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Gravitational Perturbations of a Radiating Spacetime

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This paper discusses the problem of gravitational perturbations of radiating spacetimes. We lay out the theoretical framework for describing the interaction of external gravitational fields with a radiating spacetime. This is done by deriving the field perturbation equations for a radiating metric. The equations are then specialized to a Vaidya spacetime. For the Hiscock ansatz of a linear mass model of a radiating blackhole the equations are found separable. Further, the resulting ordinary differential equations are found to admit analytic solutions. We obtain the solutions and discuss their characteristics.

1. INTRODUCTION

The study of gravitational perturbations can be traced back to the famous Einstein–Infeld–Hoffman paper of 1938[(1)] which pioneered the treatment of the two body problem in general relativity. In 1957 Regge and Wheeler[(2)] addressed the problem of the stability of a Schwarzschild black hole. Later, in his study of perturbations of a rotating black hole (Ref. 3 and later papers), Teukolsky was able to put the discipline on a stronger footing. However, little progress has been made at extending this success to cover the radiating cases. The problem of perturbing a radiating spacetime with integral spin fields has not received the attention it deserves. This, despite the fact that most astrophysical objects radiate. From regular stars to supernovae, from quasars to primordial black holes one finds that the inhabitants of our universe are generally non-static.

In the present paper, we develop a framework for discussing the problem of how external gravitational fields may interact with radiating

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spacetimes. This is done by deriving the field perturbation equations. It is found that two such equations are sufficient to describe all the non-trivial features of the perturbing gravitational field. We find that one of these equations decouples completely and is homogeneous in one of the field components. The equation for the other field component contains, in its source terms, several perturbed and therefore undetermined quantities. Using a systematic approach we are able to determine all these quantities completely, in terms of the former field component. The result is that all the perturbations are described by only two field components which satisfy two partial differential equations. The equations are then specialized to a Vaidya spacetime. For a particular model of a radiating black hole the equations are found separable. Interestingly, the resulting ordinary differential equations are found to admit analytic solutions. We obtain these solutions and discuss their characteristics.

The mathematical framework used in this paper is the null tetrad formalism of Newman and Penrose (hereafter NP formalism).[(6)] In Sec. 2 we give a brief description of the background geometry, the radiating spacetime of Vaidya. In Sec. 3 we derive the perturbation field equations for a general non-vacuum type D spacetime and adapt these equations to the Vaidya spacetime. In Sec. 4 we calculate the perturbed quantities in the source terms to arrive at the final working field equations. It is demonstrated in Sec. 5 that these equations are separable for the Hiscock linear model[(12)] of a radiating black hole. In Sec. 6 we obtain and discuss some of the solutions and we conclude the discussion in Sec. 7.

2. THE VAIDYA SPACETIME

2.1. The Metric

In this analysis we perturb the Vaidya spacetime[(9)] with incoming external gravitational fields. The Vaidya geometry, the simplest of the radiating spacetimes, is non-rotating and spherically symmetric. The energy-momentum tensor

$$T_{\mu\nu} = \rho k_\mu k_\nu$$

describes a null fluid, ($k_\mu k^\mu = 0$), of density ρ with radial flow, $k^2 = k^3 = 0$. Using this energy-momentum tensor to solve the Einstein field equations one obtains[(10)] a line element, in retarded coordinates, given by

$$ds^2 = \left[1 - \frac{2m(u)}{r}\right] du^2 + 2\, du\, dr - r^2\, d\Omega^2 \tag{1}$$

Here, u is the retarded time coordinate and $m(u)$, the mass, is a monotonically decreasing function of u.

It is convenient to introduce a null tetrad basis $z_{m\mu} = (l_\mu, n_\mu, m_\mu, \bar{m}_\mu)$ at every point this spacetime. The metric tensor $g_{\mu\nu}$ then becomes

$$g_{\mu\nu} = z_{m\mu} z_{n\nu} \eta^{mn} = l_\mu n_\nu + n_\mu l_\nu - m_\mu \bar{m}_\nu - \bar{m}_\mu m_\nu \tag{2}$$

where η^{mn} is the flat spacetime metric. Following Carmeli and Kaye[8] we choose the covariant form of the null tetrad basis as

$$\begin{aligned} l_\mu &= \delta^0_\mu \\ n_\mu &= \frac{1}{2}\left[1 - \frac{2m(u)}{r}\right] \delta^0_\mu + \delta^1_\mu \\ m_\mu &= -\frac{r}{\sqrt{2}}\,[\delta^2_\mu + i \sin\theta \delta^3_\mu] \\ \bar{m}_\mu &= -\frac{r}{\sqrt{2}}\,[\delta^2_\mu - i \sin\theta \delta^3_\mu] \end{aligned} \tag{3}$$

The contravariant vectors, z_m, considered as tangent vectors, define the directional derivatives as

$$\vec{z}_m = z^\mu_m \nabla_\mu \tag{4}$$

In the Vaidya spacetime these directional derivatives are given (from Eq. (3)) by

$$\begin{aligned} D &= l^\mu \nabla_\mu = \frac{\partial}{\partial r} \\ \Delta &= n^\mu \nabla_\mu = \frac{\partial}{\partial u} - \frac{1}{2}\left[1 - \frac{2m(u)}{r}\right] \frac{\partial}{\partial r} \\ \delta &= m^\mu \nabla_\mu = \sqrt{2}\, r \left[\frac{\partial}{\partial \theta} + i \csc\theta \frac{\partial}{\partial \varphi}\right] \\ \bar{\delta} &= \bar{m}^\mu \nabla_\mu = \sqrt{2}\, r \left[\frac{\partial}{\partial \theta} - i \csc\theta \frac{\partial}{\partial \varphi}\right] \end{aligned} \tag{5}$$

One finds that the only surviving spin (Ricci rotation) coefficients[8] of the Vaidya spacetime are

$$\rho = -\left(\frac{1}{r}\right), \qquad \alpha = -\frac{1}{2\sqrt{2}\,r}\cot\theta, \qquad \beta = -\alpha$$
$$\mu = -\frac{1}{2r}\left[1 - \frac{2m(u)}{r}\right], \qquad \text{and} \qquad \gamma = \frac{m(u)}{2r^2} \tag{6}$$

The only surviving tetrad component of Ricci tensor[8] is

$$\Phi_{22} = \frac{-\dot{m}(u)}{r^2} \tag{7}$$

where $\dot{m} = dm/du$. With this component, it is easily shown that

$$T_{\mu\nu} = 2\Phi_{22} l_\mu l_\nu = -\left[2\frac{\dot{m}(u)}{r^2}\right] l_\mu l_\nu \tag{8}$$

Further, the only non-vanishing tetrad component of the of the Weyl tensor is

$$\Psi_2 = \frac{-m(u)}{r^3} \tag{9}$$

The Vaidya spacetime is then[6] said to be Petrov type D with repeated principal null vectors l^μ and n^μ. The three optical scalars are found, to be $\sigma = \omega = 0$, $\theta = -1/r$. The metric contains two shear-free, twistless and diverging geodetic null congruencies.

3. THE PERTURBED FIELD EQUATIONS

3.1. The Type D Spacetime

In this section we develop the gravitational field perturbation equations for a general Petrov type D spacetime. We start with the Newman–Penrose (NP) equations. The full set of the NP equations can be found in several publications.[8] We only mention here that the set is made up of first order differential equations which, in the NP formalism, replace the Einstein field equations. The equations link together the null tetrad basis, the spin (Ricci rotation) coefficients, the Weyl tensor, the Ricci tensor and the scalar curvature. In using this formalism to do perturbation analysis one first specifies the perturbations of the geometry. Here we shall write the tetrad of the perturbed spacetime as

$$\boldsymbol{l}=\boldsymbol{l}^{b}+\boldsymbol{l}^{p}, \qquad \mathbf{n}=\mathbf{n}^{b}+\mathbf{n}^{p}, \qquad \mathbf{m}=\mathbf{m}^{b}+\mathbf{m}^{p}, \qquad \bar{\mathbf{m}}=\bar{\mathbf{m}}^{b}+\bar{\mathbf{m}}^{p} \tag{10}$$

where the superscripts b and p refer to the background and the perturbing quantities, respectively. Since all the other field quantities are expressible in terms of the tetrad[7] their perturbed forms can be written down. For example: $\Psi_{a=(0,1,2,3,4)}=\Psi^{b}_{a}+\Psi^{p}_{a}$.

We shall, in general, assume that the perturbations in the basis vectors are sufficiently small so that only their first order contributions axe significant. The field equations are, then, first order in the perturbing fields, linearized about the background quantities.

In type D spacetimes one can choose the tetrad vectors (see Sec. 2) $\boldsymbol{l}$ and $\mathbf{n}$ so that certain spin coefficients and certain Weyl scar components

$$\kappa^{b}=\sigma^{b}=\nu^{b}=\lambda^{b}=0, \qquad \Psi^{b}_{0}=\Psi^{b}_{1}=\Psi^{b}_{3}=\Psi^{b}_{4}=0 \tag{11}$$

We start our analysis from three of the NP[8] field equations. From the Bianchi identities we consider the two equations,

$$\begin{aligned}\bar{\delta}\Psi_{3}-D\Psi_{4}+\bar{\delta}\Phi_{21}-\Delta\Phi_{20}&\\ =3\lambda\Psi_{2}-2(\alpha+2\pi)\,\Psi_{3}+(4\varepsilon-\rho)\,\Psi_{4}-2\nu\Phi_{10}+2\lambda\Phi_{11}&\\ +(2\gamma-2\bar{\gamma}+\bar{\mu})\,\Phi_{20}+2(\bar{\tau}-\alpha)\,\Phi_{21}-\bar{\sigma}\Phi_{22}&\end{aligned} \tag{12}$$

and

$$\begin{aligned}\Delta\Psi_{3}-\delta\Psi_{4}+\bar{\delta}\Phi_{22}-\Delta\Phi_{21}&\\ =3\nu\Psi_{2}-2(\gamma+2\mu)\,\Psi_{3}+(4\beta-\tau)\,\Psi-2\nu\Phi_{11}-\bar{\nu}\Phi_{20}+2\lambda_{12}&\\ +2(\gamma+\bar{\mu})\,\Phi+(\bar{\tau}-2\bar{\beta}-2\alpha)\,\Phi_{22}&\end{aligned} \tag{13}$$

from the spin coefficient system of equations we take

$$\Delta\lambda-\bar{\delta}\nu=-(\mu+\bar{\mu})\,\lambda-(3\gamma-\bar{\gamma})\,\lambda+(3\alpha+\bar{\beta}+\pi-\tau)\,\nu-\Phi_{4} \tag{14}$$

We have complete knowledge of the geometry of the background spacetime, so we write down the equations only in those terms that make first order contributions to the perturbed field quantities. Making use of

Eqs. (11) on Eqs. (12), (13) and (14), respectively, it is seen that for a perturbed Petrov type D spacetime

$$-3\lambda^p\Psi_2+(\bar{\delta}+2\alpha+4\pi)\,\Psi_3^p-(D+4\varepsilon-\rho)\,\Psi_4^p$$
$$=(\Delta+2\gamma-2\bar{\gamma}+\mu)\,\Phi_{20}^p-(2\alpha-2\bar{\tau}+\bar{\delta})\,\Phi_{21}^p-\bar{\sigma}^p\Phi_{22} \tag{15}$$

$$-3\nu^p\Psi_2+(\Delta+2\gamma+4\mu)\,\Psi_3^p-(4\beta-\tau+\delta)\,\Psi_4^p$$
$$=(2\gamma+2\bar{\mu}+\Delta)\,\Phi_{21}^p+(-\bar{\delta}+\bar{\tau}-2\bar{\beta}-2\alpha)^p\,\Phi_{22}$$
$$+(-\bar{\delta}+\bar{\tau}-2\bar{\beta}-2\alpha)\,\Phi_{22}^p \tag{16}$$

and

$$(\Delta+3\gamma-\bar{\gamma}+\mu+\bar{\mu})\,\lambda^p=(\bar{\delta}+3\alpha+\bar{\beta}+\pi-\bar{\tau})\,\nu^p+\Psi_4^p=0 \tag{17}$$

We note that the perturbed equations are coupled both in the Weyl tensor components and the Ricci tensor components. They also contain unknown spin coefficients and directional derivatives. In attempting to decouple the equations above we use an approach akin to Teukolsky.[3] After some algebra we find, on eliminating Ψ_3 between Eqs. (15) and (16), we are left with

$$[(\Delta+3\gamma-\bar{\gamma}+4\mu+\bar{\mu})(D+4\varepsilon-\rho)]\,\Psi_4^p$$
$$-[(\bar{\delta}+3\alpha+\bar{\beta}+4\pi-\bar{\tau})(4\beta-\tau+\delta)-3\Psi_2]\,\Psi_4^p=Q_4 \tag{18}$$

Now, under the interchange $\boldsymbol{l}\rightleftarrows\mathbf{n}$, $\mathbf{m}\rightleftarrows\bar{\mathbf{m}}$, the full set of the NP equations is invariant.[11] This symmetry is not destroyed by the choice of $\boldsymbol{l}$ and $\mathbf{n}$ which gave Eqs. (11). One finds under this interchange that[7]

$$\Psi_0\rightleftarrows\Psi_4^*,\quad \Psi_3\rightleftarrows\Psi_1^*,\quad \Psi_2\rightleftarrows\Psi_2^*$$

$$\kappa\rightleftarrows-\bar{\nu},\quad \rho\rightleftarrows-\bar{\mu},\quad \sigma\rightleftarrows-\bar{\lambda},\quad \alpha\rightleftarrows-\bar{\beta},\quad \varepsilon\rightleftarrows-\bar{\gamma}\quad\text{and}\quad \pi\rightleftarrows-\bar{\tau} \tag{19}$$

Applying this to Eq. (18) we obtain the following equation for Ψ_0^p

$$(D-3\varepsilon-\bar{\varepsilon}-4\rho-\bar{\rho})(\delta-3\beta-\bar{\alpha}-4\tau+\bar{\pi})(\bar{\delta}-4\alpha+\pi)\,\Psi_0^p$$
$$-(D-3\varepsilon-\bar{\varepsilon}-4\rho-\bar{\rho})[(\Delta-4\gamma+\mu)-3\Psi_2]\,\Psi_0^p=Q_0 \tag{20}$$

Here, the source term Q_0 is given by

$$\begin{aligned}Q_0 = &(\delta - 3\beta - \bar{\alpha} - 4\tau + \bar{\pi})(D - 2\bar{\rho} - 2\varepsilon)\, \Phi^p_{02}\\ &- (\delta - 3\beta - \bar{\alpha} - 4\tau + \bar{\pi})(\delta + \bar{\pi} - 2\bar{\alpha} - 2\beta)\, \Phi^p_{00}\\ &+ (\delta - 3\beta - \bar{\alpha} - 4\tau + \bar{\pi})(\delta + \bar{\pi} - 2\bar{\alpha} - 2\beta)^p\, \Phi_{00}\\ &+ (D - 3\varepsilon + \bar{\varepsilon} - 4\rho - \bar{\rho})(\delta - 2\beta + 2\bar{\pi})\, \Phi^p_{02} - (D - 2\varepsilon + 2\bar{\varepsilon} - \rho)\, \Phi^p_{22}\\ &+ (D - 3\varepsilon + \bar{\varepsilon} - 4\rho - \bar{\rho})\, \sigma^p(\Delta - \mu)\, \Phi_{00} \qquad (21)\end{aligned}$$

Equations (18) and (20) describe the gravitational field perturbation equations for a general type D spacetime with sources.

3.2. The Perturbed Vaidya Spacetime

We shall now adapt the equations above to the problem of perturbing a Vaidya spacetime with gravitational fields. The important features to deal with here are the perturbations in the source terms Q_0 and Q_4 of Eqs. (18) and (20).

The energy-momentum tensor associated with the radiation in the Vaidya spacetime is usually interpreted via geometrical optics. Carmeli and Kaye[8] have, in fact, shown that the associated radiation field which has a monopole structure can not be identified as a source-free electromagnetic field. Therefore in addressing the problem of perturbing this radiation, one can only discuss direct perturbations on the energy-momentum tensor. Now, this energy-momentum tensor is quadratic in the metric.[17] It follows from these considerations, then, that the lowest order perturbations in the scalar components Φ_{mn} will be quadratic in the metric perturbations. However, we assumed from the beginning that the perturbations in the basis vectors are sufficiently small so that only their first order contributions are significant. In our linear theory we shall, therefore, disregard these perturbations in the scalar components Φ_{mn} contributions. Nevertheless, we still have various perturbed spin coefficients and directional derivatives to deal with in the source terms.

It is recalled (see Sec. 2) that the only quantities which survive in the Vaidya space-time background are: the spin coefficients α, β, γ, ρ, μ, the Weyl tensor component Ψ_2 and the Ricci tensor component Φ_{22}. Moreover, all these spin coefficients are real and $\alpha = -\beta$. On applying the observations made above to the field perturbation Eqs. (18) and (20) we find that

$$[(\Delta + 2\gamma + 5\mu)(D - \rho) - (\bar{\delta} - 2\beta)(\delta + 4\beta) - 3\Psi_2]\, \Psi^p_4 = Q'_4 \qquad (22)$$

$$[(D - 5\rho)(\Delta - 4\gamma + \mu) - (\delta - 2\beta)(\bar{\delta} + 4\beta) - 3\Psi_2]\, \Psi^p_0 = 0 \qquad (23)$$

where the source terms are now given by

$$Q'_4 = [(\Delta + 2\gamma + 5\mu)\,\bar{\sigma}^p - (\bar{\delta} - 2\beta)(\bar{\delta}^p - \bar{\tau}^p + 2\bar{\beta}^p + 2\alpha^p) + \lambda^p(D + \rho)]\,\Phi_{22} \tag{24}$$

and clearly

$$Q'_0 = 0 \tag{25}$$

Note that in Eq. (24) the derivative operators Δ and D must act on Φ_{22}.

We now have two Eqs. (22) and (23) which describe the two Weyl tensor components Ψ_0^p and Ψ_4^p. It can be shown that these two components are sufficient to describe all the non-trivial features of the perturbing fields. The proof of this sufficiency is achieved by showing[3] that only Ψ_0^p and Ψ_4^p are invariant under gauge transformations and infinitesimal tetrad rotations. This invariance, in turn, makes them completely measurable physical quantities.

The source terms Q'_4 in Eq. (22) still contain (see Eq. (24)) perturbations in as many as seven different quantities. In the next section we shall derive equations that describe these unknowns. In doing so, we shall discover that those perturbed quantities that do not vanish are all completely describable in terms of the one of the two field components.

4. THE PERTURBED QUANTITIES IN THE SOURCE TERMS

The aim of this section is to calculate (in terms of the known background quantities and the fields Ψ_0^p and Ψ_4^p), the perturbed quantities, $\bar{\Phi}_{12}^p$, α^p, β^p, τ^p, σ^p, λ^p and δ^p which appear in the source terms Q'_4 of Eq. (22), (see also Eq. (24)). It is worthwhile to point out that the system of equations we have constructed in the previous section must be consistent with the freedom we have in the choice of both the tetrad frame and the coordinates. In particular,[7] from the 6-parameter group of homogeneous Lorentz transformations, we have six degrees of freedom to make infinitesimal rotations of the local tetrad-frame. Further, we have four degrees of freedom to make infinitesimal coordinate transformations. Thus, altogether we have a total of ten degrees of gauge freedom. We are free to exercise these available degrees of freedom as convenience and occasion may dictate.

As has been shown,[3] in a linear perturbation theory, Ψ_0^p and Ψ_4^p are gauge invariant while Ψ_1^p, Ψ_2^p and Ψ_3^p are not. This means that we can subject the tetrad null-basis to an infinitesimal rotation in which Ψ_1^p and

Ψ_3^p vanish without affecting Ψ_0^p and Ψ_4^p. In making this choice of gauge, we have used up only four, out of ten, degrees of freedom. It can be shown (see, for example, Ref. 7 Chapter 1, Eqs. (342) and (346)) that in linearized type 1 rotations both Ψ_1^p and $\bar{\Phi}_{12}^p$ vanish while in type 2 rotations both do not. It follows, then, that a gauge in which both Ψ_1^p, Ψ_3^p and $\bar{\Phi}_{12}^p$ vanish can be chosen. Further, in this gauge, Ψ_2^p clearly vanishes. Under these circumstances, the NP equations[7] show that the linearized Bianchi identities take on a simpler form. For our purposes, the equations we need to consider from this set are:

$$\Delta \Psi_0^p = (4\gamma - \mu)\, \Psi_0^p + 3\sigma^p \Psi_2 \tag{26}$$

$$-3\delta^p \Psi_2 = -9\tau^p \Psi_2 - 2\kappa^p \Phi_{22} \tag{27}$$

$$-D\Psi_4^p = 3\lambda^p \Psi_2 - \rho \Psi_4^p - \bar{\sigma}^p \Phi_{22} \tag{28}$$

$$0 = -\kappa^p \Phi_{22} \tag{29}$$

4.1. Calculation of $\bar{\sigma}^p$ and λ^p

From Eq. (26) and the fact that all the background quantities here are real we find that

$$\bar{\sigma}^p = \left(\frac{\Delta - 4\gamma + \mu}{3\Psi_2} \right) \bar{\Psi}_0^p \tag{30}$$

Further, using Eq. (30) on Eq. (28) gives

$$\lambda^p = \left(\frac{\rho - D}{3\Psi_2} \right) \Psi_4^p - \Phi_{22} \left(\frac{\Delta - 4\gamma + \mu}{9(\Psi_2)^2} \right) \bar{\Psi}_0^p \tag{31}$$

We note that the perturbed spin coefficients λ^p and $\bar{\sigma}^p$ display a definite dependence on Ψ_4^p and/or $\bar{\Psi}_0^p$. In the rest of this section we shall derive expressions for the remaining perturbed quantities.

4.2. The Perturbation Matrix for the Basis Vectors

In order to determine the perturbed quantities τ^p, α^p, β^p and δ^p and their relations, it is necessary to study the effects of the perturbations on the basis vectors $(l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$. For compactness, it is convenient to introduce the following index notation:

$$l^1 = l^\mu, \qquad l^2 = n^\mu, \qquad l^3 = m^\mu, \qquad l^4 = \bar{m}^\mu \tag{32}$$

We can write[7] the perturbations $l^{(p)\,i}$ $(i=1, 2, 3, 4)$ in the vectors, as linear combinations of the unperturbed basis vectors l^i. Thus

$$l^{(p)\,i} = P^i_j l^j \tag{33}$$

where the P^i_j are elements of a matrix $\mathbf{P}$ that describes, completely, the perturbations in the basis vectors. Explicitly,

$$\mathbf{P} = \begin{bmatrix} P^1_1 & P^1_2 & P^1_3 & P^1_4 \\ P^2_1 & P^2_2 & P^2_3 & P^2_4 \\ P^3_1 & P^3_2 & P^3_3 & P^3_4 \\ P^4_1 & P^4_2 & P^4_3 & P^4_4 \end{bmatrix} \tag{34}$$

The l^1 and l^2 are real while the l^3 and l^4 are complex conjugates. It follows, then, that the matrix elements P^1_1, P^1_2, P^2_1 and P^2_2 are real while the remaining elements of $\mathbf{P}$ are complex. Moreover, the elements in which the indices 3 and 4 replace one another, are complex conjugates. For example, $P^2_3 = \overline{(P^2_4)}$.

4.3. Perturbations in the Angular Functions, δ^p, τ^p, α^p, and β^p

The perturbations in the directional derivative $\bar{\delta}^p$, are given from Eqs. (5) and (33), by

$$\bar{\delta}^p = (l^4 \nabla_4)^p \tag{35}$$

But from Eqs. (33) and (34) we see that

$$l^{p(4)} = P^4_j l^j = P^4_1 l^1 + P^4_2 l^2 + P^4_3 l^3 + P^4_4 l^4 \tag{36}$$

Using Eq. (36) on Eq. (35) shows that

$$\bar{\delta}^p = (l^4 \nabla_4)^p = P^4_1 D + P^4_2 \Delta + P^4_3 \delta + P^4_4 \bar{\delta} \tag{37}$$

Thus, if we operate with $\bar{\delta}^p$ on the background Ψ_2 we get

$$\bar{\delta}^p \Psi_2 = P^4_1 D\Psi_2 + P^4_2 \Delta\Psi_2 + P^4_3 \delta\Psi_2 + P^4_4 \bar{\delta}\Psi_2 \tag{38}$$

Now, in the background

$$\Psi_2 = \frac{-m(u)}{r^3}$$

Substituting for Ψ_2 in Eq. (38) and using definitions of the operators in Eqs. (5) we find that

$$\bar{\delta}^p \Psi_2 = P_1^4 \frac{[3m(u)]}{r^4} + P_2^4 \left(\frac{-\dot{m}}{r^3}\right) - P_2^4 \left(1 - \frac{2m(u)}{r}\right) \frac{3m(u)}{r^4} \tag{39}$$

Recall that in the Vaidya spacetime background, the component $\Phi_{22} = -\dot{m}(u)/r^2$ is non-vanishing. Using this on Eq. (29) shows that κ^p must vanish. It follows, then that Eq. (27) becomes,

$$\bar{\delta}^p \Psi_2 = 3\bar{\tau}^p \Psi_2 \tag{40}$$

We see immediately, that the results expressed in Eq. (39) will be inconsistent with the eigenvalue Eq. (40) unless the P_2^4 vanish, so that

$$\bar{\delta}^p \Psi_2 = P_1^4 \frac{[3m(u)]}{r^4} = -3P_1^4 \left(\frac{1}{r}\right) \Psi_2 \tag{41}$$

Equations (40) and (41), then, show that

$$\bar{\tau}^p = -P_1^4 \left(\frac{1}{r}\right) \tag{42}$$

Moreover, Eq. (37) along with the condition that the P_2^4 vanish means that whenever $\bar{\delta}^p$ acts on a function with no angular dependence its only contribution is

$$\bar{\delta}^p = P_1^4 \frac{\partial}{\partial r} = -\bar{\tau}^p r \frac{\partial}{\partial r} \tag{43}$$

We now have, in Eq. (43), a general relationship between $\bar{\delta}^p$ and $\bar{\tau}^p$.

Next, we need to deal with α^p and $\bar{\beta}^p$. It is known[6] that if the null vectors l_μ are tangent to the geodesics and equal to a gradient field, then

$$\rho = \bar{\rho} \qquad \text{and} \qquad \tau = \bar{\alpha} + \beta \tag{44}$$

These conditions are fulfilled in all Type D space-times. In particular, the unperturbed Vaidya space-time satisfies

$$\bar{\alpha} + \beta = \tau = 0$$

Consequently, in our linear perturbation analysis we should have

$$(\alpha + \bar{\beta})^p = \bar{\tau}^p \tag{45}$$

Consider, now, the second of the source terms in Eq. (24) which reads as

$$(\bar{\delta} - 2\beta)(\bar{\delta}^p - \bar{\tau}^p + 2\bar{\beta}^p + 2\alpha^p)\, \Phi_{22}$$

Our results in Eqs. (43) and (45) when applied to the above expression show that

$$(\bar{\delta} - 2\beta)(\bar{\delta}^p - \bar{\tau}^p + 2\bar{\beta}^p + 2\alpha^p)\, \Phi_{22} = 3(\bar{\delta} - 2\beta)\, \bar{\tau}^p \Phi_{22} \tag{46}$$

This achieves the purpose of expressing the effects of the four perturbed quantities τ^p, α^p, β^p and δ^p in terms of one of these quantities τ^p. The next task is to express this perturbed quantity in terms of the fields. To this end we utilize one of the equations from the spin coefficient set.[8] Consider the equation

$$\delta\tau - \Delta\sigma = (\mu\sigma + \bar{\lambda}\rho) + (\tau + \beta - \bar{\alpha})\, \tau - (3\gamma - \bar{\gamma})\, \sigma - \kappa\bar{v} + \Phi_{02} \tag{47}$$

Specializing Eq. (47) to the Vaidya spacetime and applying our linear perturbations approach we obtain, on rearranging terms, an equation whose complex conjugate is

$$(\bar{\delta} - 2\beta)\, \bar{\tau}^p = (\Delta + \mu - 2\gamma)\, \bar{\sigma}^p + \rho\lambda^p \tag{48}$$

Equation (48) when substituted into Eq. (46) gives (recall $\bar{\delta}$ does not operate on Φ^{22})

$$(\bar{\delta} - 2\beta)(\bar{\delta}^p - \bar{\tau}^p + 2\bar{\beta}^p + 2\alpha^p)\, \Phi_{22} = 3[(\Delta + \mu - 2\gamma)\, \bar{\sigma}^p + \rho\lambda^p]\, \Phi_{22} \tag{49}$$

4.4. The Final Field Perturbation Equations

We are now in position to apply the results of our discussions in this section to the source terms of Eq. (22) as given by Eq. (24). Using Eq. (49) on Eq. (24) and rewriting the sources Q'_4 as Q' we obtain,

$$Q' = [(\Delta + 2\gamma + 5\mu)\, \bar{\sigma}^p - 3(\Delta + \mu - 2\gamma)\, \bar{\sigma}^p - 3\rho\lambda^p + \lambda^p(D + \rho)]\, \Phi_{22} \tag{50}$$

We see that in the above equation the terms in λ^p cancel yielding the simpler result,

$$Q' = -2(\Delta - 4\gamma - \mu)\, \bar{\sigma}^p \Phi_{22} \tag{51}$$

Substituting for $\bar{\sigma}^p$ from Eq. (30) we find that

$$Q' = -2(\Delta - 4\gamma - \mu) \left[\Phi_{22} \left(\frac{\Delta - 4\gamma + \mu}{3\Psi_2} \right) \bar{\Psi}_0^p \right] \tag{52}$$

Equation (52) forms the result of our analysis in this section. All the perturbations in the sources have now been expressed in terms of the perturbed field $\bar{\Psi}_0^p$ only. The working field equations (see (22) and (23)) have become

$$[(\Delta+2\gamma+5\mu)(D-\rho)-(\bar{\delta}-2\beta)(\delta+4\beta)-3\Psi_2]\,\Psi_4^p=Q' \tag{53}$$

and

$$[(D-5\rho)(\Delta-4\gamma+\mu)-(\delta-2\beta)(\bar{\delta}+4\beta)-3\Psi_2]\,\Psi_0^p=0 \tag{54}$$

where, now, Q' is given by Eq. (52).

Equations (53), (54), along with (52) form the main result of our perturbation analysis. These equations give the essential features of a gravitationally perturbed Vaidya space time. All the non-trivial perturbations are sufficiently described by two tetrad scalar components of the Weyl tensor, Ψ_0^p and Ψ_4^p, which components represent the extreme helicity states of the gravitational field.

We now can rewrite the equations in a form that reveals the dependence of the fields on the physical variables of spacetime. Thus, using Eqs. (5), (6), (7) and (9) on (54), (53), and (52) respectively, we find that

$$\begin{aligned}&\left[\frac{\partial^2}{\partial r\,\partial u}+\frac{5}{r}\frac{\partial}{\partial u}-\frac{1}{2}\left(1-\frac{2m(u)}{r}\right)\frac{\partial^2}{\partial r^2}-\frac{3}{r}\left(1-\frac{m(u)}{r}\right)\frac{\partial}{\partial r}-\frac{2}{r^2}\right]\Psi_0^p\\&\quad+\frac{1}{2r^2}\left[\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}-2(\csc^2\theta+\cot^2\theta)\right.\\&\quad\left.+\csc^2\theta\frac{\partial}{\partial\varphi^2}+4i\csc\theta\cot\theta\frac{\partial}{\partial\varphi}\right]\Psi_0^p=0\end{aligned} \tag{55}$$

and

$$\begin{aligned}&\left[\frac{\partial^2}{\partial u\,\partial r}+\frac{1}{r}\frac{\partial}{\partial u}-\frac{1}{2}\left(1-\frac{2m(u)}{r}\right)\frac{\partial^2}{\partial r^2}-\frac{1}{r}\left(3-\frac{7m(u)}{r}\right)\frac{\partial}{\partial r}\right.\\&\quad\left.-\frac{2}{r^2}\left(1-\frac{4m(u)}{r}\right)\right]\Psi_4^p-\frac{1}{2r^2}\left[\frac{\partial^2}{\partial\theta^2}-\cot\theta\frac{\partial}{\partial\theta}-2(\csc^2\theta+\cot^2\theta)\right.\\&\quad\left.+\csc^2\theta\frac{\partial^2}{\partial\varphi^2}-4i\csc\theta\cot\theta\frac{\partial}{\partial\varphi}\right]\Psi_4^p=Q'\end{aligned} \tag{56}$$

where now

$$Q' = -2\left[\frac{\partial}{\partial u} - \frac{1}{2}\left(1 - \frac{2m(u)}{r}\right)\frac{\partial}{\partial r} - \frac{3m(u)}{r^2} + \frac{1}{2r}\right]$$
$$\star\left\{\left(\frac{\dot{m}(u)}{3m(u)}r\right)\left[\frac{\partial}{\partial u} - \frac{1}{2}\left(1 - \frac{m(u)}{r}\right)\frac{\partial}{\partial r} - \frac{m(u)}{r} - \frac{1}{2r}\right]\bar{\Psi}_0^p\right\} \tag{57}$$

and the star $\star$, here, indicates the terms before it operate on the terms after it.

5. SEPARATION OF VARIABLES

In this section we seek to separate the equations that were derived in the previous section. This separation of variable is effected in two phases. In Phase I we deal with the angular variables while in Phase II we deal with the retarded time and radial variables. We shall, for now, concentrate on the homogeneous parts of the equations. The contribution due to the source terms can always be constructed later once a solution for Ψ_0^p has been obtained. Incidentally, one notices (see Eq. (57)) that the luminosity-mass ratio $L/3m(u)$, $(L = -\dot{m})$ which scales the source term will almost always be vanishingly small since for most radiating objects the mass being radiated at any given time is much smaller than the rest of the body mass.

5.1. Phase I: The Spin-Weighted Angular Functions

We suppose that the gravitational fields entering the spherically symmetric background spacetime are plane waves so that the problem has azimuthal symmetry. With this, we then assume that the field equations are separable in the angular variables admitting solutions of the form

$$\Psi_{i=(0,4)}(u, r, \theta, \varphi) = \phi_{i=(0,4)}(u, r, \theta)\, e^{im\varphi}$$
$$= R_{p=(\pm 2)}(u, r)\, S_{p=(\pm 2)}(\theta)\, e \tag{58}$$

Here the subscript p is used to identify a particular spin-s field component by the spin weight. For our purposes, the spin weight p only takes on the extreme values of $\pm s$ corresponding to the extreme helicity states of the field. Explicitly, Ψ_0 has a spin weight of 2 while Ψ_4 has a spin weight of -2. Note, to avoid confusion in notation, here and henceforth we discard the superscript p previously used to identify the perturbed quantities.

Substituting (58) in the field Eqs. (55) and (56) yields the following general equation in the angular variables:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) + \left(p - p^2\cot\theta - \frac{2mp\cos\theta}{\sin^2\theta} - \frac{m^2}{\sin^2\theta} - K\right) S_p(\theta)\, e^{im\varphi} = 0 \tag{59}$$

This equation along with boundary conditions of regularity at $\theta = 0$ and $\theta = \pi$ constitute a Sturm–Liouville eigenvalue problem for the separation constant $K = {}_pK_l^m$. For fixed p and m values, the eigenvalues can be labelled by l. The smallest eigenvalue has $l = \max(p, |m|)$. For each p and m the eigenfunctions ${}_pS_l^m(\theta)$ are complete and orthogonal on the interval $0 \leqslant \theta \leqslant \pi$, as required by the Sturm Liouville theory. In our case, where the background is non-rotating the eigenfunctions are, the well known,[13] spin-weighted spherical harmonics:

$$_{p}Y_l^m(\theta, \varphi) = {}_pS_l^m(\theta)\, e^{im\varphi} \tag{60}$$

and the separation constant K is found to be given by

$$K = {}_pK_l = (l - p)(l + p + 1) \tag{61}$$

5.2. Phase II: The Radial-Null Equations

The separation of variables effected in the last section leaves us with two equations for the functions R_{+2} and R_{-2}. These functions are coefficients of ${}_2Y_l^m(\theta, \varphi)$ and ${}_{-2}Y_l^m(\theta, \varphi)$, respectively, in the spin-2 fields Ψ_0 and Ψ_2 and are each dependent on u and r only. On substituting Eq. (60) into Eqs. (55) to (57) one finds that R_{+2} satisfies

$$\left[\frac{\partial^2 R_2(u,r)}{\partial r\,\partial u} + \frac{5}{r}\frac{\partial R_2(u,r)}{\partial u} - \frac{1}{2}\left(1 - \frac{2m(u)}{r}\right)\frac{\partial^2 R_2}{\partial r}\right] - \left[\frac{3}{r}\left(1 - \frac{m(u)}{r}\right)\frac{\partial R_2(u,r)}{\partial r} - \frac{({}_2K_l - 4)}{2r^2} R_2(u,r)\right] = 0 \tag{62}$$

and R_{-2} satisfies

$$\left[\frac{\partial^2 R_{-2}}{\partial u\,\partial r} + \frac{1}{r}\frac{\partial R_{-2}}{\partial u} - \frac{1}{2}\left(1 - \frac{2m(u)}{r}\right)\frac{\partial^2 R_{-2}}{\partial r^2} - \left(\frac{3}{r} - \frac{7m(u)}{r^2}\right)\frac{\partial R_{-2}}{\partial r^2}\right] + \left(\frac{1}{2r^2}({}_{-2}K - 4) + \frac{8m(u)}{r^3}\right) R_{-2}(u,r) = 0 \tag{63}$$

We deal with Eqs. (62) and (63) separately. First, we shall seek to separate Eq. (62) in $R_2(u, r)$. By adopting a change of variables we show that the equation is separable for a specific choice of mass function. Thereafter, we shall apply this approach to Eq. (63).

5.2.1. *Change of Variables*

Equation (62), as it stands, is not separable. We shall, therefore, find it convenient to introduce the following change of variables: let us set

$$\tau = \frac{1}{u} \qquad \text{and} \qquad \xi = \frac{2m(u)}{r} \tag{64}$$

Then it is seen that

$$\begin{aligned} \frac{\partial}{\partial u} &= -\tau^2 \frac{\partial}{\partial \tau} + \frac{\dot{m}}{m(\tau)}\, \xi \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial r} &= -\frac{\xi^2}{2m(\tau)} \frac{\partial}{\partial \xi} \end{aligned} \tag{65}$$

and

$$\frac{\partial^2}{\partial r^2} = \frac{\xi^3}{2[m(\tau)]^2} \frac{\partial}{\partial \xi} + \frac{\xi^4}{4[m(\tau)]^2} \frac{\partial^2}{\partial \xi^2}$$

where, now, m is a function of τ but $\dot{m}$ still means dm/du.

5.3. Equation for R_{+2}

On substituting Eqs. (64) and (65) into (62) and rearranging we find that

$$\begin{aligned} &\left(-5\xi \frac{\partial R_2}{\partial \tau} + \xi^2 \frac{\partial^2 R_2}{\partial \xi\, \partial \tau}\right) 4\tau^2 m(\tau) - 4\dot{m}\left(\xi^3 \frac{\partial^2 R_2}{\partial \xi^2} - 4\xi^2 \frac{\partial R_2}{\partial \xi}\right) \\ &\qquad + (\xi^5 - \xi^4)\frac{\partial^2 R_2}{\partial \xi^2} - (\xi^4 - 4\xi^3)\frac{\partial R_2}{\partial \xi} + \varepsilon_l \xi^3 R_2(\tau, \xi) = 0 \end{aligned} \tag{66}$$

where we have set

$$\varepsilon_l = (l-p)(l+p+1)-4 = (l-2)(l+3)-4 \tag{67}$$

We would like, as an example, to apply our analysis on an evaporating blackhole. In general, Eq. (66) is not separable for an arbitrary mass function $m(u)$. However it can be shown to separate for one particular model of such a radiating blackhole.

5.3.1. Vaidya Model for a Linearly Radiating Blackhole

In the Vaidya model of a radiating blackhole,[(12)] the spacetime is, initially, Minkowski flat for $u<0$. Then at $u=0$ an imploding δ-function-like null fluid with a total positive mass M forms a blackhole. Hereafter, $0<u<u_0$ negative energy null fluid then falls into the blackhole evaporating the latter in the process. One known consequence[(12)] is that the spacetime violates the weak energy condition. Eventually the blackhole vanishes so that for $u \geqslant u_0$ the spacetime becomes Minkowski flat again.

One of the popular models of radiating black holes is the so-called self-similar model originally developed by Hiscock.[(12)] Popular, because from it one can construct the quantum energy stress tensor for the entire spacetime. The model has been extensively used lately, (see, for example, Refs. 14 and 15). In this model the mass is a linear function of the retarded time coordinate u.

We shall show, presently, that for the Hiscock linear mass function ansatz the above equation is separable.

Suppose

$$m(u) = \begin{cases} 0, & u<0 \\ M_0(1-\lambda u), & \dfrac{1}{\lambda} > u > 0 \\ 0, & u > \dfrac{1}{\lambda} \end{cases} \tag{68}$$

so that

$$\dot{m} = -\lambda M_0 \tag{69}$$

where m_0 is the initial mass at $u=0$ and λ is some positive parameter $0<\lambda<1/u$ that scales the radiation rate.

We shall find it convenient to institute a change of variable $u = v_0 + v$ where v_0 is some fixed value of u.

$$m(v)=\begin{cases} m_0(1+\lambda v), & -v_0<v<0 \\ m_0, & v=0 \\ m_0[1-\lambda v], & \frac{1}{\lambda}>v>0 \\ 0, & v>\frac{1}{\lambda} \end{cases} \tag{70}$$

This seemingly trivial change is important for the following reasons. In our problem we would like to discuss the behavior of the gravitational fields in a radiating blackhole background. However as we noted above just before $u=0$ there is no black hole and yet we need the ingoing fields to have been moving in a non-minkowski background. This change, therefore, makes it physically possible for us to introduce the external gravitational fields into a spacetime that already contains the black hole. Mathematically the change makes it possible, as we find out soon, to construct complete solutions that include a description of ingoing fields.

Substituting Eq. (68) into (66) and making use of Eq. (64) we find that

$$\begin{aligned} &\left(-5\xi\frac{\partial R_2}{\partial\tau}+\xi^2\frac{\partial^2}{\partial\xi}\frac{R_2}{\partial\tau}\right)4\tau^2 m(\tau)+(\xi^5-\xi^4+4\lambda m_0\xi^3)\frac{\partial^2 R_2}{\partial\xi^2} \\ &\quad\times\left(-5\xi\frac{\partial R_2}{\partial\tau}+\xi^2\frac{\partial^2}{\partial\xi}\frac{R_2}{\partial\tau}\right)4\tau^2 m(\tau)+(\xi^5-\xi^4+4\lambda m_0\xi^3)\frac{\partial^2 R_2}{\partial\xi^2} \\ &\quad-(\xi^4-4\xi^3+16\lambda m_0\xi^2)\frac{\partial R_2}{\partial\xi}+\varepsilon_l\xi^2 R_2(\tau,\xi)=0 \end{aligned} \tag{71}$$

5.3.2. *Separability*

We now ask whether Eq. (71) is separable (in the variables τ and ξ) admitting a solution of the form

$$R_2(\tau,\xi)=X_2(\xi)\ Y_2(\tau) \tag{72}$$

where X_2 and Y_2 are each functions of one variable, only. Substituting for R_2 in Eq. (71) yields two ordinary differential equations: a first order differential equation for $Y_2(\tau)$,

$$m\tau^2\left(\frac{1}{Y(\tau)}\frac{dY_2}{d\tau}\right)=-\alpha \tag{73}$$

and a second order ordinary differential equation for $X_2(\xi)$,

$$\xi^2(\xi^2-\xi+4\lambda m_0)\frac{d^2X_2}{d\xi^2}-\xi[\xi^2-4\xi+4\alpha+16\lambda m_0]\frac{dX_2}{d\xi}$$
$$+(\varepsilon_l\xi+20\alpha)\,X_2(\xi)=0 \tag{74}$$

Here, α is an arbitrary separation constant. Its characteristics are discussed in the next section.

5.4. The Function R_{-2}

Following the same approach as above we find that the equation for the function R_{-2} separates into two ordinary differential equations:

$$m(\tau)\,\tau^2\frac{1}{Y_{-2}}\frac{dY_{-2}}{d\tau}=-\gamma \tag{75}$$

and

$$\xi^2(\xi^2-\xi+4m_0\lambda)\frac{d^2X_{-2}}{d\xi^2}-\xi[5\xi^2-4\xi+4\gamma]\frac{dX_{-2}}{d\xi}$$
$$+(\xi^2+2\varepsilon_l\xi+4\gamma)\,X_{-2}=0 \tag{76}$$

The general characteristics of the separation constant γ are not different from those of α and discussed in the next section.

6. SOLUTIONS

We have achieved the separation of the original Eqs. (62) and (63) respectively, into (73), (74), (75) and (76) for the particular case of a linear mass function. In the following sections we shall seek to solve these resulting ordinary differential equations and to discuss the solutions in an attempt to draw some physical information from them.

6.1. The Functions $Y_p(\tau)$

The first order differential equations for $Y_2(\tau)$ and $Y_{-2}(\tau)$ above can be integrated immediately. Thus from Eq. (73) we find that

$$Y_2(v)=\exp\left(-\alpha\int_0^u\frac{dv}{m_0(1-\lambda v)}\right)=e^{(\alpha/\lambda m_0)\ln(1-\lambda v)} \tag{77}$$

Now $0 \leqslant \lambda < 1$ and in fact for most radiating bodies $\lambda \ll 1$. This allows us to expand the logarithmic expression $\ln(1-\lambda u)$ in the solution so that

$$Y_2(v) = e^{-\Omega v \sum_{n=1}^{\infty} 1/(n+1)\, \lambda^n v^{n+1}} \tag{78}$$

where the separation constant now takes the form Ω in which we absorb the Schwarzschild mass m_0

$$\alpha = m_0 \Omega \tag{79}$$

Consider, now, the case in which the background is not radiating. It is clear either from the solution above or from the original differential equation that for such a case the solution reduces to

$$Y_2(v_s) = e^{-\Omega_s v_s} \tag{80}$$

where the subscript s indicates quantities associated with the Schwarzschild geometry. One notices that in such a static background the quantity $Y_2(v_s)$ above constitutes the only time dependent part of the Ψ_0 field. It follows then that to be consistent with the known[3] solutions we should require

$$\Omega_s = i\omega \tag{81}$$

where ω is the frequency of the gravitational waves. This suggests that in the case of the radiating background we should expect the parameter Ω to be a complex function of λ and ω such that

$$\lim_{\lambda \to 0} \Omega(\lambda, \omega) = i\omega \tag{82}$$

The integration of the differential equation for $Y_{-2}(v)$ follows the same trend and we find that

$$Y_{-2}(v) = e^{-\Gamma v} e^{-\Gamma \sum_{n=1}^{\infty} 1/(n+1)\, \lambda^n v^{n+1}} \tag{83}$$

where

$$\gamma = m_0 \Gamma \tag{84}$$

Γ being a complex function of λ and ω such that

$$\lim_{\lambda \to 0} \Gamma = i\omega \tag{85}$$

As would be expected from the theory of deferential equations the separation constants Ω and Γ can not have unique values. The individual solutions we obtain will therefore be representatives of classes of solutions. The range of these solutions is described in terms of the frequency spectrum of the gravitational field which, in our classical treatment, takes on continuous values. It will, later, be shown that by using certain conditions on the solutions the functional form of these separation constants can be more rigidly fixed.

6.2. The Functions $X_p(\xi)$

Following the integrations of the first order differential equations for Y_p we are now left with the two equations for X_2 and X_{-2} to solve. These are respectively,

$$\xi(\xi^2-\xi+4\lambda m_0)\frac{d^2X_2}{d\xi^2}-\xi[\xi^2-4\xi-4m_0(4\lambda-\Omega)]\frac{dX_2}{d\xi}$$
$$+(\varepsilon_l\xi-20m_0\Omega)\,X_2(\xi)=0 \tag{86}$$

and

$$\xi^2(\xi^2-\xi+4m_0\lambda)\frac{d^2X_{-2}}{d\xi^2}-\xi[5\xi^2-4\xi+4m_0\Gamma]\frac{dX_{-2}}{d\xi}$$
$$+(\xi^2+2\varepsilon_l\xi+4m_0\Gamma)\,X_{-2}==0 \tag{87}$$

It is clear that at $\xi=0$ (or $r=\infty$) both the equations above have regular singularities.[16] This encourages us to seek for analytic solutions. Such solutions at $\xi=0$ ($r=\infty$) should be useful in discussing the asymptotic fall-offs of the fields and the question of energy flux.

6.2.1. *The Peeling Behavior*

Our initial goal is to develop asymptotic solutions for the functions $X_2(\xi)$ and $X_{-2}(\xi)$. Consider a zero rest mass spin-s field ψ_p in a helicity state p. According to the peeling theorem by Roger Penrose,[6] the quantities $r^{(s+p+1)}\psi_p$ and $r^{(s-p+1)}\psi_p$ have a limit at null-infinity. In the case of gravitational fields we expect the outgoing components of the solutions to fall off as

$$\psi_{(p=\pm2)}\sim\frac{1}{r^{(s+p+1)}}=\frac{1}{r^{(2\pm2+1)}} \tag{88}$$

while the ingoing solutions should fall off as

$$\psi_{(p=\pm 2)} \sim \frac{1}{r^{(s-p+1)}} = \frac{1}{r^{(2\mp 2+1)}} \tag{89}$$

It is necessary, therefore, that the solutions to our differential equations display the above asymptotic behavior. This, indeed, will be one of the tests for their validity.

6.2.2. The Indicial Equations

It has been pointed out that at $\xi = 0$ we have a regular singularity in both Eqs. (86) and (87). Therefore it seems natural to attempt developing solutions about this point. Such solutions will be valid at far distances from the black hole. This class of solutions at such distances is useful if one is to engage, as we shall later, in a meaningful discussion of the gravitational energy flux.

Let us assume that Eq. (86) admits, as a solution, a series expansion about $\xi = 0$ of the form

$$X_2(\xi) = \sum_{n=0}^{\infty} a_n \xi^{n+k} \tag{90}$$

where k is some value to be determined. Using Eq. (90) in Eq. (86) gives

$$\begin{aligned}
&\sum_{n=0}^{\infty} a_n(n+k)(n+k-2)\, \xi^n + k + 2 \\
&\quad - \sum_{n=0}^{\infty} a_n[(n+k)(n+k-5) - \varepsilon_l]\, \xi^{n+k+1} \\
&\quad + \sum_{n=0}^{\infty} 4a_n\{(n+k)[(n+k-5)\, \lambda_m - m_0\Omega] + 5m_0\Omega\}\, \xi^{n+k} = 0
\end{aligned} \tag{91}$$

For $n = 0$, $a_0 \neq 0$ we get the indicial equation

$$k[(k-5)\, \lambda - \Omega] + 5\Omega = 0 \tag{92}$$

which has two distinct roots,

$$k = \left(5, \frac{\Omega}{\lambda}\right) \tag{93}$$

Similarly, for $X_{-2}(\xi)$, we can assume a solution of the form

$$X_{-2}(\xi) = \sum_{n=0}^{\infty} b_n \xi^{n+s} \tag{94}$$

where, again s is some value to be determined. Substituting Eq. (94) into Eq. (87), we find that

$$\begin{aligned}
&\sum_{n=2}^{\infty} b_{n-2}[(n+s-2)(n+s-8)+1]\, \xi^{n+s} \\
&\quad - \sum_{n=1}^{\infty} b_{n-1}[(n+s-1)(n+s-6)+2\varepsilon_l]\, \xi^{n+s} \\
&\quad + 4m_0 \sum_{n=0}^{\infty} b_n[\lambda(n+s)(n+s-1) - \Gamma(n+s) + \Gamma]\, \xi^{n+s} = 0
\end{aligned} \tag{95}$$

From the equation above and the condition that $b_0 \neq 0$ we obtain, for $n = 0$, the indicial equation

$$s^2 - \left(1 + \frac{\Gamma}{\lambda}\right) s + \frac{\Gamma}{\lambda} = 0 \tag{96}$$

whose roots are

$$s = \left(1, \frac{\Gamma}{\lambda}\right) \tag{97}$$

Equations (93) and (97) (and the fact that Ω and Γ are complex quantities) indicate that we can expect two linearly independent solutions for each of the fields.

Asymptotic Conditions. The solutions to the indicial equations given in Eqs. (93) and (97) fix for us the leading terms for the functions $X_2(\xi)$ and $X_{-2}(\xi)$, respectively. Thus from Eq. (93) we see that

$$X_2(\xi) \sim \xi^5 \qquad \text{or} \qquad \xi^{\Omega/\lambda} \tag{98}$$

and from Eq. (97)

$$X_{-2}(\xi) \sim \xi \qquad \text{or} \qquad \xi^{\Gamma/\lambda} \tag{99}$$

Both Eqs. (98) and (99) show that the first solutions axe consistent with the peeling theorem and can, in fact be recognized as outgoing fields (recall $\xi = 2m(v)/r = 2m_0(1 - \lambda v)/r$).

On the other hand the second solutions are scaled by the quantities Ω and Γ, respectively. These are the same arbitrary separation constants which, in the last section, we showed to be complex. Since, physically, our solutions represent gravitational fields these constants must now be chosen to conform with the known boundary values for such ingoing waves. Consequently, in order to satisfy the peeling theorem, it is clear that we must have Re $\Omega \sim \lambda$, so that $X_2 \sim 1/r$ and Re $\Gamma \sim 5\lambda$, so that $X_{-2} \sim 1/r^5$. Moreover, the imaginary parts of these quantities must reduce to the limiting cases, $\lim_{\lambda\to 0} \Omega = i\omega$ and $\lim_{\lambda\to 0} \Gamma = i\omega$ as was shown to be the case. These two conditions dictate that we set

$$\Omega = \lambda + i\omega \tag{100}$$

and

$$\Gamma = 5\lambda + i\omega \tag{101}$$

The roots to the indicial Eqs. (93) and (97), respectively, now become

$$k = \left[5, \left(1 + \frac{i}{\lambda}\omega\right)\right] \tag{102}$$

and

$$s = \left[1, \left(5 + \frac{i}{\lambda}\omega\right)\right] \tag{103}$$

So that as $\xi \to 0$,

$$X_2(\xi) \to \xi^5 \qquad \text{or} \qquad X_2(\xi) \to \xi\xi^{(i/\lambda)\,\omega} \tag{104}$$

and

$$X_{-2}(\xi) \to \xi \qquad \text{or} \qquad X_{-2}(\xi) \to \xi^5\xi^{(i/\lambda)\,\omega} \tag{105}$$

The full functions $X_2(\xi)$ are readily obtained by writing down recurrence relations using Eqs. (91) and (95). The general solutions are, in each case, found to be linear combinations of the outgoing component $X_p^{(out)}$ and the ingoing component $X_p^{(in)}$.

Thus

$$X_p(\xi) = A_p X_p^{(out)} + B_p X_p^{(in)} \tag{106}$$

Here the A_p and B_p are arbitrary constants of integration. These solutions are also found to converge.

7. THE ASYMPTOTIC SOLUTIONS AND PHYSICAL INFORMATION

7.1. Significance

The principal aim of our study is to understand how gravitational waves are scattered by a background radiating spacetime. In particular, we axe interested in the measurable physical results of this process, such as the energy flux and the manner in which the waves are reflected and absorbed by a radiating black hole. To this end we have, in the preceding discussions, developed field equations that describe the effects of these waves on the back-ground spacetime. The physical quantities that we seek should, in principle, be calculated from the solutions of these equations. As can easily be shown, however, the series solutions obtainable are a result, in each case, of a three term recursion relation and so contain various coefficients that are not easy to relate. This feature of our solutions would seem to make inconvenient, their use in calculating a number of other physical quantities. It turns out, though, that for the features of our interest it is sufficient to consider the form of the solutions at certain special points. For example Chandrasekhar[7] shows that a knowledge of the incident and reflected wave amplitudes can be deduced from the form of the solution at null-infinity. Moreover, one can also engage in a meaningful discussion pertaining to energy flux at these points. This means that we need only consider the leading terms in the solutions.

7.2. The Source Terms

In creating the function Ψ_4 we have, so far, only considered solutions for the homogeneous part of the original differential equation (see Eqs. (56) and (57)). However, the full equation for Ψ_4 is inhomogeneous so that the complete solution should contain a contribution due to the sources. We recall that the source term is scaled by the luminosity $L = -dm/dv = \lambda m_0$ which obviously vanishes as the background radiation is switched off, $\lambda \to 0$. Since, in the first place, $\lambda \ll 1 \Rightarrow \dot{m}(v)/3m(v) \ll 1$, we shall presently assume that at large r values the source terms do not contribute significantly to the solution. Consequently we shall consider the asymptotic solutions from the homogeneous equation to be a sufficient representation of the general asymptotic solutions. With this we now write down the asymptotic form of the entire solutions.

7.3. The Solutions

It was shown, in Eq. (78), that for Ψ_0,

$$Y_2(u) = e^{-(\Omega/\lambda)\ln(1-\lambda v)} = e^{-\Omega u} e^{-\sum_{n=1}^{\infty} 1/(n+1)\, \lambda^n v^{n+1}} \tag{107}$$

For small λ values, the expression for the logarithmic expansion can be written to first order in λ. This gives

$$Y_2(u) = e^{(\Omega/\lambda)\ln(1-\lambda v)} \simeq e^{-\Omega v} \tag{108}$$

And using the conditions spelled in Eq. (100) to satisfy the peeling property for the ingoing field, we find that

$$Y_2(u) \simeq e^{-\lambda v - i\omega v} \tag{109}$$

Similarly, going through the same treatment for $Y_{-2}(u)$ and applying Eq. (101), we find that

$$Y_{-2}(u) = e^{-5\lambda v - i\omega v} \tag{110}$$

The above results along with the angular solutions ${}_pY_l^m(\theta, \varphi) = {}_pS_l^m(\theta)\, e^{im\varphi}$ of Eq. (60) can now be joined to the functions of Eqs. (104) and (105) (recall $\xi = 2m(v)/r = 2m_0(1-\lambda v)/r$) to give the following asymptotic solutions for the functions Ψ_0 and Ψ_4:

	outgoing	*ingoing*
$\Psi_0 \sim$	${}_2Y_l^m(\theta, \varphi)\, \dfrac{[2m(u)]^5}{r^5} \times e^{-(L/m_0)\, v} e^{-i\omega v}$	${}_2Y_l^m(\theta, \varphi)\, \dfrac{2m(u)}{r}\, e^{-(L/m_0)\, v} e^{ip} e^{-2i\omega v}$
$\Psi_4 \sim$	${}_{-2}Y_l^m(\theta, \varphi)\, \dfrac{2m(u)}{r} \times e^{-5(L/m_0)\, v} e^{-i\omega v}$	${}_{-2}Y_l^m(\theta, \varphi)\, \dfrac{[2m(u)]^5}{r^5}\, e^{-5(L/m_0)\, v} e^{ip} e^{-2i\omega v}$

(111)

Here, $p(r) = (\omega/\lambda)\ln(2m_0/r)$ and where for physical reasons we find it useful to express the solutions in terms of the luminosity L as given by $L = -dm/dv = \lambda m_0$.

8. CONCLUSIONS

We have obtained analytic solutions to the problem of gravitational fields propagating in a radiating spacetime. These solutions satisfy all the known conditions for the propagation of spin-s, zero rest mass fields. From their asymptotic form in Eqs. (111) it is seen that the solutions are completely consistent with the peeling theorem of Penrose and fall off in the manner predicted by this theorem. Moreover, one observes, further, that as the background radiation is switched off (i.e., in the limit $\lambda \to 0$), the theory recovers the known solutions (see, for example, Ref. 3) for the perturbed static geometry of Schwarzschild. We consider the passing of these two tests a validation of our analysis.

One new significant feature this analysis brings to surface is that the solutions (see Eq. (111)) are scaled by factors of the form $e^{-(s+p+1)(L/m_0)\,u}$, where $s = 2$ and $p = \pm 2$. But L/m_0 is positive definite. Consequently, these factors indicate that when gravitational fields propagate in a radiating spacetime they suffer an attenuation, and this attenuation can be quantitatively described. The attenuation weight seems in turn to be directly related to the spin weight of the perturbed fields. It is also scaled by the luminosity L of the background. Further, as one notices from the solutions, the attenuation persists independent of whether the fields are ingoing or outgoing. It is of course fair to ask whether this character of our solutions is not, in the first place, a reflection of the mass function that we chose. Recalling that general radiative mass function $m(v)$ is a monotonic decreasing function in v an expansion of the $m(v) = m_0 - Lm_0 - (dL/2!\,dv)\,m_0 - \cdots$ about $v = 0$ indicates that the first order term in the luminosity would seem to make the significant contribution. This seems to suggest that the attenuations manifested in our solutions are independent of the manner in which the blackhole radiates and may persist for any mass function chosen. As far as we know this seems to be a new feature in the literature of this branch of general relativity; one that may, indeed, have some interesting astrophysical implications.

A persistent attenuation of this sort would seem to suggest the possibility that energy is being dumped into the host spacetime. Over large time scales, this could have significant implications on the evolution of such a radiating system. This question can, however, only be resolved by a rigorous calculation of the energy flux. For such a calculation and an extension of this discussion see Ref. 18. We intend to follow up this issue in future discussions.

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