BRIEF NOTES

ON THE DECAY OF VORTICES IN A SECOND GRADE FLUID

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1. INTRODUCTION

G. I. Taylor [1] showed that the flow representing a double array of vortices which has the same periodicity in both the x and y directions is a solution to the equations of motion in two dimensions of a linearly viscous fluid. It was shown in [2] that such a result is also true for 'second order' fluids if time scale which characterizes the memory of the fluid and the size of the vortices satisfy certain apriori restrictions.

In this note we show that the results established by Taylor [1] for the linearly viscous fluid are *unconditionally* true, irrespective of the time scale which characterizes the fluid or the size of the vortices in the case of incompressible second grade fluids provided they are thermodynamically compatible. Also, in this analysis we investigate the relationship between the rate of decay of the vortices, and the periodicity of the vortices. It is found that if the periodicity is increased in the x or y directions, the vortices decay faster. It is also found, as is to be expected, that the vortices decay faster as the coefficient of viscosity μ increases, while the decay is slower if the normal stress moduli α_1 is larger.

The Cauchy stress $\underset{\sim}{T}$ in an incompressible second grade fluid is assumed to be related to the fluid motion in the following manner [3]

$$\underset{\sim}{T} = -p\underset{\sim}{1} + \mu\underset{\sim}{A}_1 + \alpha_1\underset{\sim}{A}_2 + \alpha_2\underset{\sim}{A}_1^2, \tag{1.1}$$

where μ is the coefficient of viscosity, α_1 and α_2 are the normal stress moduli, $-p\underset{\sim}{1}$ the spherical stress due to the constraint of incompressibility and $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ are the first two Rivlin- Ericksen tensors defined through

$$\underset{\sim}{A}_1 = \text{grad}\,\underset{\sim}{v} + (\text{grad}\,\underset{\sim}{v})^T, \tag{1.2}$$

and

$$\underset{\sim}{A}_2 = \dot{\underset{\sim}{A}}_1 + \underset{\sim}{A}_1(\text{grad}\,\underset{\sim}{v}) + (\text{grad}\,\underset{\sim}{v})^T\underset{\sim}{A}_1. \qquad (1.2)_2$$

In the above equations $\underset{\sim}{v}$ denotes the velocity field and the dot denotes material time differentiation. We shall take the point of view that the constitutive relation (1.1) is an exact relation and not an approximate constitutive model for non-Newtonian fluid behavior for retarded flows or slow flows (c.f. [4], [5] and [6]). Since the problem being considered is not necessarily one which is «slow» this point of view would be appropriate.

If the fluid is to be compatible with thermodynamics, in the sense that all arbitrary motions of the fluid meet the Clausius-Duhem inequality and the assumption that the specific Helmholtz free energy of the fluid be a minimum when the fluid is in equilibrium, it then follows that the material moduli which characterize the fluid have to satisfy the following restrictions [4] (1):

$$\mu \geqslant 0, \qquad \alpha_1 \geqslant 0 \qquad \text{and} \qquad \alpha_1 + \alpha_2 = 0. \qquad (1.3)_{1-3}$$

We shall henceforth assume that the strict inequalities hold in questions $(1.3)_{1,2}$, for otherwise if $\alpha_1 = 0$, we would not have a second order model by virtue of $(1.3)_3$. We shall show that $(1.3)_{1,2}$ are sufficient to ensure the extension of Taylor's result to a fluid modeled by relation (1.1).

2. DECAY OF VORTICES

The equations of motion of an incompressible second grade fluid in plane motion reduces to the following form:

$$\begin{gathered}\mu\Delta\underset{\sim}{v} + \alpha_1\Delta\underset{\sim}{v}_t + \alpha_1(\Delta\underset{\sim}{w}\times\underset{\sim}{v}) + \alpha_1\,\text{grad}\,\{\underset{\sim}{v}\cdot\Delta\underset{\sim}{v} + 1/4\,|\underset{\sim}{A}_1|^2\}\\ -\rho\underset{\sim}{v}_t - \rho(\underset{\sim}{w}\times\underset{\sim}{v}) - \rho\,\text{grad}\,(1/2\,|\underset{\sim}{v}|^2) - \text{grad}\,p = 0,\end{gathered} \tag{2.1}$$

In the above equation $\underset{\sim}{v} = u(x,y)\underset{\sim}{i} + v(x,y)\underset{\sim}{j}$, Δ denotes the two-dimensional Laplacian operator, $(\cdot)_t$ denotes the partial derivative of $(\cdot)$ with respect to t and $\underset{\sim}{w} \equiv \text{curl}\,\underset{\sim}{v} = \doteqdot w\underset{\sim}{k}$. On employing the curl operator to equation (2.1), and using the fact that the fluid is undergoing plane motion one obtains that

$$\begin{gathered}\mu\Delta w + \alpha_1\Delta w_t + \alpha_1\left\{u\,\frac{\partial(\Delta w)}{\partial x} + v\,\frac{\partial(\Delta w)}{\partial y}\right\} -\\ -\rho w_t - \rho\left\{u\,\frac{\partial w}{\partial x} + v\,\frac{\partial w}{\partial y}\right\} = 0.\end{gathered} \tag{2.2}$$

(1) The restiction that $\mu > 0$ is a restriction which is found acceptable due to physical consideration. The restriction on coefficients α_1 and α_2 however have been the subject matter of some controversy. In this context we refer the reader to [5] where it is shown that a violation of $(1.3)^2$ leads to unacceptable behavior for fluids of rheological interest. The restriction $(1.3)_3$ which is reached on the basis of the Clausius-Duhem inequality is not required in the following analysis since only plane motions are considered and terms involving α_2 do not appear in the equations of motion for plane flows.

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In the case of the linearly viscous fluid Taylor [1] assumed the stream function be of the form

$$\psi(x, y, t) = A \cos \frac{\pi x}{d} \cos \frac{\pi y}{d} e^{-\lambda t} \tag{2.3}$$

and, showed that such a solution existed and that the vortices decayed, where $\lambda = 2\mu\pi^2/\rho d^2$. Equation (2.3) represents a system of eddies each rotating in the opposite direction to that its four neighbors, d is the length of the square containing one complete eddy. We note that the problem in question is an unsteady problem and also the intertial terms are not neglected. Thus, the uniqueness theorem established for the flow of second grade fluids in steady creeping motion [6] which would imply the Stokesian solution in the case of second grade fluid do not apply here. However, as shown in [2], a solution of the form (2.3) which works in the case of a linearly viscous fluid does indeed work here. Assuming a solution of the form

$$\psi(x, y, t) = A \cos \frac{m\pi x}{d} \cos \frac{n\pi y}{d} e^{-\lambda t}, \tag{2.4}$$

we find after a lengthy but straightforward manipulation that the stream function represented by an equation of the form of (24) would be a solution to equation (2.2) if

$$\lambda = \frac{\mu\pi^2(m^2 + n^2)}{\rho d^2 + \alpha_1\pi^2(m^2 + n^2)} \tag{2.5}$$

Firstly, we observe that since $\mu > 0$ and $\alpha_1 > 0$ by equation $(1.3)_{1,2}$ and the density $\rho > 0$, we find that $\lambda > 0$, which implies that the vortices decay away asymptotically in time. On rewriting (2.5) in the form

$$\lambda = \frac{\mu}{\dfrac{d^2}{(m^2 + n^2)\pi^2} + \alpha_1}$$

we observe that an increase in the periodicity in the x or y direction increases λ and thus the vortices decay faster. Finally, it is also obvious from the above definition for λ, that λ increases as μ increases, and λ decreases as α_1 increases.

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