THE UNIVERSITY OF MICHIGAN

COLLEGE OF LITERATURE, SCIENCE, AND THE ARTS
Department of Mathematics

Technical Note

ASYMMETRIC PRIME ENDS

E. F. Collingwood
G. Piranian

UMRI Project 2913

under contract with:

MATHEMATICAL SCIENCES DIRECTORATE
AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
CONTRACT NO. AF 49(638)-633
WASHINGTON 25, D. C.

administered by

THE UNIVERSITY OF MICHIGAN RESEARCH INSTITUTE ANN ARBOR

October 1960

SUMMARY

Each simply connected domain in the plane has at most countably many prime ends whose right and left wings do not coincide. On the other hand, to each countable set E on the unit circle C there corresponds a function which is holomorphic and univalent in the unit disk D and which has the property that it carries each point of E and no point of $C \setminus E$ onto a prime end with unequal wings.

1. CLUSTER SETS

Let f denote a function which maps the unit disk D: $|z| < 1$ onto the Riemann sphere R, and let $e^{i\theta}$ be a fixed point on the unit circle C: $|z| = 1$. A point w on R belongs to the _cluster set_ $C(f, e^{i\theta})$ of f at $e^{i\theta}$ provided there exists a sequence $\{z_n\} = \{r_n e^{i\theta_n}\}$ such that $r_n \nearrow 1$, $\theta_n \to \theta$, $f(z_n) \to w$. It belongs to the _radial cluster set_ $C_\rho(f, e^{i\theta})$ provided the sequence $\{z_n\}$ can be chosen with $\theta_n = \theta$ for all n. It belongs to the _right cluster set_ $C_R(f, e^{i\theta})$ (or to the _left cluster set_ $C_L(f, e^{i\theta})$) provided the sequence $\{z_n\}$ can be chosen with $\theta_n \nearrow \theta$ (or with $\theta_n \searrow \theta$).

If f is continuous in D, then the set of points $e^{i\theta}$ where $C_\rho(f, e^{i\theta})$ does not contain $C(f, e^{i\theta})$ is a set of first category [1]; but even if f is required to be holomorphic and univalent in D, the point set in question may have measure 2π, as can be seen, for example, from the construction described in [4, Section 2]. In other words, the set where the radial cluster set is a proper subset of the complete cluster set is thin topologically but may be large geometrically. The following theorem implies that the point set where the right and left cluster sets do not coincide is subject to much more severe restrictions, even if no conditions whatever are imposed on the function f.

THEOREM 1. _If f maps the unit disk into the Riemann sphere, then_

$$C_R(f, e^{i\theta}) = C(f, e^{i\theta}) = C_L(f, e^{i\theta})$$

for all except at most countably many points $e^{i\theta}$.

Since the right (left) cluster set of a function contains the set which has been called the right (left) boundary cluster set, Theorem 1 is a corollary of Collingwood's recent theorem [2] to the effect that the right and left boundary cluster sets at $e^{i\theta}$ coincide with $C(f, e^{i\theta})$, except possibly at countably many points $e^{i\theta}$. However, for the sake of completeness and simplicity, we give here a proof which is independent of the concept of boundary cluster sets.

We cover the Riemann sphere R with a succession of nets N_K $(k = 1,2,\ldots)$, each with finitely many (closed) triangular meshes $m_{kn}(n = 1, 2,\ldots, n_k)$ of diameter less than $1/k$. By means of any convenient ordering, we arrange all the meshes m_{kn} into a sequence $\{m_j\}$. For each index j, we consider the set S_j of those points $e^{i\theta}$ for which the mesh m_j contains a point of $C(f, e^{i\theta})$ but does not meet the set $C_R(f, e^{i\theta})$. Clearly, the set S_R of points $e^{i\theta}$ for which $C(f, e^{i\theta}) \setminus C_R(f, e^{i\theta})$ is not empty is the union of the sets S_j.

If $e^{i\theta} \in S_j$, then $e^{i\theta}$ is the endpoint of an arc $I(\theta) = (e^{i\phi}, e^{i\theta})(\phi < \theta)$ such that $C(f, e^{i\psi})$ does not meet the mesh m_j if $e^{i\psi}$ lies on $I(\theta)$; for otherwise m_j would meet the set $C_R(f, e^{i\theta})$, contrary to the hypothesis that $e^{i\theta} \in S_j$. Since the arc $I(\theta)$ cannot contain points of S_j, the set S_j, is at most countable, and therefore S_R is at most countable. Likewise, the corresponding set S_L is at most countable, and the theorem is proved.

2. ASYMMETRIC PRIME ENDS

Let the function f be holomorphic and univalent in the unit disk D, and let B denote the image of D under f. If P is the prime end of B that corresponds to the point $e^{i\theta}$, the two cluster sets $C_R(f, e^{i\theta})$ and $C_L(f, e^{i\theta})$ coincide with what Ursell and Young [6, p. 14] have called the _right_ and _left wings_ of P. We shall say that the prime end P is _asymmetric_ if its two wings are not identical. The following theorem is an immediate consequence of Theorem 1.

THEOREM 2. _Each simply connected plane domain has at most countably many asymmetric prime ends._

In order to formulate our next theorem, we partition the set of asymmetric prime ends of a fixed simply connected domain into three classes U_R, U_L, and U_{RL}, as follows: a prime end belongs to U_R if its left wing is a proper subset of its right wing, to U_L if its right wing is a proper subset of its left wing, and to U_{RL} if neither of its wings is a subset of the other.

THEOREM 3. _Let E_R, E_L, and E_{RL} be three disjoint sets on the unit circle C, each at most countable. Then there exists a function f which maps the unit disk conformally onto a domain B in such a way that each point in E_R, E_L, or E_{RL} corresponds to a prime end in U_R, U_L, or U_{RL}, respectively, while all other points of C correspond to symmetric prime ends of B._

To prove Theorem 3, we order the points of $E_R \cup E_L \cup E_{RL}$ into a sequence $\{z_j\}$ $(j = 1, 2, \ldots)$. With each point z_j we associate a sequence of points z_{jp} lying on C and converging to z_j. We construct a function of the form

$$f(z) = z + \sum_{j,p} A_{jp}\{1 - (1 - z/\rho_{jp}z_{jp})^{k_{jp}}\},$$

where the A_{jp} are appropriate complex constants, $k_{jp} \searrow 0$ and $\rho_{jp} \searrow 1$ as $p \to \infty$, and where the symbol in braces represents that branch of the corresponding function which takes the value 0 at $z = 0$.

The constants that determine the function f can be chosen in such a way that f is holomorphic and univalent in D, such that the radial limit $f(e^{i\theta})$ of f exists, for each value θ, and such that the boundary of the image of D, in the neighborhood of a point $f(z_j)$, appears roughly as in Figure 1, 2, or 3, according as z_j belongs to E_R, E_L, or E_{RL}. We say "roughly" because each of the tooth-like extensions of the domain B may carry further extensions (indeed, the deformations may be everywhere dense on the boundary; difficulties that might arise from unwanted condensations of singularities can be avoided by subjecting the constants A_{jp} to the restriction $|A_{jp}| \leq 1/j!$, in other words, by requiring that the teeth associated with the point z_j have length at most $1/j!$. The details of the proof are similar to the details in [3, Sections 3 and 4], and we omit them.

Theorem 3 makes no mention of Carathéodory's four kinds of prime ends. Our construction leads to a domain whose asymmetric prime ends are of the second kind and whose symmetric prime ends are of the first kind. No matter what construction is used, the asymmetric prime ends must be of the second or fourth kind, since the prime ends of the first and third kinds have no subsidary points The following theorem shows that no other topological restriction on the distribution of the asymmetric prime ends exists.

THEOREM 4. *Let* $C = E_1 \cup E_2 \cup E_3 \cup E_4$ *be a decomposition of the unit circle such that some homeomorphic mapping of the unit disk onto a simply connected domain* B *induces a homeomorphism* ϕ *of the sets* E_1, E_2, E_3, E_4 *onto the sets of prime ends of the first, second, third, and fourth kind, respectively, of* B. *Let* S_R, S_L, *and* S_{RL} *be disjoint countable subsets of* $E_2 \cup E_4$. *Then the domain*

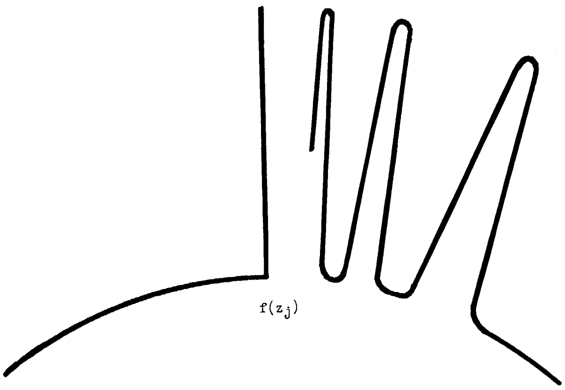

Figure 1

Figure 2

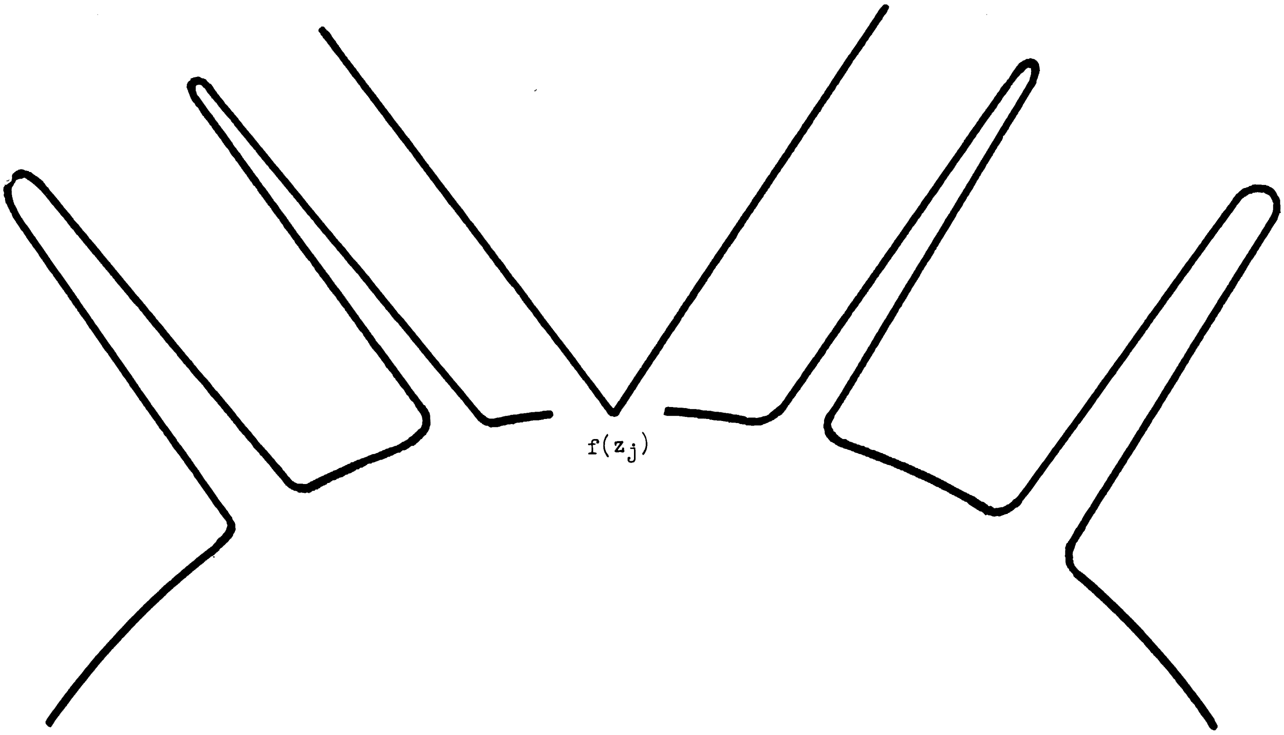

Figure 3

B *can be chosen in such a way that* S_R, S_L, *and* S_{RL} *correspond to the sets* U_R, U_L, *and* U_{RL}, *respectively, under the homeomorphism* ϕ.

A detailed proof of Theorem 4 would be tedious, and we give only a brief sketch to show how the construction in Sections 3 to 8 of [5] can be modified so as to yield the desired result.

In [5], the set E_{234} of points corresponding to prime ends of the second, third, or fourth kind is represented as the union of a sequence of disjoint sets $E_{234.i}$ ($i = 1, 2, \ldots$). The set $E_{234.1}$ is open, and each of the sets $E_{234.i}$ ($i = 2, 3, \ldots$) is closed and nowhere dense. The unit disk is subjected to certain deformations determined by the set $E_{234.2}$; the process is repeated (on a successively smaller scale) with reference to $E_{234.i}$ ($i = 3, 4, \ldots$). Thereafter, a special program of further deformations is launched with reference to $E_{234.1}$.

For our present purpose, we denote by $S_{R.i}$, $S_{L.i}$, and $S_{RL.i}$ the intersections of S_R, S_L, and S_{RL} with $E_{234.i}$. First we order the set $S_{R.2} \cup S_{L.2} \cup S_{RL.2}$ into a finite or infinite sequence $\{z_{2j}\}$. In the neighborhood of z_{21}, we deform the boundary of the unit disk in the manner indicated by Figures 4, 5, or 6, according as z_{21} belongs to S_R, S_L, or S_{RL}, and we map the set $C \setminus z_{21}$ onto the curved portion of the deformed boundary. We proceed similarly (but on smaller scales) with regard to $z_{22}, z_{23}, \ldots$. The unit disk is then mapped onto the star-domain which has been obtained, in such a way that the radii of D are deformed as is indicated in Figure 7. Clearly, the transformation can be constructed in such a way that it is continuous throughout the disk $|z| < 3/4$ and on every radius of D that does not terminate at one of the points z_{2j}. The

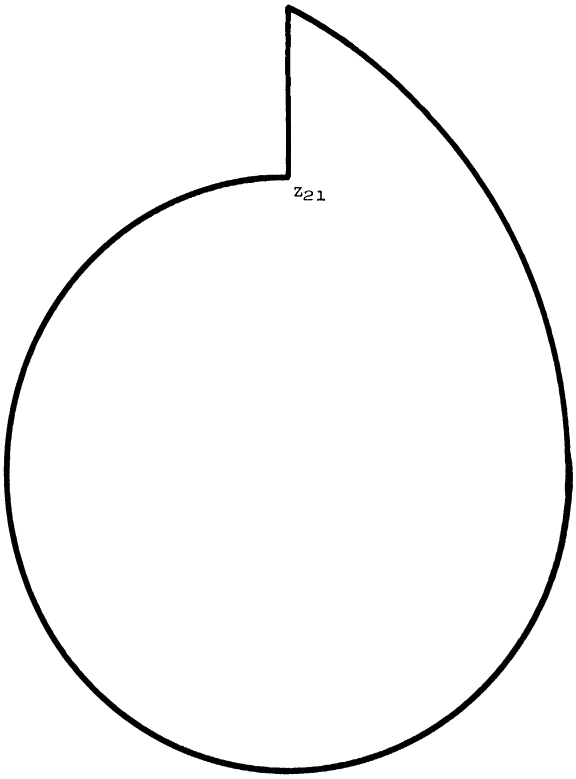

Figure 4

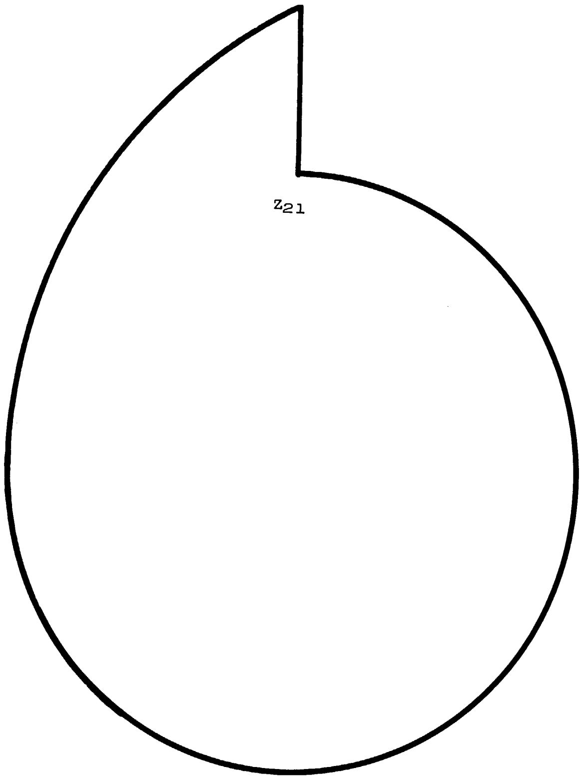

Figure 5

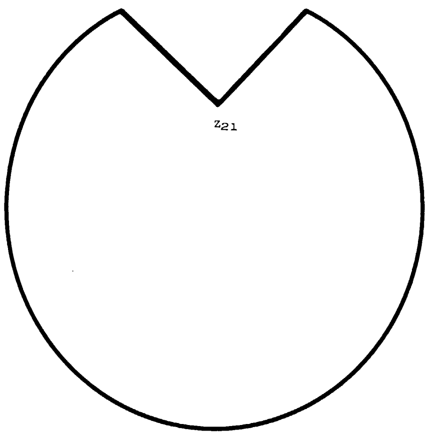

Figure 6

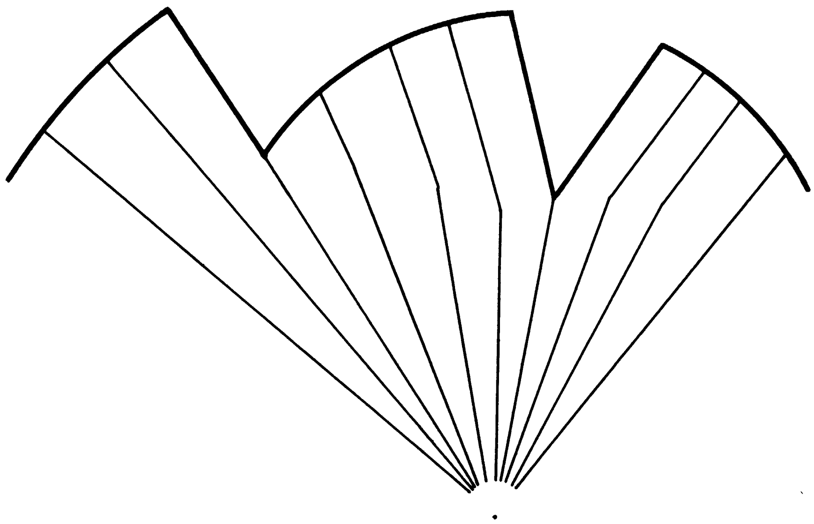

Figure 7

transformed unit disk is then further deformed according to the program in Sections 5 and 6 of [5].

The analogous deformations which have to be carried out with reference to the sets $E_{234.i}$ $(i = 3, 4, \ldots)$ are obvious (see Section 7 of [5]).

The deformations of the disk D that are used in [5] for the sake of the set $E_{234.1}$ are somewhat more complicated, because $E_{234.1}$ consists of interior points. However, the intersection of $E_2 \cup E_4$ with $E_{234.1}$ is partitioned into sets that are nowhere dense, and each of these is treated separately. Therefore we can proceed as above, provided we exercise one precaution: In the third and fourth paragraphs of Section 8 of [5], a certain denumerable set and a certain near-perfect set are extracted from $E_2 \cap E_{234.1}$ and used for special purposes. These special purposes would interfere with our technique of creating asymmetric prime ends at the points of the extracted sets, and therefore the extracted sets must be selected in such a way that they contain no points of S_R, S_L, or S_{RL}. Since E_2 is locally uncountable in $E_{234.1}$, the latter requirement presents no special difficulty.

This sketch must suffice, because any exposition which covered all details would have to include the laborious description in [5]; and it is easier for the reader to acquaint himself first with the description in [5], and to supply thereafter the modifications which we have now sketched, than it would be to struggle through a detailed description in which the various operations are presented simultaneously.

REFERENCES

1. E. F. Collingwood, _Sur le comportement à la frontiere, d'une fonction méromorphe dans le cercle unité_, C. R. Acad. Sci. Paris 240 (1955), 1502-1504.

2. E. F. Collingwood, _Cluster sets of arbitrary functions_ (to appear).

3. F. Herzog and G. Piranian, _Sets of convergence of Taylor series, II_, Duke Math. J. 20 (1953), 41-54.

4. A. J. Lohwater and G. Piranian, _The boundary behavior of functions analytic in a disk_, Ann. Acad. Sci. Fenn. Ser. A. I. no. 239 (1957), 1-17.

5. G. Piranian, _The distribution of prime ends_, Michigan Math. J. 7 (1960), 83-95.

6. H. D. Ursell and L. C. Young, _Remarks on the theory of prime ends_, Mem. Amer. Math. Soc. no. 3 (1951), 1-29.

DISTRIBUTION LIST

(One copy unless otherwise noted)

Commander
Army Rocket and Guided Missile Agency
Redstone Arsenal, Alabama
Attn: ORDXR-OTL

Commander (3 copies)
European Office, ARDC
47 Rue Cantersteen
Brussels, Belgium

Applied Mathematics and Statistics Laboratory
Stanford University
Stanford, California

Department of Mathematics
University of California
Berkeley, California

Commander, Attn: Technical Library
Air Force Flight Test Center
Edwards Air Force Base, California

The Rand Corporation (2 copies)
Technical Library
1700 Main Street
Santa Monica, California

Commander
1st Missile Division
Vandenburg Air Force Base, California
Attn: Operations Analysis Office

Department of Mathematics
Yale University
New Haven, Connecticut

Office of Naval Research (2 copies)
Department of the Navy
Washington 25, D. C.
Attn: Code 432

Director
Department of Commerce
Office of Technical Services
Washington 25, D. C.

Administrator (6 copies)
National Aeronautics and Space Administration
1520 H Street, N. W.
Washington 25, D. C.
Attn: Documents Library

Library
National Bureau of Standards
Washington 25, D. C.

Data Processing Systems Division
National Bureau of Standards
Washington 25, D. C.
Attn: Mr. Russell A. Kirsch

Applied Mathematics Division
National Bureau of Standards
Washington 25, D. C.

Diamond Ordnance Fu e Laboratories
Washington 25, D. C.
Attn: Virginia A. Johnson
Transducer Information Center

Headquarters, USAF
Assistant for Operations Analysis
Deputy Chief of Staff,
Operations, AFOOA
Washington 25, D. C.

Laboratories for Applied Sciences
University of Chicago
Museum of Science and Industry
Chicago 37, Illinois
Attn: Library, W-305

DISTRIBUTION LIST (Continued)

Commander (2 copies)
Air Force Office of Scientific Research
Washington 25, D. C.
Attn: SRM

Director
U. S. Naval Research Laboratory
Washington 25, D. C.
Attn: Library

Commander, AFRD (2 copies)
Washington 25, D. C.
Attn: Technical Library

National Science Foundation
Program Director for Mathematical Sciences
Washington 25, D. C.

Canadian Joint Staff
2450 Massachusetts Avenue, N. W.
Washington 25, D. C.
Attn: DRB/DSIS

Department of Mathematics
Northwestern University
Evanston, Illinois

Department of Mathematics
University of Chicago
Chicago 37, Illinois

Department of Mathematics
University of Illinois
Urbana, Illinois

Department of Mathematics
Purdue University
Lafayette, Indiana

Institute for Fluid Dynamics and Applied Mathematics
University of Maryland
College Park, Maryland

Mathematics and Physics Library
The Johns Hopkins University
Baltimore, Maryland

Director
National Security Agency
Fort George G. Meade, Maryland
Attn: Dr. H. H. Campaign

Department of Mathematics
Harvard University
Cambridge 38, Massachusetts

Department of Mathematics
Massachusetts Institute of Technology
Cambridge 38, Massachusetts

Commander
Detachment 2, AFRD
L. G. Hanscom Field
Bedford, Massachusetts
Attn: Technical Library

Department of Mathematics
Wayne State University
Detroit 1, Michigan

Department of Mathematics
Folwell Hall
University of Minnesota
Minneapolis, Minnesota

Department of Mathematics
Institute of Technology
Engineering Building
University of Minnesota
Minneapolis, Minnesota

Institute for Mathematical Sciences
New York University
25 Waverly Place
New York, New York
Attn: Professor M. Kline

DISTRIBUTION LIST (Continued)

Department of Mathematics
Washington University
St. Louis 8, Missouri

Department of Mathematics
University of Missouri
Columbia, Missouri

Commander
Strategic Air Command
Offutt Air Force Base
Omaha, Nebraska
Attn: Operations Analysis

The James Forrestal Research Center
Library
Princeton University
Princton, New Jersey

Library
Institute for Advanced Study
Princeton, New Jersey

Department of Mathematics
Fine Hall
Princeton University
Princton, New Jersey

Commanding General
Signal Corps Engineering Laboratory
Ft. Monmouth, New Jersey
Attn: SIGFM/EL-RPO

Commander
Air Force Missile Development Center
Holloman Air Force Base, New Mexico
Attn: Technical Library, HDOI

Commander
Air Force Special Weapons Center
Kirtland Air Force Base
Albuquerque, New Mexico
Attn: Technical Library, SWOI

Professor J. Wolfowitz
Mathematics Department
White Hall
Cornell University
Ithaca, New York

Department of Mathematics
Syracuse University
Syracuse, New York

Institute for Aeronautical Sciences
2 East 64th Street
New York 16, New York
Attn: Librarian

Department of Mathematics
University of North Carolina
Chapel Hill, North Carolina

Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

Office of Ordnance Research (2 copies)
Box CM
Duke Station
Durham, North Carolina

Department of Mathematics
Duke University
Duke Station
Durham, North Carolina

P. O. Box AA
Wright-Patterson Air Force Base
Ohio

Commander
Wright Air Development Division
Wright-Patterson Air Force Base
Ohio
Attn: WCOSI

DISTRIBUTION LIST (Concluded)

Commander
Aeronautical Research Laboratories
Wright-Patterson Air Force Base
Ohio
Attn: Technical Library

USAF Institute of Technology Library
Building 125, Area B (2 copies)
Wright-Patterson Air Force Base
Ohio
Attn: MCLI-ITLIB

Department of Mathematics
Carnegie Institute of Technology
Pittsburgh, Pennsylvania

Department of Mathematics
University of Pennsylvania
Philadelphia, Pennsylvania

Department of Mathematics
University of Wisconsin
Madison, Wisconsin

Mathematics Research Center
U. S. Army
University of Wisconsin
Madison, Wisconsin
Attn: R. E. Langer

AEDC Library
ARO, Inc.
Arnold AF Station, Tennessee

U. S. Atomic Energy Commission
Technical Information Service
Extension
P. O. Box 62
Oak Ridge, Tennessee

Applied Mechanics Review (2 copies)
Southwest Research Institute
8500 Culebra Road
San Antonio 6, Texas

Department of Mathematics
Rice Institute
Houston, Texas

Armed Services Technical (10 copies)
Information Agency
Arlington Hall Station
Arlington 12, Virginia
Attn: TIPDR

Institute for Mathematical Sciences
New York University
25 Waverly Place
New York, New York
Attn: Professor M. Kline