A NOTE ON CLUSTER EXPANSIONS, TREE GRAPH IDENTITIES, EXTRA $1/N!$ FACTORS!!!*

G.A. BATTLE and P. FEDERBUSH

Department of Mathematics, Texas A & M University, College Station, TE 77843, U.S.A.

and

Department of Mathematics, The University of Michigan, Ann Arbor, MI 48109, U.S.A.

ABSTRACT. We draw attention to a new tree graph identity which substantially improves on the usual tree graph method of proving convergence of cluster expansions in statistical mechanics and quantum field theory. We can control expansions that could not be controlled before.

The purpose of this note is to publicize some of the results on tree graph estimation from [1], Section 8 and Appendices A and B. In particular, we exhibit an *extra factor of* $1/N!$ for each local pile-up of N differentiations in a cluster expansion, over the factors obtained using usual tree graph estimation. This improvement was used by the authors in [1], and will be used by one of the authors in the study of surface effects in Debye screening [2]. It would enable Magnen and Seneor to weaken their covariance bound from $\leqslant (|x-y|^{3.5+\epsilon}+1)^{-1}$ to $\leqslant (|x-y|^{2+\epsilon}+1)^{-1}$ in the recent preprint, 'A Note on Cluster Expansions' [3]. It should have many future applications.

Following the usual Glimm–Jaffe–Spencer development we let η be a tree graph, establishing some notation

$$|\eta| = n, \qquad \eta(i) < i, \quad i = 2, \ldots, n. \tag{1}$$

K, a quantity developed in a cluster expansion, then has a typical form (see [4], for example)

$$K = \sum_n \sum_{\substack{n \\ |\eta| = n}} \int \mathrm{d}\sigma\, f(\eta, \sigma) k(\sigma, \eta) \tag{2}$$

with $f(\eta, \sigma)$ a universal function of the interpolation parameters. The usual estimate is as follows

*This work was supported in part by the National Science Foundation under Grants No. PHY 01011 and MCS 83 01116.

Letters in Mathematical Physics 8 (1984) 55–57. 0377–9017/84/0081–0055 $00.45.

$$|K| \leqslant \sum_{n} e^{n} \sup_{\substack{\sigma \\ |\eta|=n}} |k(\sigma, \eta)|. \tag{3}$$

Typically one has an estimate for k of the form

$$|k(\sigma, \eta)| \leqslant c_1 \sum_{X(2)} \cdots \sum_{X(n)} \prod_{i=2}^{n} C(X(i), X(\eta(i))) \tag{4}$$

with C positive and

$$\sum_{X} C(X, Y) \leqslant c_2. \tag{5}$$

We then get from (3) – (5)

$$|K| \leqslant c_1 e \sum_{0}^{\infty} (ec_2)^n. \tag{6}$$

We let $d_\eta(i)$ be defined by

$$d_\eta(i) = \#\{j, \eta(j) = i\}. \tag{7}$$

We now consider estimate (4) replaced by

$$|k(\sigma, \eta)| \leqslant c_1 \sum_{X(2)} \cdots \sum_{X(n)} \prod_{i=2}^{n} C(X(i), X(\eta(i))) \cdot \prod_{j=1}^{n-1} (d_\eta(j)!). \tag{8}$$

If k satisfies (8), we cannot use the usual procedure to prove convergence of the cluster expansion.

However, the procedure of [1] yields from (2), (5) and (8) the estimate

$$|K| \leqslant c_1 c_3 \sum_{0}^{\infty} (c_3 c_2)^n. \tag{9}$$

Actually c_3 may be picked equal 4. We have implicitly assumed, as is true in the usual cluster expansion situation, that the $X(i)$ are distinct.

We may give a little insight into the line of development in [1]. We view a sum

$$\sum_{n} \sum_{\eta} \int \mathrm{d}\sigma f(\eta, \sigma) \left(\sum_{X(2)} \cdots \sum_{X(n)} \right)^{R} g \tag{10}$$

where R indicates the restriction that the $X(i)$ are distinct. Each term in the sum determines a graph★ with vertices $X(i)$, and lines $(X(i), X(\eta(i)))$. Different terms may determine the same graph.

★This is called an unordered connectivity graph in [1].

If one adds up all the terms determining the same graph, one finds the graph occurs with total weight 1. If g is a function only of the graph; then (10) may be replaced by a sum of distinct graphs T, with vertices $X(i)$ and lines $(X(i), X(\eta(i)))$ ($X(1)$ is fixed, T topologically is a connected tree graph)

$$\sum_T g. \tag{11}$$

We 'disregard ordering, and sum over 'structurally distinct" graphs, each with weight one'. Substituting (8) into

$$|K| \leqslant \sum_n \sum_\eta \int \mathrm{d}\sigma f(\eta, \sigma) \, |k(\sigma, \eta)| \tag{12}$$

and using the reduction from (10) to (11), (9) follows without difficulty. Reference [1] contains other useful results. The reduction from (10) to (11) and its generalizations are powerful and elegant.

REFERENCES

1. Battle, G.A. and Federbush, P., *Ann. Phys.* 142, 95–139 (1982).
2. Federbush, P., 'Surface Effects in Debye Screening, A Model Example', work in progress.
3. Magnen, J. and Seneor, R., 'A Note on Cluster Expansions', Preprint.
4. Brydges, D. and Federbush, P., *J. Math. Phys.* 19, 2064 (1978).

(Received August 12, 1983)