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λ-Normal Forms in an Intensional Logic for English*

Abstract. Montague [7] translates English into a tensed intensional logic, an extension of the typed λ-calculus. We prove that each translation reduces to a formula without λ-applications, unique to within change of bound variable. The proof has two main steps. We first prove that translations of English phrases have the special property that arguments to functions are modally closed. We then show that formulas in which arguments are modally closed have a unique fully reduced λ-normal form. As a corollary, translations of English phrases are contained in a simply defined proper subclass of the formulas of the intensional logic.

Introduction. In this paper we consider λ-normal forms in an intensional logic motivated by its relevance to natural language. The system investigated is that of Montague's *The proper treatment of quantification in ordinary English* [7], here referred to as [PTQ]. [PTQ] correlates English phrases with logical expressions. Each derivation of an English phrase is mapped, or translated, into an expression in the intensional logic, ***IL***. ***IL*** extends the usual typed λ-calculus; it includes operators for intension, extension, necessity, and tense. This logic, but without tense, was introduced by Montague earlier in [6] and axiomatized and investigated by Gallin [5].

We investigate the question of when expressions in ***IL*** have a unique λ-reduced form. A counterexample shows that this uniqueness does not hold for ***IL*** in general. However, it does hold for all expressions with a simple property that depends only on the form of the expression. Montague's translations all have this property.

These questions arose because the translations given by the rules of [PTQ] directly are long and difficult to comprehend. One reason is that substitution rules in the English syntax are translated into applications of λ-functions in the logical syntax. It is natural to attempt to simplify the translations by eliminating all the λ-applications by λ-reduction. In [PTQ] Montague gives his examples in fully reduced form, although he gives no discussion of reduction. The reduced forms are more comprehensible to the reader and easier to interpret in a model. This increased ease also carries over if, as in our case, the entire process is implemented on a computer (see [2], [3], and [4]).

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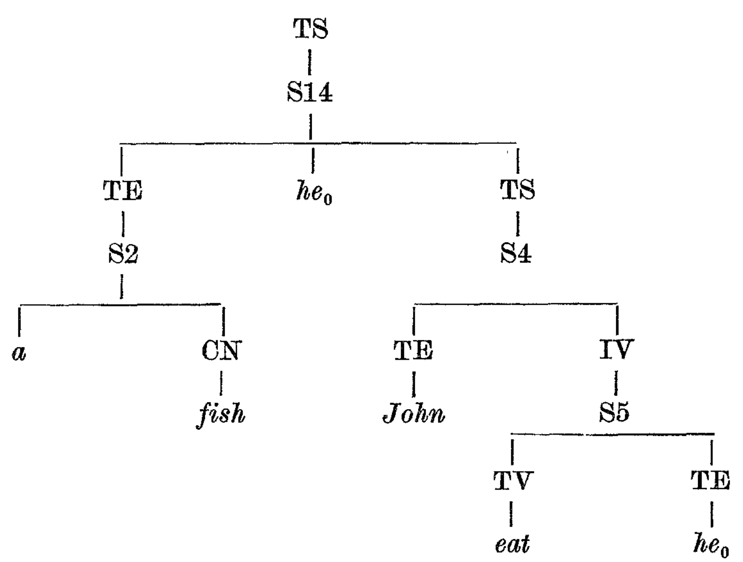
We obtain the main result in two steps. The first is a theorem on the form of translations of English phrases. We show that in translations of English phrases and in their reductions, the arguments to functions are always modally closed, hence λ-applications are always contractible.

The second step is a theorem on the intensional logic itself. It is well-known that for the usual typed λ-calculus there is a fully reduced normal form, unique to within change of bound variable. In intensional logic, in contrast, λ-reduction does not yield a unique normal form. We show, however, that if all arguments to functions are modally closed, there is a unique normal form. This combines with the result on translations to yield the main result.

Example. Before proceeding to the formal development, we give an example illustrating the stages of the process to be investigated.

Consider the English sentence *John eats a fish.* One of the derivations given for this sentence in [PTQ] is displayed in the figure. Nonterminal nodes are labeled by syntactic rules and syntactic categories, using TS for sentence and TE for term. The derivation uses the rule S14, which substitutes a term phrase (*a fish*) for a variable (he_0) in a sentence form (*John eats* him_0).



The word-for-word or direct translation is

$$[\lambda P (\exists x)[fish(x) \wedge [{}^{\vee}P](x)]]\Big({}^{\wedge}\lambda y\Big[[\lambda Q[{}^{\vee}Q]({}^{\wedge}j)]\Big({}^{\wedge}[eat({}^{\wedge}\lambda R[{}^{\vee}R](y))]\Big)\Big]\Big).$$

Reducing the outermost λ-application yields

$$(\exists x)\Big[fish(x) \wedge \Big[{}^{\vee}{}^{\wedge}\lambda y\Big[[\chi Q[{}^{\vee}Q]({}^{\wedge}j)]\Big({}^{\wedge}[eat({}^{\wedge}\lambda R[{}^{\vee}R](y))]\Big)\Big]\Big](x)\Big].$$

The subexpression $[{}^{\vee}{}^{\wedge}\lambda y\ldots]$ can now be reduced to $[\lambda y\ldots]$. Reducing

this λ next yields
$(\exists x)\big[fish(x)\wedge[\lambda Q[{}^{\vee}Q]({}^{\wedge}j)]\big({}^{\wedge}\big[eat({}^{\wedge}\lambda R[{}^{\vee}R](x))\big]\big)\big]$.
Continuing in this way leads to the reduced form
$(\exists x)\big[fish(x)\wedge\big[eat({}^{\wedge}\lambda R[{}^{\vee}R](x))\big]({}^{\wedge}j)\big]$,
in which the only λ-function lacks an argument.

Montague further transforms the formulas by introducing extensional constants where possible and by replacing certain unary functions by binary predicates. For the formula under consideration the result is
$(\exists u)[fish^{*}(u)\wedge find^{*}(j,u)]$.
The extensional forms are equivalent to the others only by assuming some meaning postulates and are not further considered here.

Basic Definitions. For convenience, we repeat Montague's definition of meaningful expression.

Types. The set of *types* is the smallest set Y such that (1) e, $t\in Y$, (2) whenever a, $b\in Y$, $\langle a,b\rangle\in Y$, and (3) whenever $a\in Y$, $\langle s,a\rangle\in Y$.

Meaningful expressions. There are denumerably many variables and infinitely many constants of each type. The set of *meaningful expressions of type a* is the smallest set such that:

(1) Every variable and constant of type a is in ME_a.
(2) If $A\in\mathrm{ME}_a$ and u is a variable of type b, then $\lambda uA\in\mathrm{ME}_{\langle b,a\rangle}$.
(3) If $A\in\mathrm{ME}_{\langle a,b\rangle}$ and $B\in\mathrm{ME}_a$, then $A(B)\in\mathrm{ME}_b$.
(4) If A, $B\in\mathrm{ME}_a$, then $A=B\in\mathrm{ME}_t$.
(5) If Q, $R\in\mathrm{ME}_t$ and u is a variable, then $\neg Q$, $[Q\wedge R]$, $[Q\vee R]$, $[Q\rightarrow R]$, $[Q\leftrightarrow R]$, $(\exists u)Q$, $(\forall u)Q$, $\Box Q$, HQ, $WQ\in\mathrm{ME}_t$.
(6) If $A\in\mathrm{ME}_a$, then $[{}^{\wedge}A]\in\mathrm{ME}_{\langle s,a\rangle}$.
(7) If $A\in\mathrm{ME}_{\langle s,a\rangle}$, then $[{}^{\vee}A]\in\mathrm{ME}_a$.

By a *meaningful expression* or *formula* of intensional logic is understood a member of ME_a for any a.

If u is a variable of type a, then λuB is understood as denoting that function from objects of type a which for any such object x, takes as value the object denoted by B when u is understood as denoting x. The expression $A(B)$ is as usual understood as denoting the value of the function denoted by A for the functional argument denoted by B. The logical symbols $\Box$, W, H may be read "it is necessary that", "it will be the case that", "it has been the case that", respectively. The expression $[{}^{\wedge}A]$ is regarded as denoting the intension of the expression A. The expression $[{}^{\vee}A]$ is well-formed only when A denotes an intension; in such a case $[{}^{\vee}A]$ denotes the corresponding extension.

Modally closed formulas. We define a subclass of meaningful expressions that have the property that their syntactic form forces their denotations to be independent of the point of reference at which they are evaluated. The class of *modally closed* (MC) *formulas* is the smallest class such that:

(1) u is MC for every variable u.
(2) $[^\wedge A]$ is MC for every formula A.
(3) $A(B)$ is MC whenever A and B are MC.
(4) $[A = B]$ is MC whenever A, B are MC.
(5) $\lambda u A$ is MC whenever A is MC.
(6) $\Box Q$ is MC for every formula Q of type t.
(7) $\neg Q$ is MC whenever Q is MC.
(8) $(\exists u)Q$ is MC and $(\forall u)Q$ is MC whenever Q is MC.
(9) $[Q \wedge R]$, $[Q \vee R]$, $[Q \rightarrow R]$, and $[Q \leftrightarrow R]$ are MC whenever Q, R are MC.

This definition of modally closed is essentially that of [5]. It differs only in the addition of (6)-(9) for formulas of sentence type (ME_t) that are introduced by Gallin only as abbreviations. (6)-(9) can easily be proved from Gallin's definitions. Formulas which contain W and H are not translatable into Gallin's system. Note that formulas WQ and HQ are not modally closed, $[^\vee A]$ is not modally closed, and constants are not modally closed.

Direct Translation. A *direct translation* of an English phrase i a translation of a derivation tree for the phrase into a meaningful expression of the logic by rules T1-T17 of [PTQ]. The relevant parts of these rules are given in the course of the proof of Theorem 1. The abbreviations used in [PTQ] are eliminated in the statements of the translation rules in order to display functional arguments more clearly.

Theorem 1 shows that in direct translations all functional arguments are variables or of the form $[^\wedge D]$, hence are modally closed.

THEOREM 1. *If a function application $C(B)$ occurs in a direct translation A, then the functional argument B is either a variable or an expression of the form $[^\wedge D]$. Hence, B is modally closed.*

PROOF: By induction on the construction of direct translations.

STATEMENT of BASIS. *If $C(B)$ occurs as part of the direct translation of a basic phrase of English, B is a variable or is of the form $[^\wedge D]$.*

PROOF of BASIS: The translations of basic phrases are given by rule T1. We verify that the result holds in each subcase of T1.

T1(a). If A is a basic expression other than *be*, *necessarily*, and the basic terms, i.e. members of BTE, then A translates into $g(A)$, where $g(A)$ is a constant and hence contains no function applications.

T1(b) *be* translates into $\lambda P\lambda x[^{\vee}P](^{\wedge}\lambda y[^{\vee}x = {}^{\vee}y])$. The only subexpression that is an argument of a function is $^{\wedge}\lambda y[^{\vee}x = {}^{\vee}y]$.

T1(c) *necessarily* translates into $\lambda p[\Box[^{\vee}p]]$, which has no function applications.

T1(d). The members of BTE, *John*, *Mary*, *Bill*, and *ninety*, translate into j^*, m^*, b^*, and n^*, respectively. By definition, A^* is $\lambda P[^{\vee}P](^{\wedge}A)$. The only functional argument is $^{\wedge}A$, which is of the proper form.

T1(e). he_n translates into $\lambda P[^{\vee}P](x_n)$, where the functional argument is a variable.

Induction step: We assume the property holds for all component expressions of the direct translation A, and show that it is true of A. The proof is by cases on the translation rules T2 through T17.

T2. This gives the translation corresponding to the rule that forms terms from common nouns by adding determiners. The three possible results are:
for *every*, $\lambda P(\forall x)[S'(x)\rightarrow[^{\vee}P](x)]$;
for *the*, $\lambda P(\exists y)[(\forall x)[S'(x)\leftrightarrow x = y]\wedge[^{\vee}P](y)]$;
and for *a*, $\lambda P(\exists x)[S'(x)\wedge[^{\vee}P](x)]$, where S' is the translation of common noun phrase. The only new function applications have variables x and y as arguments, and by induction all functional arguments of S' are variables or are of the form $[^{\wedge}A]$.

T3. This translates the rule that adds a relative clause to a common noun phrase. The result is $\lambda x[E'(x)\wedge Q']$, where E' and Q' are translations. As above, x is a variable, and E' and Q' have the property by induction. We note that [8] (p. 261 fn.) corrects T3 to avoid collision of variables by replacing x in Q' by a new variable y. This clearly does not affect the property.

T4 through T10. These give translations for the rules of function application. The translations are all fo the form $E'(^{\wedge}B')$. The only new argument is $^{\wedge}B'$.

T11. Sentence conjunction and disjunction, $[Q'\wedge R']$ and $[Q'\vee R']$, introduce no new function applications.

T12. Verb phrase conjunction and disjunction have the translations $\lambda x[C'(x)\wedge E'(x)]$ and $\lambda x[C'(x)\vee E'(x)]$. The new function applications have the variable x as argument.

T13. Term phrase disjunction translates into $\lambda P[A'(P)\vee B'(P)]$, where the only new argument is the variable P.

T14. Quantification over sentences has the translation $A'({}^{\wedge}\lambda xQ')$. The only new argument is of the form $[{}^{\wedge}D]$.

T15 and T16. Quantification over a common noun phrase or an intransitive verb phrase has the translation $\lambda yA'({}^{\wedge}\lambda x[E'(y)])$. The argument of E' is a variable and the argument of A' is of the proper form.

T17. The translations of the rules of tense and sign are $\neg A'({}^{\wedge}E')$, $WA'({}^{\wedge}E')$, $\neg WA'({}^{\wedge}E')$, $HA'({}^{\wedge}E')$, and $\neg HA'({}^{\wedge}E')$. The new function application in each of them is $A'({}^{\wedge}E')$, which has an argument of the proper form.

Definitions of Reductions. The results to be proved are about the formulas that can be obtained from direct translations by application of reductions of the λ-calculus. Some definitions are needed.

λ-contraction. A *λ-application* is a formula of the form $[\lambda xA](B)$. The λ-application is *contractible* if either (i) B is modally closed or (ii) no free occurrence of x in A lies in an intensional context of A, that is, within the scope of a ${}^{\wedge}$, $\square$, H, or W in A. If $[\lambda xA](B)$ is a contractible part of a formula, then its *contraction* is any result of first changing bound variables in A to avoid variable collisions and then substituting B for each free occurrence of x in the modified A.

EI-contraction. An *EI-formula* is a formula of the form $[{}^{\vee}[{}^{\wedge}C]]$. It is always *contractible*; its *contraction* is C.

The only *contractible parts* of a formula are the contractible λ-applications and the *EI*-formulas.

Reduction. Let A be a formula that contains a contractible part B with contraction C. Then a contraction of A is the result E of replacing B by C in A. We say that A reduces to E, or *red*(A, E). We denote by *contr*(A, E) the formula B that is contracted. The relation *red** is the reflexive transitive closure of *red* and change of bound variable.

Reduced forms. A formula is in *reduced form* if it contains no contractible parts. It is *fully reduced* if it is reduced and contains no λ-applications.

Translations. By a *translation* of an English phrase we mean its direct translation A by the rules of [PTQ], or any formula E such that *red**(A, E), that is, any E resulting from A by reduction.

Modal closure. We now prove that reduction of translations preserves modal closure of functional arguments. We begin with some lemmas about properties preserved by substitution.

Notation. Let Sub(B, x, A) be the result of substituting B for all free occurrences of x in A.

LEMMA 1. *Let B be a subexpression of A. Let A′ be the result of replacing an occurrence of B in A by an expression B′. Let A be* MC. *If B is not* MC *or if both B and B′ are* MC, *then A′ is* MC.

PROOF: The proof is by induction on the construction of the modally closed formula A. We follow the numbering of the definition of modally closed.

We note first that if A' is B', then A' is MC. For in this case A is B, A is MC, that is, B is MC. So by hypothesis B' is MC, that is, A' is MC. This will be used in each case of the induction below.

BASIS:
(1) *A is a variable. Then A is B and A′ is B′. So A′ is* MC *by the argument above.*

Induction step:
(2) A is $[{}^\wedge C]$. A' is either B' or $[{}^\wedge C']$ where C' is the result of replacing B by B' in C and thus MC by definition.
(3) A is $C(D)$. A' is B' or $C'(D)$ or $C(D')$. Since A is MC, C and D are MC by definition. If A' is $C'(D)$ then C' is MC by induction, so A' is MC by definition. Similarly, if A' is $C(D')$. (5) A is λuC. A' is B' or $\lambda uC'$. By induction C' is MC so by definition A' is MC.
The remaining arguments are similar: (4), (7), and (9) are like (3); (6) is like (2); and (8) is like (5).

LEMMA 2. *Substitution of a modally closed expression for all free occurrences of a variable in a modally closed expression yields a modally closed expression.*

PROOF: The proof is a finite induction on the number of occurrences of the variable. At each step we are substituting a modally closed expression for a modally closed expression, so the result is modally closed by Lemma 1.

LEMMA 3. (Substitution preserves modal closure of arguments). *If all functional arguments of A are modally closed, and B and all of its functional arguments are modally closed, then all functional arguments of Sub(B, x, A) are modally closed.*

PROOF:
(1) If the occurrence of x replaced by B is not in a functional argument then the new functional arguments introduced by the substitution are just the functional arguments of B, which are modally closed by hypothesis.

(2) If x occurs in a functional argument of A, then we are substituting a modally closed expression B for a variable in a modally closed expression. Hence by Lemma 2 the result is modally closed. The other new functional arguments in the result are those of B and are modally closed by hypothesis.

One more lemma is needed before we can present Theorem 2.

LEMMA 4. (Reduction preserves modal closure of functional arguments). *If all functional arguments of A are modally closed and A reduces to E, then all functional arguments of E are modally closed.*

PROOF: We show that each type of contraction preserves modal closure of functional arguments.

(1) λ-contraction consists of first a possible change of bound variables and then substitution. Change of bound variables replaces variables with other variables. Clearly this preserves modal closure of all expressions and subexpressions. The substitution replaces a subexpression $[\lambda xC](B)$ by $Sub(B, x, C)$. By Lemmas 2 and 3, all arguments that are subexpressions of $Sub(B, x, C)$ are MC, and by Lemma 1, any modally closed expression that contains the expression $[\lambda xC](B)$ that has been replaced by $Sub(B, x, C)$ remains MC.

(2) EI-contraction replaces a subexpression of the form $[{}^{\vee}{}^{\wedge}C]$ with C. The result holds by Lemma 1 and because the subexpressions of C are unchanged.

The theorem now follows immediately.

THEOREM 2. *In translations, the arguments to functions are modally closed.*

PROOF: By an induction on translations. The basis is provided by Theorem 1 on direct translations. The induction step is by Lemma 4.

The next theorem shows that an even stronger property holds of the arguments of translations: they are either variables or of the form $[{}^{\wedge}C]$. Even though Theorem 2 would obviously follow from this, we have deliberately chosen to prove it separately, using Lemma 4. Proving it in that way makes it clear that Theorem 2 would apply to any system in which the arguments of direct translations are modally closed, a weaker constraint than that required for Theorem 3.

LEMMA 5. *If all functional arguments of A are either variables or of the form $[{}^{\wedge}C]$, and B and all of its arguments are also of these two forms, then all arguments of $Sub(B, x, A)$ are of these forms.*

PROOF: The proof is similar to that of Lemma 3.

(1) If the occurrence of x replaced by A is not a functional argument then the new functional arguments introduced by the substitution are just the arguments of B, which have the desired property by hypothesis.

(2) If x occurs in a functional argument, then there are two cases. If the argument is x, substitution preserves the property, because B and its arguments have the property. If the argument is $[{}^{\wedge}C]$ it remains in that form, and the other functional arguments introduced have the property because they are the arguments of B.

LEMMA 6. *Reduction preserves the property that functional arguments are either variables or of the form* $[{}^{\wedge}C]$.

PROOF: As for Lemma 4. The only change is in part (1), λ-reduction, where Lemma 5 is used in place of Lemmas 2 and 3.

THEOREM 3. *In translations, functional arguments are either variables or of the form* $[{}^{\wedge}C]$.

PROOF: By an induction, with the basis by Theorem 1 and the induction step by Lemma 6.

λ-Normal forms. For the typed λ-calculus it is well-known that every expression has an irreducible λ-normal form, unique to within change of bound variable. It is natural to ask whether this is also true for the intensional logic ***IL***. One might suspect a problem for several reasons: the definition of contractible part is more complicated in ***IL***; the intension operator provides λ-abstraction over points of reference, but since there is no actual variable over points of reference no change of bound variable is possible; and there can be complex interactions of the intension operator with λ's.

By Theorem 2, in translations all functional arguments are modally closed. Thus, for application to English, we can use this modal closure as a hypothesis and show that translations have a unique fully reduced normal form. (Following our main result we show the necessity of this hypothesis).

The proof extends the proofs given in [1] and [9] for the typed λ-calculus. Modifications for ***IL*** appear in the definition of order of the contractible part $[{}^{\vee}[{}^{\wedge}A]]$ and in Lemma 7.

Order of contractible parts. We introduce a measure of the complexity of contractible parts, based on the types occurring. The *order of the type* a, $\#a$, is simply the number of left angle-brackets occurring in the symbol for a. It is immediate that for all a and b, $\#a < \#\langle a, b\rangle$, and $\#a < \#\langle s, a\rangle$. The *order of a contractible part* is defined by $\#[\lambda xA](B) = \#\langle b, a\rangle$ and $\#[{}^{\vee}[{}^{\wedge}A]] = \#\langle s, a\rangle$, where A is of type a and B is of type b.

Minimal contractible part. A contractible part is minimal if it contains no proper subformulas that are contractible.

LEMMA 7. *Let A be a formula in which all functional arguments are modally closed. If A has a minimal contractible part B of order k, then the contraction of B contains no contractible part of order k or greater.*

PROOF: Immediate for EI-contraction.

For λ-contraction we note that if $[\lambda xC](D)$ is minimal, then its contraction can contain (or be) a contractible part in only two ways:
(i) D is $[\lambda yE]$ of type $\langle f, e\rangle = d$ and x occurs in C in a subformula $x(F)$. The order of the resulting contractible part $[\lambda yE](F)$ is $\#\langle f, e\rangle = \#d < \#\langle d, c\rangle = k$.
(ii) D is $[{}^{\wedge}E]$ where $\langle s, e\rangle = d$ and x occurs in C in a subformula $[{}^{\vee}x]$. The order of the resulting contractible part $[{}^{\vee}[{}^{\wedge}E]]$ is $\#\langle s, e\rangle = \#d < \#\langle d, c\rangle = k$.

D can contain no λ-applications, since by the hypothesis they would be contractible and hence D would not be minimal.

REMARK. Lemma 7 does not require its hypothesis, but without it the proof is more difficult. The proof requires that the minimal contractible part B cannot contain an uncontractible λ-application that becomes contractible when B is reduced. This follows from two observations: (i) If D is not modally closed, neither is $Sub(B, x, D)$, and (ii) If C contains y in an intensional context, so does $Sub(B, x, C)$, provided x is not y.

LEMMA 8. *Let A be a formula in which all functional arguments are modally closed. If A reduces to D by contraction of B to C, all new contractible parts of D are contained in C.*

PROOF: There are no new λ-applications not contained in C. Under the hypothesis, all old λ-applications are already contractible.

REMARK. Lemma 8 needs the hypothesis, as the following example shows.

$[\lambda u^{\wedge}u]({}^{\vee\wedge}v)$ has only one contractible part $[{}^{\vee\wedge}v]$, which is of order 1. The λ-application is not contractible because $[{}^{\vee\wedge}v]$ is not modally closed and $[{}^{\wedge}u]$ is an intensional context. After contraction of the argument, the formula is $[\lambda u^{\wedge}u](v)$, which is a contractible part of order $\#\langle e, \langle s, e\rangle\rangle = 2$, where both u and v are of type e.

LEMMA 9. *Let A be a formula in which all functional arguments are modally closed. If D results from A by contraction of a minimal contractible part of order k, then any new contractible parts of D are of order less than k.*

PROOF: By Lemmas 7 and 8.

Let N^* be the set of all finite tuples of natural numbers, ordered by the relation $>$ as follows:
$\langle x_1, \ldots, x_n\rangle > \langle y_1, \ldots, y_m\rangle$ iff (a) $n > m$ or (b) $n = m$ and there exists k such that for all i $(0 < i < k)$, $x_{n-i} = y_{m-i}$ and $x_{n-k} < y_{m-k}$. (This is sometimes called "reverse lexicographic order".)

LEMMA 10. *If $X_1, X_2, \ldots$ is a sequence of elements of N^* such that $X_i < X_{i+1}$, then this sequence is finite.*

As Pietrzykowski observes, the lemma is easily proved by a double induction on the length of X_i and on the value of its rightmost components.

Definition of L. We define a mapping L of formulas into N^*: for any formula A, $L(A) = \langle i_1, \ldots, i_k \rangle$, where i_j is the number of contractible parts of order j in A.

LEMMA 11. *Let A be a formula in which all functional arguments are modally closed. If D results from A by contraction of a minimal contractible part B, then $L(A) > L(D)$.*

PROOF: Let $L(A) = \langle i_1, \ldots, i_k, \ldots, i_m \rangle$ and let $\# B = k$ $(1 \leqslant k \leqslant m)$. Comparing D with A, D has one fewer contractible part of order k, possibly some new contractible parts of order less than k, but no new contractible parts of higher order, by Lemma 9. Hence, $L(D) = \langle j_1, \ldots, j_{k-1}, i_k - 1, i_{k+1}, \ldots, i_m \rangle$. By the definition of the ordering relation, $L(A) < L(D)$.

LEMMA 12. *Let $A\langle 1 \rangle$ be a formula in which all functional arguments are modally closed. Then there is a fully reduced formula $A\langle n \rangle$ such that red*$(A\langle 1 \rangle, A\langle n \rangle)$.*

PROOF: Let $A\langle 1 \rangle, A\langle 2 \rangle, \ldots$ be a sequence of formulas such that *red*$(A\langle i \rangle, A\langle i+1 \rangle)$ $(i \geqslant 1)$, and *contr*$(A\langle i \rangle, A\langle i+1 \rangle)$ is minimal. If $A\langle i \rangle$ is not fully reduced, then it has at least one minimal contractible part, so there exists an $A\langle i+1 \rangle$ such that *red*$(A\langle i \rangle, A\langle i+1 \rangle)$. But by Lemma 11, $L(A\langle i \rangle) > L(A\langle i+1 \rangle)$ for all $i \geqslant 1$. Hence by Lemma 10, the sequence $L(A\langle 1 \rangle), L(A\langle 2 \rangle), \ldots$ must be finite. Thus there exists a fully reduced formula $A\langle n \rangle$ and *red**$(A\langle 1 \rangle, A\langle n \rangle)$.

COROLLARY. *Let A be a formula in which all functional arguments are modally closed. Then reduction by contraction of minimal parts yields a fully reduced formula B such that red*(A, B).*

Definition: Let $/A/$ denote the equivalence class defined by change of bound variables on A.

LEMMA 13 (Uniqueness). *Let $A\langle 1 \rangle$ be a formula in which all functional arguments are modally closed. If red*$(A\langle 1 \rangle, A\langle n \rangle)$, *red*$(A\langle 1 \rangle, B)$, and both $A\langle n \rangle$ and B are fully reduced, then $A\langle n \rangle$ and B are the same up to change of bound variable, i.e. $/A\langle n \rangle/ = /B/$.*

Proof: Let *red*$(A\langle i \rangle, A\langle i+1 \rangle)$ for $1 \leqslant i \leqslant n-1$. We assume that $/A\langle n \rangle/ \neq /B/$ and obtain a contradiction.

Let k be a number such that *red**$(A\langle k \rangle, B)$ and not *red**$(A\langle k+1 \rangle, B)$. Obviously such a k exists and $(1 \leqslant k \leqslant n-1)$. Since *red**$(A\langle k \rangle, B)$, there exists a sequence $A\langle k\rangle = B\langle 1\rangle, B\langle 2\rangle, \ldots, B\langle m\rangle = B$ $(m \geqslant 2)$ such that *red*$(B\langle i\rangle, B\langle i+1\rangle)$ holds for $1 \leqslant i \leqslant m-1$. (The reduction sequence here might include some non-minimal contractions).

We introduce the definition of the image(s) of an expression E under reduction. Let *red*(B, C) and let E be a subexpression of B. If *contr*(B, C) is equal to E, then there are no images of E in C. If E does not overlap *contr*(B, C), then the image of E in C is just the corresponding occurrence of E in C. If *contr*(B, C) is a proper subexpression of E, then the image of E in C is E', where *red*(E, E') by *contr*(B, C). If E is a proper subexpression of *contr*(B, C), then there are three cases. If *contr*(B, C) is $[{}^{\vee\wedge}D]$, then the image of E is the corresponding occurrence of E in C. If *contr*(B, C) is $[\lambda x D](F)$, E occurs in D, and *red*(D, D') by *contr*(B, C), then the image of E is the corresponding expression *Sub*(F, x, E') in D', where $/E/ = /E'/$. If contr(B, C) is $[\lambda x D](F)$ and E occurs in F, there may be zero or more images of E in C. They are the occurrences of E in the copies of F that are introduced for the free x in D (after change of bound variable in D). If *red*(B, C) and *red*(C, D) and E is a subexpression of B, then the images of E in D are the images of the images of E in C. We note that the image of a contractible part is always a contractible part.

Now let $E =$ *contr*$(A\langle k\rangle, A\langle k+1\rangle)$ and let $E\langle 1, 1\rangle = E$ and for $1 < i < m$ let $E\langle i, 1\rangle, \ldots, E\langle i, n_i\rangle$ be the subparts of $B\langle i\rangle$ that are the images of the $E\langle i, j\rangle$ under the replacement induced by the reduction of $B\langle i-1\rangle$ to $B\langle i\rangle$. (Note that E might be duplicated, or might disappear, so that n_l may be 0).

There must exist $B\langle p\rangle$ $(1 < p \leqslant m)$ such that no image $E\langle p\rangle$ of E is present, for otherwise B would have a contractible part. Now define the sequence $B'\langle 1\rangle, \ldots, B'\langle m-1\rangle$ as follows: for $1 \leqslant i < p$, let $B'\langle i\rangle$ be the result of replacing each $E\langle i, j\rangle$ in $B\langle i\rangle$ by its contraction. For $p \leqslant i < m$, let $B'\langle i\rangle = B\langle i+1\rangle$. It can be seen that *red**$(B'\langle i\rangle, B'\langle i+1\rangle)$ for $1 \leqslant i \leqslant m-2$ and that $/B'\langle m-1\rangle/ = /B\langle m\rangle/$. There are two cases in showing that *red**$(B'\langle p-1\rangle, B'\langle p\rangle)$. If *contr*$(B\langle p-1\rangle, B\langle p\rangle)$ is an image of E, then $B'\langle p-1\rangle$ is $B\langle p\rangle$. If *contr*$(B\langle p-1\rangle, B\langle p\rangle)$ is not itself an image of E, then it must be a λ-contraction $[\lambda x C](D)$ where the final images of E occur in D and where x does not occur free in C. Then *red*$(B'\langle p-1\rangle, B\langle p\rangle)$ by the contraction of $[\lambda x C](D')$ where *red**(D, D') by contracting all images of E in D. Moreover, since $A\langle k+1\rangle$ is $B'\langle 1\rangle$ we have that *red**$(A\langle k+1\rangle, B'\langle 2\rangle)$. These two results imply that *red*$(A\langle k+1\rangle, B)$ holds, which contradicts the definition of k, and completes the proof.

THEOREM 4. *If all functional arguments of A are modally closed, there is a fully reduced formula B such that red*(A, B) and B is unique to within change of bound variable.*

PROOF: Lemmas 7 through 12 prove existence; Lemma 13 proves uniqueness.

The main result now follows.

THEOREM 5. *Translations of English phrases have a fully reduced λ-normal form, unique to within change of bound variable.*

PROOF: By Theorem 2 and Theorem 4.

Remarks on *IL*. An example shows that the unique normal form result does not extend to *IL* in general. Consider the formula

$$\big[\lambda x\big[\lambda y[{}^{\wedge}y] = [u(x)]\big](x)\big](c)$$

where x and y are variables of type a, c is a constant of type a, and u is a variable of type $\langle a, \langle s, a\rangle\rangle$. Both λ-applications are contractible. Contracting the λx yields $\big[\lambda y[{}^{\wedge}y] = [u(c)]\big](c)$, which cannot be further contracted because c is not modally closed and y occurs in the intensional context $[{}^{\wedge}y]$. Contracting the λy first instead yields $\big[\lambda x[{}^{\wedge}x] = [u(x)]\big](c)$, which cannot be further contracted because c is not modally closed and x occurs in the intensional context $[{}^{\wedge}x]$. Both of these formulas are therefore reduced forms and they are not the same.

The example depends on the particular definition chosen for λ-contraction. Each of the reduced forms obtained is equivalent to $\big[\lambda x[{}^{\wedge}x]\big](c) = u(c)$, which is in some sense further reduced. If we were to redefine λ-contraction to get this result, uniqueness might be provable. Some combination and modification of the axiom schemata AS4.1 through AS4.7 and AS6 of [5] (pp. 19-20) could be used in this way.

Conclusions. Our results are a first step toward the characterization of the subset of formulas of intensional logic that are obtained as translations of English sentences. These formulas have unique fully reduced normal forms, in contrast to the general case for the intensional logic.

The main theorems have immediate application in computer processing of the expressions obtained as translations of English phrases from the [PTQ] fragment. The unique normal form can be obtained by contractions of minimal parts, and this process will always terminate. The resulting form is easier to comprehend than the direct translation and is an appropriate form for display or for evaluation.

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