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Curve veering and mode localization in a buckling problem

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1. Introduction

In structural dynamics, the vibrational characteristics of periodic structures have been shown to be highly sensitive to small (and often unavoidable) periodicity-destroying structural irregularities. Specifically, under conditions of weak internal coupling for the structure of interest, the drastic phenomena of mode localization and eigenvalue loci veering have been evidenced (see the paper by Pierre [1] and the references cited therein). Typical examples of structures featuring such an extreme sensitivity to irregularities include blade assemblies, multi-span structures, and truss-like space structures.

The purpose of this Note is to investigate whether similar phenomena occur in buckling problems as well. The underlying motivation for the study is that both free vibration and buckling problems are governed by eigenvalue problems; thus one can speculate that their eigensolutions are affected by irregularities in a similar way. This would result, under certain conditions, in the high sensitivity of the buckling loads and corresponding buckling patterns to irregularities, or imperfections, and more specifically in the localization of the buckling modes to a small geometric region of the structure.

The occurrence of buckling pattern localization and load loci veering is examined in the present Note using a simple example of a nearly periodic structure, namely a two-span column with a torsional spring acting at the intermediate support. This spring is used to vary the amount of coupling between the two spans, which has been shown to govern the sensitivity to imperfections in vibration problems [1]. Here it is clear that the interspan coupling decreases as the spring stiffness increases. Furthermore, the location of the intermediate support determines whether the column is ordered or disordered: if the two spans have exactly the same length, the column is ordered, or tuned; otherwise, it is disordered, or mistuned. Pierre [1] has shown that, in the weak coupling (weak disorder) case, strong localization of the vibration modes and veering of the natural frequency loci occur for this structure. In this Note the two lowest buckling loads are plotted as a function of the support location and the buckling patterns are studied for different values of the interspan coupling. An asymptotic analysis of the difference between these loads is also conducted for the central support case.

2. Analysis

Consider the beam shown in Fig. 1, with length l and constant bending stiffness EI. The torsional spring has stiffness c. In terms of the nondimensional quantities

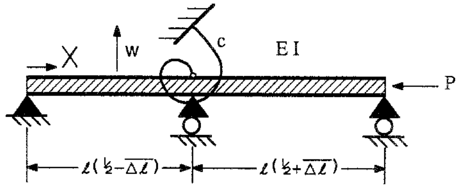

Figure 1
Geometry of two-span column.

$$\begin{cases} \alpha = 0.5 - \overline{\Delta l}, \quad \beta = 0.5 + \overline{\Delta l}, \quad \gamma = \dfrac{cl}{EI} \\ \lambda = \sqrt{\dfrac{Pl^2}{EI}}, \quad x = \dfrac{X}{l}, \quad \xi = x - \alpha \end{cases} \tag{1}$$

the characteristic equation for the buckling loads is given by

$$\begin{aligned}(\lambda^2\alpha + \gamma)\lambda\beta \sin(\lambda\alpha)\cos(\lambda\beta) - (\lambda^2 + \gamma)\sin(\lambda\alpha)\sin(\lambda\beta) \\ + (\lambda^2\beta + \gamma)\lambda\alpha\cos(\lambda\alpha)\sin(\lambda\beta) - \lambda^2\alpha\beta\gamma\cos(\lambda\alpha)\cos(\lambda\beta) = 0\end{aligned} \tag{2}$$

and the buckling modes have the form

$$w(x) = \begin{cases} K_1\left[\sin(\lambda x) - \dfrac{x}{\alpha}\sin(\lambda\alpha)\right] & 0 \le x \le \alpha \\ K_2[\xi - \beta(1 - \cos(\lambda\xi) + \cot(\lambda\beta)\sin(\lambda\xi)] & 0 \le \xi \le \beta \end{cases} \tag{3}$$

where

$$\begin{cases} K_1 = K[\gamma\sin(\lambda\beta) - \gamma\lambda\beta\cos(\lambda\beta) + \beta\lambda^2\sin(\lambda\beta)] \\ K_2 = K\lambda^2\sin(\lambda\alpha)\sin(\lambda\beta) \end{cases} \tag{4}$$

Equation (2) is solved numerically, and the lowest two (nondimensional) buckling loads are denoted λ_1 and λ_2.

For the special case $\gamma = 0$ (no torsional spring, that is, strong interspan coupling), λ_1 and λ_2 are plotted as a function of $\overline{\Delta l}$ in Fig. 2. When the intermediate support is at

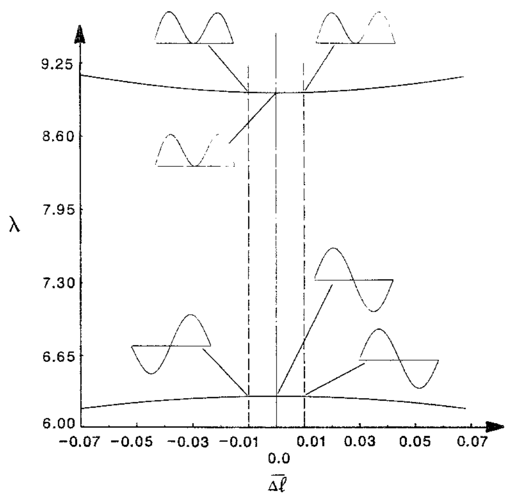

Figure 2
Loci of lowest two buckling loads near $\overline{\Delta l} = 0$ for $\gamma = 0$.

the center of the column (i.e., $\overline{\Delta l} = 0$), $\lambda_1 = 2\pi$ (with a sinusoidal, antisymmetric buckling mode) and $\lambda_2 = 8.9868$ (corresponding to each half of the column buckling as if it were fixed at the center). The buckling modes for $\overline{\Delta l} = 0$, 0.01, and -0.01 are sketched in Fig. 2. The two loci of the eigenvalues do not approach each other in this case, and there is no localization of the modes about one of the spans. The buckling loads and modes for the slightly disordered column are indeed small perturbations of those for the ordered system ($\overline{\Delta l} = 0$). Thus in this strong interspan coupling case the sensitivity of the buckling modes to irregularities is small.

The case $\gamma = 600$ (i.e., small coupling between spans) is illustrated in Fig. 3. At $\overline{\Delta l} = 0$, $\lambda_2 = 8.9868$ again, while $\lambda_1 = 8.9277$; thus, the lowest two buckling loads of the ordered column are very close. On a larger scale, the two eigenvalue curves would appear to intersect, when actually they veer away from each other abruptly at the ordered state ($\overline{\Delta l} = 0$) with very large local curvature—a phenomenon referred to as *eigenvalue loci veering*. The corresponding mode shapes for small $\overline{\Delta l}$ exhibit *strong localization*, in which the amplitude on one side of the intermediate support is much larger than that on the other side. In this weak interspan coupling case (that is, when the loads of the ordered column are close), the introduction of small disorder, or asymmetry, $\overline{\Delta l}$, drastically alters the buckling modes as well as the loci of the critical loads. These phenomena are similar to those observed in free vibration problems [1].

The gap $\lambda_2 - \lambda_1$, at $\overline{\Delta l} = 0$, is the key to the occurrence of mode localization and curve veering: it becomes smaller as the torsional stiffness, γ, increases and thus can be regarded as a measure of the interspan coupling. This is shown by the curve labelled "exact" in Fig. 4, where the abscissa is $1/\gamma$. One can also perform an asymptotic analysis for large γ. In Eq. (2), let

$$\alpha = \beta = 0.5, \quad \gamma = \frac{1}{\varepsilon}, \quad \lambda = \mu + a\varepsilon + b\varepsilon^2 + O(\varepsilon^3) \tag{5}$$

where $\mu = 8.9868$. Setting coefficients of like powers of ε equal to zero, one obtains $a = -4\mu$ and $b = 32\,\mu$ for $\lambda = \lambda_1$. The first-order result ($\lambda_2 - \lambda_1 = -a\varepsilon$) and second-order result are depicted in Fig. 4. In the range shown, the second-order approximation is very close to the exact solution.

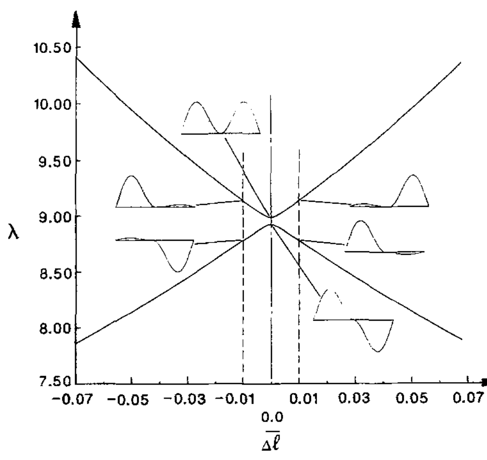

Figure 3
Loci of lowest two buckling loads near $\overline{\Delta l} = 0$ for $\gamma = 600$.

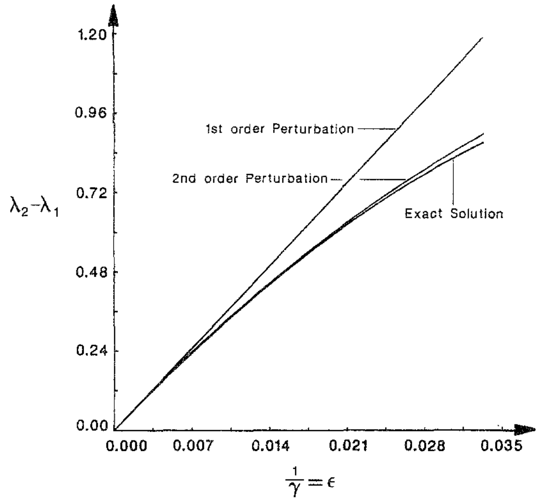

Figure 4
Gap $\lambda_2 - \lambda_1$ at $\overline{\Delta l} = 0$ as function of $1/\gamma$.

Acknowledgements

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Reference

[1] C. Pierre, *Mode localization and eigenvalue loci veering phenomena in disordered structures.* J. Sound & Vibration *126* (3), 485–502 (1988).

Summary

It is shown on a simple example that small disorder, or mistuning, may alter drastically the eigensolution of buckling problems in nearly periodic structures with weak internal coupling. Specifically, the phenomena of eigenvalue loci veering and mode shape localization, which are known to occur for free vibration problems, are evidenced in the case of structural buckling as well.

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