Production variability in manufacturing systems: Bernoulli reliability case *

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The problem of production variability in serial manufacturing lines with unreliable machines is addressed. Bernoulli statistics of machine reliability are assumed. Three problems are considered: the problem of production variance, the problem of constant demand satisfaction, and the problem of random demand satisfaction generated by another (unreliable) production line. For all three problems, bounds on the respective variability measures are derived. These bounds show that long lines smooth out the production and reduce the variability. More precisely, these bounds state that the production variability of a line with many machines is smaller than that of a single machine system with production volume and reliability characteristics similar to those of the longer line. Since all the variability measures for a single machine line can be calculated relatively easily, these bounds provide analytical tools for analysis and design of serial production lines from the point of view of the customer demand satisfaction.

Keywords: production systems, unreliable machines, production variability

1. Introduction

1.1. Problems addressed

This paper is devoted to a study of production variability in manufacturing systems with unreliable machines. Due to machine breakdowns, the number of parts produced by such systems during a fixed interval of time is a random variable. Its expected value and variance characterize the production volume and the production variability, respectively.

The production volume has been the subject of study in a plethora of publication (see, for instance, review [3]). In contrast, the production variance has been addressed in just a few recent articles (see, for instance, [4,6,12,13]); they are briefly reviewed in subsection 1.3 below.

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Although the production variance is a useful performance measure, it gives only a general, theoretical characterization of the production variability. Indeed, the knowledge of the variance does not yet specify the probability with which a fixed shipping schedule is satisfied, although a fixed shipping schedule is a standard "supplier–customer" relation in most large volume industries (e.g., the automotive production). Intuitively, the smaller the variance, the larger the chance to meet the shipping schedule. However, a more direct measure, e.g., the probability to produce a fixed number of parts during a given shipping period, referred to as "due time performance", is required in practice. For very long shipping periods, this probability can be estimated asymptotically based on the production volume and its variance (using the Central Limit Theorem). However, in many practical situations the shipping intervals are short, and the central limit approach does not apply. Therefore, a direct method for evaluating the "due time performance" is necessary.

Another problem where the production variance does not directly quantify the quality of the supplier is the problem of satisfying a random demand. Here the "customer" is another production line connected to the "supplier" line by a material handling system. Since the "customer" can be either up or down, the demand is random, and the problem is to quantify how the random production of the "supplier" satisfies the random demand of the "customer".

In this paper, we address all three problems:

1. The production variance problem – a theoretical characterization of the production variability;

2. The Due Time Performance problem – a characterization of the quality of the "supplier" in a fixed demand environment;

3. The Random Demand Satisfaction problem – a characterization of the quality of the "supplier" in a random demand environment.

The main result obtained is as follows: in each of the above problems, the variability measures can be quantified by simple bounds. These bounds are analytical in nature and could be used as a tool for design and analysis of production lines from the point of view of the customer demand satisfaction. In addition, these bounds characterize the nature of the production variability in serial lines. Specifically, they show that longer lines reduce production variability in the following sense: the production variability in lines with many machines is smaller than the production variability of a single machine system with production rate equal to that of the longer line and the breakdown statistics analogous to those of the machines used in the longer line.

The remainder of this paper is structured as follows: the problem formulation and a review of the available literature are given in subsections 1.2 and 1.3, respectively. The production variance problem, the Due Time Performance problem and the Random Demand Satisfaction problem are discussed in sections 2–4. The conclusions are formulated in section 5. All the proofs are given in the appendix.

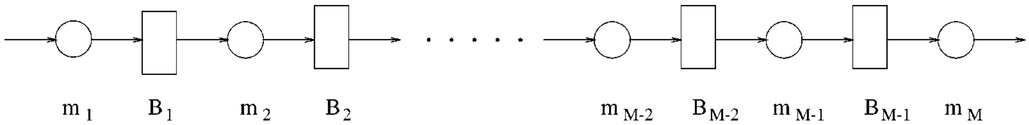

Figure 1. Serial production line.

1.2. *Problem formulations*

In this paper, we study exclusively the simplest, but practical, production system – the serial production line. The basic structure of this system is shown in figure 1, where the circles represent the machines and the rectangles are the buffers. Two variations of this structure, to accommodate problems 2 and 3, will be discussed in the subsequent subsections. In order to introduce the production variance problem, only the basic structure is required.

1.2.1. Production variance problem

The assumptions formulated below define the machines, the buffers, the interactions between machines, and the demand.

Machines:

(i) Each machine requires a fixed unit of time to process a part. This unit is referred to as the cycle time. All machines have identical cycle time. The time axis is slotted with the slot duration equal to the cycle time.

(ii) During a cycle time, each machine can be in one of two states: "up" or "down". When up, the machine can process a part. When down, no processing can take place.

(iii) The state of the machine in each cycle time is determined by the process of Bernoulli trials. In other words, it is assumed that during each slot machine m_i, $i = 1, \ldots, M$, is up with probability p_i and down with probability $1 - p_i$; the state of the machine is determined at the beginning of each cycle, independent of the state of this machine in the previous cycle.

Remark 1.1. Assumption (iii) defines the Bernoulli statistics of machine breakdowns. In our experience, many assembly systems obey this reliability model. The reason is that machine downtime in assembly operations is often of the duration comparable with that of the cycle time (i.e., the time necessary to accomplish an operation). Physically, this happens due to the fact that parts have to be assembled with the highest possible quality and, to do this, operational conveyers are sometimes stopped for a short period of time. Another frequent perturbation is pallets jam on the conveyors; to correct this problem also a short period of time is required. In many assembly systems these are predominant perturbation. These situations lead to the Bernoulli reliability model. In contrast, the Markovian model (see, for instance, [1,5]) implies that the downtime is due to a machine's physical breakdowns which often require long, relative to the cycles, time to repair. The Markovian model is more appropriate for machine operations.

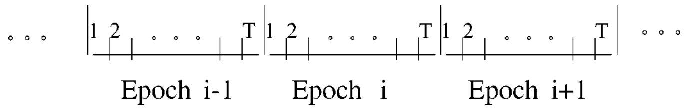

Figure 2. Epochs.

Buffers:

(iv) Each buffer, B_i, $i = 1, \ldots, M-1$, is capable of storing N_i parts, $1 \leqslant N_i < \infty$

Starvation rule:

(v) If B_i, $i = 1, \ldots, M-1$, is empty at the beginning of the time slot, then m_{i+1}, $i = 1, \ldots, M-1$, is starved during this time slot. The first machine is never starved.

Blockage rule:

(vi) If B_i, $i = 1, \ldots, M-1$, is full at the beginning of a time slot and m_{i+1}, $i = 1, \ldots, M-1$, does not take a part from B_i at the beginning of this slot, then m_i, $i = 1, \ldots, M-1$, is blocked during this time slot.

Remark 1.2. As follows from assumptions (iii), (v) and (vi), a machine can be down even if it is starved or blocked. This is referred to as the "time dependent failures", in contrast to the "operation dependent failures" considered in [5]. The time dependent failures are assumed here to simplify the analysis.

Demand:

(vii) From the point of view of the demand, the time axis is divided into "epochs", each containing T time slots (figure 2).

(viii) At the end of each epoch, a shipment of D parts has to be available for the customer. If $p_a = \mathit{PR}(p_1, \ldots, p_M, N_1, \ldots, N_{M-1})$ is the production rate of the system, i.e., the average number of parts produced by the last machine, m_M, during a time slot, then

$$D \leqslant Tp_a. \tag{1.1}$$

Remark 1.3. A method for calculating the production rate in the system defined by (i)–(vi) has been developed in [8]. Thus, the upper bound of D is readily available.

Demand satisfaction policy:

(ix) All parts produced per epoch are shipped to the customer, i.e., no backloging takes place and no parts produced are stored.

Remark 1.4. Obviously, the above demand satisfaction policy is not practical. This is exactly why the production variance problem does not have a direct industrial interpretation. Assumption (ix) is introduced here only for the purpose of formulating this problem.

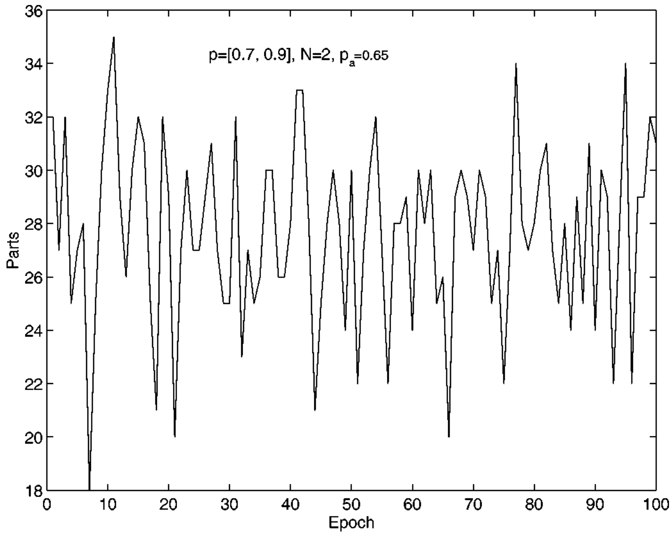

Figure 3. Variation in serial production system output.

Assumptions (i)–(ix) define the system under consideration. In the time scale of the time slot, these assumptions define a stationary, ergodic Markov chain. Only the steady state of this chain (i.e., the invariant measure or the stationary distribution) is considered in this work. We refer to this steady state as the "normal system operation".

Let t be the number of parts produced by the last machine, m_M (and according to assumption (ix), shipped to the customer), during the epoch in the normal system operation. Obviously, $E(t) = Tp_a$ characterizes the production volume per epoch, but only on the *average*; in each particular epoch either more or fewer parts may be produced. Figure 3 gives a simulation result for a two-machine system where $p_a = 0.65$ and $T = 40$ (i.e., $E(t) = 26$). The production volume per epoch for 100 successive epochs in the normal system operation is shown. The volume varies from 18 to 35 parts, i.e., the variability is substantial. If $D = 25$ parts/epoch, then in only about 80% of epochs the demand is met. A similar figure for the case of machines with Markovian reliability characteristics can be found in [5].

The problem of evaluating this variability is formulated as follows:

Problem 1. Given the production system (i)–(ix), develop a method for evaluating the variance of t as a function of the system parameters.

A solution of this problem is given in section 2 below.

1.2.2. Due time performance problem

Problem 1 is not practical mainly because no accumulation of parts produced per epoch is allowed in order to satisfy the demand in the subsequent epochs. To enable this possibility, we modify the structure of figure 1 by introducing the Finished Goods

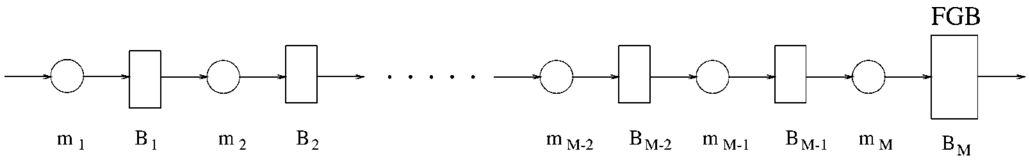

Figure 4. Serial production line with finished goods buffer.

Buffer (FGB) as shown in figure 4. The assumptions concerning this system are as follows:

Machines:
Remain the same as in assumptions (i)–(iii).

Buffers:
A small modification is required:

(iv′) The system has M buffers, $B_1, \ldots, B_M$. Buffers $B_1, \ldots, B_{M-1}$ are the in-process buffers, $1 \leqslant N_i < \infty$, $i = 1, \ldots, M-1$; buffer B_M is the finished goods buffer (FGB), $0 \leqslant N_M < \infty$.

Starvation rule:
Remains the same as in assumption (v).

Blockage rule:
Again, a modification is required, due to the presence of the FGB.

(vi′) It is assumed that (vi) holds with one exception: the last machine, m_M, can be blocked during a time slot if the FGB is full at the beginning of this time slot.

Demand:
Remain the same as in assumptions (vii), (viii).

Demand satisfaction policy:

(ix′) At the beginning of epoch i, parts are removed from the FGB in the amount of $\min(H(i-1), D)$, where $H(i-1)$ is the number of parts that remained in the FGB at the end of the $(i-1)$th epoch. If $H(i-1) \geqslant D$, the shipment is complete; if $H(i-1) < D$, the balance of the shipment, i.e., $D - H(i-1)$ parts, is to be produced by m_M. Parts produced are immediately removed from the FGB and prepared for shipment, until the shipment is complete, i.e., D parts are available. If the shipment is complete before the end of the epoch, the system continues operating, but with the parts being accumulated in the FGB, either until the end of the epoch or until the last machine, m_M, is blocked, whichever occurs first. If the shipment is not complete by the end of the epoch, an incomplete shipment is sent to the customer. No backlog is allowed.

Remark 1.5. The values of D and N_M are assumed to be two independent parameters.

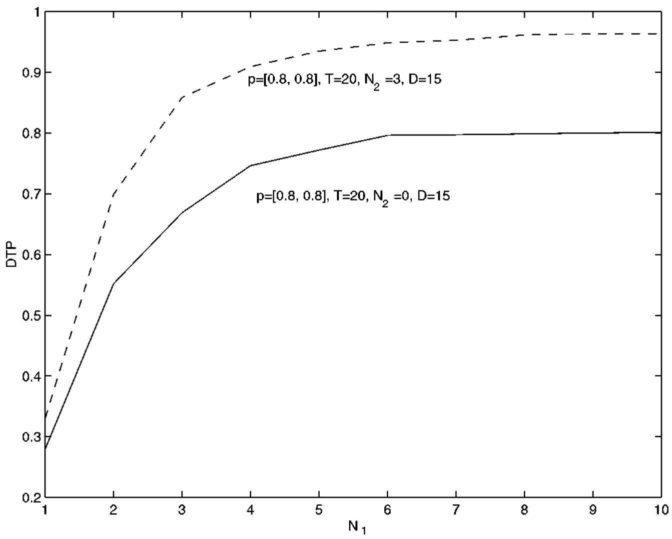

Figure 5. Due time performance of serial production line.

In an appropriately defined state space, the system specified by (i)–(iii), (iv′), (v), (vi′), (vii), (viii), and (ix′) is again a stationary, ergodic Markov chain. Referring to its steady state, let $\bar{t}_i$ be the number of parts produced during the ith epoch. Introduce the Due Time Performance measure (*DTP*) as follows:

$$DTP = Pr\big(\bar{t}_i + H(i-1) \geqslant D\big). \tag{1.2}$$

Figure 5 illustrates the behavior of this performance measure for a two-machine line in two cases, with and without the FGB. The *DTP* is shown as a function of the capacity of the in-process buffer. If N_2, the capacity of the finished goods buffer, is zero (i.e., no finished goods buffer is present), the *DTP* is always below 0.8. With $N_2 = 3$, the *DTP* is 0.9 when N_1 is 4 and goes up to 0.96 when N_1 is 10. Obviously, for any value of N_1, the effect of FGB is substantial. In other words, FGB reduces the production variability more effectively than the in-process buffers.

To analyze this and a number of related phenomena, we formulate

Problem 2. Given the production system, (i)–(iii), (iv′), (v), (vi′), (vii), (viii), and (ix′), develop a method for evaluating *DTP* as a function of the system's parameters.

A solution to the simplest case of this problem and a hypothesis concerning the general case are given in section 3 below.

1.2.3. Meeting a random demand problem

In many industrial situations, one production line feeds another through a finished goods buffer. A typical structure of this situation is shown in figure 6. Since the

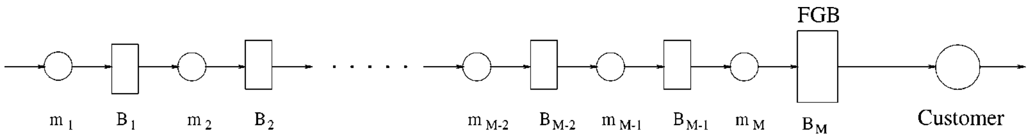

Figure 6. Serial production system with random demand.

"customer production line" could be either up or down, the problem of satisfaction of random demand arises. To formulate this problem, the following is assumed:

Machines:
Remain the same as in assumptions (i)–(iii).

Buffers:
Remain the same as in assumption (iv').

Starvation rule:
Remain the same as in assumption (v).

Blockage rule:
Remain the same as in assumption (vi').

Demand:

(vii′) Demand, d, is a random variable which takes value 1 ("up") with probability p_c and 0 ("down") with probability $1 - p_c$.

Demand satisfaction policy:

(viii′) If $d(i) = 1$, one part should be delivered to the customer at the beginning of time slot i; if the FGB is empty at the beginning of this time slot, the demand is not satisfied. If $d(i) = 0$, no parts are removed from the FGB at the beginning of slot i and the demand is met.

The system defined by assumptions (i)–(iii), (iv′), (v), and (vi′)–(viii′) is again a stationary, ergodic Markov chain. Let $h(i)$ be the occupancy of the FGB at the beginning of the slot i during the normal system operation. Introduce the Random Demand Satisfaction measure (*RDS*) as follows:

$$RDS = Pr\big(h(i) \geqslant d(i)\big). \tag{1.3}$$

Figure 7 illustrates the behavior of this performance measure for a two-machine system with and without the FGB. In both cases, the customers have the same random demand rate. The *RDS*, obtained by simulations, is plotted against the capacity of the in-process buffer. As follows from this figure, if N_2, the capacity of the finished goods buffer, is zero (i.e., no finished goods buffer is present), the *RDS* is always below 0.85. With $N_2 = 2$, the *RDS* is 0.87 if $N_1 = 1$, 0.93 if $N_1 = 3$ and up to 0.95 when N_1 is 10. Obviously, for any value of N_1, the effect of FGB is again substantial. In other words, FGB improves the random demand satisfaction more effectively than the in-process buffers.

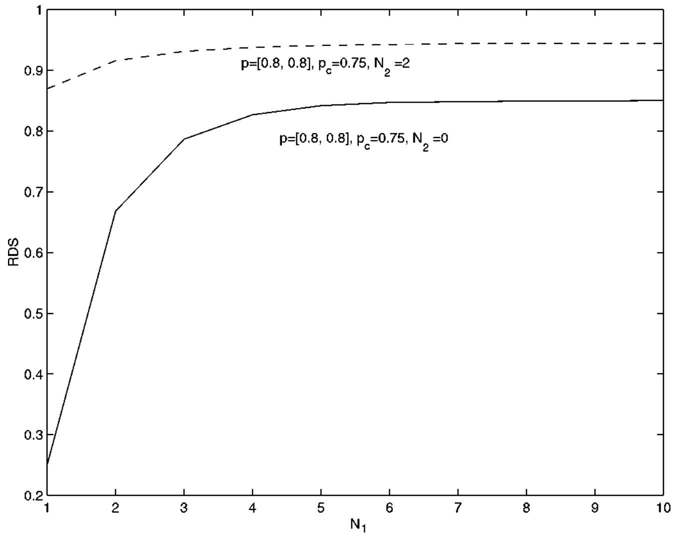

Figure 7. Random demand satisfaction of serial production system.

To investigate this phenomenon in more detail, we formulate

Problem 3. Given the production system (i)–(iii), (iv′), (v), and (vi′)–(viii′), develop a method for calculating *RDS* as a function of the system's parameters.

A solution of this problem is given in section 4 of this paper.

1.3. Literature review

As has been pointed out above, the number of publications on production variability due to machine breakdowns is very limited. Most of them address only the problem of the production variance. Among those, paper [12] is perhaps the first one in the area. It presents a numerical technique for production variance evaluation. Papers [6] and [7] developed an analytical technique and its numerical implementation. Both approaches, however, are numerically intensive, which precludes the analysis of systems with many machines and large buffers. The latest result in this line of research, [13], overcomes this problem and provides a numerically efficient algorithm for the production variance evaluation in a large class of two-machine lines with any capacity of the intermediate storage. Paper [14] provides a closed form expression for the variance rate of the output from multistation production lines with no interstation buffers and time-dependent failures.

Another direction of research was initiated in [4]. Problem 1 of subsection 1.2 is motivated by this formulation. The approach of [4] is based on the exact calculation of the production variance for a single machine with the Markovian reliability characteristics and a decomposition technique for longer lines. These ideas have been extended in [2] and [9].

The study of the Due Time Performance problem was initiated in [9]. The case of a single machine with Markovian reliability statistics has been analyzed both exactly and asymptotically (with respect to the length of the shipping period). The hypothesis advanced in [4] concerning the normality of the number of parts produced during a shipping period has been justified. Some properties of the DTP as a function of system parameters have been analyzed. Paper [10] studies the distribution and the variance of time to produce a fixed lot size with single unreliable machine. In addition, Due Time Performance was analyzed in [15,16]. It should be pointed out that all studies mentioned above do not consider a finished goods buffer. To the best of our knowledge, the literature offers no results on *DTP* in systems with FGB.

There seem to be no publications available in the literature directly addressing the Random Demand Satisfaction problem.

This brief review indicates that the production variability is a largely unexplored area of research. Given its practical importance in the framework of the customer demand satisfaction (or customer delight, as some Japanese companies refer to it), variability of production can be viewed as an important topic for research, both from the industrial and theoretical perspectives.

2. Production variance problem

Consider the M-machine line $\{p_1, \dots, p_M, N_1, \dots, N_{M-1}\}$ (figure 8). Following the procedure developed in [8], aggregate this line into a single machine defined by $p_a = PR(p_1, \dots, p_M, N_1, \dots, N_{M-1})$, where $PR(\cdot)$ is the production rate of the line under consideration calculated using the technique developed in [8].

Let $Var_M(t)$ be the production variance of the M-machine line $\{p_1, \dots, p_M, N_1, \dots, N_{M-1}\}$ during the epoch of length T. Let $Var_1(t)$ be the production variance of the one-machine system with $p = p_a$. Obviously, due to the Bernoulli statistics of the breakdowns,

$$Var_1(t) = Tp_a(1 - p_a). \tag{2.1}$$

Theorem 2.1. Under assumptions (i)–(ix),

$$Var_M(t) \leqslant Var_1(t). \tag{2.2}$$

Proof. See the appendix. □

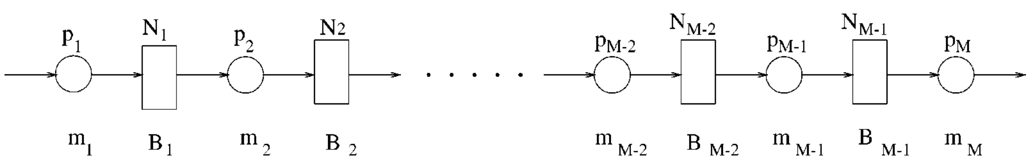

Figure 8. M-machine serial production line.

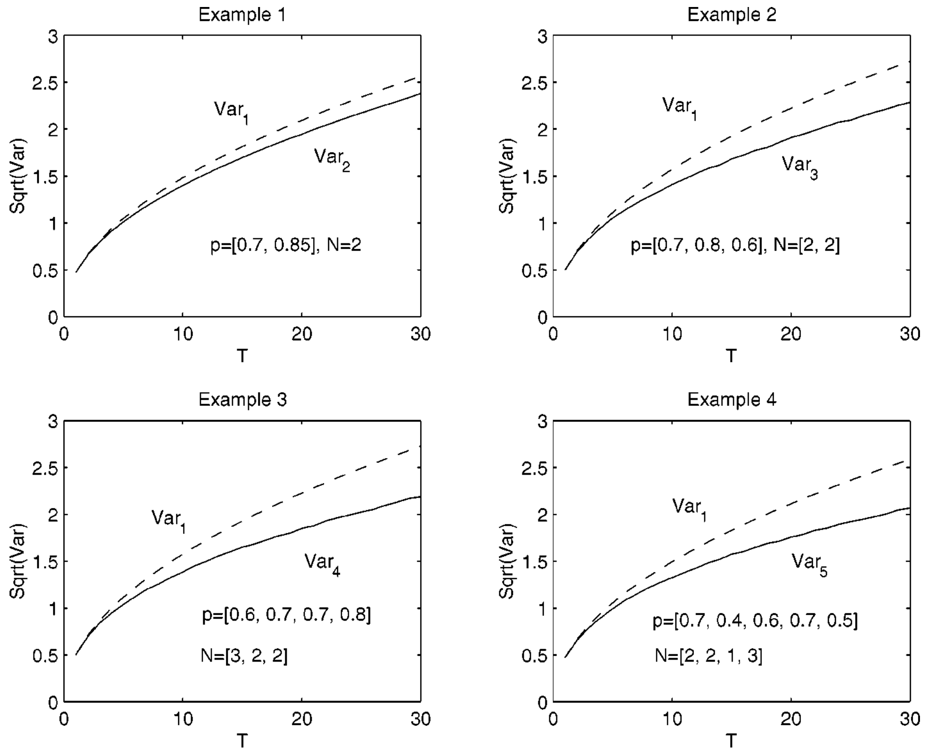

Figure 9. Upper bound of production variance.

The tightness of the bound of theorem 2.1 is illustrated in figure 9 for two-, three-, four- and five-machine lines. Here $Var_M(t)$ is obtained by simulations and $Var_1(t)$ is calculated according to (2.1).

Remark 2.1. All numerical simulations mentioned in this paper have been carried out as follows: in each run of the discrete event model, zero initial conditions for all buffers have been assumed and a $5000T$ time slots “warm up” period have been carried out. The next $25000T$ slots have been used for the statistical analysis. The confidence intervals have been calculated using the methodology of [17] with 50 runs. The t distribution was used. It turns out that the confidence intervals obtained are too small to be shown in the figures. For instance, for the system of figure 9, example 1, the 95% confidence interval of $\sqrt{Var_2}$ for $T = 10$ is $[1.3975, 1.3997]$, for $T = 20$ is $[1.9471, 1.9505]$, and for $T = 30$ is $[2.3718, 2.3769]$. Similar numbers are obtained for all other examples.

Along with the variance, it may be of interest to consider another measure of variability – the *coefficient of variation*. In the case of t's, the coefficient of variation, $v(t)$, is defined as

$$v(t) = \frac{\sqrt{Var(t)}}{E(t)} = \frac{\sqrt{Var(t)}}{Tp_a}. \tag{2.3}$$

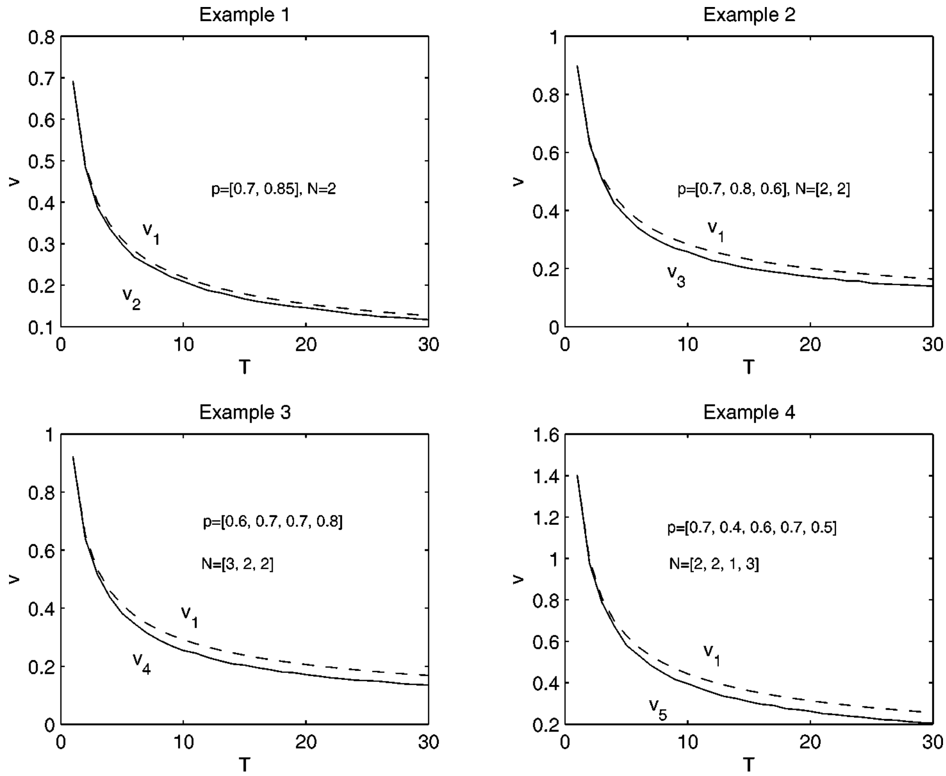

Figure 10. Upper bound of coefficients of variation.

Let $v_M(t)$ and $v_1(t)$ denote the coefficients of variation for the M- and one-machine lines, respectively, i.e.,

$$v_M(t) = \frac{\sqrt{Var_M(t)}}{Tp_a},$$
$$v_1(t) = \frac{\sqrt{Var_1(t)}}{Tp_a} = \sqrt{\frac{1-p_a}{Tp_a}}.$$

Then, from theorem 2.1, follows

Corollary 2.1. Under assumptions (i)–(ix),

$$v_M(t) \leqslant v_1(t) = \sqrt{\frac{1-p_a}{Tp_a}}. \tag{2.4}$$

The tightness of bound (2.4) is illustrated in figure 10.

Concluding this section, it should be pointed out that the bounds obtained above could be used for design of production lines from the point of view of the customer demand satisfaction. Indeed, if the system is designed so that the upper bounds meet the requirements, then the variance and the coefficient of variation, observed during the system operation, will also satisfy the specifications. Obviously, this approach may result in system “over-design” (due to the relative lack of tightness of the bounds), however, at least from the point of view of the coefficient of variation this over-design may not be too excessive.

Finally, it should be pointed out that since this formulation contains no finished goods buffer, the design of the system from the point of view of its variance amounts to the design of the system from the point of view of its production rate: since the right hand side of (2.2) is just the variance of the binomially distributed random variable, a decrease of this variance is accomplished by the increase of the production rate, provided that the production rate is greater than 0.5. In contrast, in the Due Time Performance problem and in the Random Demand Satisfaction problem, the capacity of the FGB is another design variable which affects the production variability in the strongest manner.

3. Due Time Performance problem

Consider the M-machine line with finished goods buffer $\{p_1, \ldots, p_M, N_1, \ldots, N_{M-1}, N_M\}$ (figure 11). The Due Time Performance measure, introduced in section 1, is defined as

$$DTP = Pr\big(\bar{t}_i + H(i-1) \geqslant D\big),$$

where $\bar{t}_i$ is the number of parts produced during the ith epoch, $H(i-1)$ is the number of parts that remained in the FGB at the end of the $(i-1)$th epoch.

Unlike the case of the production variance, the calculation of *DTP* even for a one-machine system is a nontrivial problem. Therefore, before addressing the general case, we study first a one-machine production system. The same approach is used in the subsequent section for the Random Demand Satisfaction problem.

3.1. *Due Time Performance measure of one-machine system*

Consider a one-machine production system with the FGB of capacity N. Introduce notations:

$$t_i = \text{number of parts produced during epoch } i \text{ if no blockage occurs}.$$

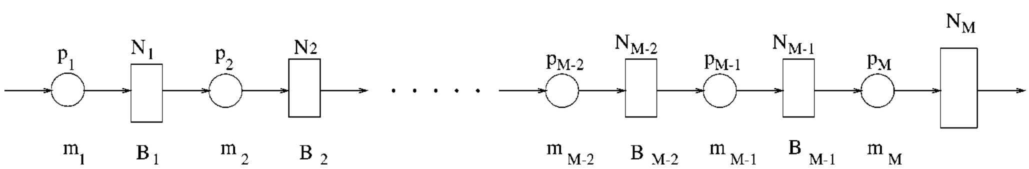

Figure 11. M-machine serial production line with FGB.

$$z_k = Pr\big(H(i-1) = k\big), \quad k = 0, 1, \ldots, N.$$
$$r_{k,j} = Pr(t_i = D + k - j), \quad k = 1, 2, \ldots, N-1, \; j = 0, 1, \ldots, N.$$
$$\overline{r}_{N,j} = Pr(t_i \geqslant D + N - j), \quad j = 0, 1, \ldots, N. \tag{3.1}$$

Probabilities $r_{k,j}$ and $\overline{r}_{N,j}$ can be easily calculated since they refer to the system without the FGB. Specifically,

$$r_{k,j} = \binom{T}{D+k-j} p^{D+k-j}(1-p)^{T-(D+k-j)}, \tag{3.2}$$

$$\overline{r}_{N,j} = \sum_{k=D+N-j}^{T} \binom{T}{k} p^k (1-p)^{T-k}. \tag{3.3}$$

Using the above notations, the *DTP* of a one-machine system with FGB can be calculated as follows:

Theorem 3.1. Let $Z = [z_1, \ldots, z_N]'$ be a vector defined by

$$Z = -R^{-1} Z_0, \tag{3.4}$$

where matrix R and vector Z_0 are as follows:

$$R = \begin{pmatrix} r_{1,1} - r_{1,0} - 1 & r_{1,2} - r_{1,0} & \cdots & r_{1,N-1} - r_{1,0} & r_{1,N} - r_{1,0} \\ r_{2,1} - r_{2,0} & r_{2,2} - r_{2,0} - 1 & \cdots & r_{2,N-1} - r_{2,0} & r_{2,N} - r_{2,0} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ r_{N-1,1} - r_{N-1,0} & r_{N-1,2} - r_{N-1,0} & \cdots & r_{N-1,N-1} - r_{N-1,0} - 1 & r_{N-1,N} - r_{N-1,0} \\ \overline{r}_{N,1} - \overline{r}_{N,0} & \overline{r}_{N,2} - \overline{r}_{N,0} & \cdots & \overline{r}_{N,N-1} - r_{N,0} & \overline{r}_{N,N} - \overline{r}_{N,0} - 1 \end{pmatrix}, \tag{3.5}$$

$$Z_0 = \begin{pmatrix} r_{1,0} \\ r_{2,0} \\ \cdots \\ r_{N-1,0} \\ \overline{r}_{N,0} \end{pmatrix}. \tag{3.6}$$

Then, under assumptions, (i)–(iii), (iv′), (v), (vi′), (vii), (viii) and (ix′), the *DTP* of the system with $M = 1$ is given by

$$DTP = \sum_{k=0}^{N} \sum_{j=D-k}^{T} z_k \binom{T}{j} p^j (1-p)^{T-j}. \tag{3.7}$$

Proof. See appendix. □

An illustration of the behavior of *DTP* as a function of the FGB capacity is given in figure 12. Obviously, *DTP* is a monotonically increasing function of N. However,

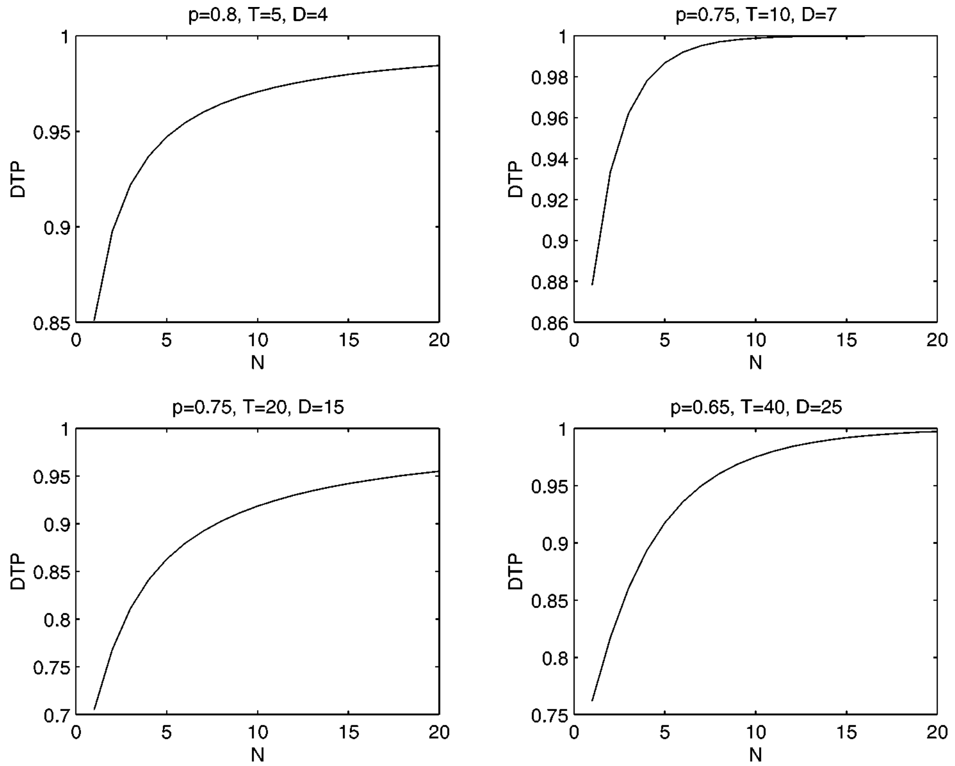

Figure 12. DTP of one-machine system.

contrary to some practitioners' belief, there is no reason to have very large FGBs: a capacity of 10–15 results in practically the same *DTP* as $N = \infty$.

3.2. *DTP of M-machine system: a lower bound*

Let DTP_M be the *DTP* measure for an M-machine line with FGB of capacity N_M, and let p_a be the production rate of this line when no FGB is present. Let DTP_1 be the *DTP* measure of a one-machine line with $p = p_a$ and the FGB of capacity N_M.

Hypothesis 3.1. Under assumptions (i)–(iii), (iv′), (v), (vi′), (vii), (viii) and (ix′), the following inequality holds:

$$DTP_M \geqslant DTP_1.$$

Unfortunately, at present we do not have a formal proof of this statement. However, in every system that we studied numerically this relationship was true. In these studies, the left hand side of the inequality was evaluated numerically and the right hand side was calculated using (3.7). An illustration is given in figure 13 for three-, four-, and five-machine systems.

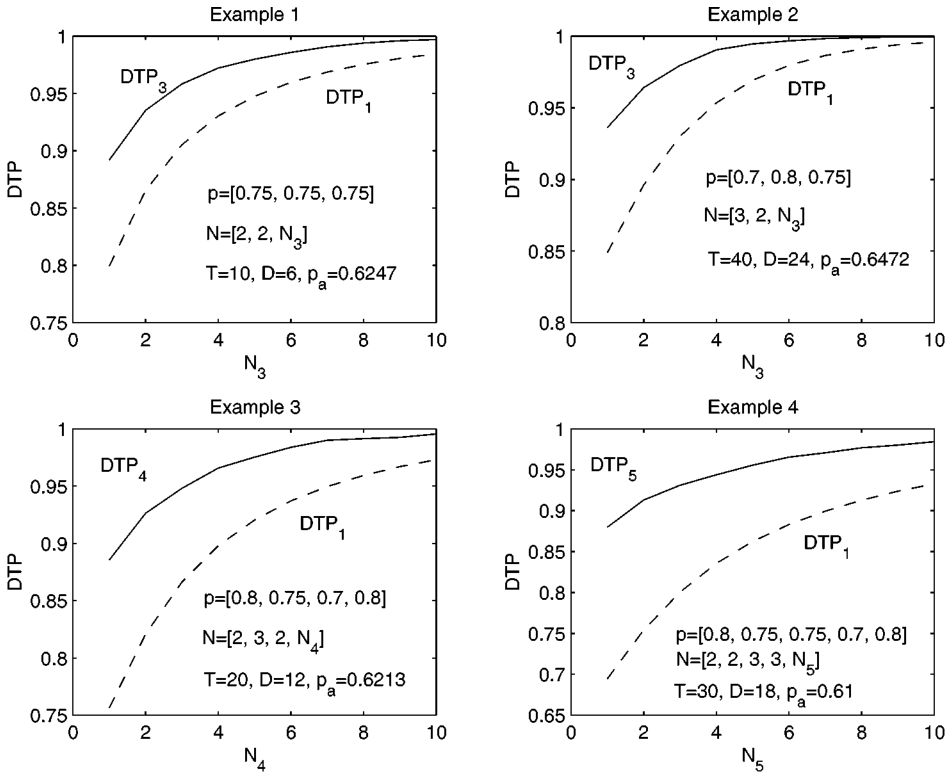

Figure 13. Lower bound of DTP.

Remark 3.1. In figure 13, example 1, the 95% confidence interval of DTP_3 for $N_3 = 2$ is $[0.9349, 0.9357]$, for $N_3 = 6$ is $[0.9855, 0.9861]$, and for $N_3 = 10$ is $[0.9969, 0.9971]$. Similar small confidence intervals have been obtained for the rest of the cases of figure 13.

The bound of hypothesis 3.1 can be used for design of the finished goods buffers in serial production lines. Indeed, since one is typically interested in having *DTP* as high as possible, if the lower bound meets the requirement, then the real *DTP* does as well. This, of course, is contingent on the validity of the hypothesis, which we believe is true: in spite of our efforts, no single counter-example has been found. We also believe that a proof of this hypothesis would be an important contribution to the area, and the interested reader is encouraged to attempt such a proof.

4. Meeting a Random Demand problem

Consider the M-machine line with random demand and finished goods buffer $\{p_1, \ldots, p_M, N_1, \ldots, N_{M-1}, N_M\}$ and p_c (figure 14). The random demand satisfaction

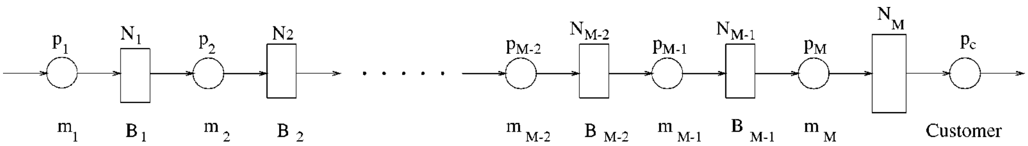

Figure 14. M-machine serial production line with random demand.

measure was defined in section 1 as

$$RDS = Pr\big(h(i) \geqslant d(i)\big),$$

where $h(i)$ is the number of parts in the finished goods buffer at the beginning of slot i and $d(i)$ is the demand in the ith slot, $d(i) = 0$ or 1.

4.1. *Random Demand Satisfaction measure of one-machine system*

Theorem 4.1. Under assumptions (i)–(vi), (vii′), and (viii′) and $M = 1$, the following holds:

$$RDS = \begin{cases} 1 - \dfrac{1-p}{N+1-p} p_c & \text{if } p = p_c, \\ 1 - \dfrac{(1-p)(1-\alpha)}{1 - \frac{p}{p_c}\alpha^N} p_c & \text{if } p \neq p_c, \end{cases} \tag{4.1}$$

where

$$\alpha = \frac{p(1-p_c)}{p_c(1-p)}.$$

Proof. See the appendix. □

It follows from this expression that *RDS* is monotonically increasing as a function of the FGB capacity and the machine's reliability and monotonically decreasing as a function of the demand, p_c. This is illustrated in figure 15.

4.2. *RDS of M-machine system: a lower bound*

Let RDS_M be the *RDS* measure for an M-machine line with FGB of capacity N_M, and let p_a be the production rate of this line when no FGB is present. Let RDS_1 be the *RDS* measure of a one-machine system with production rate p_a and FGB of capacity N_M. In both systems, the demand rate, p_c, is assumed to be the same.

Theorem 4.2. Under assumptions (i)–(iii), (iv′), (v), and (vi′)–(viii′),

$$RDS_M > RDS_1. \tag{4.2}$$

Proof. See the appendix. □

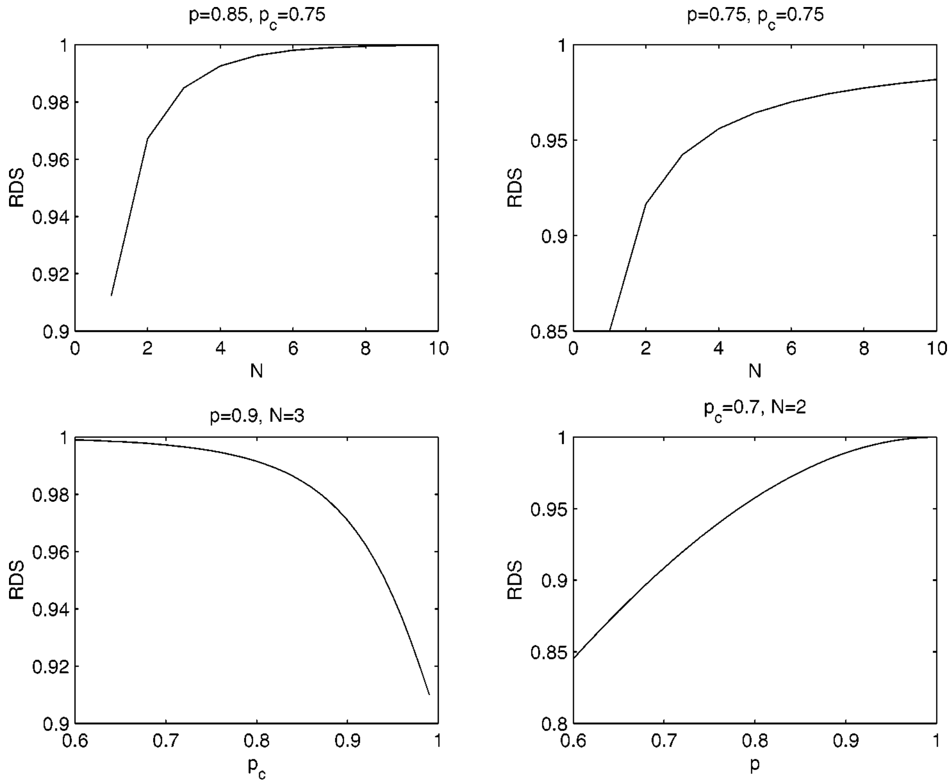

Figure 15. RDS of one-machine system.

The tightness of the bound (4.2) is illustrated in figure 16 for two-, three-, four-, and five-machine lines. In this figure, RDS_M has been obtained by simulations and RDS_1 was calculated according to theorem 4.1.

Remark 4.1. In figure 16, example 1, the 95% confidence interval of RDS_2 for $N_2 = 2$ is $[0.9526, 0.9534]$, for $N_2 = 6$ is $[0.9903, 0.9909]$, and for $N_2 = 10$ is $[0.9969, 0.9971]$. Similar small confidence intervals have been obtained for the rest of the cases of figure 16.

Theorem 4.2 may also be useful for design of production systems: if the system is structured so that the lower bound meets the *RDS* requirement, then RDS_M surely meets this demand.

5. Conclusions

This paper addresses the problem of production variability in serial production lines. The measures of variability discussed are the production variance, the Due Time Performance, and the Random Demand Satisfaction. Even in the simplest case of

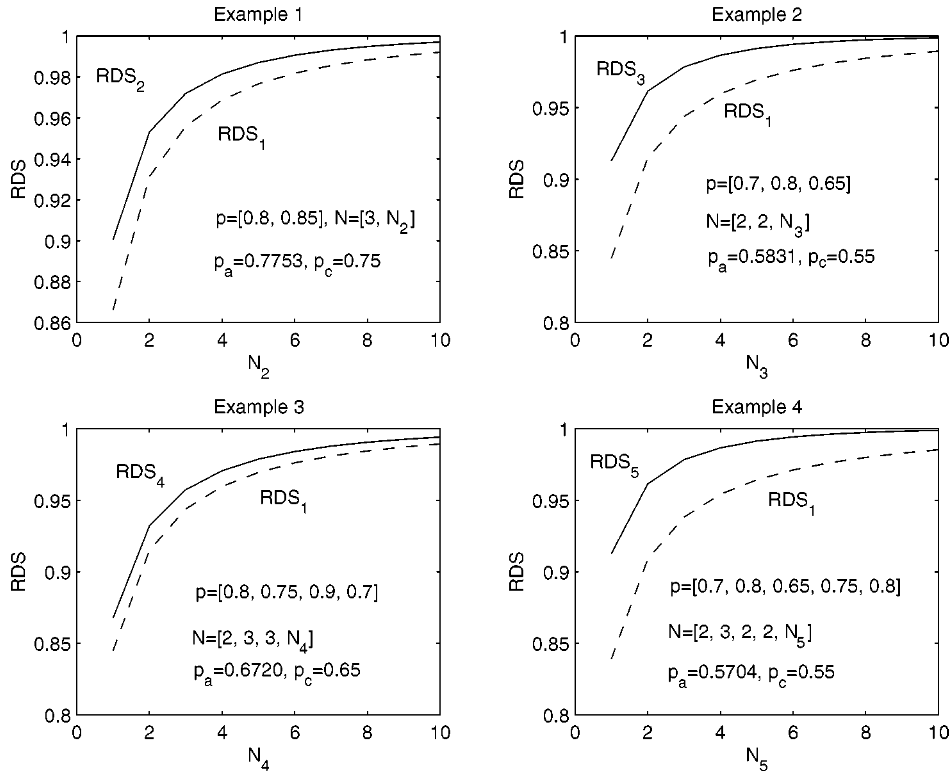

Figure 16. Lower bound of RDS.

machines with Bernoulli reliability statistics, considered here, the exact calculation of these variability measures for production systems with many machines is a formidable problem. Fortunately, however, these measures could be bounded relatively easily and in the desired direction, so that the bounds obtained could be used for design of finished goods buffers in a number of manufacturing situations. This provides a practical significance of the work.

The theoretical contribution is that, as the bounds show, longer lines smooth out the production and result in a variability lower than that of one-machine systems with similar production rates and machine reliability characteristics. This could be viewed as an argument for distributing the material processing among several stations and against using a single machine capable of performing numerous operations.

Appendix

Due to space limitations, we provide here only sketches of most proofs. The complete proofs can be found in [11].

Proof of theorem 2.1. We begin with the two-machine system and then extend the result to the M-machine case.

Introduce the following notations:

$$x(i) = \begin{cases} 1, & \text{machine } m_M \text{ produces a part at time slot } i, \quad i = 1, 2, \ldots, T, \\ 0, & \text{machine } m_M \text{ fails to produce a part at time slot } i, \quad i = 1, 2, \ldots, T. \end{cases}$$

$t = \sum_{i=1}^{T} x(i)$ the number of parts produced by machine m_M during an epoch in the steady state operation.

$h(i)$ is the occupancy of the last buffer B_{M-1} at the beginning of slot i.

The proof of theorem 2.1 consists of the following steps:

Step 1. Express the variance of the number of parts produced per epoch in the steady state of the system's operation in terms of appropriate conditional probabilities:

$$\begin{aligned} Var_M(t) = 2 \sum_{i=1}^{T-1} \sum_{j=i+1}^{T} Pr\bigl(x(j) = 1 \mid x(i) = 1\bigr) Pr\bigl(x(i) = 1\bigr) \\ + \sum_{i=1}^{T} E\bigl([x(i)]^2\bigr) - T^2 p_a^2. \end{aligned} \tag{A.1}$$

Step 2. Show that in a two-machine system the probability of the buffer being non-empty at the beginning of slot $i+1$, given that a part was produced by machine m_2 at slot i, is less than or equal to the probability of the buffer being non-empty at the beginning of slot $i+1$, given that a part was not produced by machine m_2 at slot i:

$$Pr\bigl(h(i+1) > 0 \mid x(i) = 1\bigr) \leqslant Pr\bigl(h(i+1) > 0\bigr) \leqslant Pr\bigl(h(i+1) > 0 \mid x(i) = 0\bigr). \tag{A.2}$$

Step 3. Show that in a two-machine system the probability of the buffer being non-empty at the beginning of slot $i+k$, $k = 1, 2, \ldots, T-i$, given that a part was produced by machine m_2 at slot i, is less than or equal to the probability of the buffer being non-empty at the beginning of slot $i+k$, given that a part was not produced by machine m_2 at slot i:

$$\begin{aligned} &Pr\bigl(h(j) > 0 \mid x(i) = 1\bigr) \leqslant Pr\bigl(h(j) > 0\bigr) \leqslant Pr\bigl(h(j) > 0 \mid x(i) = 0\bigr), \\ &\quad j = i+1, i+2, \ldots, T. \end{aligned} \tag{A.3}$$

Step 4. Show that in an M-machine system, the probability of the last buffer B_{M-1} being non-empty at the beginning of slot $i+k$, $k = 1, 2, \ldots, T-i$, given that a part was produced by the last machine m_M at slot i, is less than or equal to the probability of the last buffer B_{M-1} being non-empty at the beginning of slot $i+k$, given that a part was not produced by the last machine m_M at slot i:

$$\begin{aligned} &Pr\bigl(h(j) > 0 \mid x(i) = 1\bigr) \leqslant Pr\bigl(h(j) > 0\bigr) \leqslant Pr\bigl(h(j) > 0 \mid x(i) = 0\bigr), \\ &\quad j = i+1, i+2, \ldots, T. \end{aligned} \tag{A.4}$$

Step 5. Shows that in an M-machine system, the probability that the last machine m_M produces a part at slot $i+k$, $k=1,2,\dots,T-i$, given that a part was produced by the last machine m_M at slot i, is less than or equal to the unconditional probability that the last machine m_M produces a part at slot $i+k$:

$$\mathit{Pr}\big(x(j)=1 \mid x(i)=1\big) \leqslant \mathit{Pr}\big(x(j)=1\big), \quad j=i+1,i+2,\dots,T. \tag{A.5}$$

Step 6. Use steps 1 and 5 to prove the statement of the theorem: for a single machine system with the isolation production rate $\mathit{Pr}(x(i)=1)=p_a$, due to the Bernoulli reliability statistics,

$$\mathit{Var}_1(t) = 2\sum_{i=1}^{T-1}\sum_{j=i+1}^{T}\mathit{Pr}\big(x(j)=1\big)\mathit{Pr}\big(x(i)=1\big) + \sum_{i=1}^{T}E\big([x(i)]^2\big) - T^2p_a^2. \tag{A.6}$$

Therefore,

$$\begin{aligned}\mathit{Var}_M(t) - \mathit{Var}_1(t) = \sum_{i=1}^{T}\sum_{j=1,\ j\neq i}^{T}&\big[\mathit{Pr}(x(j)=1 \mid x(i)=1) - \mathit{Pr}(x(j)=1)\big]\\ &\times \mathit{Pr}(x(i)=1) \leqslant 0.\end{aligned} \tag{A.7}$$

Thus,

$$\mathit{Var}_M(t) \leqslant \mathit{Var}_1(t). \qquad \square$$

Proof of theorem 3.1. The logic of this proof is as follows:

Step 1. Express the due time performance in terms of the probability of the finished goods buffer occupancy at the end of the epoch (equation (A.8)):

$$\mathit{DTP} = \sum_{k=0}^{N}\mathit{Pr}\big(\bar{t}_i \geqslant D-k\big)\mathit{Pr}\big(H(i-1)=k\big). \tag{A.8}$$

Step 2. Derive the characterization of the probability mass function, $\mathit{Pr}(H(i)=k)$, for $k=1,2,\dots,N-1$, where $H(i)$ is the number of parts in the Finished Goods Buffer (FGB) at the end of the ith epoch:

$$\mathit{Pr}\big(H(i)=k\big) = \sum_{j=0}^{N}\mathit{Pr}\big(t_i = D+k-j\big)\mathit{Pr}\big(H(i)=j\big), \quad k=1,\dots,N-1, \tag{A.9}$$

i.e.,

$$z_k = \sum_{j=0}^{N} r_{k,j}z_j, \quad k=1,\dots,N-1. \tag{A.10}$$

Step 3. Same as step 2 but for $k = N$:

$$Pr(H(i) = N) = \sum_{j=0}^{N} Pr\big(t_i \geqslant N + D - j\big) Pr\big(H(i-1) = j\big), \tag{A.11}$$

i.e.,

$$z_N = \sum_{j=0}^{N} \overline{r}_{N,j} z_j. \tag{A.12}$$

Step 4. Same as step 2 but for $k = 0$:

$$Pr\big(H(i) = 0\big) = 1 - \sum_{k=1}^{N} Pr\big(H(i) = k\big), \tag{A.13}$$

i.e.,

$$z_0 = 1 - \sum_{k=1}^{N} z_j. \tag{A.14}$$

Step 5. Combine the results of steps 2–4 into matrix–vector form and solve for the probability mass function, $Pr(H(i) = k)$, $k = 0, 1, \ldots, N$:

$$\begin{aligned}
&(r_{1,1} - r_{1,0} - 1)z_1 + (r_{1,2} - r_{1,0})z_2 + \cdots + (r_{1,N} - r_{1,0})z_N = -r_{1,0},\\
&(r_{2,1} - r_{2,0})z_1 + (r_{2,2} - r_{2,0} - 1)z_2 + \cdots + (r_{2,N} - r_{2,0})z_N = -r_{2,0},\\
&\qquad\qquad \cdots \qquad\qquad \cdots \qquad\qquad \cdots\\
&(r_{N-1,1} - r_{N-1,0})z_1 + (r_{N-1,1} - r_{N-1,0})z_2 + \cdots\\
&\quad + (r_{N-1,N-1} - r_{N-1,0} - 1)z_{N-1} + z_N = -r_{N-1,0},\\
&\big(\overline{r}_{N,1} - \overline{r}_{N,0}\big)z_1 + \big(\overline{r}_{N,2} - \overline{r}_{N,0}\big)z_2 + \cdots + (\overline{r}_{N,N} - \overline{r}_{N,0} - 1)z_N = -\overline{r}_{N,N},
\end{aligned} \tag{A.15}$$

or,

$$RZ = -Z_0. \tag{A.16}$$

Thus,

$$Z = -R^{-1} Z_0. \tag{A.17}$$

Step 6. From the above calculation, obtain the claim of the theorem:

$$DTP = \sum_{k=0}^{N} z_k Pr\big(t_i \geqslant D - k\big) = \sum_{k=0}^{N} \sum_{j=D-k}^{T} z_k \binom{T}{j} p^j (1-p)^{T-j}. \qquad \square$$

Proof of theorem 4.1. By definition of *RDS*, we have

$$\begin{aligned} RDS &= Pr\big(h(i) \geqslant d(i)\big) = 1 - Pr\big(h(i) < d(i)\big) = 1 - Pr\big(d(i) = 1, h(i) = 0\big) \\ &= 1 - Pr\big(d(i) = 1\big) \cdot Pr\big(h(i) = 0\big) = 1 - p_c \cdot Pr\big(h(i) = 0\big). \end{aligned} \tag{A.18}$$

By considering the system as a two-machine line, from [8] we obtain

$$Pr\big(h(i) = 0\big) = \begin{cases} \dfrac{1-p}{N+1-p} & \text{if } p = p_c, \\ \dfrac{(1-p)(1-\alpha)}{1-\frac{p}{p_c}\alpha^N} & \text{if } p \neq p_c, \end{cases} \tag{A.19}$$

where $\alpha = \frac{p(1-p_c)}{p_c(1-p)}$. Therefore,

$$RDS = \begin{cases} 1 - \dfrac{1-p}{N+1-p}p_c & \text{if } p = p_c, \\ 1 - \dfrac{(1-p)(1-\alpha)}{1-\frac{p}{p_c}\alpha^N}p_c & \text{if } p \neq p_c. \end{cases} \qquad \square$$

Proof of theorem 4.2. Consider two systems: system 1 with parameters $\{p_1, p_2, \ldots, p_M, p_c, N_1, N_2, \ldots, N_{M-1}, N_M\}$, and system 2 with parameters $\{p_a, p_c, N_M\}$ where $p_a = PR(p_1, p_2, \ldots, p_M, N_1, N_2, \ldots, N_{M-1})$. Introduce the following notations:

- $s_a =$ probability that machine m_M is starved in a system where no blockage of m_M takes place.
- $s_M =$ probability that machine m_M is starved in the system defined by assumptions (i)–(iii), (iv′), and (vi′)–(viii′).
- $h_M =$ occupancy of the Finished Goods Buffer in system 1 at the beginning of the time slot.
- $h_a =$ occupancy of the Finished Goods Buffer in system 2 at the beginning of the time slot.

From (A.18), we have,

$$\begin{aligned} RDS_M &= 1 - p_c \cdot Pr(h_M = 0), \\ RDS_1 &= 1 - p_c \cdot Pr(h_a = 0). \end{aligned} \tag{A.20}$$

Since

$$Pr(h_M = 0) = Pr(h_M = 0)(1 - p_M + p_M s_M) + Pr(h_M = 1)p_c(1 - p_M + p_M s_M),$$

then

$$Pr(h_M = 1) = \frac{p_M(1 - s_M)}{p_c(1 - p_M(1 - s_M))}Pr(h_M = 0).$$

As in [8],

$$Pr(h_M = k) = \frac{p_M^k (1 - s_M)^k (1 - p_c)^{k-1}}{p_c^k (1 - p_M(1 - s_M))^k} Pr(h_M = 0).$$

Then

$$\begin{aligned}\sum_{k=0}^{N_M} Pr(h_M = k) &= Pr(h_M = 0)\left[1 + \frac{p_M(1 - s_M)}{p_c(1 - p_M(1 - s_M))} + \frac{p_M^2(1 - s_M)^2(1 - p_c)}{p_c^2(1 - p_M(1 - s_M))^2}\right. \\ &\quad \left. + \cdots + \frac{p_M^{N_M}(1 - s_M)^{N_M}(1 - p_c)^{N_M - 1}}{p_c^{N_M}(1 - p_M(1 - s_M))^{N_M}}\right] \\ &= Pr(h_M = 0) A_M = 1,\end{aligned}$$

where

$$\begin{aligned}A_M = 1 &+ \frac{p_M(1 - s_M)}{p_c(1 - p_M(1 - s_M))} + \frac{p_M^2(1 - s_M)^2(1 - p_c)}{p_c^2(1 - p_M(1 - s_M))^2} + \cdots \\ &+ \frac{p_M^{N_M}(1 - s_M)^{N_M}(1 - p_c)^{N_M - 1}}{p_c^{N_M}(1 - p_M(1 - s_M))^{N_M}}.\end{aligned} \tag{A.21}$$

Analogously, we write

$$\begin{aligned}&\sum_{k=0}^{N_M} Pr(h_a = k) \\ &\quad = Pr(h_a = 0)\left[1 + \frac{p_a}{p_c(1 - p_a)} + \frac{p_a^2(1 - p_c)}{p_c^2(1 - p_a)^2} + \cdots + \frac{p_a^{N_M}(1 - p_c)^{N_M - 1}}{p_c^{N_M}(1 - p_a)^{N_M}}\right] \\ &\quad = Pr(h_a = 0) A_a = 1,\end{aligned}$$

where

$$A_a = 1 + \frac{p_a}{p_c(1 - p_a)} + \frac{p_a^2(1 - p_c)}{p_c^2(1 - p_a)^2} + \cdots + \frac{p_a^{N_M}(1 - p_c)^{N_M - 1}}{p_c^{N_M}(1 - p_a)^{N_M}}. \tag{A.22}$$

Therefore, we obtain

$$A_M Pr(h_M = 0) = A_a Pr(h_a = 0). \tag{A.23}$$

Since

$$s_a > s_M,$$

then

$$p_M(1 - s_M) > p_M(1 - s_a) = p_a,$$

which implies

$$\frac{p_M(1-s_M)}{1-p_M(1-s_M)} > \frac{p_a}{1-p_a}.$$

Substituting this into (A.21) and (A.22), we have

$$A_M > A_a.$$

From (A.23), it follows that

$$Pr(h_M = 0) < Pr(h_a = 0).$$

Substituting this into (A.20), we finally obtain

$$RDS_M > RDS_1. \qquad \square$$

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