THE UNIVERSITY OF MICHIGAN

COLLEGE OF ENGINEERING
Department of Engineering Mechanics

Technical Report

RADIAL EXPANSION IN PLATES COMPOSED OF IDEAL COULOMB MATERIAL

M. D. Coon
R. M. Haythornthwaite

ORA Project 05894

under contract with:

DEPARTMENT OF THE ARMY
ORDNANCE TANK-AUTOMOTIVE COMMAND
DETROIT ORDNANCE DISTRICT
CONTRACT NO. DA-20-018-AMC-0980T
DETROIT, MICHIGAN

administered through:

OFFICE OF RESEARCH ADMINISTRATION ANN ARBOR

September 1965

TABLE OF CONTENTS

LIST OF FIGURES

Figure

SUMMARY

Analysis for the plastic expansion of an annular sheet subject to in-plane radial forces is extended to ideally plastic materials in which the yield criterion is sensitive to the mean stress. Upper and lower bounds to the yield pressures, are found for the general initial motion problem where differential pressures are applied to the inner and outer perimeters. The subsequent motion problem is discussed for the particular case where the outer edge remains unloaded.

I. INTRODUCTION

The radial expansion of thin sheets in the plastic range has been the subject of intensive investigation. Taylor [1], Hill [2], Prager [3], Hodge and Sankaranarayanan [4], and Sokolovsky [5] have provided various elements of plane stress solutions based on the rigid, ideally plastic materials, and Ford and Alexander [6] have done extensive numerical analysis for the elastic-plastic material with strain hardening. Nordgren and Naghdi [7,8] have extended the work to the case of combined twist and expansion.

All the above work has been based on a 'structural' approximation for plane stress in which no attempt is made to follow in detail the nature of the lateral displacements. Some of the flows implied are impossible of realization and in consequence the associated loads will be lower bounds rather than exact yield loads. In the present work, reference will be made in the first instance to the more general analysis for the three-dimensional axi-symmetric body due to Shield [9]. Furthermore, the analysis will be carried out for a material which follows the yield criterion of Coulomb [10], where the strength is dependent on the intensity of the mean stress. As a consequence of the latter generalization, the results may have some relevance to the strength of sheets composed of soil-like materials and ice.

COULOMB YIELD CRITERION

This criterion has been discussed by Shield [11] in a manner relevent to the present work. Being based on the concept of internal friction, the criterion rests on the postulate that flow can occur whenever the shear stress developed across any plane reaches a value $c - \sigma \tan \varphi$, where σ is the normal (tensile) stress on the plane and c, φ are intrisically positive material constants. The criterion can also be expressed in the form

$$\sigma_1 = N^2\sigma_3 - 2cN \quad (\sigma_3 \geq \sigma_2 \geq \sigma_1) \tag{1}$$

where σ_1, σ_2, σ_3 are the principal stresses and $N = \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)$. The six planes obtained by interchanging σ_1, σ_2 and σ_3 in Equation (1) define the Coulomb yield surface in principal stress space.

In the theory of ideally plastic solids, a stress strain relation is introduced in a form which ensures uniqueness of applied loads in boundary value problems with force boundary conditions. It is sufficient to postulate that an element of the material cannot release work in any closed loading cycle [12]. The yield surface must be convex, and the vector formed by the increments of plastic strain associated with a particular stress state on the yield surface must lie in the direction of the outwards drawn normal.

The vector formed by the principal strain increments is defined uniquely when the yield surface has a continuously turning tangent but at corners it can lie perpendicular to any supporting plane. The yield surface for the Coulomb criterion comprises a series of six planes which intersect to form corners, and the flow rule will vary according to the position of the stress state point on the surface. For example, on the side represented by Equation (1)

$$\dot{\epsilon}_1 : \dot{\epsilon}_2 : \dot{\epsilon}_3 = -1 : 0 : N^2 . \tag{2}$$

On the adjacent side $(\sigma_2 \geq \sigma_3 \geq \sigma_1)$

$$\dot{\epsilon}_1 : \dot{\epsilon}_2 : \dot{\epsilon}_3 = -1 : N^2 : 0 \tag{3}$$

and so at the corner common to these sides

$$\dot{\epsilon}_1 : \dot{\epsilon}_2 : \dot{\epsilon}_3 = -\alpha - \beta N^2 : \alpha N^2 \tag{4}$$

where α, β are positive numbers.

The material constant φ (and hence $N = \tan(\frac{\pi}{4} + \frac{\varphi}{2})$) is found to be positive in practice, and the flow rules for all the various sides and corners then imply dilation of the material during flow.

II. INITIAL MOTION

An annular disc occupying the region

$$r_2 \geq r \geq r_1 \quad ; \quad h_0/2 \geq z \geq - h_0/2 \tag{5}$$

initially is subject to the boundary tractions

$$\sigma_r]_{r=a_0} = P_1 \quad ; \quad \sigma_r]_{r=b_0} = - P_2 \quad ; \quad \sigma_z]_{z=+h_0/2} = \sigma_z]_{z=-h_0/2} = 0 \ . \tag{6}$$

Upper and lower bounds for the combinations of pressure which induce initial deformation will be sought. If h_0 is either very large when compared with r_1 or very small when compared with r_2, the direction of the principal stresses will approach the axial, circumferential and radial directions, by symmetry. The bounds to be obtained are based on stress distribution in which these directions are maintained throughout, and in particular the axial stress is zero throughout, but it is to be noted that certain of them are nevertheless valid for plates of any thickness, including those of intermediate thickness where the actual stress states may be quite different.

The cross-section of the yield surface, Equation (1), formed by the plane $\sigma_z = 0$ is shown in Figure (1). Plastic regimes associated with the various sides and corners will be investigated.

REGIME AB

The ordering of the principal stresses is $\sigma_r \geq \sigma_z \geq \sigma_\Theta$ and Equation (1) becomes

$$\sigma_\Theta = N^2\sigma_r - P_o \tag{7}$$

where $P_o = 2cN$ is the compressive yield strength. For radial equilibrium

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\Theta}{r} = 0 \tag{8}$$

and on integration, after substitution for σ_Θ from Equation (7):*

$$\left.\begin{aligned} \sigma_r &= \frac{P_o}{N^2-1} + C_1 r^{N^2-1} \\ \sigma_\Theta &= \frac{P_o}{N^2-1} + C_1 N^2 r^{N^2-1}, \end{aligned}\right\} \quad (T_o \geq \sigma_r \geq 0) \tag{9}$$

The flow rule associated with Equation (7) is

*Due to the singularity, the solution for the Tresca yield criterion cannot be obtained by substituting N = 1; however it can be retrieved by setting $N = 1+\epsilon$ where $\epsilon \to 0$.

$$\dot{\epsilon}_r : \dot{\epsilon}_\theta : \dot{\epsilon}_z = N^2 : -1 : 0 \qquad (10)$$

which, when written in terms of the velocities u, w, becomes

$$\left.\begin{aligned} \frac{\partial u}{dr} + N^2 \frac{u}{r} &= 0 \\ \frac{\partial w}{\partial z} &= 0 \\ \frac{\partial u}{\partial z} + \frac{\partial w}{dr} &= 0 \end{aligned}\right\} \qquad (11)$$

where the last equation expresses the necessary and sufficient condition for coincidence of the principal directions of stress and strain rates. Setting $w]_{z=0} = 0$, $w = 0$ everywhere, by the second of (11) and by the third of (11), u is then a function of r only. The first of (11) is now an ordinary differential equation and the solution is

$$u = C_2 r^{-N^2} . \qquad (12)$$

REGIME BC

The ordering is $\sigma_r > \sigma_\theta > \sigma_z$. Equation (1) becomes

$$\sigma_z = N^2\sigma_r - 2CN \qquad (13)$$

so the flow rule is $\dot{\epsilon}_r : \dot{\epsilon}_\theta : \dot{\epsilon}_z = N^2 : 0 : -1$.

In terms of velocities

$$\left.\frac{\partial u}{dr} + N^2 \frac{\partial w}{dz} = 0 \right\}$$

$$\left.\begin{aligned} \frac{u}{r} &= 0 \\ \frac{\partial u}{dz} + \frac{\partial w}{dr} &= 0 \end{aligned}\right\} \qquad (14)$$

and also

The velocities are zero everywhere, so no solution is possible.

REGIME CD

The ordering is $\sigma_\Theta > \sigma_r > \sigma_z$. Equation (1) becomes

$$\sigma_z = N^2\sigma_\Theta - 2CN \qquad (15)$$

so the flow rule is $\dot{\epsilon}_r : \dot{\epsilon}_\Theta : \dot{\epsilon}_z = 0 : N^2 : -1$. In terms of velocities

$$\left.\begin{aligned} \frac{\partial u}{dr} &= 0 \\ \frac{u}{r} + N^2 \frac{\partial w}{\partial z} &= 0 \\ \frac{\partial u}{\partial w} + \frac{\partial w}{dr} &= 0 \end{aligned}\right\} \qquad (16)$$

and also

The displacements are zero everywhere and no solution is possible.

REGIME DE

The analysis parallels that given above for regime AB, and it is found that

$$\left.\begin{aligned} \sigma_r &= \frac{P_o}{N^2-1} + C_3 r^{N^{-2}-1} \\ \sigma_\Theta &= \frac{P_o}{N^2-1} + \frac{C_3}{N^2} r^{N^{-2}-1} \end{aligned}\right\} \qquad (17)$$

$$u = C_4 r^{-1/N^2}. \tag{18}$$

REGIMES EF and FA

These regimes are analogous to regimes BC and CD respectively and they also can be discarded because no compatible motion is possible.

CORNERS A,B,D and E

In view of radial equilibrium, Equation (8), these state points can be reached only at discrete radii.

CORNER C

This corner lies at the intersection of the facets BC, Equation (13) and CD, Equation (15), where $\sigma_r = \sigma_\Theta = T_o$ and the flow rule is

$$\dot{\epsilon}_r : \dot{\epsilon}_\Theta : \dot{\epsilon}_z = \alpha N^2 : \beta N^2 : -\alpha - \beta$$

where α,β are positive numbers, so $\dot{\epsilon}_r + \dot{\epsilon}_\Theta + N^2\dot{\epsilon}_z = 0$ which becomes, when written in terms of the velocities

$$\left.\begin{aligned} \frac{\partial u}{dr} + \frac{u}{r} + N^2 \frac{\partial w}{\partial z} &= 0 \\ \text{and also} \qquad \frac{\partial u}{dz} + \frac{\partial w}{dr} &= 0 \end{aligned}\right\} . \tag{19}$$

Several solutions have been given previously [13]; one of the simplest is

$$\left.\begin{aligned} u &= \frac{2}{\pi} \sqrt{N^2 - z^2/r^2} \\ w &= \frac{2}{\pi} \tan^{-1} \sqrt{N^2 r^2/z^2 - 1} \end{aligned}\right\} \tag{20}$$

where $z \geq 0$. The deforming zone is restricted to the sector $N \geq z/r \geq -N$.

CORNER F

This corner is analogous to corner C discussed above. The stresses are $\sigma_r = \sigma_\theta = -P_o$ and a compatible velocity field, [13], is

$$\left.\begin{aligned} u &= -\frac{2}{\pi}\sqrt{1/N^2 - z^2/r^2} \\ w &= -\frac{2}{\pi}\tan^{-1}\sqrt{r^2/z^2N^2 - 1} \end{aligned}\right\} \qquad (21)$$

The deforming zone is restricted to the sector $1/N \geq z/r \geq 1/N$.

It is immediately evident that exact solutions will be available for certain ratios of the inside and outside pressure by the simple expedient of consigning the entire plate to the regimes AB or DE, or to the corners C or F. The solutions are exact for all plate thicknesses. For other ratios of r_1 and r_2 these regimes can be used as the basis for upper bounds.

When regime AB applies to the entire plate, substitution of the boundary conditions, Equation (16), in Equation (9) leads to

$$p_2 = p_1\beta^{N^2-1} - \frac{1-\beta^{N^2-1}}{N^2-1} \qquad (22)$$

$$(0 \geq p_1,\ p_2 \geq -1/N^2)$$

where $p_1 = P_1/P_o$; $p_2 = P_2/P_o$ and $\beta = b_o/a_o$. When regime DE applies throughout, Equation (17) leads to

$$p_2 = p_1\beta^{N^{-2}-1} - \frac{1-\beta^{N^{-2}-1}}{N^2-1} \qquad (23)$$

$$(1 \geq p_1,\ p_2 \geq 0).$$

The singular points C, F give rise to complete solutions for the case $P_1 = P_2$. The above cases appear to exhaust the exact solutions available when no restriction is placed on the thickness h_o. Exact solutions for other ratios of P_1 and P_2 presumably require stress distributions where the principal stresses are no longer in the directions r, θ, z. These solutions will lie within the range of the general equations for axi-symmetric plastic flow [9, 14].

THE THIN PLATE

In the case of the thin plate, the solutions given above still apply, and they can also be used as a basis for quite close inner and outer bounds which are correct bounds within the framework of three-dimensional theory (Bounds obtained by the 'structural' theory of plates in which displacement conditions in the lateral direction are often ignored are of dubious validity). Figure (2) shows the results of computations for the particular case where $N^2 = 3$, $\beta^2 = 3$. Equation (22) becomes line fgh. Point g represents a complete solution but at other points the inequalities in Equation (22) are not satisfied and the points lie outside the yield surface, by the upper bound theorem of limit analysis [15]. Equation (23) becomes line labc. In this case the inequalities are satisfied along ab and the remainder of the line represents an upper bound.

The velocity fields associated with the singular points C,F can be employed to give very useful upper bounds. Placing the thin plate at z = 0, and equating the work done by the forces of the known solution to the work done by a general set of pressures P_1, P_2, we obtain the outer bound

$$p_2 = p_1/\beta - 1/N^2 + 1/\beta N^2 \tag{24}$$

from corner C and the outer bound

$$p_2 = p_1/\beta + 1 - 1/\beta \qquad \text{from corner F} . \tag{25}$$

These are shown as lines jkl and def respectively in Figure (2).

Placing the thin plate at a height such that the radial velocity is zero at the inner edge for velocity fields (20) and (21), we obtain the outer bounds

$$p_2 = -1/N^2 \qquad \text{from corner C} \tag{26}$$

and

$$p_2 = +1 \qquad \text{from corner F} .$$

These are shown as lines hj and dc in Figure (2). Some inner bound lines are available at once by noting the property of convexity of the yield surface in the space of generalized stresses p_1, p_2. As a consequence of this property, any lines joining known solution points are inner bounds. Examples are bd and gj. Another bound can be obtained by assuming all stress state points lie in regime BC, Figure (1), where $\sigma_\theta = T_0$. On integrating Equation (8), and substituting the boundary conditions (5), we obtain Equation (24) once more, subject to the requirement that

$$0 \geq p_1, \; p_2 \geq -1/N^2 . \tag{27}$$

Coincident upper and lower bounds have been found, so it follows that jk, Figure (2) represents the true yield line, even though a complete solution has not been found. (The velocity field, Equation (20) is not compatible with stress state points on side BC, Figure (1).)

A similar procedure based on side FE, Figure (1) confirms that Equation (25) is a lower bound, subject to the requirement

$$1 \geq p_1, \ p_2 \geq 0 \qquad (28)$$

and so line de, Figure (2) is also a true yield line.

Points e and k, Figure (2) are evidently on the yield curve, so further inner bound lines may be obtained by joining eg and ka.

III. CONTINUED MOTION*

The general problem of continued motion presents great difficulty because the particular velocity field must be selected which allows the deforming body to remain in equilibrium at every instant [16]. For the expanding sheet, it so happens that the continued motion solution is straightforward when there is expansion without thickening (side ab and point g of Figure(2)). This is the only case where an incipient velocity field is known and for other cases resource must be made to the approximation of conventional plane stress theory.

For the purposes of illustration, attention will be confined to an annular plate subject to internal pressure $P_1 = P$ and zero external pressure ($P_2 = 0$), as shown in Figure(3a). Equation (23) reduces to

*This section formed part of the Ph.D. dissertation of M.D. Coon at The University of Michigan.

$$p = \frac{\beta^{1-N^{-2}}-1}{N^2-1} \quad . \tag{29}$$

This expression is valid when $1 \geq p \geq 0$ or, in terms of β,

$$N^{2N^2/(N^2-1)} \geq \beta \geq 1 \quad . \tag{30}$$

For larger values of β, the yield load is not known, but a lower bound, $p = 1$, is established at once by noting that the stress field for the plate where

$$\beta = N^{2N^2/(N^2-1)} \tag{31}$$

is statically admissible for any larger plate. The expression for p in Equation (29) is shown as the solid lines in Figure(4). The dotted lines indicate the lower bound, $p = 1$, obtained above.

Equation (31) gives a β_{cr} such that for $\beta \leq \beta_{cr}$ the plate expands without thickening as shown in Figure (3b) but for $\beta \geq \beta_{cr}$ the determination will involve thickening as shown in Figure (3c).

FIRST SOLUTION: $\beta \leq \beta_{cr}$

The exact initial motion solution given above will be the starting point for this analysis. However, in this problem the stresses, velocities, and strain rates will be functions of time. Because there is no viscosity, time can be replaced by any variable which increases monotonically as the applied load increases. The coordinates will represent the current geometry, therefore, it will be convenient to take a (the inner radius) as a measure of time.

The velocity for this case can be found from Equation (18) by noting that when a is taken as the measure of time, the new variable r/a replaces the variable r. Therefore, Equation (18) gives

$$u = C_4\left(\frac{a}{r}\right)^{1/N^2} . \qquad (32)$$

To determine u completely the velocity of some point must be specified and for convenience u = 1 at r = a is chosen. This implies that $C_4 = 1$.

The displacement, $\bar{u}$, can be found from the definition of velocity, which becomes

$$\frac{dr}{da} = u = \left(\frac{a}{r}\right)^{1/N^2} \qquad (33)$$

after making use of Equation (32).

The boundary condition is

$$r = r_o \text{ at } a = a_o \qquad (34)$$

where the initial coordinate of a point is given by r_o. Its final coordinate will be r. The initial and final coordinates of a point can be related by

$$r_o = r - \bar{u}(r,a) . \qquad (35)$$

By separating Equation (33) and using the above boundary conditions, one obtains

$$\int_{r-\bar{u}}^{r} r^{1/N^2} dr = \int_{o}^{a} a^{1/N^2} da \qquad (36)$$

which can be integrated to give

$$\bar{u} = r - \left[r^{(N^2+1)/N^2} - a^{(N^2+1)/N^2} + a_o^{(N^2+1)/N^2} \right]^{N^2/(N^2+1)} . \tag{37}$$

Equation (37) implies that

$$\frac{b}{a} < \beta = \frac{b_o}{a_o} \quad \text{for all} \quad a > a_o . \tag{38}$$

Therefore, as the motion continues the plate becomes smaller (i.e. β becomes smaller) which implies that the entire plate remains on side DE of Figure(1).

The stresses can be found from Equations (17) by noting that the constant C_3 is now a function of time. This function of time can be obtained by using the boundary condition $\sigma_r = 0$ at $r = b$ which leads to

$$\left. \begin{aligned} \sigma_r &= \frac{P_o}{N^2-1} \left[1 - \left(\frac{b}{r}\right)^{(N^2-1)/N^2} \right] \\ \sigma_\theta &= \frac{P_o}{N^2(N^2-1)} \left[N^2 - \left(\frac{b}{r}\right)^{(N^2-1)/N^2} \right] \end{aligned} \right\} . \tag{39}$$

The time variation in the stresses is expressed in terms of b which can be related to a through Equation (37).

In this solution, all of the equations for the three-dimensional problem have been satisfied and this is an exact solution to the continued motion problem.

SECOND SOLUTION: $\beta > \beta_{cr}$

For this solution, the mode of deformation must involve thickening of the plate and it appears that there are no exact three-dimensional solutions for this case. A solution can be obtained by averaging the strain rate $\dot{\epsilon}_z$ through the thickness. If this is done $\dot{\epsilon}_z$, is given by

$$\dot{\epsilon}_z = \frac{1}{h} \frac{Dh}{Dt} \tag{40}$$

where
$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial r} \quad . \tag{41}$$

Equation (41) denotes the rate of change following an element, i.e. the material derivative. When this is done, the third of Equations (11) will not be satisfied, which implies that the principal directions of stress and strain rate do not coincide. Also, for the plate of variable thickness, the equilibrium equation will be

$$\frac{\partial}{\partial r} (h\sigma_r) + \frac{h}{r} (\sigma_r - \sigma_\Theta) = 0 \tag{42}$$

where h is a function of r. As in the solution for $\beta \leq \beta_{cr}$, the stresses, velocities, and strain rates will be functions of time.

Statically admissible stresses can be found by introducing a rigid region on the range $r_2 \leq r \leq b$ (Figure(3c)). For $r_1 \leq r < r_2$ the stresses can be taken on regime DE of Figure (1) with $\sigma_r = 0$ at r_2 and $\sigma_r = -P_o$ at r_1. Similar to the case for $\beta \leq \beta_{cr}$, it is convenient to choose some radius as a measure of time and r_1 will be chosen. On the range $r_1 \leq r < r_2$ the actual stress distribution can be found, as for the case $\beta \leq \beta_{cr}$, from Equation (17) using the boundary condition $\sigma_r = 0$ at $r = r_2$, this gives

$$\sigma_r = \frac{P_o}{N^2-1} \left[1 - \left(\frac{r_2}{r}\right)^{(N^2-1)/N^2} \right] \tag{43}$$

and $$\sigma_\Theta = \frac{P_o}{N^2(N^2-1)}\left[N^2 - \left(\frac{b}{r}\right)^{(N^2-1)/N^2}\right] .$$

The time variation of the stresses is now expressed through r_2 which can be related to r_1 by noting that $\sigma_r = -P_o$ at $r = r_1$, and this leads to

$$r_2 = r_1 \, N^{2N^2/(N^2-1)} \quad . \tag{44}$$

On the range $a \leq r < r_1$ the stresses can be taken at point E of Figure(1) where

$$\sigma_r = -P_o \text{ and } \sigma_\Theta = 0 \; . \tag{45}$$

For the stresses to be statically admissible, the plate must thicken in this range and the thickening can be found from Equation (42) to be

$$h = \frac{h_o r_1}{r} \tag{46}$$

where the boundary condition $h = h_o$ (h_o is the initial thickness of the plate) at $r = r_1$ has been used. The stress solution is now complete.

For the range $r_1 \leq r < r_2$ the velocity can be found, as in the case of $\beta \leq \beta_{cr}$, from Equation (18). However, in this case, $u = 0$ at $r = r_2$ and Equation (18) gives u identically 0 on this range. On the range $a \leq r < r_1$ the velocity is found from the equation for the strain rates at point E, which is

$$\dot{\epsilon}_\Theta + \dot{\epsilon}_z + N^2 \dot{\epsilon}_r = 0 \; . \tag{47}$$

This equation can be written in terms of u and h to give

$$N^2 \frac{\partial u}{\partial r} + \frac{u}{h} \frac{\partial h}{\partial r_1} + \frac{1}{h} \frac{\partial h}{\partial r_1} + \frac{u}{r} = 0 \quad . \tag{48}$$

Combining Equations (46) and (48) and using the boundary condition $u = 0$ at $r = r_1$ one obtains

$$u = \frac{1}{N^2} \quad \frac{r}{N^2 r_1} \quad . \tag{49}$$

The displacement $\bar{u}$ is found from the definition of velocity to be

$$\bar{u} = r - \frac{1}{N^2+1} \left\{ \left[(N +1)r - r_1 \right] \left(\frac{r_1}{a_o} \right)^{1/N^2} + a_o \right\} . \tag{50}$$

The solution for $\beta > \beta_{cr}$ is now complete except that the inner radius a must be related to r_1. This can be accomplished by rewriting Equation (47) in the following form

$$\dot{\epsilon}_\theta + \dot{\epsilon}_z + \dot{\epsilon}_r = (1-N^2)\dot{\epsilon}_r \quad . \tag{51}$$

The right side of Equation (51) is known because ϵ_r is known. The left side of Equation (51) is the time rate of change of the change of the volume per unit volume. Thus

$$\dot{\epsilon}_\theta + \dot{\epsilon}_z + \dot{\epsilon}_r = (1-N^2)\dot{\epsilon}_r = \frac{\Delta \dot{V}}{V} \tag{52}$$

where V is the current volume of the deformed material, i.e., all of the material inside the radius r_1 and $\Delta\dot{V}$ is the time rate of change of the volume inside this radius.

Because a_0 is the initial radius of the hole and a its present radius the change in volume is given by

$$\Delta V = \int_a^{r_1} 2\pi hrdr - \pi h_o[r_1^2 - a_o^2] \ . \tag{53}$$

The rate of volume change per unit volume is then

$$\Delta \dot{V} = \frac{d}{dr_1}\left[\int_a^{r_1} 2\pi hrdr - \pi h_o[r_1^2 - a_o^2\right] \ . \tag{54}$$

An alternative expression for $\Delta \dot{V}$ can be found by substituting the values of the strain rates found from Equation (49) into Equation (52):

$$\Delta \dot{V} = \frac{N^2-1}{N^2 r_1}\int_a^{r_1} 2\pi hrdr \ . \tag{55}$$

Hence

$$\frac{d}{dr_1}\left[\int_a^{r_1} 2hrdr - h_o[r_1^2 - a_o^2]\right] = \frac{N^2-1}{N^2 r_1}\int_a^{r_1} 2hrdr \ . \tag{56}$$

The thickness h is given by Equation (46), and the radius a is a function of r_1 which is to be determined. Equation (56) is equivalent to

$$\frac{da}{dr_1} = \frac{1}{N^2}\left[1 - \frac{a}{r_1}\right] \tag{57}$$

and with the boundary condition $a = a_o$ when $r_1 = a_o$, the solution is

$$a = \frac{r_1}{N^2+1}\left[1 + N^2\left(\frac{a_o}{r_1}\right)^{(N^2+1)/N^2}\right]. \tag{58}$$

The first solution ($\beta \leq \beta_{cr}$) is an exact solution to three-dimensional problems for both the limit load and for finite displacement. The second solution ($\beta > \beta_{cr}$) is of the structural type used by previous authors for analogous problems. In terms of the three-dimensional point of view, it gives a lower bound to the limit load. This solution cannot be exact in terms of classical plasticity theory, because the principal directions of stress and strain rate do not coincide. Also, the thickness changes in the plate are accounted for in the equilibrium equation, but the stress boundary conditions have not been satisfied on the part of the plate in which thickening occurs.

REFERENCES

1. Taylor, G. I., "The formation and enlargement of a circular hole in a thin plastic sheet," Q. J. Mech., Appl. Math., Vol. 1, 1948, pp. 103-124.

2. Hill, R., "Plastic distortion of non-uniform sheets," Phil. Mag., Vol. 40, 1949, pp. 971-983.

3. Prager, W., "On the use of singular yield conditions and associated flow rules," J. Appl. Mech., Vol. 20, 1953, pp. 317-320.

4. Hodge, P. G., Jr. and R. Sankaranarayanan, "On finite expansion of a hole in a thin infinite plate," Q. Appl. Math., Vol. 16, 1958, pp. 73-80.

5. Sokolovsky, V. V., "Expansion of a circular hole in a rigid/plastic plate," Appl. Math., Mech. (Transl. PMM), Vol. 25, 1961, pp. 809-815.

6. Alexander, J. M. and H. Ford, "On expanding a hole from zero radius in a thin infinite plate," Proc. Roy. Soc. London Ser. A 226, 1954, pp. 543-561.

7. Nordgren, R. P. and P. M. Naghdi, "Finite twisting and expansion of a hole in a rigid/plastic plate," J. Appl. Mech., Vol. 30, 1963, pp. 605-612.

8. Nordgren, R. P. and P. M. Naghdi, "Loading and unloading solutions for an elastic/plastic annular plate in the state of plane stress under confined pressure and couple," Intern. J. Engin. Sci., Vol. 1, 1963, pp. 33-70.

9. Shield, R. T., "On the plastic flow of metals under conditions of axial symmetry," Proc. Roy. Soc. London Ser A 233, 1955, pp. 267-286.

10. Coulomb, C. A., "Essai sur une application des regles de maximis et minimis a quelqnes problemes de Statique, relatifs a l'Architecture," Mem. Acad. Sci. (Savants Estrangeis), Vol. 7, 1773, pp. 343-382.

11. Shield, R. T., "On Coulomb's law of failure in soil," J. Mech. Phys. Solid, Vol. 4, 1955, pp. 10-14.

12. Drucker, D. C., "A more fundamental approach to plastic stress-strain relations," Proc. 1st U.S. Nat. Congr. Appl. Mech., (ASME 1951), pp. 487-491.

13. Haythornthwaite, R. M., "The mechanics of the triaxial test for soils," Proc. Am. Soc. Civil Engrs., Vol. 86 (SM5), 1960, pp. 35-62.

14. Cox, A. B., Eason, G. and Hopkins, H. G., "Axially symmetric plastic deformation of soils," Phil. Trans. Roy. Soc. Ser. A 254, 1961, pp. 1-45.

15. Drucker, D. C., Greenberg, H. J., and Prager, W., "Extended limit design theorems for continuous media," Q. Appl. Math., Vol. 9, 1952, pp. 381-389.

16. Hill, R., "On the problem of uniqueness in the theory of a rigid-plastic solid-III," J. Mech. Phys. Solids, Vol. 4, 1957, pp. 153-161.

FIGURE CAPTIONS

Figure 1 Plane stress section of Coulomb yield criterion.

Figure 2 Yield curve for $\beta^2 = 3$ and $N^2 = 3$.

Figure 3 Annular plate a) Undeformed, b) Deformed without thickening: $\beta \leq \beta_{cr}$, c) Deformed with thickening: $\beta > \beta_{cr}$.

Figure 4 Limit load as a function of plate size.

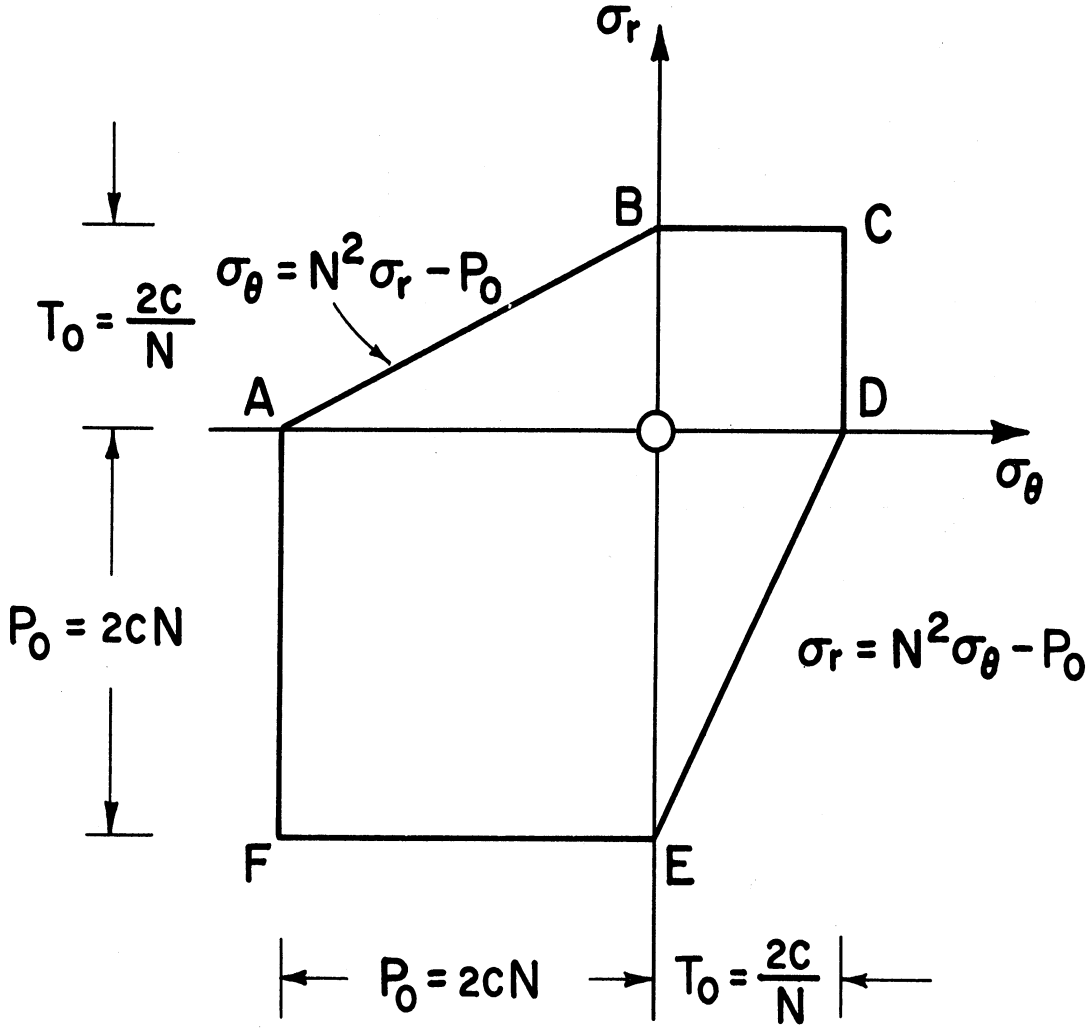
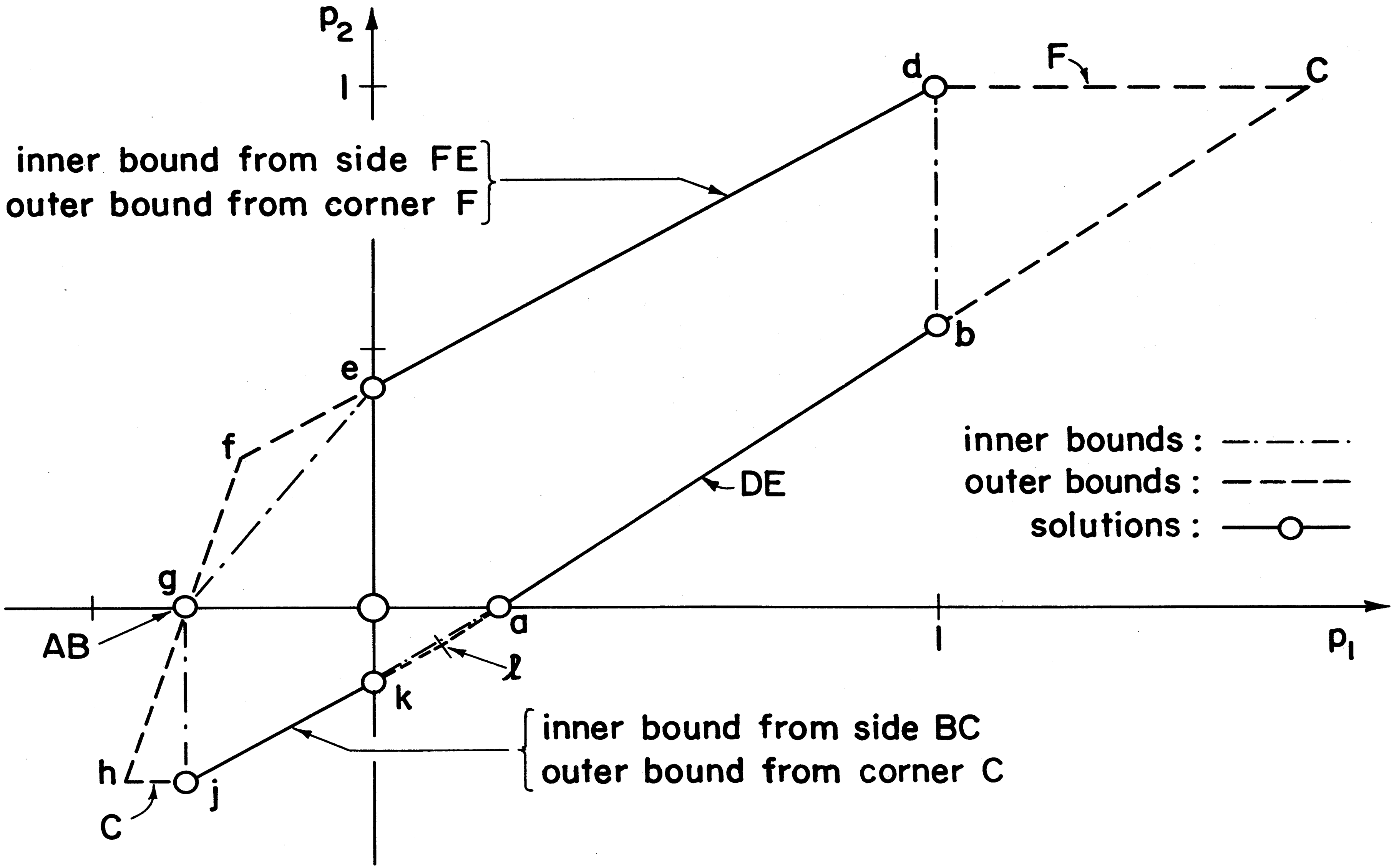

σ_r
B
C
$\sigma_\theta = N^2\sigma_r - P_o$
$T_o = \frac{2c}{N}$
A
D
σ_θ
$P_o = 2cN$
$\sigma_r = N^2\sigma_\theta - P_o$
F
E
$P_o = 2cN$
$T_o = \frac{2c}{N}$

Figure 1



p2
1
d
F
C
inner bound from side FE
outer bound from corner F
b
e
f
inner bounds :
outer bounds :
solutions :
DE
g
AB
a
1
p1
ℓ
k
inner bound from side BC
outer bound from corner C
h
j
C

Figure 2

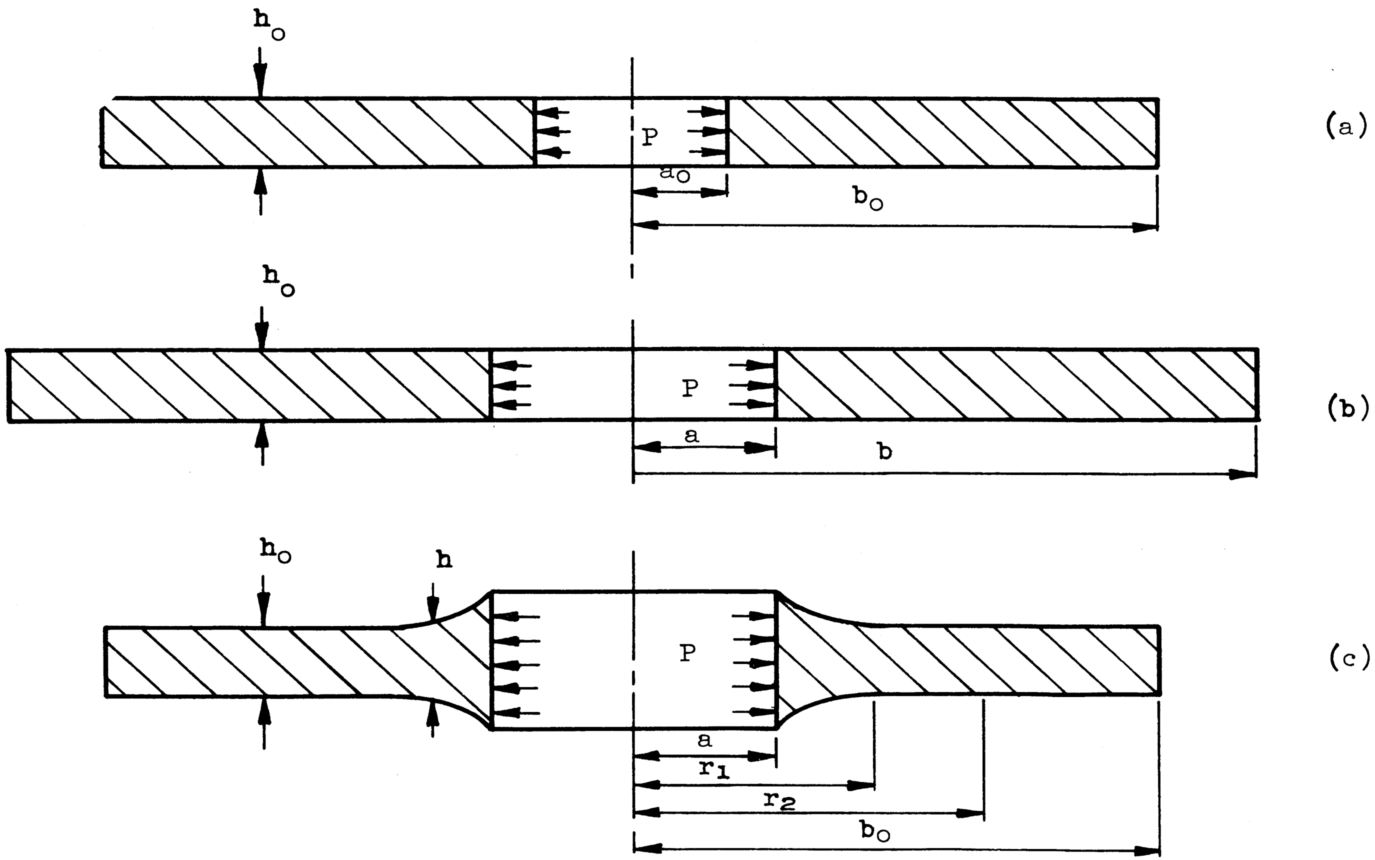


Figure 3

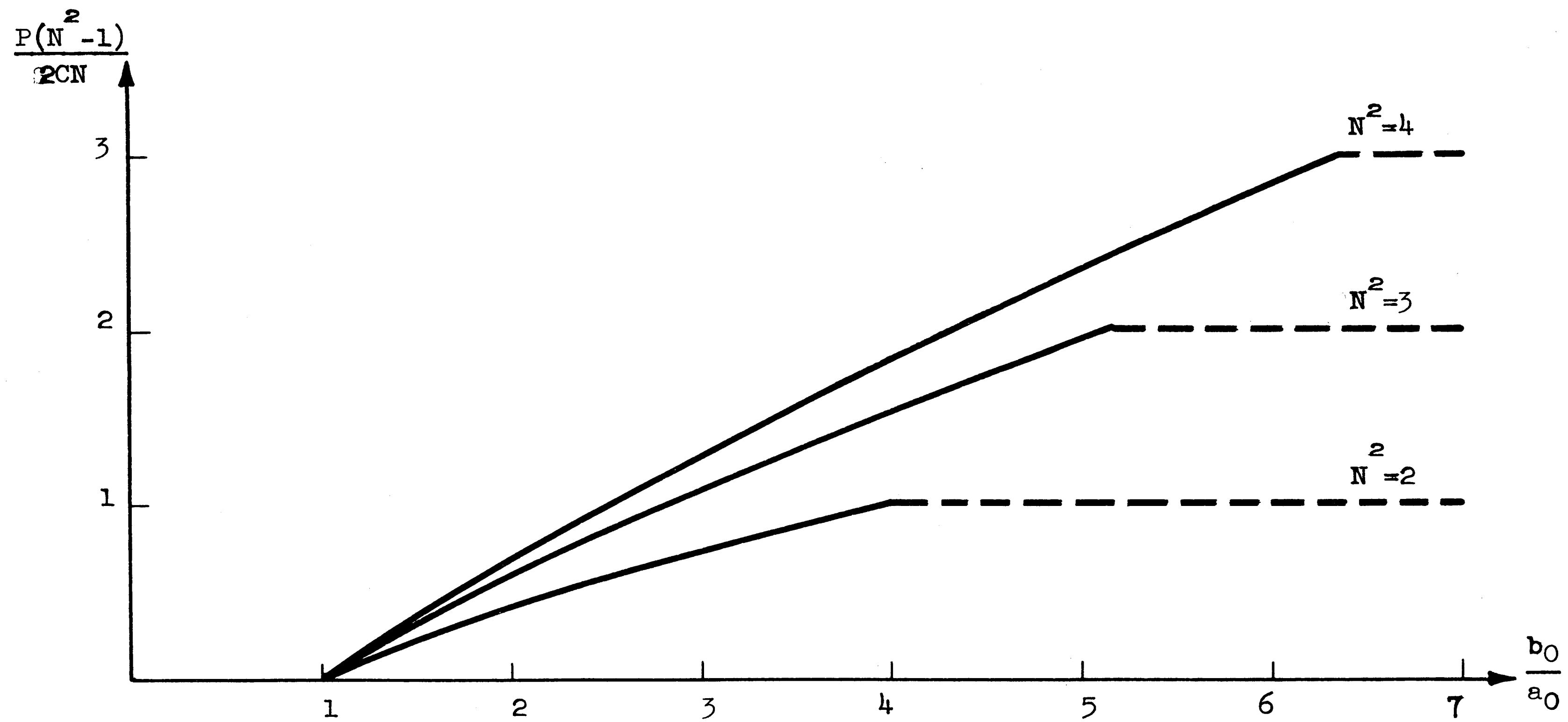


Figure 4

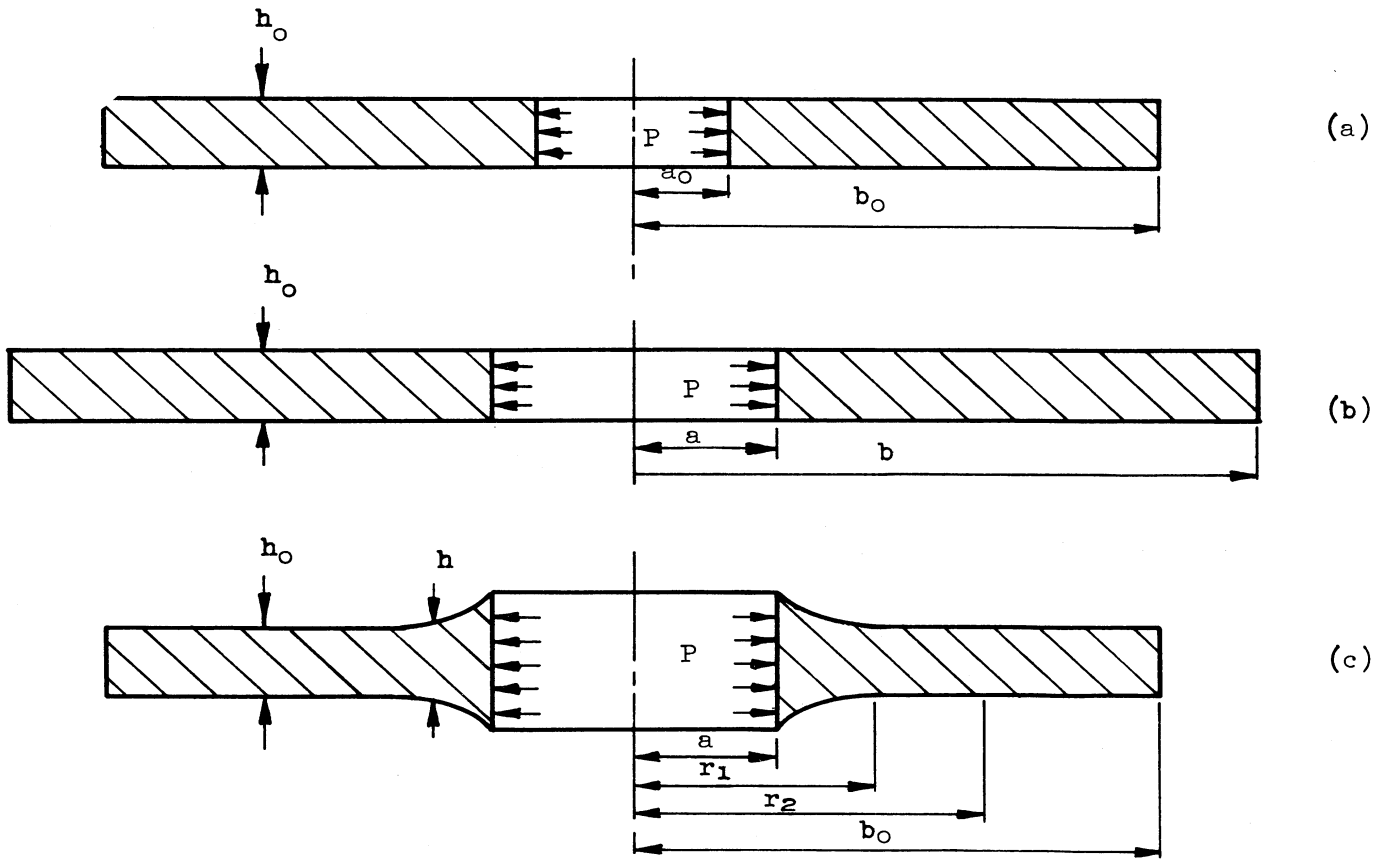


Figure 3