\mathbf{n}}\rangle} n_\alpha \bar{n}_\beta s^r \right) \Bigg|_{s=1}. \tag{3}$$

Here $\mathbf{n}$ is a unit vector in $\mathbb{C}^r$, and $W_{ij} = n_i \bar{n}_j$, a rank one Hermitian matrix. $\int d\Omega$ denotes a normalized integral over all such $\mathbf{n}$, an integral over the unit sphere in

$\mathbb{C}$ with unitary-invariant measure. That the support of the Fourier transform lies on the complex projective space of such W is the content of Nelson's theorem.

In fact the formulas in (2) and (3) do not coincide with formulas in [5] when $r = 2$, 3 or 4, but formulas *of such type* are not unique. We do not know the full scope of such nonuniqueness.

We first sketch a derivation/proof of formulas (2) and (3), especially emphasizing the *ideas*. We note the relation between Gaussian integrals in $d = 2r$ real dimensions, and integrals over the corresponding unit sphere S^{d-1}.

$$\frac{1}{\mathcal{N}}\int \mathrm{d}x_i \mathrm{e}^{-\Sigma x_i^2}\prod^{2N} x_{\alpha(i)} = \frac{1}{\mathcal{N}}\int r^{d-1}r^{2N}\mathrm{e}^{-r^2}\mathrm{d}r\int \mathrm{d}\Omega'\prod^{2N} n_{\alpha(i)} \tag{4}$$

$$= \int_{S^{d-1}} \mathrm{d}\Omega \prod^{2N} n_{\alpha(i)} \cdot \frac{\int \mathrm{d}r \mathrm{e}^{-r^2} r^{2N+d-1}}{\int \mathrm{d}r \mathrm{e}^{-r^2} r^{d-1}} \tag{5}$$

$$= \int_{S^{d-1}} \mathrm{d}\Omega \prod^{2N} n_{\alpha(i)} \frac{\Gamma\left(\frac{2N+d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}. \tag{6}$$

Here n_i is the unit vector parallel to x_i, and $\int \mathrm{d}\Omega'$ is integral over the sphere in its usual measure and $\int \mathrm{d}\Omega$ the normalized spherical measure. From (6) we see *the integral over a unit sphere of a homogeneous polynomial of degree* $2N$ *is 'approximately'* $1/N!$ *the gaussian integral of the same polynomial.*

We note that multiplying the term in A^k by $(1/k!)$ induces a transform (formally) as follows

$$1 + A + A^2 + \cdots = \frac{1}{1-A} \longrightarrow \mathrm{e}^A\ . \tag{7}$$

We consider the gaussian integral formula (for $|A| < 1$):

$$\frac{\mathrm{Det}(1-A)}{\mathcal{N}}\int \mathrm{d}x_i \mathrm{e}^{-\Sigma|x_i^2| + \langle A\mathbf{x},\bar{\mathbf{x}}\rangle} x_\alpha \bar{x}_\beta = \left(\frac{1}{1-A}\right)_{\alpha\beta}. \tag{8}$$

In the expansion of the integrand on the left side of (8) each power of A has associated to it two powers of x. Thus converting from a gaussian integral to an integral over a unit sphere approximately multiplies each power of A^N by $1/N!$, which would convert $1/(1-A)$ to e^A. The *wrong* formula we get putting these ideas together would yield

$$\mathrm{Det}(1-A)\int \mathrm{d}\Omega \mathrm{e}^{\langle A\mathbf{n},\bar{\mathbf{n}}\rangle} n_\alpha \bar{n}_\beta = (\mathrm{e}^A)_{\alpha\beta}. \tag{9}$$

We turn to the easy task of converting the above careless argument leading to the wrong formula (9), to the detailed correct computation that turns (9) into (2). (Hitherto we have trodden a path redolent with the creative epiphanies of mathematical research, we now segue to the ineluctable concomitant consecration to inferential syntax.) We start from the right side of Equation (3), and assume for the moment $|A| < 1$.

$$\frac{1}{\Gamma(r)}\sum_{j=0}^{r} P_j(A)\frac{\mathrm{d}^{r-j}}{\mathrm{d}s^{r-j}}\left(\int \mathrm{d}\Omega \mathrm{e}^{s\langle A\mathbf{n},\bar{\mathbf{n}}\rangle} n_\alpha \bar{n}_\beta s^r\right)\Bigg|_{s=1} \tag{10}$$

We expand the exponent and perform the operations on s, getting

$$\sum_{j=0}^{r} P_j(A)\int \mathrm{d}\Omega \sum_{k=0}^{\infty}\Big(\langle A\mathbf{n},\bar{\mathbf{n}}\rangle\Big)^k n_\alpha \bar{n}_\beta \cdot \frac{1}{\Gamma(r)}\cdot\frac{1}{k!}\cdot\frac{(r+k)!}{(k+j)!} \tag{11}$$

Now we use the equality of Equations (4), (5), (6) to convert (11) to

$$\sum_{j=0}^{r} P_j(A)\frac{1}{\mathcal{N}}\int \mathrm{d}x_i \mathrm{e}^{-\Sigma x_i^2}\sum_{k=0}^{\infty}\Big(\langle A\mathbf{x},\bar{\mathbf{x}}\rangle\Big)^k x_\alpha \bar{x}_\beta \frac{1}{\Gamma(r)}\frac{1}{k!}\frac{(r+k)!}{(k+j)!}\frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{2(k+1)+d}{2})}. \tag{12}$$

We rewrite the equality of Equation (8) in expanded form

$$\sum_{j=0}^{r} P_j(A)\frac{1}{\mathcal{N}}\int \mathrm{d}x_i \mathrm{e}^{-\Sigma x_i^2}\sum_{k=0}^{\infty}\Big(\langle A\mathbf{x},\bar{\mathbf{x}}\rangle\Big)^k x_\alpha \bar{x}_\beta \frac{1}{k!} = 1 + A + A^2 + \cdots \tag{13}$$

We mark the fact that Equation (13) is a separate equality for each homogeneous degree in powers of A. On each side of the equation we multiply terms homogeneous of degree ℓ by $1/\ell!$, arriving at

$$\begin{aligned}&\sum_{j=0}^{r} P_j(A)\frac{1}{\mathcal{N}}\int \mathrm{d}x_i \mathrm{e}^{-\Sigma x_i^2}\sum_{k=0}^{\infty}\Big(\langle A\mathbf{x},\bar{\mathbf{x}}\rangle\Big)^k x_\alpha \bar{x}_\beta \frac{1}{k!}\frac{1}{(j+k)!}\\ &\quad = 1 + A + \frac{A^2}{2!} + \cdots = \mathrm{e}^A.\end{aligned} \tag{14}$$

The equality of the left side of (14) with (12) follows from

$$\frac{1}{\Gamma(r)}\frac{1}{k!}\cdot\frac{(r+k)!}{(k+j)!}\cdot\frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{2(k+1)+d}{2})} = \frac{1}{k!}\frac{1}{(j+k)!} \tag{15}$$

using $r = 2d$.

We have thus established our equalities of Equations (2) and (3) for $|A| < 1$. But each side of these equations is analytic in the elements of A, so the equalities hold for all hermitian A.

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