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SUPERCONNECTIONS AND HIGHER INDEX THEORY

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Abstract

Let M be a smooth closed spin manifold. The higher index theorem, as given for example in Proposition 6.3 of [CM], computes the pairing between the group cohomology of $\pi_1(M)$ and the Chern character of the "higher" index of a Dirac-type operator on M. Using superconnections, we give a heat equation proof of this theorem on the level of differential forms on a noncommutative base space. As a consequence, we obtain a new proof of the Novikov conjecture for hyperbolic groups.

I. Introduction

Let M be a smooth closed connected spin manifold. Let V be a Hermitian vector bundle on M. If M is even-dimensional, the Atiyah-Singer index theorem identifies the topological expression $\int_M \widehat{A}(M) \wedge \mathrm{Ch}(V)$ with the index of the Dirac-type operator acting on L^2-sections of the bundle $S(M) \otimes V$, where $S(M)$ is the spinor bundle on M [AS1].

When M is not simply-connected, one can refine the index theorem to take the fundamental group into account [Co3, Kas, Mo]. Let Γ denote the fundamental group of M. Let $\nu : M \to B\Gamma$ be the classifying map for the universal cover $\widetilde{M}$ of M. For $[\eta] \in H^*(B\Gamma; \mathbf{C})$, higher index theory attempts to identify $\int_M \widehat{A}(M) \wedge \mathrm{Ch}(V) \wedge \nu^*[\eta]$ with an analytic expression. The main topological and geometric applications of higher index theory are to Novikov's conjecture on homotopy-invariants of non-simply-connected manifolds [No], and to questions of the existence of positive-scalar-curvature metrics on M [Ro].

In order to motivate the statement of the higher index theorem, let us first recall how Lusztig used the index theorem for families of operators to prove a higher index theorem in the case of $\Gamma = \mathbb{Z}^k$ [Lu]. Let $T^k = \mathrm{Hom}(\Gamma, U(1))$ be the dual group to Γ and let L_θ be the flat unitary line bundle over M whose holonomy is specified by $\theta \in T^k$. Consider the product fibration $M \to M \times T^k \to T^k$. Suppose for simplicity that M is even-dimensional; then there is a bundle $\mathcal{H}$ over T^k of $\mathbb{Z}_2$-graded Hilbert spaces, where $\mathcal{H}_\theta$, the fiber over $\theta \in T^k$, consists of the L^2-sections of $S(M) \otimes V \otimes L_\theta$. There is also a family Q of vertical Dirac-type operators parametrized by T^k, where Q_θ acts on $\mathcal{H}_\theta$. The analytic index Index(Q) of the family of elliptic operators, as defined in [AS2], lies in $K^0(T^k)$. An element $[\eta]$ of the group cohomology $H^\ell(\mathbb{Z}^k; \mathbb{C})$ gives a homology class $\tau_\eta \in H_\ell(T^k; \mathbb{C})$, against which the Chern character $\mathrm{Ch}\left(\mathrm{Index(Q)}\right) \in H^*(T^k; \mathbb{C})$ can be paired. The families index theorem [AS2] then implies

$$\int_{\tau_\eta} \mathrm{Ch}\left(\mathrm{Index(Q)}\right) = const.(l) \int_M \widehat{A}(M) \wedge \mathrm{Ch}(V) \wedge \nu^*[\eta] \, , \qquad (*)$$

giving the desired analytic interpretation of the right-hand-side. The purpose of [Lu] was to apply $(*)$ to the Novikov conjecture.

In order to extend these methods to nonabelian Γ, let us note some algebraic properties of the above construction. The algebra of continuous functions $C(T^k)$ acts on the vector space $C(\mathcal{H})$ of continuous sections of $\mathcal{H}$ by multiplication. Upon performing Fourier transform over T^k, $C(\mathcal{H})$ maps to a certain subspace of the L^2-sections of the pullback bundle $S(\widetilde{M}) \otimes \widetilde{V}$ on $\widetilde{M}$, this subspace thus being a $C(T^k)$-Hilbert module in the sense of [Kas].

The generalization of Lüsztig's method to nonabelian Γ is based on a "fibration" $M \to P \to B$ which exists only morally, where B is a noncommutative space whose "algebra of continuous functions" is taken to be the algebra $\Lambda = C_r^*\Gamma$, the reduced group C^*-algebra [Co3]. (When $\Gamma = \mathbb{Z}^k$, $\Lambda \cong C(T^k)$.) Mishchenko and Kasparov define a Hilbert Λ-module of L^2-sections of $S(\widetilde{M}) \otimes \widetilde{V}$, upon which a Dirac-type operator $\widetilde{D}$ acts. The analytic index of $\widetilde{D}$ lies in "$K^0(B)$", or more precisely in $K_0(\Lambda)$ [Mi, Kas]. The Mishchenko-Fomenko index theorem identifies the analytic index with a topological index [MF].

In order to pair these indices with the group cohomology of Γ, one needs additional structure on B. Let $\mathfrak{B}^\infty$ be a dense subalgebra of Λ containing $\mathbb{C}\Gamma$ which is stable under the holomorphic functional calculus of Λ [Co1]. (For example, if $\Gamma = \mathbb{Z}^k$, one can take $\mathfrak{B}^\infty$ to be $C^\infty(T^k)$.) Then $K_0(\Lambda) \cong K_0(\mathfrak{B}^\infty)$. One can think of the image of $\mathrm{Index}(\widetilde{D})$ under this isomorphism as being a "smoothing" of $\mathrm{Index}(\widetilde{D})$.

One can then use the fact that $K_0(\mathfrak{B}^\infty)$ pairs with the cyclic cohomology $HC^*(\mathfrak{B}^\infty)$ of $\mathfrak{B}^\infty$ [Co1] to extract numbers from $\mathrm{Index}(\widetilde{D})$. In loose but more familiar terms, the Chern character $\mathrm{Ch}\left(\mathrm{Index}(\widetilde{D})\right)$ lies in the "cohomology" of B. More precisely, it lies in the cyclic homology group $HC_*(\mathfrak{B}^\infty)$ [Co1, Ka]. One then wants to define a "homology class" of B which one can pair with $\mathrm{Ch}\left(\mathrm{Index}(\widetilde{D})\right)$. The correct notion of homology for B is given by the (periodic) cyclic cohomology of $\mathfrak{B}^\infty$. In particular, given a group cocycle $\eta \in Z^l(\Gamma; \mathbf{C})$, one obtains an cyclic cocycle $\tau_\eta \in ZC^l(\mathbf{C}\Gamma)$ (eqn. (62)). If τ_η extends to an element of $ZC^l(\mathfrak{B}^\infty)$ then Proposition 6.3 of [CM] gives

$$\left\langle \mathrm{Ch}\left(\mathrm{Index}(\widetilde{D})\right), \tau_\eta \right\rangle = const.(l) \int_M \widehat{A}(M) \wedge \mathrm{Ch}(V) \wedge \nu^*[\eta] \,. \qquad (**)$$

The special case when $l = 0$ is the L^2-index theorem [At].

An equivalent and more concrete description of the above "fibration" is given by a vector bundle $\mathcal{E}$ over M whose fibers are finitely-generated right projective $\mathfrak{B}$-modules for an appropriate algebra $\mathfrak{B}$ [Mi]. We will use this latter description in making things precise, although we will move back and forth freely between the two pictures.

In another direction, using Quillen's theory of superconnections [Q], Bismut gave a heat equation proof of the Atiyah-Singer families index theorem on the level of differential forms on the base space [Bi]. Equation $(*)$ is a consequence.

Analogously, we wish to give a heat equation proof of $(**)$. Our original purpose was to study higher versions of spectral invariants, such as the eta invariant [Lo1]. These higher eta invariants should enter into a higher index theorem for manifolds with boundary. However, it turned out to be necessary to first understand the case of closed manifolds, i.e. equation $(**)$, in terms of superconections. This is what we present here.

As in [Bi], we wish to produce an explicit differential form on B which represents $\mathrm{Ch}\left(\mathrm{Index}(\widetilde{D})\right)$. First, one needs to know what a form on the noncommutative space B should mean. A differential complex $\overline{\Omega}_*(\mathfrak{B})$ was defined in [Ka], and its homology can be identified with a subspace of the cyclic homology of the relevant algebra $\mathfrak{B}$. In Section II we briefly review this theory. In this section we also consider integral operators on sections of $\mathcal{E}$ and define their traces and supertraces.

In the case at hand, the relevant vector bundles $\mathcal{E}$ come from a flat $\mathfrak{B}$-bundle over M. There is some choice in exactly which subalgebra $\mathfrak{B}$ of Λ is taken. In Section III we consider a subalgebra $\mathfrak{B}^\omega$ of Λ consisting of elements whose coefficients decay faster than any exponential in a word-length metric. If $\Gamma = \mathbf{Z}$ then $\mathfrak{B}^\omega$ is isomorphic to the restrictions of holomorphic functions on $\mathbf{C} - 0$ to the unit circle, and so $\mathfrak{B}^\omega$ is like an algebra of "analytic" functions on B. (The technical reason for the appearance of this algebra is the existence of finite-propagation-speed estimates for heat kernels on $\widetilde{M}$.) The smooth sections $\Gamma^\infty(\mathcal{E}^\omega)$ of the corresponding vector bundle $\mathcal{E}^\omega$ are shown to correspond to smooth sections of $S(\widetilde{M}) \otimes \widetilde{V}$ with rapid decay. Using this description, we make the trace of Section II more explicit.

By construction, the vector space of smooth sections of $\mathcal{E}^\omega$ is a right $\mathfrak{B}^\omega$-module. Let $\nabla : \Gamma^\infty(\mathcal{E}^\omega) \to \Gamma^\infty\big(\mathcal{E}^\omega \otimes_{\mathfrak{B}^\omega} \Omega_1(\mathfrak{B}^\omega)\big)$ be a connection on $\mathcal{E}^\omega$. This is, in a sense, a connection in the vertical direction of $\mathcal{E}^\omega$, when thought of as a vector bundle over M. Let Q be the Dirac-type operator on $\Gamma^\infty(\mathcal{E}^\omega)$. Applying Quillen's formalism [Q], for any $\beta, s > 0$, the Chern character of $\mathcal{E}^\omega$ is defined to be

$$\mathrm{ch}_{\beta,s}(\mathcal{E}^\omega) = \mathrm{STR}\exp\big(-\beta(\nabla + sQ)^2\big) \in \overline{\Omega}_*(\mathfrak{B}^\omega)\ . \qquad (***)$$

To make this expression useful, one needs an explicit description of a connection on $\mathcal{E}^\omega$. In Section IV we show that the simplest such connection comes from a function $h \in C_0^\infty(\widetilde{M})$ with the property that the sum of the translates of h is 1. Then $(***)$ is a well-defined closed element of $\overline{\Omega}_*(\mathfrak{B}^\omega)$, and its homology class is independent of s.

Given a group cocycle $\eta \in Z^l(\Gamma;\mathbf{C})$, if the corresponding cyclic cocycle $\tau_\eta \in ZC^l(\mathbf{C}\Gamma)$ extends to an element of $ZC^l(\mathfrak{B}^\omega)$ then the pairing

$$\langle \mathrm{ch}_{\beta,s}(\mathcal{E}^\omega), \tau_\eta\rangle \ \in \mathbf{C} \qquad (****)$$

is well-defined and independent of s. As usual with heat equation approaches to index theory, the $s \to 0$ limit of $(****)$ becomes the integral of a local expression on M. In Section V we compute this limit. (The local analysis is easier than in [Bi], as there is no need to use a Levi-Civita superconnection.) The limit must involve $\nu^*[\eta]$, and it may seem strange that this could become a local expression on M, but this is where the choice of h enters. In Proposition 12 we find

$$\lim_{s\to 0}\langle \mathrm{ch}_{\beta,s}(\mathcal{E}^\omega), \tau_\eta\rangle = \beta^{l/2}/(l!)\int_M \widehat{A}(M) \wedge \mathrm{Ch}(V) \wedge \omega\ ,$$

where ω is a closed l-form on M whose pullback to $\widetilde{M}$ is given by

$$\pi^*\omega = \sum R^*_{g_1} dh \wedge \ldots \wedge R^*_{g_l} dh \ \eta(e, g_1, \ldots, g_l) \in \Lambda^l(\widetilde{M}) \ .$$

We then show that ω represents $\nu^*[\eta] \in H^l(M; \mathbf{C})$.

It remains to show that

$$\langle \mathrm{ch}_{\beta,s}(\mathcal{E}^\omega), \tau_\eta \rangle = \left\langle \mathrm{Ch}_\beta \left(\mathrm{Index}(\widetilde{D}) \right), \tau_\eta \right\rangle \ . \qquad (*****)$$

For this, we find it necessary to work with the algebra $\mathfrak{B}^\infty$ and assume that τ_η extends to a cyclic cocycle of $\mathfrak{B}^\infty$. In Section VI we sketch a proof of $(*****)$. We reduce to the case of invertible $\widetilde{D}$, and then use a trick of [Bi] to show the equality. This completes the proof of $(**)$.

One application of $(**)$ is to the Novikov conjecture. Taking $\widetilde{D}$ to be the signature operator, the right-hand-side of $(**)$ becomes $const.(l) \int_M L(M) \wedge \nu^*[\eta]$, where $L(M) \in H^*(M; \mathbf{C})$ is the Hirzebruch L-polynomial. The Novikov conjecture states that this "higher" signature is an (orientation-preserving) homotopy invariant of M. One can show that $\mathrm{Index}(\widetilde{D}) \in K_0(\Lambda)$ is a homotopy invariant of M [Mi, Kas, HS]. If the group Γ is such that one can apply $(**)$ then the validity of the Novikov conjecture follows. In particular, in [CM] it was shown that if Γ is hyperbolic in the sense of Gromov [GH] then $(**)$ applies. Thus our proof of $(**)$ gives a new proof of the validity of the Novikov conjecture for hyperbolic groups. One can also apply $(**)$ to find obstructions to the existence of positive-scalar-curvature metrics on M [Ro]. If one takes $\widetilde{D}$ to be the pure Dirac operator then if M has positive scalar curvature, $\mathrm{Index}(\widetilde{D})$ vanishes. Thus if the group Γ is such that one can apply $(**)$, $\int_M \widehat{A}(M) \wedge \nu^*[\eta]$ is an obstruction to the existence of a positive-scalar-curvature metric on M.

In [Lo1] a bivariant Chern character was proposed in the case of finitely-generated projective modules. The obstacle to defining a bivariant Chern character for more general projective modules was the lack of a good trace theory for Hilbert modules. In the present case there is such a trace. The smooth sections of $\mathcal{E}^\infty = \mathcal{E}^\omega \otimes_{\mathfrak{B}^\omega} \mathfrak{B}^\infty$ form a $\left(C^\infty(M), \mathfrak{B}^\infty\right)$-bivariant module, and the pairing $\langle \mathrm{ch}_{\beta,s}, \tau_\eta \rangle$ of the bivariant Chern character with τ_η is a cocycle in the space $C^*_\epsilon\left(C^\infty(M)\right)$ of entire cyclic cochains [Co2]. In Section VII we compute the $s \to 0$ limit of $\langle \mathrm{ch}_{\beta,s}, \tau_\eta \rangle$.

Heat equation methods were also used in the paper of Connes and Moscovici [CM] to attack the Novikov conjecture, and it is worth comparing the two approaches. One difference is that we use heat kernels to form the Chern character of a superconnection as in $(***)$, whereas in [CM] the heat kernels are used to form an idempotent matrix over an algebra of smoothing operators [CM, Section 2]. Theorem 5.4 of [CM] is similar to our Corollary 2, but is stronger in that it is a statement about $\mathbf{C}\Gamma$, whereas Corollary 2 is a statement about $\mathfrak{B}^\omega$. We believe that there is some point to taking a superconnection approach to these questions, as there should be interesting extensions.

This paper is an extension of [Lo1], in which the finite-dimensional analog was worked out. An exposition of the Mischenko-Fomenko theorem and related results appears in [Hi].

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II. Algebraic Preliminaries

Let $\mathfrak{B}$ be a Fréchet locally m-convex algebra with unit, i.e. the projective limit of a sequence of Banach algebras with unit [Mal]. We first define a graded differential algebra (GDA) $\widehat{\Omega}_*(\mathfrak{B})$. This will be an appropriate completion of

$$\Omega_*(\mathfrak{B}) = \bigoplus_{k=0}^{\infty} \Omega_k(\mathfrak{B}) \;, \tag{1}$$

the universal GDA of $\mathfrak{B}$ [Co1, Ka]. As a vector space, $\Omega_k(\mathfrak{B})$ is given by

$$\Omega_k(\mathfrak{B}) = \mathfrak{B} \otimes \left(\otimes^k (\mathfrak{B}/\mathbf{C})\right) . \tag{2}$$

As a GDA, $\Omega_*(\mathfrak{B})$ is generated by $\mathfrak{B}$ and $d\mathfrak{B}$ with the relations

$$d1 = 0 \;, \quad d^2 = 0 \;, \quad d(\omega_k \omega_\ell) = (d\omega_k)\omega_\ell + (-1)^k \omega_k (d\omega_\ell) \tag{3}$$

for $\omega_k \in \Omega_k(\mathfrak{B})$, $\omega_\ell \in \Omega_\ell(\mathfrak{B})$. It will be convenient to write an element ω_k of $\Omega_k(\mathfrak{B})$ as a finite sum $\sum b_0 db_1 \dots db_k$. Recall that the homology of the differential complex $\overline{\Omega}_*(\mathfrak{B}) = \Omega_*(\mathfrak{B})/[\Omega_*(\mathfrak{B}), \Omega_*(\mathfrak{B})]$ is isomorphic to a subspace of the reduced cyclic homology of $\mathfrak{B}$ [Ka]. (This statement must be modified in degree zero, for which we refer to [Ka].)

Let $\Theta_*(\mathfrak{B})$ denote the GDA

$$\Theta_*(\mathfrak{B}) = \bigoplus_{k=0}^{\infty} (\otimes^{k+1} \mathfrak{B}) \;, \tag{4}$$

with the product given by

$$(b_0 \otimes b_1 \otimes \ldots \otimes b_k)(c_0 \otimes c_1 \otimes \ldots \otimes c_\ell) = b_0 \otimes b_1 \otimes \ldots \otimes b_k c_0 \otimes c_1 \otimes \ldots \otimes c_\ell \quad (5)$$

and the differential given by

$$\begin{aligned} d(b_0 \otimes b_1 \otimes \ldots \otimes b_k) = 1 \otimes b_0 \otimes b_1 \otimes \ldots \otimes b_k - b_0 \otimes 1 \otimes b_1 \otimes \ldots \\ \otimes b_k + \ldots + (-1)^{k+1} b_0 \otimes b_1 \otimes \ldots \otimes b_k \otimes 1 \ . \end{aligned} \quad (6)$$

Give $\Theta_k(\mathfrak{B})$ the projective tensor product topology, with closure $\widehat{\Theta}_k(\mathfrak{B})$. Let

$$\widehat{\Theta}_*(\mathfrak{B}) = \prod_{k=0}^{\infty} \widehat{\Theta}_k(\mathfrak{B}) \quad (7)$$

denote the completion of $\Theta_*(\mathfrak{B})$ in the product topology.

PROPOSITION 1. *$\widehat{\Theta}_*(\mathfrak{B})$ is a Fréchet* GDA.

There is a natural embedding $\mathfrak{e}$ of $\Omega_*(\mathfrak{B})$, as a graded differential algebra, in $\widehat{\Theta}_*(\mathfrak{B})$, with

$$\mathfrak{e}(b) = b \ , \quad \mathfrak{e}(db) = 1 \otimes b - b \otimes 1 \ . \quad (8)$$

Let $\widehat{\Omega}_*(\mathfrak{B})$ denote the closure of $\mathfrak{e}(\Omega_*(\mathfrak{B}))$ in $\widehat{\Theta}_*(\mathfrak{B})$.

COROLLARY 1. *$\widehat{\Omega}_*(\mathfrak{B})$ is a Fréchet* GDA.

Define $\overline{\widehat{\Omega}}_*(\mathfrak{B})$ to be $\widehat{\Omega}_*(\mathfrak{B})/\overline{[\widehat{\Omega}_*(\mathfrak{B}), \widehat{\Omega}_*(\mathfrak{B})]}$. Let $\overline{H}_*(\mathfrak{B})$ denote the homology of the differential complex $\overline{\widehat{\Omega}}_*(\mathfrak{B})$.

Let $\mathfrak{E}$ be a Fréchet space which is a (continuous) right $\mathfrak{B}$-module. If $\mathfrak{F}$ is a Fréchet space which is a (continuous) left $\mathfrak{B}$-module, let $\mathfrak{E} \widehat{\otimes} \mathfrak{F}$ be the projective topological tensor product of $\mathfrak{E}$ and $\mathfrak{F}$. Let $\mathfrak{H}$ be the closure in $\mathfrak{E} \widehat{\otimes} \mathfrak{F}$ of

$$\operatorname{span}\{eb \otimes f - e \otimes bf : e \in \mathfrak{E} \ , \ f \in \mathfrak{F} \ , \ b \in \mathfrak{B}\} \ . \quad (9)$$

We put $\mathfrak{E} \widehat{\otimes}_{\mathfrak{B}} \mathfrak{F}$ to be the Fréchet space $(\mathfrak{E} \widehat{\otimes} \mathfrak{F})/\mathfrak{H}$.

With this definition, $\mathfrak{E} \widehat{\otimes}_{\mathfrak{B}} \widehat{\Omega}_k(\mathfrak{B})$ is isomorphic to the closure of the algebraic tensor product $\mathfrak{E} \otimes_{\mathfrak{B}} \Omega_k(\mathfrak{B}) \subset \mathfrak{E} \otimes_{\mathfrak{B}} (\otimes^{k+1} \mathfrak{B}) = \mathfrak{E} \otimes (\otimes^k \mathfrak{B})$ in $\mathfrak{E} \widehat{\otimes} (\widehat{\otimes}^k (\mathfrak{B}))$, where the latter has the projective tensor product topology.

For the rest of this section, we assume that $\mathfrak{E}$ is a finitely generated right projective $\mathfrak{B}$-module. Let $\mathfrak{F}$ be a Fréchet $\mathfrak{B}$-bimodule. Then there is a trace

$$\mathrm{Tr} : \mathrm{Hom}_{\mathfrak{B}}(\mathfrak{E}, \mathfrak{E} \widehat{\otimes}_{\mathfrak{B}} \mathfrak{F}) \to \mathfrak{F}/\overline{[\mathfrak{B}, \mathfrak{F}]} \,. \tag{10}$$

To define Tr, write $\mathfrak{E}$ as $e\mathfrak{B}^n$, with e a projector in $M_n(\mathfrak{B})$. Then an operator $T \in \mathrm{Hom}_{\mathfrak{B}}(\mathfrak{E}, \mathfrak{E} \widehat{\otimes}_{\mathfrak{B}} \mathfrak{F}) = \mathrm{Hom}_{\mathfrak{B}}(e\mathfrak{B}^n, e\mathfrak{F}^n)$ can be represented by a matrix $T \in M_n(\mathfrak{F})$ satisfying $eT = Te = T$. Put

$$\mathrm{Tr}(T) = \sum_{i=1}^{n} T_{ii} \qquad \left(\mathrm{mod}\,\overline{[\mathfrak{B}, \mathfrak{F}]}\right) \,. \tag{11}$$

This is independent of the choices made. (We quotient by the closure of $[\mathfrak{B}, \mathfrak{F}]$ to ensure that the trace takes value in a Fréchet space.)

LEMMA 1. *Suppose that $\mathfrak{E}$ and $\mathfrak{E}'$ are finitely generated right projective $\mathfrak{B}$-modules and $\mathfrak{F}$ is a Fréchet algebra containing $\mathfrak{B}$. Given $T \in \mathrm{Hom}_{\mathfrak{B}}(\mathfrak{E}, \mathfrak{E}' \widehat{\otimes}_{\mathfrak{B}} \mathfrak{F})$ and $T' \in \mathrm{Hom}_{\mathfrak{B}}(\mathfrak{E}', \mathfrak{E} \widehat{\otimes}_{\mathfrak{B}} \mathfrak{F})$, let $T'T \in \mathrm{Hom}_{\mathfrak{B}}(\mathfrak{E}, \mathfrak{E} \widehat{\otimes}_{\mathfrak{B}} \mathfrak{F})$ and $TT' \in \mathrm{Hom}_{\mathfrak{B}}(\mathfrak{E}', \mathfrak{E}' \widehat{\otimes}_{\mathfrak{B}} \mathfrak{F})$ be the induced products. Then $\mathrm{Tr}(T'T) = \mathrm{Tr}(TT') \in \mathfrak{F}/\overline{[\mathfrak{F}, \mathfrak{F}]}$.*

We omit the proof.

In the case that $\mathfrak{E}$ is $\mathbb{Z}_2$-graded by an operator $\Gamma_{\mathfrak{E}} \in \mathrm{End}_{\mathfrak{B}}(\mathfrak{E})$ satisfying $\Gamma_{\mathfrak{E}}^2 = 1$, we can extend the trace to a supertrace by $\mathrm{Tr}_s(T) = \mathrm{Tr}(\Gamma_{\mathfrak{E}} T)$.

Let M be a closed connected oriented smooth Riemannian manifold. Let $\mathcal{E}$ be a smooth $\mathfrak{B}$-vector bundle on M with fibers isomorphic to $\mathfrak{E}$. This means that if $\mathcal{E}$ is defined using charts $\{U_\alpha\}$, then a transition function is a smooth map $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{End}_{\mathfrak{B}}(\mathfrak{E})$. We will denote the fiber over $m \in M$ by $\mathcal{E}_m$. If $\mathfrak{F}$ is a Fréchet algebra containing $\mathfrak{B}$, let $\mathcal{E} \widehat{\otimes}_{\mathfrak{B}} \mathfrak{F}$ denote the $\mathfrak{B}$-vector bundle with fibers $(\mathcal{E} \widehat{\otimes}_{\mathfrak{B}} \mathfrak{F})_m = \mathcal{E}_m \widehat{\otimes}_{\mathfrak{B}} \mathfrak{F}$ and transition functions $\phi_{\alpha\beta} \widehat{\otimes}_{\mathfrak{B}} Id_{\mathfrak{F}} \in \mathrm{End}_{\mathfrak{B}}(\mathfrak{E} \widehat{\otimes}_{\mathfrak{B}} \mathfrak{F})$. Let $\Gamma^\infty(\mathcal{E})$ denote the right $\mathfrak{B}$-module of smooth sections of $\mathcal{E}$.

DEFINITION: Let $\mathrm{Hom}^\infty_{\mathfrak{B}}(\mathcal{E}, \mathcal{E} \widehat{\otimes}_{\mathfrak{B}} \mathfrak{F})$ be the algebra of integral operators $T : \Gamma^\infty(\mathcal{E}) \to \Gamma^\infty(\mathcal{E} \widehat{\otimes}_{\mathfrak{B}} \mathfrak{F})$ with smooth kernels $T(m_1, m_2) \in \mathrm{Hom}_{\mathfrak{B}}(\mathcal{E}_{m_2}, \mathcal{E}_{m_1} \widehat{\otimes}_{\mathfrak{B}} \mathfrak{F})$. That is, for $s \in \Gamma^\infty(\mathcal{E})$,

$$(Ts)(m_1) = \int_M T(m_1, m_2) s(m_2) dvol(m_2) \in \mathcal{E}_{m_1} \widehat{\otimes}_{\mathfrak{B}} \mathfrak{F} \,. \tag{12}$$

DEFINITION: For $T \in \mathrm{Hom}^{\infty}_{\mathfrak{B}}(\mathcal{E}, \mathcal{E}\,\widehat{\otimes}_{\mathfrak{B}} \mathfrak{F})$,

$$\mathrm{TR}(T) = \int_M \mathrm{Tr}\left(T(m,m)\right) dvol(m) \in \mathfrak{F}/\overline{[\mathfrak{F},\mathfrak{F}]} \,. \tag{13}$$

PROPOSITION 2. TR *is a trace.*

Proof: We have

$$(TT')(m,m') = \int_M T(m,m'')\; T'(m'',m') dvol(m'') \,. \tag{14}$$

Then

$$\mathrm{TR}(TT') = \int_M \mathrm{Tr}\left(T(m,m'')T'(m'',m)\right) dvol(m'') dvol(m) = \\ \int_M \mathrm{Tr}\left(T'(m'',m)T(m,m'')\right) dvol(m) dvol(m'') = \mathrm{TR}(T'T) \,. \tag{15}$$

□

If the fibers of $\mathcal{E}$ are $\mathbf{Z}_2$-graded, we can extend TR to a supertrace STR on $\mathrm{Hom}^{\infty}_{\mathfrak{B}}(\mathcal{E}, \mathcal{E}\,\widehat{\otimes}_{\mathfrak{B}} \mathfrak{F})$ by

$$\mathrm{STR}(T) = \int_M \mathrm{Tr}_s\left(T(m,m)\right) dvol(m) \in \mathfrak{F}/\overline{[\mathfrak{F},\mathfrak{F}]} \,. \tag{16}$$

III. $\mathfrak{B}^{\omega}$-Bundles

Let Γ be a finitely-generated discrete group and let $\| \circ \|$ be a right-invariant word-length metric on Γ. For $q \in \mathbf{Z}$, define the Hilbert space

$$\ell^{2,q}(\Gamma) = \left\{ f : \Gamma \to \mathbf{C} : |f|_q^2 = \sum_g \exp\left(2q\|g\|\right) \left|f(g)\right|^2 < \infty \right\} \tag{17}$$

and let $\mathfrak{B}^{\omega}$ be the vector space

$$\mathfrak{B}^{\omega} = \bigcap_q \ell^{2,q}(\Gamma) \,. \tag{18}$$

LEMMA 2.

$$\mathfrak{B}^{\omega} = \left\{ f : \Gamma \to \mathbf{C} : \text{ for all } q \in \mathbf{Z} \,,\; \sup_g \left(\exp\left(q\|g\|\right) \left|f(g)\right| \right) < \infty \right\} .$$

Proof: If $f \in \mathfrak{B}^\omega$ then for all $q \in \mathbf{Z}$, $\exp\left(2q\|g\|\right)\left|f(g)\right|^2$ is bounded in g, and so $\exp\left(q\|g\|\right)\left|f(g)\right|$ is bounded in g. Suppose that $f : \Gamma \to \mathbf{C}$ is such that for all $r \in \mathbf{Z}$,

$$\sup_g \left(\exp\left(r\|g\|\right)|(g)|\right) = C_r < \infty \ .$$

Then $\sum_g \exp\left(2q\|g\|\right)\left|f(g)\right|^2 \leq C_r^2 \sum_g \exp\left(2(q-r)\|g\|\right)$. As Γ has at most exponential growth, by taking r large enough we can ensure that the last sum is finite. □

PROPOSITION 3. *$\mathfrak{B}^\omega$ is independent of the choice of $\| \circ \|$, and is an algebra with unit under convolution.*

Proof: As all word-length metrics are quasi-isometric [GH], the independence follows. If $T \in \mathfrak{B}^\omega$ and $f \in \ell^{2,q}(\Gamma)$, we will show that

$$|T * f|_q \ \leq const.(q,T)|f|_q. \tag{19}$$

If we then take both T and f in $\mathfrak{B}^\omega$, the proposition will follow.

Let f_h denote $f(h)$. Then

$$\begin{aligned} &\left|\sum_h T_{gh^{-1}} f_h\right|^2 \leq \\ &\left(\sum_h \exp\left(-q\|h\|\right)|T_{gh^{-1}}|^{1/2}|T_{gh^{-1}}|^{1/2}\exp\left(q\|h\|\right)|f_h|\right)^2 \leq \\ &\left(\sum_h \exp\left(-2q\|h\|\right)|T_{gh^{-1}}|\right)\left(\sum_{h'}|T_{gh'^{-1}}|\exp\left(2q\|h'\|\right)|f_{h'}|^2\right) . \end{aligned} \tag{20}$$

Thus

$$\begin{aligned} &|T * f|_q^2 = \sum_g \exp\left(2q\|g\|\right)\left|\sum_h T_{gh^{-1}} f_h\right|^2 \leq \\ &\sum_g \left(\sum_h \exp\left(2q(\|g\| - \|h\|)\right) |T_{gh^{-1}}|\right)\left(\sum_{h'}|T_{gh'^{-1}}|\exp\left(2q\|h'\|\right)|f_{h'}|^2\right) \leq \\ &\sum_g \left(\sum_h \exp\left(2q\|gh^{-1}\|\right)|T_{gh^{-1}}|\right)\left(\sum_{h'}|T_{gh'^{-1}}|\exp\left(2q\|h'\|\right)|f_{h'}|^2\right) \leq \\ &\left(\sum_k \exp\left(2q\|k\|\right)|T_k|\right)\left(\sum_\ell |T_\ell|\right)\left(\sum_{h'}\exp\left(2q\|h'\|\right)|f_{h'}|^2\right) = \\ &\left(\sum_k \exp\left(2q\|k\|\right)|T_k|\right)\left(\sum_\ell |T_\ell|\right)|f|_q^2 \ . \end{aligned} \tag{21}$$

□

Let Λ denote the reduced group C^*-algebra of Γ, namely the completion of $\mathbb{C}\Gamma$ with respect to the operator norm on $B(\ell^2(\Gamma))$, where $\mathbb{C}\Gamma$ acts on $\ell^2(\Gamma)$ by convolution.

There is a Fréchet topology on $\mathfrak{B}^\omega$ coming from its definition as a projective limit of Hilbert spaces. There is also a description of $\mathfrak{B}^\omega$ as a Fréchet locally m-convex algebra. Namely, put

$$\mathcal{P} = \{T \in \Lambda : \text{for all } q \in \mathbb{Z}, T \text{ acts as a bounded operator by convolution on } \ell^{2,q}(\Gamma)\} . \tag{22}$$

By its definition, $\mathcal{P}$ is equipped with a sequence of norms.

PROPOSITION 4. *As topological vector spaces,* $\mathfrak{B}^\omega = \mathcal{P}$.

Proof: By the proof of Proposition 3, $\mathfrak{B}^\omega$ injects continuously into $\mathcal{P}$. Applying an element T of $\mathcal{P}$ to the element $e \in \bigcap_q \ell^{2,q}(\Gamma)$ gives a continous injection of $\mathcal{P}$ into $\mathfrak{B}^\omega$. These two maps are clearly inverses of each other.□

It follows that $\mathfrak{B}^\omega$ has a holomorphic functional calculus.

Note. $\mathfrak{B}^\omega$ is generally not holomorphically closed in Λ. For example, if $\Gamma = \mathbb{Z}$ then an element T of $\mathfrak{B}^\omega$ can be identified with its Fourier transform $T = \sum T_g z^g$, a holomorphic function on $\mathbb{C} - 0$. This identification gives $\mathfrak{B}^\omega \cong H(\mathbb{C} - 0)$. On the other hand, in this case $\Lambda \cong C(S^1)$. Taking for example $T = z \in H(\mathbb{C} - 0) \subset C(S^1)$, the spectrum of T in $C(S^1)$ consists of the unit circle. If f is the holomorphic function defined on a neighborhood of the unit circle by $f(w) = (w-2)^{-1}$, $f(T)$ is well-defined in $C(S^1)$, but does not lie in $H(\mathbb{C} - 0)$.

Let Γ denote the fundamental group of M. Let $\widetilde{M}$ denote the universal cover of M, on which $g \in \Gamma$ acts on the right by $R_g \in \text{Diff}(\widetilde{M})$. Denote the covering map by $\pi : \widetilde{M} \to M$. As Γ acts on $\mathfrak{B}^\omega$ on the left, we can form $\widetilde{M} \times_\Gamma \mathfrak{B}^\omega$, a flat $\mathfrak{B}^\omega$-bundle over M. Let E be a Hermitian vector bundle with Hermitian connection on M and let $\widetilde{E}$ be the pullback of E to $\widetilde{M}$, with the pulled-back connection. Let $R_g^* \in \text{Aut}(\widetilde{E})$ denote the action of $g \in \Gamma$ on $\widetilde{E}$.

DEFINITION: $\mathcal{E}^\omega = (\widetilde{M} \times_\Gamma \mathfrak{B}^\omega) \otimes E$, a $\mathfrak{B}^\omega$-bundle over M.

Fix a base point $x_0 \in \widetilde{M}$.

PROPOSITION 5. *There is an isomorphism*

$$L : \Gamma^\infty(\mathcal{E}^\omega) \to \Big\{ f \in C^\infty(\widetilde{M}, \widetilde{E}) : \text{ for all } q \in \mathbf{Z} \text{ and all multi-indices } \alpha \ ,$$

$$\sup_x \left(\exp\left(qd(x_0,x)\right)\left|\nabla^\alpha f(x)\right|\right) < \infty \Big\} \ .$$

Proof: By the construction of $\mathcal{E}^\omega$, $\Gamma^\infty(\mathcal{E}^\omega)$ consists of the Γ-equivariant elements of $C^\infty(\widetilde{M}, \widetilde{E} \otimes \mathfrak{B}^\omega)$. Writing $s \in \Gamma^\infty(\mathcal{E}^\omega)$ as $\sum_g s_g g$ with $s_g \in C^\infty(\widetilde{M}, \widetilde{E})$, the equivariance means that

$$R^*_\gamma \gamma s = s \text{ for all } \gamma \in \Gamma. \tag{23}$$

This becomes $\sum_g (R^*_\gamma s_g)\gamma g = \sum_g s_{\gamma g}\gamma g$, and so $R^*_\gamma s_g = s_{\gamma g}$ for all $\gamma, g \in \Gamma$. Thus $s_g = R^*_g s_1$, and so $s = \sum_g \left(R^*_g s_1\right) g$.

Let L be the map which takes s to s_1. We will show that L is the desired isomorphism. First, if $\widetilde{m} \in \widetilde{M}$ then

$$s(\widetilde{m}) = \sum_g (R^*_g s_1)(\widetilde{m}) \ g \in \widetilde{E}_{\widetilde{m}} \otimes \mathfrak{B}^\omega \ . \tag{24}$$

Thus for all $q \in \mathbf{Z}$, $\sup_g \left(\exp\left(q\|g\|\right)\left|s_1(\widetilde{m}g)\right|\right) < \infty$. By the smoothness of s, we have such an estimate uniformly for $\widetilde{m}$ lying within a fundamental domain of $\widetilde{M}$ containing x_0. As $\widetilde{M}$ is quasi-isometric to Γ [GH], there are constants $A > 0$ and $B \geq 0$ such that for all $x \in \widetilde{M}$ and $g \in \Gamma$,

$$A^{-1}\|g\| - B \leq d(xg^{-1}, x) \leq A\|g\| + \mathrm{B} \ . \tag{25}$$

Then

$$\begin{aligned} &\exp\left(qd(x_0,x)\right)\left|s_1(x)\right| \leq \\ &\exp\left(qd(x_0,xg^{-1})\right)\exp\left(qd(xg^{-1},x)\right)\left|s_1(xg^{-1}g)\right| \leq \\ &const.\exp\left(qd(x_0,xg^{-1})\right)\exp\left(qA\|g\|\right)\left|s_1(xg^{-1}g)\right| \ . \end{aligned} \tag{26}$$

By choosing g so that xg^{-1} lies within a fundamental domain containing x_0, we obtain from (26) that $\exp\left(qd(x_0,x)\right)\left|s_1(x)\right|$ is uniformly bounded in x. The same argument applies to the covariant derivatives of s_1.

Now suppose that $f \in C^\infty(\widetilde{M}, \widetilde{E})$ is such that for all $q \in \mathbf{Z}$ and all multi-indices α,

$$\sup_x \left(\exp\left(qd(x_0,x)\right)\left|\nabla^\alpha f(x)\right|\right) < \infty \ . \tag{27}$$

Put $L'(f) = \sum_g (R_g^* f)\, g$. We must show that $L'(f) \in \Gamma^\infty(\mathcal{E}^\omega)$. It will then follow that L' is an inverse to L.

By construction, $L'(f)$ is Γ-equivariant. Let $\{V_\alpha\}$ be a collection of charts on M over which E is trivialized. Then we can reduce to the case that E is a trivial $\mathbf{C}$-bundle and $f \in C^\infty(V_\alpha \times \Gamma, \mathbf{C})$, with the above decay conditions. It is enough to show that when restricted to $V_\alpha \times \{e\}$, $\sum_g (R_g^* f) g$ represents a smooth map from V_α to $\mathfrak{B}^\omega$. For $\widetilde{m} \in V_\alpha \times \{e\}$,

$$\Big(\sum (R_g^* f) g\Big)(\widetilde{m}) = \sum f(\widetilde{m} g) g \ , \tag{28}$$

and so for all $q \in \mathbf{Z}$,

$$\begin{aligned}
&\exp\big(q\|g\|\big)\big|f(\widetilde{m}g)\big| \leq \\
&const. \exp\big(qA\, d(\widetilde{m}, \widetilde{m}g)\big)\big|f(\widetilde{m}g)\big| \leq \\
&const. \exp\big(qA\, d(\widetilde{m}, x_0)\big) \exp\big(qA\, d(x_0, \widetilde{m}g)\big)\big|f(\widetilde{m}g)\big| \leq \\
&const. \sup_x \big(\exp\big(qA\, d(x_0, x)\big)\big|f(x)\big|\big) < \infty \ .
\end{aligned} \tag{29}$$

Thus $\sum_g (R_g^* f)\, g$ is a map from V_α to $\mathfrak{B}^\omega$. Doing the same estimates using covariant derivatives gives the smoothness. □

PROPOSITION 6. *The algebra* $\mathrm{End}^\infty_{\mathfrak{B}^\omega}(\mathcal{E}^\omega) \equiv \mathrm{Hom}^\infty_{\mathfrak{B}^\omega}(\mathcal{E}^\omega, \mathcal{E}^\omega)$ *is isomorphic to the algebra of* Γ*-invariant integral operators* T *on* $L^2(\widetilde{M}, \widetilde{E})$ *with smooth kernels* $T(x,y) \in \mathrm{Hom}(\widetilde{E}_y, \widetilde{E}_x)$ *such that for all* $q \in \mathbf{Z}$ *and multi-indices* α *and* β,

$$\sup_{x,y} \big(\exp\big(qd(x,y)\big)\big|\nabla_x^\alpha \nabla_y^\beta T(x,y)\big|\big) < \infty \ .$$

We omit the proof, which is similar to that of Proposition 5.

Let $\phi \in C_0^\infty(\widetilde{M})$ be such that

$$\sum_g R_g^* \phi = 1 \ . \tag{30}$$

Let tr denote the local trace on $\mathrm{End}(\widetilde{E}_x)$.

Note. We now have defined three traces: tr is the trace on $\mathrm{End}(\widetilde{E}_x)$, Tr is the trace on $\mathrm{End}_{\mathfrak{B}^\omega}(\mathcal{E}^\omega_m)$ and TR is the trace on $\mathrm{End}^\infty_{\mathfrak{B}^\omega}(\mathcal{E}^\omega)$. If E is $\mathbb{Z}_2$-graded, the corresponding supertraces are denoted $\mathrm{tr}_s, \mathrm{Tr}_s$ and STR.

PROPOSITION 7. *Representing an element $T \in \mathrm{End}^\infty_{\mathfrak{B}^\omega}(\mathcal{E}^\omega)$ by an operator $\widetilde{T} \in B\big(L^2(\widetilde{M}, \widetilde{E})\big)$ as in Proposition 6, its trace is given by*

$$\mathrm{TR}(T) = \sum_g \left[\int_{\widetilde{M}} \phi(x)\, \mathrm{tr}\left((R_g^* \widetilde{T})(x,x)\right) dvol(x) \right] g \quad \left(\mathrm{mod}\, \overline{[\mathfrak{B}^\omega, \mathfrak{B}^\omega]}\right) \tag{31}$$

$$= \sum_g \left[\int_{\widetilde{M}} \phi(x)\, \mathrm{tr}\left(\widetilde{T}(xg,x)\right) dvol(x) \right] g \quad \left(\mathrm{mod}\, \overline{[\mathfrak{B}^\omega, \mathfrak{B}^\omega]}\right) . \tag{32}$$

Proof: The proof is a matter of unraveling the isomorphisms of Propositions 5 and 6. Let $\{V_\alpha\}$ be a collection of charts on M over which E is trivialized. Then we can reduce to the case that E is a trivial $\mathbb{C}$-bundle. We have $\pi^{-1}(V_\alpha) \cong V_\alpha \times \Gamma$. For $\widetilde{m}_1, \widetilde{m}_2 \in V_\alpha \times \{e\}$, we can use isomorphisms to represent

$$T(m_1, m_2) \in \mathrm{Hom}^\infty_{\mathfrak{B}^\omega}(\mathcal{E}^\omega_{m_2}, \mathcal{E}^\omega_{m_1}) \cong \mathrm{Hom}_{\mathfrak{B}^\omega}(\mathfrak{B}^\omega, \mathfrak{B}^\omega) \cong \mathfrak{B}^\omega \tag{33}$$

by $\sum_g \widetilde{T}(\widetilde{m}_1 g, \widetilde{m}_2) g$. Then

$$\int_{V_\alpha} \mathrm{Tr}\left(T(m,m)\right) dvol(m) =$$

$$\int_{V_\alpha} \sum_g \widetilde{T}(mg,m) g\, dvol(m) \quad \left(\mathrm{mod}\, \overline{[\mathfrak{B}^\omega, \mathfrak{B}^\omega]}\right) =$$

$$\int_{V_\alpha} \sum_g \sum_\gamma \phi(m\gamma)\, \widetilde{T}(mg\gamma, m\gamma) g\, dvol(m) \quad \left(\mathrm{mod}\, \overline{[\mathfrak{B}^\omega, \mathfrak{B}^\omega]}\right) =$$

$$\int_{V_\alpha} \sum_g \sum_\gamma \phi(m\gamma)\, \widetilde{T}(m\gamma\gamma^{-1} g\gamma, m\gamma) g\, dvol(m) \quad \left(\mathrm{mod}\, \overline{[\mathfrak{B}^\omega, \mathfrak{B}^\omega]}\right) =$$

$$\int_{V_\alpha} \sum_g \sum_\gamma \phi(m\gamma)\, \widetilde{T}(m\gamma g, m\gamma)\, \gamma g \gamma^{-1}\, dvol(m) \quad \left(\mathrm{mod}\, \overline{[\mathfrak{B}^\omega, \mathfrak{B}^\omega]}\right) =$$

$$\int_{V_\alpha} \sum_g \sum_\gamma \phi(m\gamma) \widetilde{T}(m\gamma g, m\gamma) \left(g + [\gamma g, \gamma^{-1}]\right) dvol(m) \; \left(\mathrm{mod}\, \overline{[\mathfrak{B}^\omega, \mathfrak{B}^\omega]}\right) =$$

$$\int_{V_\alpha} \sum_\gamma \sum_g \phi(m\gamma)\, \widetilde{T}(m\gamma g, m\gamma) g\, dvol(m) \quad \left(\mathrm{mod}\, \overline{[\mathfrak{B}^\omega, \mathfrak{B}^\omega]}\right) =$$

$$\int_{\pi^{-1}(V_\alpha)} \sum_g \phi(x)\widetilde{T}(xg,x)g\, dvol(x) \quad \left(\mathrm{mod}\,\overline{[\mathfrak{B}^\omega,\mathfrak{B}^\omega]}\right) . \tag{34}$$

Using a partition of unity subordinate to $\{V_\alpha\}$ and adding the contributions of the various charts gives (31). □

We now give the extension of the previous propositions to form-valued sections of $\mathcal{E}^\omega$. With the notation of Section II, put $\mathfrak{F}^\omega = \widehat{\Omega}_*(\mathfrak{B}^\omega)$. As in Proposition 5, we can represent an element f of $\Gamma^\infty(\mathcal{E}^\omega \widehat{\otimes}_{\mathfrak{B}^\omega} \mathfrak{F}^\omega)$ of degree k as $\sum f_{g_1\ldots g_k} dg_1 \ldots dg_k$, with each $f_{g_1\ldots g_k} \in C^\infty(\widetilde{M},\widetilde{E})$ a smooth rapidly decreasing section of $\widetilde{E}$. As in Proposition 6, we can represent an element K of $\mathrm{Hom}^\infty_{\mathfrak{B}^\omega}(\mathcal{E}^\omega, \mathcal{E}^\omega \widehat{\otimes}_{\mathfrak{B}^\omega} \mathfrak{F}^\omega)$ of degree k by smooth rapidly decreasing kernels $K_{g_1\ldots g_k}(x,y) \in \mathrm{Hom}(\widetilde{E}_y, \widetilde{E}_x)$ such that $K = \sum K_{g_1\ldots g_k} dg_1 \ldots dg_k$ is Γ-invariant. Then for $f \in \Gamma^\infty(\mathcal{E}^\omega)$ we have

$$(Kf)(x) = \sum \int_{\widetilde{M}} K_{g_1\ldots g_k}(x,y) f(y) dvol(y) dg_1 \ldots dg_k . \tag{35}$$

As in Proposition 7, we have

$$\begin{aligned}\mathrm{TR}(K) = \sum \int_{\widetilde{M}} \phi(x)\ \mathrm{tr}\left(K_{g_1\ldots g_k}(xg_0,x)\right) dvol(x)\, g_0 dg_1 \ldots dg_k \\ \left(\mathrm{mod}\,\overline{[\widehat{\Omega}_*(\mathfrak{B}^\omega), \widehat{\Omega}_*(\mathfrak{B}^\omega)]}\right) .\end{aligned} \tag{36}$$

IV. The Chern Character

Now suppose in addition that M^n is even-dimensional and spin. Let S be the $\mathbf{Z}_2$-graded spinor bundle on M, with the Levi-Civita connection, and let V be a Hermitian bundle on M with Hermitian connection. Take E to be $S \otimes V$. Let Q denote the self-adjoint extension of the Dirac-type operator acting on $C_0^\infty(\widetilde{M},\widetilde{E})$ [At]. In terms of a local framing of the tangent bundle,

$$Q = -i \sum_{\mu=1}^{n} \gamma^\mu D_\mu , \tag{37}$$

with the Dirac matrices $\{\gamma^\mu\}_{\mu=1}^n$ satisfying

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\delta^{\mu\nu} . \tag{38}$$

Proposition 8. *For $T > 0$, $e^{-TQ^2} \in \mathrm{End}^\infty_{\mathfrak{B}^\omega}(\mathcal{E}^\omega)$.*

Proof: First, e^{-TQ^2} is a Γ-invariant operator. By elliptic regularity, $e^{-TQ^2}(x,y)$ is smooth. Put $N = [n/4]+1$. Let ϵ be a fixed sufficiently small number. If $d(x,y) > \epsilon$, put $R = d(x,y) - \epsilon$. By the finite-propagation-speed estimates of [CGT], we have the estimate [Lo2]

$$\begin{aligned}&\left|(Q^{2k}e^{-TQ^2}Q^{2\ell})(x,y)\right| \leq \\ &const.(R^2/T)^{-1/2}[R^{-2(k+\ell)} + R^{-2(k+\ell)-4N} + \\ &R^{2(k+\ell)}T^{-2(k+\ell)} + R^{2(k+\ell)+4N}T^{-2(k+\ell)-4N}]\, e^{-R^2/4T}\ .\end{aligned} \tag{39}$$

The requisite bounds on the covariant derivatives of $e^{-TQ^2}(x,y)$ follow by standard methods. Then the proposition follows from Proposition 6. □

Note. In the "fibration" picture, the fact that e^{-TQ^2} commutes with $\mathfrak{B}^\omega$ means that it corresponds to a family of vertical operators.

Let $h \in C_0^\infty(\widetilde{M})$ be such that

$$\sum_g R_g^* h = 1\ . \tag{40}$$

Given $f \in \Gamma^\infty(\mathcal{E}^\omega)$, considering it as an element of $C^\infty(\widetilde{M}, \widetilde{E})$ by Proposition 5, define its covariant derivative to be

$$\nabla_g f = h\ R_g^* f \in C^\infty(\widetilde{M}, \widetilde{E})\ . \tag{41}$$

Note that $C^\infty(M)$ acts on sections of $\Gamma^\infty(\mathcal{E}^\omega \widehat{\otimes}_{\mathfrak{B}^\omega} \mathfrak{F}^\omega)$ by multiplication.

PROPOSITION 9.

$$\nabla f = \sum_g \nabla_g f \,\widehat{\otimes}_{\mathfrak{B}^\omega} dg$$

defines a connection

$$\nabla : \Gamma^\infty(\mathcal{E}^\omega) \to \Gamma^\infty(\mathcal{E}^\omega \,\widehat{\otimes}_{\mathfrak{B}^\omega} \widehat{\Omega}_1(\mathfrak{B}^\omega)) \tag{42}$$

which commutes with the action of $C^\infty(M)$.

Proof: We first show that ∇ formally commutes with the action of $C^\infty(M)$. Given $\alpha \in C^\infty(M)$, α acts on $C^\infty(\widetilde{M}, \widetilde{E})$ by multiplication by $\pi^*(\alpha)$. Then

$$\begin{gathered}\nabla(\alpha \cdot f) = \nabla\big(\pi^*(\alpha) f\big) = \sum_g h\, R_g^*\big(\pi^*(\alpha) f\big) \,\widehat{\otimes}_{\mathfrak{B}^\omega} dg = \\ \sum_g h\, \pi^*(\alpha)\, R_g^* f \,\widehat{\otimes}_{\mathfrak{B}^\omega} dg = \alpha \cdot \nabla f \ .\end{gathered} \tag{43}$$

Thus ∇ acts fiberwise on the vector bundle $\mathcal{E}^\omega$. To make this explicit, as in the proof of Proposition 5 we can consider the element s of $\Gamma^\infty(\mathcal{E}^\omega)$ corresponding to f to be a sum $s = \sum s_g\, g$, where $s_g \in C^\infty(\widetilde{M}, \widetilde{E})$ and $s_g = R_g^* f$. Then ∇s becomes

$$\sum_{g,k} R_g^*(h R_k^* f)\, g dk = \sum_{g,k} R_g^* h (R_{gk}^* f)\, g dk = \sum_{g,k} R_g^* h\, s_{gk}\, g dk \ . \tag{44}$$

Applied to a point $\widetilde{m} \in \widetilde{M}$, we have

$$\nabla\Big(\sum_g s_g(\widetilde{m})\, g\Big) = \sum_{g,k} h(\widetilde{m} g)\ s_{gk}(\widetilde{m})\, g dk \ . \tag{45}$$

Let

$$\nabla_m : \mathcal{E}_m^\omega \to \mathcal{E}_m^\omega \,\widehat{\otimes}_{\mathfrak{B}^\omega} \widehat{\Omega}_1(\mathfrak{B}^\omega) \tag{46}$$

be the restriction of ∇ to the fiber $\mathcal{E}_m^\omega \cong E_m \otimes \mathfrak{B}^\omega$ over $m = \pi(\widetilde{m})$. Then ∇_m can be represented by

$$\nabla_m\Big(\sum_g t_g\, g\Big) = \sum_{g,k} h(\widetilde{m} g)\, t_{gk}\, g dk \ , \tag{47}$$

where $t_g \in \widetilde{E}_{\widetilde{m}}$.

By hypothesis, $t = \sum_g t_g\ g \in \mathcal{E}_m^\omega \cong E_m \otimes \mathfrak{B}^\omega$. We must show that $\nabla_m(t)$ is in $\mathcal{E}_m^\omega \,\widehat{\otimes}_{\mathfrak{B}^\omega} \widehat{\Omega}_1(\mathfrak{B}^\omega) \cong E_m \otimes \widehat{\Omega}_1(\mathfrak{B}^\omega)$.

As in Section II, let us think of $E_m \otimes \widehat{\Omega}_1(\mathfrak{B}^\omega)$ as embedded in $E_m \otimes \mathfrak{B}^\omega \,\widehat{\otimes} \mathfrak{B}^\omega$. Then $\nabla_m(t)$ is formally represented as

$$\nabla_m(t) = \sum_{g,k} h(\widetilde{m} g)\ t_{gk}\ g(1 \otimes k - k \otimes 1) =$$

$$\sum_{g,k} h(\widetilde{m}g)\ t_{gk}\ g \otimes k - \sum_{g,k} h(\widetilde{m}g)\ t_{gk}\ gk \otimes 1 =$$

$$\sum_{g,k} h(\widetilde{m}g)\ t_{gk}\ g \otimes k - \sum_{g,k} h(\widetilde{m}g)\ t_k\ k \otimes 1 =$$

$$\left(\sum_{g} h(\widetilde{m}g)g \otimes \sum_{k}\ t_{gk}\ k\right) - t \otimes 1 =$$

$$\left(\sum_{g} h(\widetilde{m}g)g \otimes (g^{-1}t)\right) - t \otimes 1\ . \tag{48}$$

As h has compact support, the g-sum in $\sum_g h(\widetilde{m}g)g \otimes (g^{-1}t)$ is finite, and it follows that (48) makes sense in $E_m \otimes \mathfrak{B}^\omega \widehat{\otimes} \mathfrak{B}^\omega$.

We now show that ∇_m is a connection. If $\gamma \in \Gamma$,

$$\nabla_m(t\gamma) = \nabla_m\left(\sum_g t_g\ g\gamma\right) = \nabla_m\left(\sum_g t_{g\gamma^{-1}}\ g\right) =$$

$$\sum_{g,k} h(\widetilde{m}g)\ t_{gk\gamma^{-1}}\ gdk = \sum_{g,k} h(\widetilde{m}g)\ t_{gk}\ gd(k\gamma) =$$

$$\sum_{g,k} h(\widetilde{m}g)\ t_{gk}\ g(dk)\gamma + \sum_{g,k} h(\widetilde{m}g)\ t_{gk}\ gkd\gamma =$$

$$\nabla_m(t)\gamma + \sum_{g,k} h(\widetilde{m}g)\ t_k\ kd\gamma = \nabla_m(t)\gamma + td\gamma\ . \tag{49}$$

Then

$$\nabla_m(tb) = (\nabla_m t)b + t \widehat{\otimes}_{\mathfrak{B}^\omega} db \tag{50}$$

for any $b \in \mathfrak{B}^\omega$.

As h is smooth, it follows that ∇ is also a connection. □

Note. There is a strong relationship between the connections ∇ considered here and the partially flat connections of [Ka, Chapitre 4].

Define the superconnection

$$D_s = \nabla + sQ \in \mathrm{Hom}^\infty\left(\mathcal{E}^\omega, \mathcal{E}^\omega \widehat{\otimes}_{\mathfrak{B}^\omega} \widehat{\Omega}_*(\mathfrak{B}^\omega)\right)\ . \tag{51}$$

Then $D_s^2 \in \mathrm{Hom}^\infty_{\mathfrak{B}^\omega}\left(\mathcal{E}^\omega, \mathcal{E}^\omega \widehat{\otimes}_{\mathfrak{B}^\omega} \widehat{\Omega}_*(\mathfrak{B}^\omega)\right)$ is given by

$$D_s^2 = s^2Q^2 + s(\nabla Q + Q\nabla) + \nabla^2\ . \tag{52}$$

Here $\nabla Q + Q\nabla$ is given explicitly by

$$(\nabla Q + Q\nabla)(f) = \sum_g (\partial h)\ R_g^* f \,\widehat{\otimes}_{\mathfrak{B}^\omega} dg \ , \tag{53}$$

where $f \in C^\infty(\widetilde{M}, \widetilde{E})$ and

$$\partial h = [Q, h] = -i \sum_\mu \gamma^\mu \partial_\mu h \ , \tag{54}$$

and ∇^2 is given by

$$\nabla^2(f) = \sum_g \sum_{g'} h\ R_g^* h\ R_{gg'}^* f \,\widehat{\otimes}_{\mathfrak{B}^\omega} dg\, dg' \ . \tag{55}$$

Put

$$\mathfrak{P} = -\big(s(\nabla Q + Q\nabla) + \nabla^2\big) \ , \tag{56}$$

and for $\beta > 0$ define

$$\exp(-\beta D_s^2) \in \mathrm{Hom}^\infty_{\mathfrak{B}^\omega}\left(\mathcal{E}^\omega, \mathcal{E}^\omega \,\widehat{\otimes}_{\mathfrak{B}^\omega} \widehat{\Omega}_*(\mathfrak{B}^\omega)\right) \tag{57}$$

to be

$$\begin{aligned}\exp(-\beta D_s^2) = \exp(-\beta s^2 Q^2) + &\int_0^\beta \exp(-u_1 s^2 Q^2)\mathfrak{P} \\ &\exp\big(-(\beta - u_1)s^2 Q^2\big)du_1 + \\ &\int_0^\beta \int_0^{u_1} \exp(-u_1 s^2 Q^2)\mathfrak{P} \exp a(-u_2 s^2 Q^2)\mathfrak{P} \\ &\exp\big(-(\beta - u_1 - u_2)s^2 Q^2\big)du_2 du_1 + \dots\end{aligned} \tag{58}$$

As only a finite number of terms of the expansion of (58) contribute to the degree-k component of $\exp(-\beta D_s^2)$, it is clear that (58) converges.

DEFINITION: For $s > 0$, the Chern character $\mathrm{ch}_{\beta,s}(\mathcal{E}^\omega) \in \overline{\widehat{\Omega}}_{even}(\mathfrak{B}^\omega)$ is given by

$$\mathrm{ch}_{\beta,s}(\mathcal{E}^\omega) = \mathrm{STR} \exp(-\beta D_s^2) \ . \tag{59}$$

PROPOSITION 10. *$\mathrm{ch}_{\beta,s}(\mathcal{E}^\omega)$ is closed.*

We omit the proof, which is straightforward.

PROPOSITION 11. *The class of $\mathrm{ch}_{\beta,s}(\mathcal{E}^\omega)$ in $\overline{H}_*(\mathfrak{B}^\omega)$ is independent of $s \in (0, \infty)$.*

Proof: Formally,

$$\frac{d}{ds}\,\mathrm{ch}_{\beta,s}(\mathcal{E}^\omega) = d(-\beta\ \mathrm{STR}\, Qe^{-\beta D_s^2})\ . \tag{60}$$

It is straightforward to check that this equation is valid. Then if $s_1, s_2 \in (0,\infty)$,

$$\mathrm{ch}_{\beta,s_1}(\mathcal{E}^\omega) - \mathrm{ch}_{\beta,s_2}(\mathcal{E}^\omega) = d\Big(-\beta \int_{s_2}^{s_1} \mathrm{STR}\, Qe^{-\beta D_s^2} ds\Big)\ . \tag{61}$$

□

Let η be an antisymmetric left-invariant (unnormalized) group k-cocycle. Then η defines a cyclic k-cocycle τ_η on $\mathbf{C}\Gamma$ by

$$\begin{aligned} &\tau_\eta(g_0,\ldots,g_k) = \eta(g_0, g_0g_1, g_0g_1g_2,\ldots, g_0g_1\ldots g_k) \text{ if } g_0g_1\ldots g_k = e \\ &\tau_\eta(g_0,\ldots,g_k) = 0 \ \text{ if } g_0g_1\ldots g_k \neq e \qquad \text{[Co1]}. \end{aligned} \tag{62}$$

Suppose that there are constants C and D so that

$$\big|\tau_\eta(g_0,\ldots,g_k)\big| \le C\exp\big(D(\|g_0\| + \ldots + \|g_k\|)\big)\ . \tag{63}$$

Then τ_η extends to a k-cocycle on $\mathfrak{B}^\omega$ and so can be paired with $\mathrm{ch}_{\beta,s}$. By Proposition 11, the pairing $\langle \mathrm{ch}_{\beta,s}(\mathcal{E}^\omega), \tau_\eta\rangle$ is independent of s.

V. Small-Time Limit

PROPOSITION 12.

$$\lim_{s\to 0}\langle \mathrm{ch}_{\beta,s}(\mathcal{E}^\omega), \tau_\eta\rangle = \beta^{k/2}/(k!)\int_M \widehat{A}(M)\wedge \mathrm{Ch}(V)\wedge\omega\ , \tag{64}$$

where ω is the closed k-form on M given by

$$\pi^*\omega = \sum R_{g_1}^* dh \wedge \ldots \wedge R_{g_k}^* dh\ \eta(e, g_1,\ldots,g_k) \in \Lambda^k(\widetilde{M})\ . \tag{65}$$

Proof: First, let us consider the contribution to $\langle \mathrm{ch}_{\beta,s}, \tau_\eta \rangle$ coming from the term

$$(-1)^k \int_0^\beta \ldots \int_0^{u_{k-1}} \exp(-u_1 s^2 Q^2) s(\nabla Q + Q\nabla) \exp(-u_2 s^2 Q^2)$$
$$s(\nabla Q + Q\nabla) \ldots s(\nabla Q + Q\nabla) \exp(-(\beta - u_1 - \ldots - u_k) s^2 Q^2) du_k \ldots du_1 \tag{66}$$

of $\exp(-\beta D_s^2)$. Written out explicitly, this will be

$$\sum (-1)^k \int_0^\beta \ldots \int_0^{u_{k-1}} \int_{\widetilde{M}} \phi(x_0) \, \mathrm{tr}_s \left[R_{g_0}^* \exp(-u_1 s^2 Q^2) s(\partial h) R_{g_1}^* \right.$$
$$\exp(-u_2 s^2 Q^2) s(\partial h) R_{g_2}^* \ldots s(\partial h) R_{g_k}^*$$
$$\left. \exp\big(-(\beta - u_1 - \ldots - u_k) s^2 Q^2\big) \right] (x_0, x_0)$$
$$dvol(x_0) du_k \ldots du_1 \, \tau_\eta(g_0, \ldots, g_k) = \tag{67}$$

$$\sum (-1)^k \int_0^\beta \ldots \int_0^{u_{k-1}} \int_{\widetilde{M}} \phi(x_0) \, \mathrm{tr}_s \left[\exp(-u_1 s^2 Q^2) s R_{g_0}^* (\partial h) \right.$$
$$\exp(-u_2 s^2 Q^2) \; s R_{g_0 g_1}^* (\partial h) \ldots s R_{g_0 \cdots g_{k-1}}^* (\partial h)$$
$$\exp\big(-(\beta - u_1 - \ldots - u_k) s^2 Q^2\big)$$
$$\left. R_{g_0 \cdots g_k}^* \right] (x_0, x_0) dvol(x_0) du_k \ldots du_1 \; \tau_\eta(g_0, \ldots, g_k) = \tag{68}$$

$$\sum (-1)^k \int_0^\beta \ldots \int_0^{u_{k-1}} \int_{\widetilde{M}} \cdots \int_{\widetilde{M}} \phi(x_0) \, \mathrm{tr}_s \left[\exp(-u_1 s^2 Q^2)(x_0, x_1) \right.$$
$$s(\partial h)(x_1 g_0) \exp(-u_2 s^2 Q^2)(x_1, x_2) \, s(\partial h)(x_2 g_0 g_1) \ldots s(\partial h)(x_k g_0 g_1 \cdots g_{k-1})$$
$$\left. \exp\big(-(\beta - u_1 - \ldots - u_k) s^2 Q^2\big)(x_k g_0 g_1 \cdots g_k, x_0) \right] dvol(x_k) \ldots dvol(x_0)$$
$$du_k \ldots du_1 \, \tau_\eta(g_0, \ldots, g_k) \; . \tag{69}$$

Because for small s the heat kernels are concentrated near the diagonal, the only terms which will survive in the $s \to 0$ limit will have $g_0 \ldots g_k = e$. Furthermore, the $s \to 0$ limit reduces to a question of local asymptotics on $\widetilde{M}$. By the Getzler calculus [G], (69) equals $(2\pi)^{-n} \int_{T\widetilde{M}} \mathrm{tr}_s (\sigma P)_{s^{-1}} dx \, d\xi$, where P denotes the operator appearing in (69), σP is its symbol in the Getzler calculus and $(\sigma P)_{s^{-1}}$ is the rescaled symbol. A straightforward calculation gives that in the limit $s \to 0$, this becomes

$$\sum(-1)^k \beta^{-k/2}\left(\int_0^\beta \cdots \int_0^{u_{k-1}} du_k \ldots du_1\right) \int_{\widetilde{M}} \phi(x)\widehat{A}(x) \wedge \mathrm{Ch}(\widetilde{V})(x) \wedge$$

$$dh(xg_0) \wedge \ldots \wedge dh(xg_0 \cdots g_{k-1})\eta(g_0, g_0g_1, g_0g_1g_2, \ldots, g_0g_1 \cdots g_{k-1}, e) = \tag{70}$$

$$\sum(-1)^k \beta^{k/2}/(k!) \int_{\widetilde{M}} \phi\ \widehat{A}(\widetilde{M}) \wedge \mathrm{Ch}(\widetilde{V}) \wedge R^*_{g_0} dh \wedge \ldots \wedge R^*_{g_0 \cdots g_{k-1}} dh$$

$$\eta(g_0, g_0g_1, g_0g_1g_2, \ldots, g_0 \cdots g_{k-1}, e) = \tag{71}$$

$$\beta^{k/2}/(k!) \int_{\widetilde{M}} \phi\ \widehat{A}(\widetilde{M}) \wedge \mathrm{Ch}(\widetilde{V}) \wedge \widetilde{\omega}\ , \tag{72}$$

where $\widetilde{\omega} \in \Lambda^k(\widetilde{M})$ is given by

$$\widetilde{\omega} = \sum R^*_{g_1} dh \wedge \ldots \wedge R^*_{g_k} dh\ \eta(e, g_1, \ldots, g_k)\ . \tag{73}$$

Now let us consider the contribution to $\langle \mathrm{ch}_{\beta,s}, \tau_\eta \rangle$ coming from a term of $\exp(-\beta D_s^2)$ which contains a ∇^2, such as, for example,

$$(-1)^k \int_0^\beta \cdots \int_0^{u_{k-1}} \exp(-u_1 s^2 Q^2)\ \nabla^2\ \exp(-u_2 s^2 Q^2)\ s(\nabla Q + Q\nabla) \cdots$$

$$s(\nabla Q + Q\nabla) \exp\big(-(\beta - u_1 - \ldots - u_k)s^2 Q^2\big)\, du_k \ldots du_1\ . \tag{74}$$

Written out explicitly, this gives

$$\sum(-1)^k \int_0^\beta \cdots \int_0^{u_{k-1}} \int_{\widetilde{M}} \cdots \int_{\widetilde{M}} \phi(x_0)\, \mathrm{tr}_s \big[\exp(-u_1 s^2 Q^2)(x_0, x_1)$$

$$h(x_1 g_0)h(x_1 g_0 g_1)\ \exp(-u_2 s^2 Q^2)(x_1, x_2) s(\partial h)(x_2 g_0 g_1 g_1') \cdots$$

$$s(\partial h)(x_k g_0 g_1 g_1' g_2 \cdots g_{k-1}) \exp\big(-(\beta - u_1 - \ldots - u_k)s^2 Q^2\big)$$

$$(x_k g_0 g_1 g_1' g_2 \cdots g_k, x_0)\big] dvol(x_k) \cdots$$

$$dvol(x_0) du_k \ldots du_1 \tau_\eta(g_0, g_1, g_1', g_2, \ldots, g_k)\ . \tag{75}$$

By the Getzler calculus, in the $s \to 0$ limit, (75) becomes

$$\sum(-1)^k \beta^{-(k-1)/2}\left(\int_0^\beta \cdots \int_0^{u_{k-1}} du_k \ldots du_1\right) \int_{\widetilde{M}} \phi(x)\widehat{A}(x)\, \mathrm{Ch}(\widetilde{V})(x)$$

$$h(xg_0)h(xg_0g_1)dh(xg_0g_1g_1') \wedge \ldots \wedge dh(xg_0g_1g_1'g_2 \cdots g_{k-1})$$

$$\eta(g_0, g_0g_1, g_0g_1g_1', \ldots, g_0g_1g_1'g_2 \cdots g_{k-1}, e) = \tag{76}$$

$$\sum(-1)^k \beta^{(k+1)/2}/(k!)$$
$$\int_{\widetilde{M}} \phi\, \widehat{A}(\widetilde{M}) \wedge \mathrm{Ch}(\widetilde{V}) \wedge R^*_{g_0} h \wedge R^*_{g_0 g_1} h \wedge R^*_{g_0 g_1 g_1'} dh \wedge \ldots$$
$$\wedge R^*_{g_0 g_1 g_1' g_2 \ldots g_{k-1}} dh\, \eta(g_0, g_0 g_1, g_0 g_1 g_1', \ldots, g_0 g_1 g_1' g_2 \cdots g_{k-1}, e) = \tag{77}$$
$$\pm \beta^{(k+1)/2}/(k!) \int_{\widetilde{M}} \phi\, \widehat{A}(\widetilde{M}) \wedge \mathrm{Ch}(\widetilde{V}) \wedge \widetilde{\omega}' , \tag{78}$$

where $\widetilde{\omega}' \in \Lambda^k(\widetilde{M})$ is given by

$$\widetilde{\omega}' = \sum R^*_{g_1} h \wedge R^*_{g_1'} h \wedge \ldots \wedge R^*_{g_k} dh\, \eta(e, g_1, g_1', g_2, \ldots, g_k) . \tag{79}$$

As $\eta(e, g_1, g_1', g_2, \ldots, g_k)$ is antisymmetric in g_1 and g_1' , it follows that $\widetilde{\omega}'$ vanishes. The same argument shows that all of the terms involving ∇^2 vanish. □

LEMMA 3. *The form $\widetilde{\omega}$ of (73) is a closed Γ-invariant form on $\widetilde{M}$.*

Proof: $\widetilde{\omega}$ is clearly closed. For all $\gamma \in \Gamma$, we have

$$\begin{aligned} R^*_\gamma \widetilde{\omega} = &\sum R^*_{\gamma g_1} dh \wedge \ldots \wedge R^*_{\gamma g_k} dh\, \eta(e, g_1, \ldots, g_k) = \\ &\sum R^*_{g_1} dh \wedge \ldots \wedge R^*_{g_k} dh\, \eta(e, \gamma^{-1} g_1, \ldots, \gamma^{-1} g_k) = \\ &\sum R^*_{g_1} dh \wedge \ldots \wedge R^*_{g_k} dh\, \eta(\gamma, g_1, \ldots, g_k). \end{aligned} \tag{80}$$

From the cocycle condition, this equals

$$\begin{aligned} \sum R^*_{g_1} dh \wedge \ldots \wedge R^*_{g_k} dh \big[&\eta(e, g_1, \ldots, g_k) - \eta(e, \gamma, g_2, \ldots, g_k) + \ldots + \\ &(-1)^k \eta(e, \gamma, g_1, \ldots, g_{k-1})\big] . \end{aligned} \tag{81}$$

But for all r,

$$\begin{aligned} &\sum R^*_{g_1} dh \wedge \ldots \wedge R^*_{g_k} dh\, \eta(e, \gamma, g_1, \ldots, \widehat{g}_r, \ldots, g_k) = \\ &\pm \Big(\sum_{g_r} R^*_{g_r} dh\Big) \wedge \sum R^*_{g_1} dh \wedge \ldots \wedge R^*_{g_{r-1}} dh \wedge R^*_{g_{r+1}} dh \wedge \\ &\ldots \wedge R^*_{g_k} dh\, \eta(e, \gamma, g_1, \ldots, \widehat{g}_r, \cdots, g_k) \end{aligned} \tag{82}$$

and

$$\sum_{g_r} R^*_{g_r} dh = d\Big(\sum_{g_r} R^*_{g_r} h\Big) = d(1) = 0 . \tag{83}$$

Thus only the first term of (81) contributes, and so

$$R_\gamma^* \widetilde{\omega} = \sum R_{g_1}^* dh \wedge \ldots \wedge R_{g_k}^* dh \; \eta(e, g_1, \ldots, g_k) = \widetilde{\omega}. \tag{84}$$

□

End of Proof of Proposition 12: From Lemma 3, there is a closed form ω on M such that $\widetilde{\omega} = \pi^*(\omega)$. Then

$$\int_{\widetilde{M}} \phi \; \widehat{A}(\widetilde{M}) \wedge \mathrm{Ch}(\widetilde{V}) \wedge \widetilde{\omega} = \int_M \widehat{A}(M) \wedge \mathrm{Ch}(V) \wedge \omega \; . \tag{85}$$

□

We now wish to show that the cohomology class of the closed form $\widetilde{\omega}$ is the pullback to M of the cohomology class $[\eta]$ on $B\Gamma$. To do so, it is convenient to first relax the smoothness conditions on $\widetilde{\omega}$.

Let h be a Lipschitz function on $\widetilde{M}$ of compact support with

$$\sum_g R_g^* h = 1 \; . \tag{86}$$

As the distributional derivatives of a Lipschitz function are L^∞-functions, it makes sense to define $\widetilde{\omega}_h$ by

$$\widetilde{\omega}_h = \sum R_{g_1}^* dh \wedge \ldots \wedge R_{g_k}^* dh \, \eta(e, g_1, \ldots, g_k) \; , \tag{87}$$

a closed Γ-invariant L^∞ k-form on $\widetilde{M}$, and let $\omega_h \in \Lambda^k(M)$ be such that $\pi^* \omega_h = \widetilde{\omega}_h$. It is known that one can compute the de Rham cohomology of M using flat forms (i.e. L^∞-forms τ such that $d\tau$ is also L^∞) [Te].

LEMMA 4. *The cohomology class of ω_h is independent of h.*

Proof: Let h' be another choice for h. Then

$$\widetilde{\omega}_h - \widetilde{\omega}_{h'} = \sum \big[R_{g_1}^* d(h - h') \wedge \ldots \wedge R_{g_k}^* dh + \ldots + \\ R_{g_1}^* dh' \wedge \ldots \wedge R_{g_k}^* d(h - h') \big] \eta(e, g_1, \ldots, g_k) \; . \tag{88}$$

Put

$$\widetilde{\sigma}_r = \sum R_{g_1}^* dh' \wedge \ldots R_{g_r}^* (h - h') \wedge \ldots \wedge R_{g_k}^* dh \; \eta(e, g_1, \ldots, g_k) \; , \tag{89}$$

a flat $(k-1)$-form on $\widetilde{M}$. Then

$$\widetilde{\omega}_h - \widetilde{\omega}_{h'} = d\Big(\sum_{r=1}^{k}(-1)^{r+1}\widetilde{\sigma}_r\Big) . \tag{90}$$

Furthermore, for all $\gamma \in \Gamma$,

$$\begin{aligned} R_\gamma^* \widetilde{\sigma}_r = &\sum R_{g_1}^* dh' \wedge \dots R_{g_r}^*(h-h') \wedge \dots \wedge R_{g_k}^* dh \ \eta(\gamma, g_1, \dots, g_k) = \\ &\sum R_{g_1}^* dh' \wedge \dots R_{g_r}^*(h-h') \wedge \dots \wedge R_{g_k}^* dh \big[\eta(e, g_1, \dots, g_k) - \\ &\eta(e, \gamma, g_2, \dots, g_k) + \dots + (-1)^k \eta(e, \gamma, g_1, \dots, g_{k-1})\big] . \end{aligned} \tag{91}$$

As

$$\sum_g R_g^* dh = \sum_g R_g^* dh' = \sum_g R_g^*(h-h') = 0 , \tag{92}$$

it follows that $\widetilde{\sigma}_r$ is Γ-invariant. Then $\omega - \omega' = d\sigma$, where $\sigma \in \Lambda^{k-1}(M)$ is such that

$$\pi^* \sigma = \sum_{r=1}^{k}(-1)^{r+1}\widetilde{\sigma}_r . \tag{93}$$

□

Let X be the simplicial complex whose ordered cochain complex is the standard complex of Γ [Br]. The k-simplices of X are $(k+1)$-tuples of distinct elements of Γ. We will take Γ to act on the right on X. Then the simplicial complex X/Γ is a model for $B\Gamma$. For a vertex v, let b_v denote the barycentric coordinate (on a simplex containing v) corresponding to v. Let j be the continuous piecewise linear function on X given by

$$\begin{aligned} j(x) = 0 &\text{ if } x \in [g_0, \dots, g_k] \text{ and } g_0 \neq e, \dots, g_k \neq e \\ b_e &\text{ if } x \in [g_0, \dots, g_k] \text{ and } g_i = e \text{ for some } i . \end{aligned} \tag{94}$$

LEMMA 5. $\sum_g R_g^* j = 1$.

Proof: Suppose that $x \in [g_0, \ldots, g_k]$. Then

$$\sum_g (R_g^* j)(x) = \sum_g j(xg) = \sum_{i=0}^k j(xg_i^{-1}) = \sum_{i=0}^k b_e(xg_i^{-1}) = \sum_{i=0}^k b_{g_i}(x) = 1. \tag{95}$$

□

Let $\widetilde{\omega}_j$ be the polynomial form on X, with coefficients in $\mathbb{C}$, given by

$$\widetilde{\omega}_j = \sum R_{g_1}^* dj \wedge \ldots \wedge R_{g_k}^* dj \ \eta(e, g_1, \ldots, g_k) \ . \tag{96}$$

Let ω_j be the polynomial form on X/Γ such that $\widetilde{\omega}_j = \pi^* \omega_j$.

We define a k-cocycle $\widetilde{\eta} \in C^k(X; \mathbb{C})$ by putting

$$\langle \widetilde{\eta}, [\gamma_0, \gamma_1, \ldots, \gamma_k] \rangle = \eta(\gamma_0^{-1}, \ldots, \gamma_k^{-1}) \ . \tag{97}$$

By the left invariance of the group cocycle η, $\widetilde{\eta}$ is right-invariant on X. With abuse of notation, let η denote the corresponding simplicial cocycle on X/Γ.

PROPOSITION 13. *As elements of $H^k(X/\Gamma; \mathbb{C})$, $[\omega_j] = [\eta]$.*

Proof: Let A denote the de Rham map from polynomial forms on X to $C^*(X)$. Then

$$\begin{aligned}
&(A\omega_j)[\gamma_0, \ldots, \gamma_k] = \\
&\sum \eta(e, g_1, \ldots, g_k) \left\langle R_{g_1}^* dj \wedge \ldots \wedge R_{g_k}^* dj, [\gamma_0, \ldots, \gamma_k] \right\rangle = \\
&\sum \eta(e, \gamma_{i_1}^{-1}, \ldots, \gamma_{i_k}^{-1}) \left\langle R_{\gamma_{i_1}{}^{-1}}{}^* dj \wedge \ldots \wedge R_{\gamma_{i_k}{}^{-1}}{}^* dj, [\gamma_0, \ldots, \gamma_k] \right\rangle \ ,
\end{aligned} \tag{98}$$

where $i_1, \ldots, i_k \in \{0, 1, \ldots, k\}$. Now (98) equals

$$\sum \eta(e, \gamma_{i_1}^{-1}, \ldots, \gamma_{i_k}^{-1}) \left\langle db_{\gamma_{i_1}} \wedge \ldots \wedge db_{\gamma_{i_k}}, [\gamma_0, \ldots, \gamma_k] \right\rangle \ . \tag{99}$$

A simple calculation gives that (99) in turn equals

$$\sum_{r=0}^k (-1)^{r+1} \eta(e, \gamma_0^{-1}, \ldots, \widehat{\gamma_r}^{-1}, \ldots, \gamma_k^{-1}) = \eta(\gamma_0^{-1}, \ldots, \gamma_k^{-1}) \ . \tag{100}$$

Thus $A(\omega_j)$ is the cochain η. As the de Rham map is an isomorphism on complex cohomology [GM], the proposition follows. □

Let ν be the canonical (up to homotopy) map $\nu : M \to B\Gamma$ classifying the universal cover $\widetilde{M}$, with lift $\widetilde{\nu} : \widetilde{M} \to E\Gamma$.

PROPOSITION 14. *As elements of $H^*(M, \mathbb{C}), [\omega] = \nu^*([\eta])$.*

Proof: Let us triangulate M. Upon subdivision, we can homotop ν to be a simplicial map. Then with $h = \widetilde{\nu}^* j$, we have $\omega_h = \nu^* \omega_j$. Thus as elements of $H^*(M, \mathbb{C})$,

$$[\omega_h] = [\nu^* \omega_j] = \nu^*[\omega_j] = \nu^*[\eta] \; . \tag{101}$$

By Lemma 4, $[\omega_h]$ is independent of the particular choice of h, and the proposition follows. □

COROLLARY 2. *For all $s > 0$,*

$$\langle \mathrm{ch}_{\beta,s}(\mathcal{E}^\omega), \tau_\eta \rangle = \beta^{k/2}/(k!) \int_M \widehat{A}(M) \wedge \mathrm{Ch}(V) \wedge \nu^*([\eta]) \; . \tag{102}$$

Note. One can equally well pair $\mathrm{ch}_{\beta,s}(\mathcal{E}^\omega)$ with any element of $HC^*(\mathfrak{B}^\omega)$. Modulo growth conditions, there is a way of producing an element $\tau \in HC^k(\mathfrak{B}^\omega)$ from a conjugacy class $\langle x \rangle$ of Γ and a k-cocycle of the group $\Gamma_x / \{x\}$, where Γ_x is the centralizer of x in Γ and $\{x\}$ is the subgroup generated by x [Bu]. (The cocycle (62) comes from the special case when $\langle x \rangle = \langle e \rangle$.) However, the cyclic cohomology classes corresponding to $\langle x \rangle \neq \langle e \rangle$ will pair with $\mathrm{ch}_{\beta,s}(\mathcal{E}^\omega)$ to give zero. The reason is that a cyclic k-cocycle τ based on $\langle x \rangle$ will have $\tau(g_0, \ldots, g_k) = 0$ if $g_0 g_1 \ldots g_k \notin \langle x \rangle$. However, by the proof of Proposition 12, in the $s \to 0$ limit one sees that the terms with $g_0 g_1 \ldots g_k \neq e$ do not contribute to $\langle \mathrm{ch}_{\beta,s}(\mathcal{E}^\omega), \tau \rangle$.

VI. Reduction to the Index Bundle

We first review some of the results of [MF]. Recall that Λ is the reduced group C^*-algebra of Γ. Let $\mathcal{E}$ denote the $\mathbb{Z}_2$-graded Λ-bundle over M given by $\mathcal{E} = (\widetilde{M} \times_\Gamma \Lambda) \otimes E$. The L^2-sections $\Gamma^0(\mathcal{E})$ of $\mathcal{E}$ form a right Λ-Hilbert module. The Dirac-type operator $\widetilde{D}$ is an odd densely-defined unbounded operator on $\Gamma^0(\mathcal{E})$. One can find finitely-generated right projective Λ-Hilbert submodules $F^\pm$ of $\Gamma^0(\mathcal{E}^\pm)$ and complementary Λ-Hilbert modules $G^\pm \subset \Gamma^0(\mathcal{E}^\pm)$ such that $\widetilde{D}$ is diagonal with respect to the decomposition $\Gamma^0(\mathcal{E}^\pm) = G^\pm \oplus F^\pm$, and writing $\widetilde{D} = \widetilde{D}_G \oplus \widetilde{D}_F$, in addition $\widetilde{D}_G : G^\pm \to G^\mp$ is invertible. By definition, the index of $\widetilde{D}$ is

$$\mathrm{Index}(\widetilde{D}) \equiv [F^+] - [F^-] \in K_0(\Lambda) \; ; \tag{103}$$

this is independent of the choice of $F^\pm$.

Now suppose that $\mathfrak{B}^\infty$ is a densely-defined subalgebra of Λ which is stable with respect to the holomorphic functional calculus on Λ, and $\mathfrak{B}^\omega \subset \mathfrak{B}^\infty \subset \Lambda$. A standard result in K-theory is that $K_0(\Lambda) \cong K_0(\mathfrak{B}^\infty)$ [Bo, Appendice]. There is a Chern character Ch_β from $K_0(\mathfrak{B}^\infty)$ to $\overline{HC_*(\mathfrak{B}^\infty)}$, the reduced cyclic homology of $\mathfrak{B}^\infty$ [Ka]. Let η be a group k-cocycle on $\mathbb{C}\Gamma$ which extends to an element τ_η of the cyclic cohomology of $\mathfrak{B}^\infty$. By the explicit formula (62), τ_η is a reduced cyclic cohomology class if $k > 0$.

We will sketch a proof of the following proposition. Many of the details are as in [Bi].

PROPOSITION 15.

$$\left\langle \mathrm{Ch}_\beta\left(\mathrm{Index}(\widetilde{D})\right), \tau_\eta \right\rangle = \beta^{k/2}/(k!) \int_M \widehat{A}(M) \wedge \mathrm{Ch}(V) \wedge \nu^*([\eta]) .$$

Proof: Define $\mathcal{E}^\infty$ to be $(\widetilde{M} \times_\Gamma \mathfrak{B}^\infty) \otimes E$. An examination of the proof of [MF] shows that $F^\pm$ and $G^\pm$ can be chosen to be of the form $F^\pm = \mathcal{F}^\pm \otimes_{\mathfrak{B}^\infty} \Lambda$ and $G^\pm = \mathcal{G}^\pm \otimes_{\mathfrak{B}^\infty} \Lambda$, where $\mathcal{F}^\pm$ and $\mathcal{G}^\pm$ are subspaces of $\Gamma^\infty(\mathcal{E}^\infty)$. (This uses the fact that $\mathfrak{B}^\infty$ is stable with respect to the holomorphic functional calculus in Λ.) Write $\widetilde{D}_{\mathcal{F}^\pm}$ and $\widetilde{D}_{\mathcal{G}^\pm}$ for the restrictions of $\widetilde{D}$ to $\mathcal{F}^\pm$ and $\mathcal{G}^\pm$ respectively. Put

$$\mathcal{H}^\pm = \mathcal{G}^\pm \oplus \mathcal{F}^\pm \oplus \mathcal{F}^\mp. \tag{104}$$

For $\alpha \in \mathbb{C}$, define $R^\pm_\alpha : \mathcal{H}^\pm \to \mathcal{H}^\mp$ by

$$R^\pm_\alpha = \begin{pmatrix} \widetilde{D}_{\mathcal{G}^\pm} & 0 & 0 \\ 0 & \widetilde{D}_{\mathcal{F}^\pm} & \alpha \\ 0 & \alpha & 0 \end{pmatrix} . \tag{105}$$

We have that $\widetilde{D}_{\mathcal{G}^\pm}$ is invertible. Put

$$S^\pm_\alpha = \begin{pmatrix} \widetilde{D}_{\mathcal{F}^\pm} & \alpha \\ \alpha & 0 \end{pmatrix} \tag{106}$$

and let

$$S^\pm_\alpha \otimes_{\mathfrak{B}^\infty} \Lambda : F^\pm \oplus F^\mp \to F^\mp \oplus F^\pm \tag{107}$$

be the extension to a bounded operator on finitely-generated Hilbert Λ-modules. As $\widetilde{D}_F$ is a bounded operator, it follows that $S^\pm_\alpha \otimes_{\mathfrak{B}^\infty} \Lambda$ is invertible for α large. Then the fact that $\mathfrak{B}^\infty$ is stable under the holomorphic functional calculus in Λ implies that $S_\alpha^\pm$ is also invertible for α large. Thus $R_\alpha^\pm$ is invertible for α large. We define $\exp(-\operatorname{TR}_\alpha^2)$ by the Duhamel expansion in α. As R_α differs from $\widetilde{D}\oplus 0$ by a finite-rank operator in the sense of [Kas], there is no problem in showing that $\exp(-\operatorname{TR}_\alpha^2)$ is well-defined.

Extend the $\mathfrak{B}^\omega$-connection ∇ on $\mathcal{E}^\omega$ to a $\mathfrak{B}^\infty$-connection on

$$\mathcal{E}^\infty = \mathcal{E}^\omega \otimes_{\mathfrak{B}^\omega} \mathfrak{B}^\infty. \tag{108}$$

Let $\nabla_{\mathcal{F}}$ be a $\mathfrak{B}^\infty$-connection on $\mathcal{F}$ and let

$$\nabla' = \nabla \oplus \nabla_{\mathcal{F}} \tag{109}$$

be the sum connection on $\mathcal{H}$. Define the Chern character

$$\operatorname{ch}_{\beta,s,\alpha}(\mathcal{H}) = \operatorname{STR}\exp\left(-\beta(\nabla' + sR_\alpha)^2\right) \in \overline{\overline{\Omega}}(\mathfrak{B}^\infty) \tag{110}$$

by a Duhamel expansion in ∇'. For $\alpha = 0$, we have

$$\operatorname{ch}_{\beta,s,0}(\mathcal{H}) = \operatorname{ch}_{\beta,s}(\mathcal{E}^\infty) - \operatorname{STR}\exp(-\beta\nabla_{\mathcal{F}}^2) . \tag{111}$$

Now $\operatorname{STR}\exp(-\beta\nabla_{\mathcal{F}}^2) \in \overline{\overline{\Omega}}(\mathfrak{B}^\infty)$ represents $\operatorname{Ch}_\beta\left([\mathcal{F}]\right)$ [Ka]. If we can show that $\operatorname{ch}_{\beta,s,0}(\mathcal{H})$ is zero in $\overline{H}_*(\mathfrak{B}^\infty)$ then we will have that as classes in $\overline{H}_*(\mathfrak{B}^\infty)$,

$$\operatorname{ch}_{\beta,s}(\mathcal{E}^\infty) = \operatorname{STR}\exp(-\beta\nabla_{\mathcal{F}}^2) = \operatorname{Ch}_\beta([\mathcal{F}]) = \operatorname{Ch}_\beta\left(\operatorname{Index}(\widetilde{D})\right) , \tag{112}$$

and the proposition will follow.

A standard homotopy argument shows that the class of $\operatorname{ch}_{\beta,s,\alpha}(\mathcal{H})$ in $\overline{H}_*(\mathfrak{B}^\infty)$ is independent of α. Take α large enough that R_α is invertible.

We define a pseudodifferential calculus as in [MF], except that the symbol $\sigma(m,\xi)$ will take value in $\operatorname{End}_{\mathfrak{B}^\infty}(\mathcal{E}_m^\infty)$. Then R_α is an elliptic first-order ψdo. (In terms of the "fibration" picture, it corresponds to a smooth family of elliptic first-order vertical ψdo's.) As in the usual calculus of ψdo's, R_α has a parametrix P_α, an order -1 ψdo, such that

$$I - R_\alpha P_\alpha = K_{1\alpha} \text{ and } I - P_\alpha R_\alpha = K_{2\alpha} , \tag{113}$$

where $K_{1\alpha}$ and $K_{2\alpha}$ are smoothing operators. It follows that

$$(R_\alpha)^{-1} = P_\alpha + K_{2\alpha}(R_\alpha)^{-1} \tag{114}$$

is also an order -1 ψdo.

Define a connection $\nabla''_{\mathcal{H}^-}$ on $\mathcal{H}^-$ by

$$\nabla''_{\mathcal{H}^-} = (R^-_\alpha)^{-1}\nabla'_{\mathcal{H}^+}\, R^-_\alpha \tag{115}$$

and define ∇'' to be $\nabla'_{\mathcal{H}^+} \oplus \nabla''_{\mathcal{H}^-}$. Then

$$\nabla''_{\mathcal{H}^+} - \nabla'_{\mathcal{H}^+} = 0 \tag{116}$$

and

$$\nabla''_{\mathcal{H}^-} - \nabla'_{\mathcal{H}^-} = (R^-_\alpha)^{-1}\ \left(\nabla'_{\mathcal{H}^+}\, R^-_\alpha - R^-_\alpha\ \nabla'_{\mathcal{H}^+}\right) \tag{117}$$

is an order -1 operator. We have a homotopy of connections on H from ∇' to ∇'' given by $\nabla' + u(\nabla'' - \nabla')$, $u \in [0,1]$. It follows as in [Bi, Prop. 2.10] that $\mathrm{ch}_{\beta,s,\alpha}(H) = \mathrm{STR}\exp\left(-\beta(\nabla' + sR_\alpha)^2\right)$ represents the same class in $\overline{H}_*(\mathfrak{B}^\infty)$ as $\mathrm{STR}\exp\left(-\beta(\nabla'' + sR_\alpha)^2\right)$.

We claim that if STR $\exp(-\beta(\nabla'' + sR_\alpha)^2)$ is expanded in ∇'', the terms vanish algebraically. To see this formally, write $\nabla'' + sR_\alpha$ in terms of the decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ as

$$\begin{aligned}\nabla'' + sR_\alpha = &\begin{pmatrix} \nabla'_{\mathcal{H}^+} & sR^-_\alpha \\ sR^+_\alpha & (R^-_\alpha)^{-1}\nabla'_{\mathcal{H}^+}R^-_\alpha \end{pmatrix} = \\ &\begin{pmatrix} I & 0 \\ 0 & s^{-1}(R^-_\alpha)^{-1} \end{pmatrix}\begin{pmatrix} \nabla'_{\mathcal{H}^+} & I \\ s^2R^-_\alpha R^+_\alpha & \nabla'_{\mathcal{H}^+} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & s(R^-_\alpha) \end{pmatrix}\end{aligned} \tag{118}$$

and so formally,

$$\begin{aligned}\mathrm{STR}\exp\left(-\beta(\nabla'' + sR_\alpha)^2\right) = \mathrm{STR}\exp\left(-\beta\begin{pmatrix} \nabla'_{\mathcal{H}^+} & I \\ s^2R^-_\alpha R^+_\alpha & \nabla'_{\mathcal{H}^+} \end{pmatrix}^2\right) \\ \in \overline{\overline{\Omega}}_*(\mathfrak{B}^\infty)\ .\end{aligned} \tag{119}$$

However, expanding (119) in $\nabla'_{\mathcal{H}^+}$, one finds that (119) vanishes for algebraic reasons.

(To see this last point, consider an analogous statement in the finite-dimensional case. For $A, B \in M_N(\mathbb{C})$ put

$$\mathrm{M} = \begin{pmatrix} A & I \\ B & A \end{pmatrix} \in M_{2N}(\mathbb{C})\ . \tag{120}$$

Then $\det(\mathrm{M}) = \det(A^2 - B)$ and if $A^2 - B$ is invertible,

$$\mathrm{M}^{-1} = \begin{pmatrix} (A^2-B)^{-1}A & -(A^2-B)^{-1} \\ I - A(A^2-B)^{-1}A & A(A^2-B)^{-1} \end{pmatrix} . \tag{121}$$

Thus $\operatorname{Str} \mathrm{M}^{-1} = 0$. If $\lambda \notin \operatorname{Spec}(\mathrm{M})$, by changing A to $A - \lambda I$, we obtain that $\operatorname{Str}(\mathrm{M} - \lambda I)^{-1} = 0$. Then by the functional calculus, if f is a holomorphic function in a neighborhood of $\operatorname{Spec}(\mathrm{M})$, $\operatorname{Str} f(\mathrm{M}) = 0$.)

This formal argument can be made rigorous as in [Bi, Prop. 2.17].

Note. If M is odd-dimensional then one can use Quillen's formalism [Q] to define the odd Chern character

$$\mathrm{ch}_{\beta,s}(\mathcal{E}^\infty) = \mathrm{Tr}_\sigma \exp\big(-\beta(\nabla + sQ\sigma)^2\big) \in \overline{\overline{\Omega}}_{odd}(\mathfrak{B}^\infty) . \tag{122}$$

The operator $\widetilde{D}$ gives an element $\operatorname{Index}(\widetilde{D})$ of $K_1(\mathfrak{B}^\infty)$ [Kas]. Using a suspension argument as in [BF], one can show that Proposition 15 also holds in the odd case.

COROLLARY 3 [CM]. *If Γ is a hyperbolic group in the sense of Gromov [GH] then for all $[\eta] \in H^*(\Gamma; \mathbf{C})$, the higher-signature $\int_M L(M) \wedge \nu^*\big([\eta]\big)$ is an (orientation-preserving) homotopy invariant of M.*

Proof: Let $\mathfrak{B}^\infty$ be the algebra

$$\mathfrak{B}^\infty = \big\{A \in \Lambda : \widetilde{\partial}^k(A) \text{ is bounded for all } k \in \mathbf{N}\big\} , \tag{123}$$

where $\widetilde{\partial}$ is the operator of [CM, p. 383]. By [CM, p. 385], if $[\eta] \in H^*(\Gamma; \mathbf{C})$ then $[\eta]$ can be represented by a group cocycle η such that τ_η extends to a cyclic cocycle on $\mathfrak{B}^\infty$. Letting $\widetilde{D}$ be the signature operator, the result of Mishchenko and Kasparov [Mi, Kas, HS] on the homotopy invariance of

$$\operatorname{Index}(\widetilde{D}) \in K_0(\Lambda) \cong K_0(\mathfrak{B}^\infty) \tag{124}$$

along with Corollary 2 implies the result. (As usual when dealing with the signature operator, it is irrelevant whether or not M is spin.) □

VII. Bivariant Extension

Let $\mathfrak{A}$ be the C^*-algebra $C(M)$. Then $\big(\Gamma^0(\mathcal{E}), \widetilde{D}\big)$ forms an unbounded $(\mathfrak{A}, \Lambda)$ Kasparov module, and so gives an element of $KK(\mathfrak{A}, \Lambda)$ [BJ]. A bivariant Chern character $\mathrm{ch}_{\beta,s}$ was defined in [Lo1] in the case of finite-dimensional projective modules, and it was indicated that the bivariant Chern character should be well-defined whenever there is a good notion of trace on the Hilbert modules. Such is the case here. The bivariant Chern character is a combination of Quillen's superconnection Chern character [Q] and the entire cyclic cocycle of [JLO]. In the setup of Section IV, given $\eta \in Z^k(\Gamma; \mathbb{C})$ such that τ_η pairs with $\mathfrak{B}^\infty$, there is a corresponding entire cyclic cocycle $\langle \mathrm{ch}_{\beta,s}, \tau_\eta \rangle \in C^*_\epsilon\big(C^\infty(M)\big)$. It is given explicitly as follows:

DEFINITION: For $a_0, \ldots, a_m \in C^\infty(M)$,

$$\langle \mathrm{ch}_{\beta,s}, \tau_\eta \rangle (a_0, \ldots, a_m) = \beta^{-m/2} \Big\langle \int_0^\beta \cdots \int_0^{u_{m-1}} \mathrm{STR}\, a_0 \exp(-u_1 D_s^2)$$
$$[D_s, a_1] \exp(-u_2 D_s^2) [D_s, a_2] \ldots [D_s, a_m] \exp\big(- (\beta - u_1 - \ldots - u_m) D_s^2 \big)$$
$$du_m \ldots du_1, \tau_\eta \Big\rangle . \tag{125}$$

(Note that the $\langle \mathrm{ch}_{\beta,s}, \tau_\eta \rangle (1)$ of equation (125) equals the $\langle \mathrm{ch}_{\beta,s}, \tau_\eta \rangle$ of Proposition 12.)

As before, the class of $\langle \mathrm{ch}_{\beta,s}, \tau_\eta \rangle$ in $H^*_\epsilon\big(C^\infty(M)\big)$ is independent of s. As in Section V, we can take the $s \to 0$ limit to obtain that $\langle \mathrm{ch}_{\beta,s}, \tau_\eta \rangle$ is cohomologous to the entire cyclic cocycle $\langle \mathrm{ch}_{\beta,0}, \tau_\eta \rangle$ given by

$$\begin{aligned} \langle \mathrm{ch}_{\beta,0}, \tau_\eta \rangle (a_0, \ldots, a_m) = &\beta^{k/2} / (k! m!) \int_M \widehat{A}(M) \wedge \mathrm{Ch}(V) \wedge \\ &\omega \wedge a_0 da_1 \wedge da_2 \wedge \ldots \wedge da_m \,. \end{aligned} \tag{126}$$

Here ω is the differential form of (65).

If $W \in K^0(M)$ is represented by a projection $p \in M_r\big(C^\infty(M)\big)$, let $\mathrm{Ch}_*(p)$ be the entire cyclic cycle of [GS]. Then we obtain that $\langle \mathrm{ch}_{\beta,s}, \tau_\eta \rangle \big(\mathrm{Ch}_*(p) \big)$ is proportionate to $\int_M \widehat{A}(M) \wedge \mathrm{Ch}(V) \wedge \omega \wedge \mathrm{Ch}(W)$. Note that in the case of the signature operator, the entire cyclic cohomology class of $\langle \mathrm{ch}_{\beta,s}, \tau_\eta \rangle$ is not a homotopy invariant, as otherwise one could take $[\eta]$ to be a 0-group cocycle and conclude that the rational L-class is a homotopy invariant, which is false.

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