Approximating Extreme Points of Infinite Dimensional Convex Sets

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Abstract

The property that an optimal solution to the problem of minimizing a continuous concave function over a compact convex set in $\mathbb{R}^n$ is attained at an extreme point is generalized by the Bauer Minimum Principle to the infinite dimensional context. The problem of approximating and characterizing infinite dimensional extreme points thus becomes an important problem. Consider now an infinite dimensional compact convex set in the nonnegative orthant of the product space $\mathbb{R}^\infty$. We show that the sets of extreme points E_N of its corresponding finite dimensional projections onto $\mathbb{R}^N$ converge in the product topology to the closure of the set of extreme points E of the infinite dimensional set. As an application, we extend the concept of total unimodularity to infinite systems of linear equalities in nonnegative variables where we show when extreme points inherit integrality from approximating finite systems. An application to infinite horizon production planning is considered.

Key words. Infinite dimensional convex sets, extreme points, projections, infinite dimensional total unimodularity.

1 Introduction

Many important problems in Operations Research are naturally phrased within the context of an infinite dimensional linear vector space (see Luenberger, 1969). An important instance is the problem of selecting a sequence of decisions over an infinite horizon that minimizes its associated discounted cost (see, for example, Bès and Sethi (1988), Schochetman and Smith (1989)). Included within this class are nonhomogeneous Markov Decision Processes (Hopp, Bean, and Smith, 1987), capacity expansion under nonlinear demand (Bean and Smith, 1985) and equipment replacement under time varying demand or technological change (Bean, Lohmann, and Smith, 1985).

By Bauer's Minimum Principle (see Roy, 1987), when the feasible region is a nonempty compact convex subset $S \subseteq \mathbb{R}^\infty$, and the minimizing objective function is a concave lower semi-continuous function on S, then the optimum is attained at an extreme point of S. The determination of the properties of extreme points of compact convex sets in $\mathbb{R}^\infty$ thereby leads to a characterization of optimal properties. We show in this paper that the extreme points of the finite dimensional projections of S arbitrarily well approximate their infinite dimensional counterparts, thus allowing for the inheritance of finite dimensional properties in the infinite dimensional case whenever such properties are preserved in the limit. We illustrate this principle by showing that the property of integer extreme points is inherited in the infinite horizon case for a classic production planning problem.

Now consider a non-empty compact and convex set S in the nonnegative orthant of the product space $\mathbb{R}^\infty$. Our interest in this paper is to approximate, and thereby characterize, the extreme points of this set. We will approximate S by its corresponding projections S_N onto $\mathbb{R}^N$ ($N = 1, 2, \ldots$). Conditions will be provided that assure that the extreme points E_N of S_N converge (with respect to the underlying product topology) to the extreme points E of S. Not only does this result allow for the finite computation of approximations of the extreme points of S, but it also, as already noted, provides for the inheritance of all finite dimensional properties of E_N that are preserved under componentwise convergence to E. As an illustration, we apply this technique to extending the notion of total unimodularity to an infinite system of linear equalities in nonnegative variables where it is shown that all extreme points must be integer valued.

The literature on the extreme point structure of infinite dimensional convex sets goes back to Minkowski (1911) who defined a point of a convex subset of a linear space as an extreme point if the subset remaining after its removal is convex. The subject became an important tool of functional analysis with the publication of the Krein-Milman theorem (Krein and Milman, 1940) which, as later extended by Milman, Kelley, and Bourbaki, established that every compact convex subset of a locally convex topological linear space is the closed convex hull of its extreme points. This result was later extended to locally compact subsets by Klee (1957). These positive results are noteworthy since convex sets can display a disconcerting number of pathological properties in the context of infinite dimensional spaces (Klee, 1951). See also Roy (1987) for an up-to-date survey of the literature.

Anderson and Nash (1987) revisited the characterization of extreme points for infinite dimensional linear systems in their path breaking book. Their motivation was to extend the simplex method to infinite dimensional linear programming; however, their task was complicated and their success limited by the pathologies inherent in such problems. For example, such linear programs may have optimal solutions but fail to have optimal basic solutions.

Our approach here is indirect, as in Romeijn, Smith, and Bean (1992), in that we establish extreme point properties by demonstrating their inheritance from their finite dimensional projections. Key to this is establishing that the extreme points of these approximating sets converge to the extreme points of the infinite system. In section 2, we establish the mathematical framework for this problem, and in section 3 we demonstrate conditions for this convergence to take place. Section 4 is an application that establishes sufficient conditions for extreme points of linear systems to be integer valued.

2 Mathematical framework

2.1 Extreme points

The following will serve as our definition of an extreme point of a convex set:

Definition 2.1 *A point $x \in S$ is called an* extreme point *of S if x is not the midpoint of any line segment contained in S. In other words, if $x = \frac{1}{2}(u+v)$, where $u, v \in S$, implies that $x = u = v$, then x is an extreme point of S.*

We assume that $I\!R^\infty$ is a product space equipped with the product topology inherited from the underlying Euclidean spaces. This means that a sequence $x^1, x^2, \ldots$, where $x^n \in I\!R^\infty$ for all n, converges to some $x \in I\!R^\infty$ precisely when its components x_j^n converge to x_j in the ordinary Euclidean metric on $I\!R$ for all j.

We will repeatedly use the following result, pertaining to the existence of extreme points:

Lemma 2.2 *Any nonempty, closed, convex subset of the non-negative orthant of a finite-dimensional Euclidean space has an extreme point.*

Proof: This follows directly from lemma 3.3 in Klee (1957). □

2.2 Projections

For each $N = 1, 2, \ldots$, define the projection function

$$p_N : I\!R^\infty \to I\!R^N$$

as

$$p_N(x) = (x_1, \ldots, x_N)$$

and the corresponding projections of S onto $I\!R^N$ as

$$S_N = \{p_N(x) : x \in S\}.$$

We will sometimes want to view S_N as a set embedded in the infinite dimensional linear space $I\!R^\infty$. Therefore, we will at times also let

$$S_N = \{(p_N(x), 0) : x \in S\}$$

where the precise meaning of S_N will be clear from the context. Now let E_N be the set of extreme points of S_N. (Thus, E_N can also be thought of as a set in either $I\!R^N$ or $I\!R^\infty$, depending on the context.)

The principal objective of this paper is to find conditions under which the sequence of sets of extreme points of S_N converges (in the Kuratowski sense to be defined below) to the set of extreme points of S.

2.3 Convergence of sets

We begin by defining Kuratowski convergence (Kuratowski, 1966) for a sequence of sets in $I\!R^\infty$. Let $K_N \subseteq I\!R^\infty$ for $N = 1, 2, \ldots$. Define:

(i) $\liminf_{N\to\infty} K_N =$ the set of points $x \in I\!R^\infty$ for which there exists $x^N \in K_N$, for N sufficiently large, such that $\lim_{N\to\infty} x^N = x$.

(ii) $\limsup_{N\to\infty} K_N$ = the set of points $x \in \mathbb{R}^\infty$ for which there exists a subsequence $\{K_{N_k}\}$ of $\{K_N\}$ and a corresponding sequence $\{x^k\}$ such that $x^k \in K_{N_k}$ for all k, and $\lim_{k\to\infty} x^k = x$.

In general,

$$\liminf_{N\to\infty} K_N \subseteq \limsup_{N\to\infty} K_N.$$

If $K \subseteq \mathbb{R}^\infty$ such that $K \subseteq \liminf_{N\to\infty} K_N$ and $\limsup_{N\to\infty} K_N \subseteq K$, i.e. $\liminf_{N\to\infty} K_N = \limsup_{N\to\infty} K_N = K$, then we write

$$\lim_{N\to\infty} K_N = K$$

and say that $\{K_N\}$ *Kuratowski converges* to K.

3 Convergence of projections

We now return to the set S. We will first show that the sequence of projections S_N (viewed as subsets of $\mathbb{R}^\infty$ by extension with zeroes) Kuratowski converges to S.

Lemma 3.1 *The sequence of projections S_N converges in the Kuratowski sense to S, i.e.*

$$\lim_{N\to\infty} S_N = S.$$

Proof: We need to show that

(i) $S \subseteq \liminf_{N\to\infty} S_N$, and

(ii) $\limsup_{N\to\infty} S_N \subseteq S$.

The first property follows directly by observing that

$$\lim_{N\to\infty} (p_N(x), 0) = x$$

for all $x \in S$. To prove the second property, we introduce, for all N, the set

$$S^N = \{x \in \mathbb{R}^\infty : x \geq 0,\ p_N(x) \in S_N\}$$

i.e. S^N can be obtained from S_N by arbitrarily extending all elements of S_N to nonnegative elements of $\mathbb{R}^\infty$. We will first show that

$$S = \bigcap_{N=1}^{\infty} S^N. \tag{1}$$

Since $S \subseteq S^N$ for all N, it is clear that

$$S \subseteq \bigcap_{N=1}^{\infty} S^N.$$

It remains to be shown that

$$\bigcap_{N=1}^{\infty} S^N \subseteq S.$$

Let $x \in S^N$ for all N. For all N, choose $y^N \in S$ such that $p_N(y^N) = (x_1, \ldots, x_N) \in I\!R^N$. Then $y_i^N = x_i$ for $i = 1, \ldots, N$. Thus

$$\lim_{N \to \infty} y^N = x.$$

Since S is closed, we have $x \in S$, so (1) follows. Now, by Kuratowski (1966),

$$\lim_{N \to \infty} S^N = \bigcap_{N=1}^{\infty} \overline{S^N}$$

since $S^{N+1} \supseteq S^N$ for all N. Since S is compact, the sets S^N are closed, and thus

$$\lim_{N \to \infty} S^N = \bigcap_{N=1}^{\infty} S^N = S.$$

Property (ii) now follows by observing that

$$\limsup_{N \to \infty} S_N \subseteq \limsup_{N \to \infty} S^N = \lim_{N \to \infty} S^N.$$

□

In section 2 we defined p_N to be the projection of points in $I\!R^\infty$ onto $I\!R^N$. Similarly, we can define a projection of points in $I\!R^M$ onto $I\!R^N$ (for $M > N$). We will denote these projections also by p_N, where the appropriate interpretation should be clear from the context.

The following lemmas show the relationship between extreme points of the projections S_N (regarded as subsets of $I\!R^N$) and the extreme points of the original set S.

Lemma 3.2 *For every extreme point x of S_N there exists an extreme point of S_{N+1} which is identical to x in its first N components.*

Proof: Let x be an extreme point of S_N. Then consider

$$T = S_{N+1} \cap \{y \in I\!R^\infty : p_N(y) = x\}.$$

Clearly, T is nonempty, since it contains $p_{N+1}(z)$, where $z \in S$ is such that $x = p_N(z)$. Now let x' be an extreme point of T. (Such a point exists by lemma 2.2.) Then the desired result follows if x' is an extreme point of S_{N+1} as well. Let $u, v \in S_{N+1}$ such that $x' = \frac{1}{2}(u+v)$. Now note that $p_N(u), p_N(v) \in S_N$, and $x = p_N(\frac{1}{2}(u+v)) = \frac{1}{2}p_N(u)+\frac{1}{2}p_N(v)$. Since x is an extreme point of S_N, we have that $p_N(u) = p_N(v) = x$, so that $u, v \in T$. But now, since x' is an extreme point of T, we have that $u = v = x'$, so x' is an extreme point of S_{N+1}, which proves the lemma. □

By invoking lemma 3.2 exactly $M - N$ times for $M > N$, we conclude

Corollary 3.3 *For every extreme point x of S_N and for every $M > N$ there exists an extreme point of S_M which is identical to x in its first N components.*

Lemma 3.4 *For every extreme point x^N of S_N there exists an extreme point of S which is identical to x^N in its first N components.*

Proof: By lemma 3.2, there exist extreme points $x^{N+1}, x^{N+2}, \ldots$ of S_{N+1}, $S_{N+2}, \ldots$ respectively, such that, for all $N \leq i < j$, x^j is identical to x^i in its first i components. So, by lemma 3.1 the convergent sequence $\{x^i\}_{i=N}^{\infty}$ (after extending its elements making them elements of S) converges to some $x \in S$. It remains to be shown that $x \in E$. Suppose not, then there exist $u, v \in S$ $(u \neq v)$ such that $x = \frac{1}{2}(u+v)$. Now consider the first component in which u and v differ, say $M > N$. Since $x = \frac{1}{2}(u+v)$, $p_M(x) = \frac{1}{2}p_M(u) + \frac{1}{2}p_M(v)$, where $p_M(x) = x^M$ and $p_M(u), p_M(v) \in S_M$. But since x^M is an extreme point of S_M, $p_M(u)$ must be equal to $p_M(v)$, implying that the M-th components of u and v are equal, which is a contradiction. Thus x is an extreme point of S. □

Lemma 3.5

$$\limsup_{N\to\infty} E_N \subseteq \overline{E}.$$

Proof: Let $x \in \limsup_{N\to\infty} E_N$, and let $x^k \in E_{N_k}$ such that $\lim_{k\to\infty} x^k = x$. Consider the set $\{y \in S : y_i = x_i^k \text{ for } i = 1, \ldots, N_k\}$. By lemma 3.4, this set contains an extreme point of S, say y^k. We now have a sequence $\{y^k\}_{k=1}^{\infty}$ in E. This sequence clearly converges to x, so we have $x \in \overline{E}$. Therefore, $\limsup_{N\to\infty} E_N \subseteq \overline{E}$. □

We can now prove the first major result of this paper.

Theorem 3.6 *The sequence of sets of extreme points E_N of the projections S_N of the compact convex set S converges, i.e. $\lim_{N\to\infty} E_N$ exists. Moreover,*

$$\lim_{N\to\infty} E_N \subseteq \overline{E}.$$

Proof: Let $\{x^{N_k}\}_{k=1}^{\infty}$ be a subsequence of points in E_{N_k}, $k = 1, 2, \ldots$, respectively, such that

$$\lim_{k\to\infty} x^{N_k} = x$$

i.e.,

$$x \in \limsup_{N\to\infty} E_N.$$

In order to show that $x \in \liminf_{N\to\infty} E_N$, we need to construct a sequence of points $\{x^N\}_{N=1}^{\infty}$ such that $x^N \in E_N$ for all N, and such that

$$\lim_{N\to\infty} x^N = x.$$

Let $N_k < N < N_{k+1}$. We now choose x^N as follows: set

$$x^N = (x_1^{N_k}, \ldots, x_{N_k}^{N_k}, y_{N_k+1}, \ldots, y_N, 0)$$

where the y_j are chosen in such a way that x^N is an extreme point of S^N, which we can do by corollary 3.3. Thus, a sequence of extreme points x^N in E_N has been created, which clearly converges to x in the product topology. So we have shown that

$$\limsup_{N\to\infty} E_N \subseteq \liminf_{N\to\infty} E_N.$$

The second claim of the theorem now follows from lemma 3.5. □

Remark: Note that all the above results remain valid if the assumption that S is compact is replaced by the assumption that S and its projections S_N are all closed. Note also that, under compactness, the statement of theorem 3.6 is nontrivial in that $\limsup_{N\to\infty} E_N \neq \emptyset$ so that $\lim_{N\to\infty} E_N \neq \emptyset$. Theorem 3.6 then tells us that $E \neq \emptyset$, which of course is also concludable from the Krein-Milman theorem since $S \neq \emptyset$ by hypothesis.

In the remainder of this section we will show that $\overline{E} \subseteq \liminf_{N\to\infty} E_N$, so that $\lim_{N\to\infty} E_N = \overline{E}$. In order to prove this result we need the notion of an *exposed point*, which Klee (1958) defines as follows:

Definition 3.7 *A point $x \in S$ is called an* exposed point *of S if S is supported at x by a closed hyperplane which intersects S only at x.*

Let $\exp(S)$ denote the set of exposed points of S. We can then prove the following lemma.

Lemma 3.8 $\exp(S) \subseteq \liminf_{N\to\infty} E_N$.

Proof: Let $x \in \exp(S)$. Then there exists a continuous linear functional c such that $\min\{c(y) : y \in S\}$ is attained uniquely by x. Now let, for all N, Q_N denote the set of points for which $\min\{c(y) : y \in S_N\}$ is attained. Then, since $\lim_{N\to\infty} S_N = S$, the Maximum Theorem (see Berge, 1963) says that $\limsup_{N\to\infty} Q_N \subseteq \{x\}$. Now choose $x^N \in Q_N$ such that $x^N \in E_N$. This is possible since, by Bauer's Minimum Principle, a continuous linear functional has an extreme point optimum when minimized over a compact set (see Roy, 1987). Now, by the compactness of S, every subsequence of $\{x^N\}$ has a convergent subsequence which converges to x. Therefore, $\lim_{N\to\infty} x^N = x$, and thus $x \in \liminf_{N\to\infty} E_N$. □

We are now able to prove the major result of this paper.

Theorem 3.9 *The sets of extreme points E_N of the projections S_N of the compact convex set S converges to the closure of the extreme points of S, i.e.*

$$\lim_{N\to\infty} E_N = \overline{E}.$$

Proof: By the previous lemma,

$$\exp(S) \subseteq \liminf_{N\to\infty} E_N.$$

Thus, by the first part of theorem 3.6

$$\exp(S) \subseteq \lim_{N\to\infty} E_N.$$

Since $\lim_{N\to\infty} E_N$ is closed (see Kuratowski, 1966), we also have

$$\overline{\exp(S)} \subseteq \lim_{N\to\infty} E_N.$$

Klee (1958) proves that, since S is compact, $E \subseteq \overline{\exp(S)}$, so

$$E \subseteq \lim_{N\to\infty} E_N.$$

Again using the fact that $\lim_{N\to\infty} E_N$ is closed, we obtain

$$\overline{E} \subseteq \lim_{N\to\infty} E_N.$$

Combining this with the second part of theorem 3.6 we conclude

$$\lim_{N\to\infty} E_N = \overline{E}.$$

□

Unfortunately, E may fail to be closed so that theorem 3.9 cannot be strengthened to $\lim_{N\to\infty} E_N = E$ without additional hypotheses. In fact, there are well-known examples of convex sets in $I\!R^3$ for which the set of all extreme points is not closed. Consider for example the convex hull of the line segment joining the points $(0,0,-1)$ and $(0,0,1)$ union the unit ball with center at $(1,0,0)$.

For a more interesting example in $I\!R^\infty$, let S be the convex hull of the set

$$E = \bigcup_{j=1}^{\infty} \{x \in I\!R^\infty : x_j \in \{0,2\},\ x_i = 1 \text{ for all } i \neq j\}.$$

Clearly, E is the set of extreme points of S, the projections of S are

$$S_N = \mathrm{co}\left(\bigcup_{j=1}^{N} \{x \in I\!R^N : x_j \in \{0,2\},\ x_i = 1 \text{ for all } i = 1,\ldots,N;\ i \neq j\} \right)$$

and their extreme points are

$$E_N = \bigcup_{j=1}^{N} \{x \in I\!R^N : x_j \in \{0,2\},\ x_i = 1 \text{ for all } i = 1,\ldots,N;\ i \neq j\}.$$

It is easy to see that $(1,1,\ldots) \in \overline{E}\backslash E$, so that E is not closed. By theorem 3.9, $\lim_{N\to\infty} E_N = \overline{E} \neq E$. This example is striking in that there exists a sequence of *extreme* points converging to the *center* of the feasible region.

In cases where E is closed, we have $\overline{E} = E$, so that theorem 3.9 becomes

Corollary 3.10 *If E is closed, then*

$$\lim_{N\to\infty} E_N = E.$$

4 Extension of total unimodularity to infinite dimensional linear systems

4.1 Lower triangular linear systems

In general, the projections S_N of S may be difficult to characterize. However, consider the case where S can be expressed as the solution set of an infinite linear system, i.e.

$$S = \{x \in \prod_{i=1}^{\infty} \mathbb{R}^{n_i} : Ax = b,\, x \geq 0\}$$

where $A = (A_{ij})$ is a doubly infinite lower block-triangular matrix, $x \in \prod_{i=1}^{\infty} \mathbb{R}^{n_i}$, and $b \in \prod_{i=1}^{\infty} \mathbb{R}^{m_i}$. Hence S is the solution set to

$$\sum_{j=1}^{i} A_{ij} x_j = b_i \qquad i = 1, 2, \ldots$$

where A_{ij} is an $(m_i \times n_j)$-matrix, $x_j \in \mathbb{R}^{n_j}$, $x_j \geq 0$, and $b_i \in \mathbb{R}^{m_i}$.

Now for each $N = 1, 2, \ldots$, consider the *algebraic projections* T_N of S formed by ignoring the (vector) variables and (vector) constraints beyond the N-th one:

$$T_N = \{x \in \prod_{i=1}^{N} \mathbb{R}^{n_i} : \sum_{j=1}^{i} A_{ij} x_j = b_i \text{ for } i = 1, \ldots, N;\, x \geq 0\}.$$

The sequence of algebraic projections $\{T_N\}$ is called *extendable* if, for all N, any solution to the first N linear equalities and nonnegativity constraints has some continuation which satisfies the infinite set of constraints. In this case, $S_N = T_N, N = 1, 2, \ldots$. In fact, extendability holds if and only if the algebraic projection T_N is equal to the ordinary projection S_N onto $\prod_{i=1}^{N} \mathbb{R}^{n_i}$ for all N. Note that in the previous sections we only considered explicitly the case where $n_i = 1$ for all i. However, it is easy to see that all results will still hold for arbitrary, but finite, values of n_i.

4.2 Total unimodularity

Consider the following extension of the concept of *total unimodularity*:

Definition 4.1 *A doubly infinite matrix $A = (a_{ij})_{i,j=1,2,\ldots}$ is called* totally unimodular *if every finite square submatrix of A has determinant $0, 1$ or -1.*

For the remainder of this section, we impose the following

Assumption 4.2 *A is a lower block-triangular matrix, and the set $S = \{x \in X : Ax = b, x \geq 0\}$ is compact and has extendable algebraic projections T_N ($N = 1, 2, \ldots$)).*

Note that a sufficient condition for S to be compact is that we have finite bounds on the variables.

We can now prove the following theorem, which is an extension of a corresponding result for the finite dimensional case (see e.g. Schrijver, 1986).

Theorem 4.3 *Let A be totally unimodular, and let the vector b have integer components. Then, under assumption 4.2, the extreme points of $S = \{x \in I\!R^{\infty} : Ax = b, x \geq 0\}$ are integer valued.*

Proof: Since A is totally unimodular and b consists of integers, the extreme points of $T_N = S_N$ are integer valued (Schrijver, 1986). From theorem 3.9,

$$\lim_{N \to \infty} E_N = \overline{E}.$$

Now suppose $x \in \overline{E}$. Then there exists a sequence of points $\{x^N\}_{N=1}^{\infty}$ such that $x^N \in E^N$ for all N, and $x^N \to x$ as $N \to \infty$. Since all x^N are integer valued, x must be integer valued as well. So all points from $\overline{E}$ are integer valued. But $E \subseteq \overline{E}$, so all points in E are integer valued. □

4.3 An application in infinite horizon production planning

Consider the following infinite horizon production planning problem.

$$\min \sum_{j=1}^{\infty} \alpha^{j-1} \left(k_j(P_j) + h_j(I_j)\right)$$

subject to

$$\begin{aligned} I_{j-1} + P_j - I_j &= d_j & j = 1, 2, \ldots \\ P_j &\leq \overline{P}_j & j = 1, 2, \ldots \\ I_j &\leq \overline{I}_j & j = 1, 2, \ldots \\ P_j, I_j &\geq 0 & j = 1, 2, \ldots \end{aligned}$$

where P_j denotes production in period j, I_j denotes net inventory at the end of period j, and d_j the demand in period j. We assume that the cost of production k_j and the cost of carrying inventory h_j are nondecreasing concave functions. Moreover, we require

$$\sum_{j=1}^{\infty} \alpha^{j-1} \left(k_j(\overline{P}_j) + h_j(\overline{I}_j)\right) < \infty$$

so that the objective function is well-defined for all feasible solutions. We then have the following result:

Theorem 4.4 *If in the above production planning problem the demands are integer, they never exceed potential production in a period, and the upper bounds on production and inventory are integers, then there exists an integer valued optimal solution to the problem.*

Proof: First of all, since demand in a period never exceeds potential production, it is easy to see that the algebraic projections of the system defining the feasible region are extendable, so that the ordinary projection of the feasible region onto $\prod_{j=1}^{N} I\!R^2$ is given by the solution set to the first N constraints (and the first N upper bounds).

Secondly, it is well-known that the constraint matrix of the feasible region is totally unimodular for any finite horizon version of the problem. But then, by definition 4.1, the constraint matrix of the infinite horizon problem is totally unimodular.

Theorem 4.3 now states that the extreme points of the feasible region of the production planning problem are integer valued. Finally, continuity of the objective function, together with compactness of the feasible region, guarantees that one of those (integer valued) extreme points is optimal. □

The same argument can be easily applied to more complex planning problems. For example, Jones, Zydiak, and Hopp (1988) introduced an infinite horizon linear programming formulation of an equipment replacement/capacity expansion problem where demand for capacity is nondecreasing over time. Key to the validity of this relaxed LP formulation is the presence of an integer valued optimal solution, which they established directly by verifying the optimality of a constructive integer valued solution. However, one can show that, since machines have finite lifetimes in Jones, Zydiak, and Hopp (1988), all decision variables can be bounded without loss of optimality, so that the feasible region can be restricted to a compact set. Moreover, since there are no a priori bounds on the number of new machines that can be bought or salvaged in any year, the property of extendability holds. Finally, total unimodularity is readily established for the finite horizon versions of the problem. We can therefore conclude from theorem 4.3 that all extreme point solutions, and hence an optimal solution, are integer. This extends the applicability of the model in Jones, Zydiak and, Hopp (1988) to the more general case of arbitrary time varying demand for capacity and time dependent costs arising in the presence of technological change.

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